Algebraic properties of some statistics on permutations

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Motivation and goals

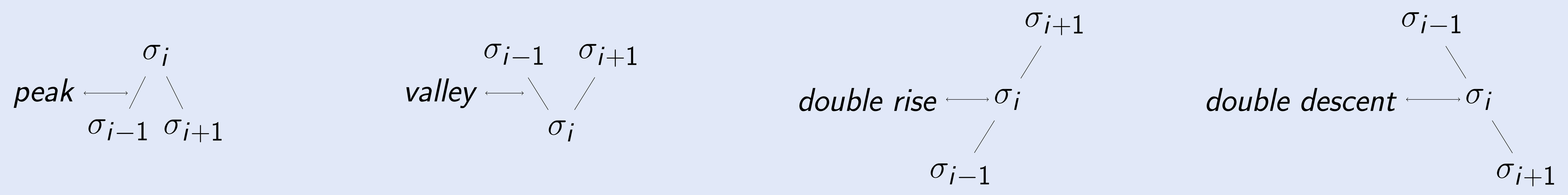
In the combinatorial approach of Laguerre polynomials, some statistics on permutations arise naturally, and they have nice interpretations in terms of Laguerre histories and increasing binary trees. On the other hand, there exists a combinatorial algebra **FQSym** whose basis elements are given by permutations. Thanks to these statistics, one can define some equivalence relations on permutations. For example, Françon and Viennot build an equivalence relation thanks to the peaks, valleys, double rises and double descents on permutations.

- Does these statistics have nice interpretations in **FQSym**?
- If so, is it related to others known combinatorial algebras?

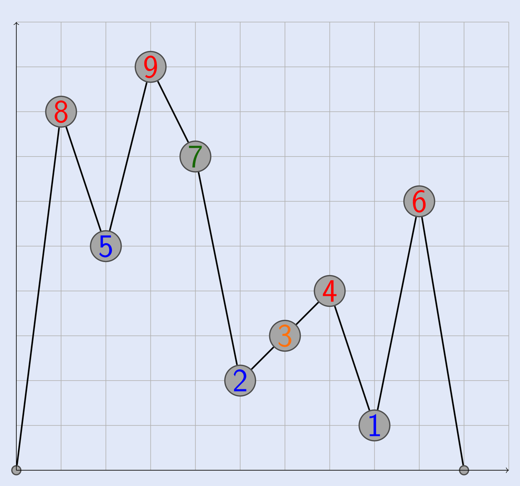
1. Bijection, combinatorial objects and statistics

Definition of some statistics on permutations

Let σ be in $\mathfrak{S}_n$. By convention, $\sigma_0 = 0$, and $\sigma_{n+1} = 0$. Let i be a position between 1 and n. The value σ_i is a:



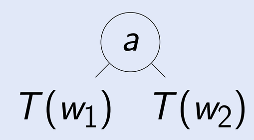
For $\sigma = 859723416$:



$$P(\sigma) = \{8, 9, 4, 6\}$$
$$V(\sigma) = \{5, 2, 1\}$$
$$Dr(\sigma) = \{3\}$$
$$Dd(\sigma) = \{7\}$$

Increasing binary trees

Let w be a word without repetition on an alphabet A totally ordered. If w is empty, the associated increasing tree is the empty tree. Otherwise, let a be the smallest letter. We have: $w = w_1 a w_2$. So we build recursively the associated increasing binary tree of w as follow:

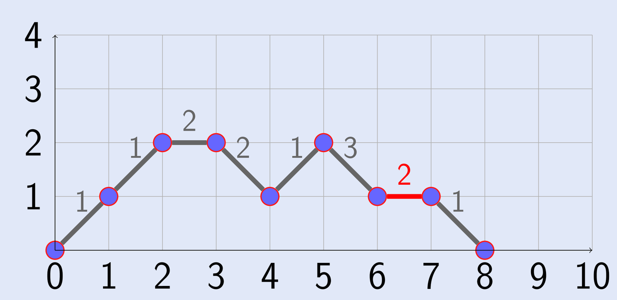


Bijection of Françon-Viennot [FV79] by example

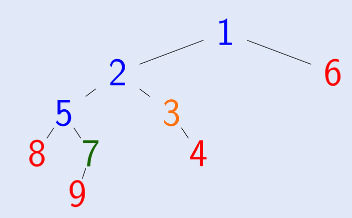
A Laguerre history is a valued two-colored Motzkin path where the i-th step has a weight between 1 and the heigth plus one of the current position (we begin at position $(0, 0)$). Let us describe the bijection by an algorithm. The input is a Laguerre history of size $n - 1$, and the output a permutation of size n. Let us begin with the word $w = \infty$. One reads a Laguerre history h from left to right, and for each step transforms w as follows: if t is the type of the i-th step of h, and j its weight, the j-th ∞ is replaced by:

$$\begin{cases} i & \text{if } t \text{ is a south-east step} \\ \infty i \infty & \text{if } t \text{ is a north-east step} \\ i\infty & \text{if } t \text{ is an east step} \\ \infty i & \text{if } t \text{ is a red east step} \end{cases}$$

Finally, replace the last ∞ by n.



Increasing binary trees and statistics [Fla80]: example



peak ⟷ leaf
valley ⟷ node with two children
double rise ⟷ node with a right child
double descent ⟷ node with a left child

2. Algebraic background and combinatorics

The algebra FQSym [DHT02]

FQSym is a graded algebra whose components of weight n have dimensions $n!$ and the bases are indexed by permutations. The product on the basis $\mathbf{F}_\sigma$ is given by the shifted shuffle:

$$\mathbf{F}_\sigma \mathbf{F}_\tau = \sum_{s \in \sigma \Cup \tau} \mathbf{F}_s.$$

For example, if $\sigma = 3142$, and $\tau = 42135$, we have:

$$\mathbf{F}_{3142} \cdot \mathbf{F}_{42135} = \mathbf{F}_{314286579} + \mathbf{F}_{314862579} + \mathbf{F}_{314865279} + \cdots + \mathbf{F}_{318654792} + \cdots + \mathbf{F}_{865793142}.$$

For $s = 4213$, and $\tau = 42135$, we have:

$$\mathbf{F}_{4213} \cdot \mathbf{F}_{42135} = \mathbf{F}_{428657913} + \mathbf{F}_{486257913} + \mathbf{F}_{486527913} + \cdots + \mathbf{F}_{865479213} + \cdots + \mathbf{F}_{421865793}.$$

The permutations σ and s have the same peaks, valleys, double rises, and double descents. We see this in their increasing binary trees:

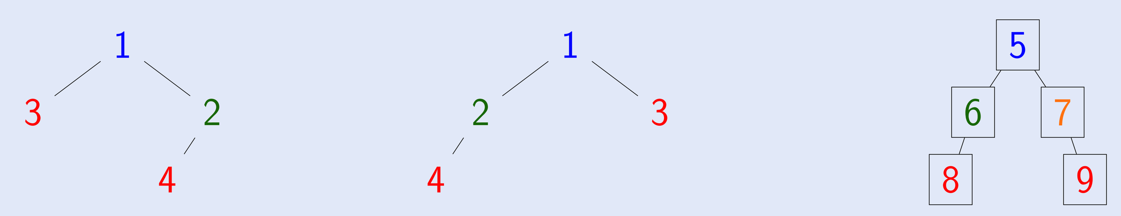
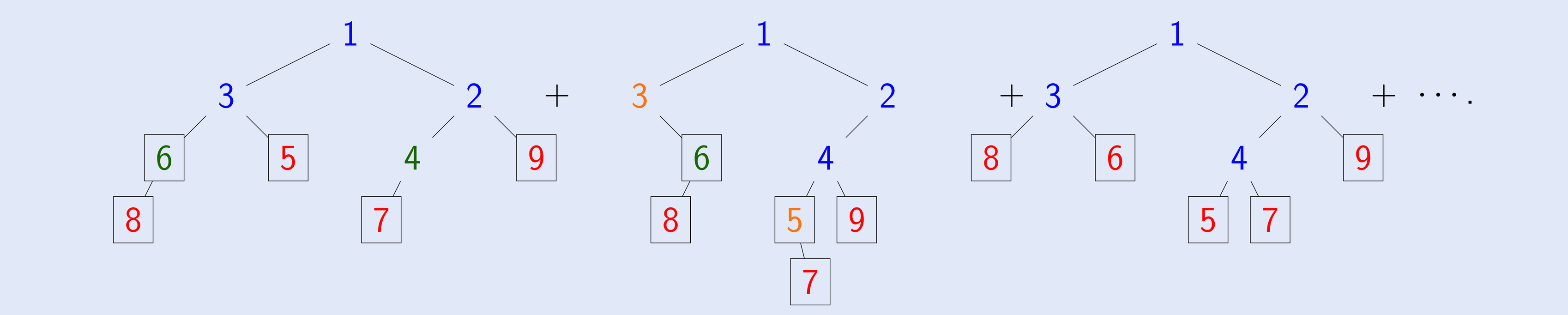


Figure: Increasing binary trees of σ, s, and τ.

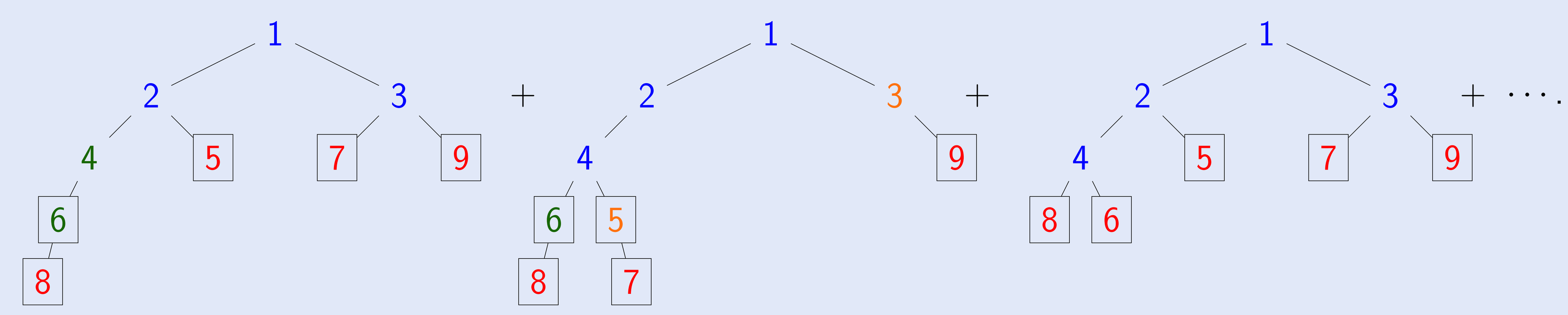
What happens when shuffling these elements with τ?

Increasing binary trees of a product and graft

Some increasing binary trees coming from $\sigma \Cup \tau$:



Some increasing binary trees coming from $s \Cup \tau$:



Observations:

- For each tree from the first product there exists a unique tree in the second such that each label has the same number of right and left children,
- the tree of τ is cut in the same way in both trees.

3. Sketch of proof by example

We set $\sigma_1 = 52341$, $\sigma_2 = 35241$, $s = 3421$, and $\sigma = 859723416$ in $\sigma_1 \Cup s$. Observe that σ_1 and σ_2 have the same four statistics. Let us build σ' in $\sigma_2 \Cup s$ such that σ' and σ have the same four statistics.

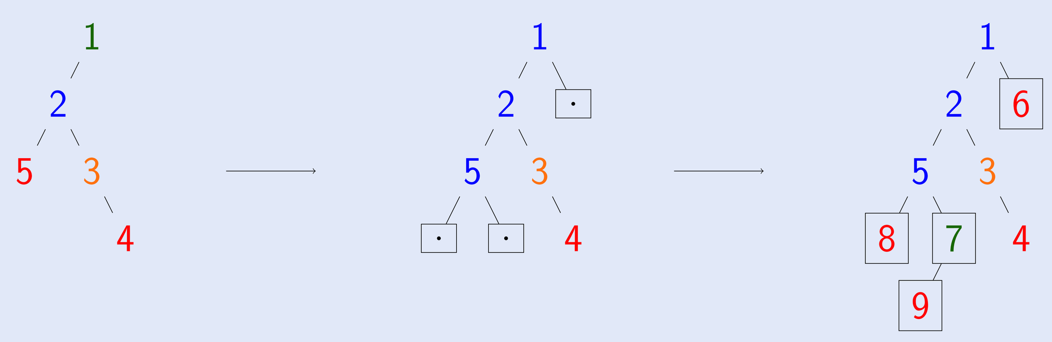


Figure: Construction of the increasing binary tree of σ from σ_1 and s.

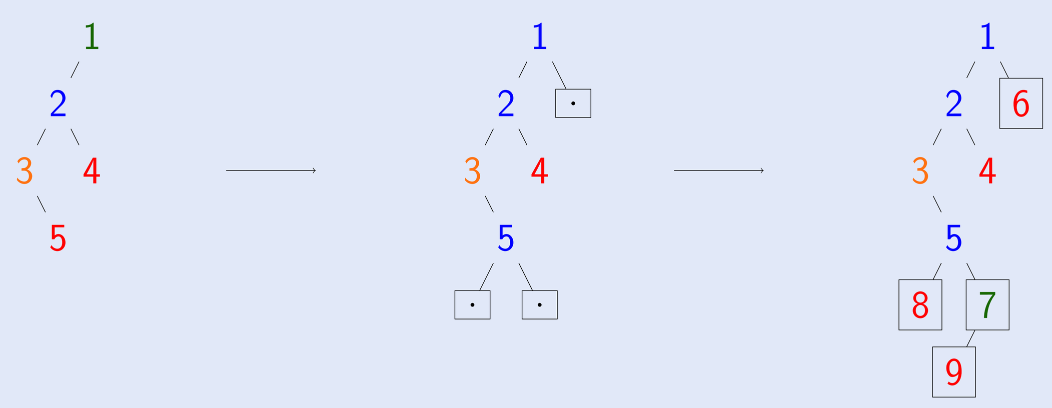


Figure: Construction of the increasing binary tree of σ' from σ_2 and s.

4. Results

Algebraic interpretation

We deduce from the previous part that if two permutations σ_1 and σ_2 have the same four statistics, then for any s there exists a bijection ϕ from $\sigma_1 \Cup s$ to $\sigma_2 \Cup s$ such that an element and its image have the same four statistics. In algebraic term, it means the equivalence relation "having the same four statistics" is a right ideal. Now, by slightly modifying the proof, we obtain that it is in fact an ideal, so we build an quotient of **FQSym**. Other quotients are built in the same way by considering different equivalence relations thanks to the statistics.

The different quotients and their properties

quotient by	dimensions	quotient algebras	free algebras
(P, V, Dr, Dd)	C_n	yes	yes
(P,V, Dr ∪ Dd)	M_{n-1}	yes	yes
(P,V ∪ Dr ∪ Dd)	$\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$	no	no
(P ∪ V ∪ Dr, Dd)	2^{n-1}	no	no
(P ∪ V, Dr, Dd)	$\frac{3^{n-1}+1}{2}$	no	no
(P, Dr, V ∪ Dd)	A_{n-1}	no	no
(P ∪ V, Dr ∪ Dd)	2^{n-2}	no	no
(P ∪ Dd, V ∪ Dr)	2^{n-1}	yes	yes
(P ∪ V ∪ Dr ∪ Dd)	1	yes	yes

References

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