# Combinatorics of Genocchi numbers using non-commutative methods

Vincent Vong

September 13, 2014

**Abstract**

In this article, we generalize the Gandhi polynomials in a non-commutative way. We show that surjective pistols arise naturally from our non-commutative polynomials.This method also enables us to explain the combinatorial interpretations of different generalizations of the Gandhi polynomials.

## 1 Introduction

The Genocchi numbers $(G_{2n})_{n\in\mathbb{N}}$, Taylor coefficients of the exponential series $x\tan(\frac{x}{2})$, have been studied in several ways. First, their relations with the Euler numbers (coefficients of $\tan(x)$) have been studied by Viennot in [Vie82] who gave an overview of their combinatorial properties. But they can also be studied from one of their refinements, the Gandhi polynomials. Gandhi in [Gan70] conjectured that the sequence of polynomials defined by $C_0 = 1$ and $C_{n+1}(x) = x^2C_n(x+1) - (x-1)^2C_n(x)$ are polynomials such that $C_n(1) = G_{2n+2}$. This statement was proved independently by Carlitz in [Car71], by Riordan and Stein in [RS73]. Later, these polynomials were generalized by Dumont and Foata in [DF76] as follows: $DF_1 = 1$, and $DF_{n+1}(x, y, z) = (x+z)(y+z)DF_n(x, y, z+1) - z^2DF_n$. In order to give a combinatorial interpretation of these, they introduced the surjective pistols, surjective maps $p$ from $\{1, \cdots, 2n\}$ onto $\{2, 4, \cdots, 2n\}$ such that $p(i) \geq i$, one of the combinatorial objects enumerated by the Genocchi numbers. They showed that the different parameters $x$, $y$, and $z$ correspond to different statistics on surjective pistols. Another interesting property of these polynomials is their symmetry on the three variables $x$, $y$, $z$. It has been proved by Dumont and Foata in [DF76], and Carlitz gave an explicit symmetric formula in [Car80]. Han in [Han96] provided another combinatorial interpretation of these polynomials, and found back the interpretation of Dumont and Foata. Then, Dumont defined a generalization with six parameters. Independently, Zeng and Randrianarivony respectively in [Zen96] and [Ran94] provided different properties about these. Finally, Han and Zeng in [HZ99] showed that the Gandhi polynomials admit a $q$-analog, and provided several combinatorial interpretations of their coefficients.

In this paper, we use the non-commutative paradigm to find back the combinatorial interpretations of Gandhi polynomials and their generalizations in terms of surjective pistols. We will see that the surjective pistols arise naturally from our non-commutative definition of Gandhi polynomials. Moreover, we obtain new interpretations of the $q$-analog.

## 2 The Gandhi polynomials and their different generalizations

In this section, we recall the definition of the Gandhi polynomials and different generalizations as the Dumont-Foata polynomials, and the polynomials with six parameters given by Dumont in [Dum95].

Instead of the original definition of the Gandhi polynomials, we introduce an equivalent definition of those. Then we give a non-commutative version of these polynomials and justify their relation with the surjective pistols, which are one of the combinatorial interpretations of the Gandhi polynomials [DF76]. Finally, we get back to the different combinatorial interpretations obtained in [DF76], [Han96], and [Ran94] about Gandhi polynomials, the Dumont-Foata polynomials and the six-parameter polynomials.

## 2.1 The Gandhi polynomials

Let us define the Gandhi polynomials, denoted by $(C_n)_{n\geq 1}$, as follows:

$$\begin{cases} C_1(x) &= 1 \\ C_{n+1}(x) &= \Delta(C_n(x)\cdot x^2) \quad \text{if } n\geq 1, \end{cases} \tag{1}$$

where $\Delta(f(x)) = f(x+1) - f(x)$, for any function $f$.

**Example 1.** For $n=1,\ 2,\ 3$, we have:

$$\begin{aligned} C_1(x) =&\ \ 1 \\ C_2(x) =&\ \ 2x+1 \\ C_3(x) =&\ \ 6x^2+8x+3. \end{aligned} \tag{2}$$

It is known from [DF76], that these polynomials have a combinatorial interpretation in terms of surjective pistols, which are defined by:

**Definition 2.1.1.** A *surjective pistol* $p$ of size $2n$ is a surjective map from $\{1,\cdots,2n\}$ onto $\{2,4,\cdots,2n\}$ such that for each $i$ in $\{1,\cdots,2n\}$, $p(i)\geq i$.

The set of surjective pistols of size $2n$ is denoted by $\mathcal{P}_{2n}$. From now on, a surjective pistol $p$ of size $2n$ is represented by a word $w$ of size $2n$ such that the $i$-th letter of $w$ is the value $p(i)$.

**Example 2.** The word $w = 226466$ represents the surjective pistol of size 6 sending 1 to 2, 2 to 2, 3 to 6, 4 to 4, 5 to 6 and 6 to 6.

The name pistol comes from one of its graphical representation:

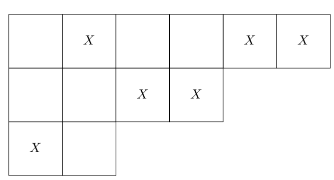

Figure 1: Graphical representation of the surjective pistol $p = 264466$.

Note that from a surjective pistol $p$ of size $2n$, we obtain a surjective pistol of size $2n+2$ by concatenating twice $2n+2$ to the right of $p$. This operation essentially mimics the multiplication by $x^2$ on the induction formula of Gandhi polynomials (1). We cannot obtain all surjective pistols of size $2n+2$ by this process. However if we substitute in the word $p\cdot(2n+2)\cdot(2n+2)$, some but not all values $2n$ by $2n+2$, we still have a surjective pistol. For example, if $p = 226466$, then 22646688, 22846688, 22648688, 22646888, 22848688, 22846888 and 22648888 are indeed surjective pistols. The substitution process essentially mimics the finite difference on words. Indeed, as we shall see, the set $\mathcal{P}_{2n+2}$ is generated in this way, each surjective pistol appearing exactly once in this construction.

The non-commutative polynomials will help us formalize the previous ideas. We proceed as follows: we build a non-commutative analog of the Gandhi polynomials and see that surjective pistols arise naturally from these.

**Definition 2.1.2.** Let $A$ be an alphabet. The $R-$algebra of *non-commutative polynomials* over $A$, denoted by $R<A>$, is the algebra generated by the elements of $A$ equipped with the concatenation product. In particular a linear basis is given by the finite words on $A$.

From now on, our alphabet is $A = \{a_i, \text{ for } i \in \mathbb{N}^*\} \cup \{a_\infty\}$, and $R = \mathbb{R}$. Let us give a sense to "having a non-commutative analog". There is a natural projection $\Pi$ from $\mathbb{R}<A>$ onto $\mathbb{R}[x]$:

$$\begin{cases} \Pi(a_\infty) & = & x, & \\ \Pi(a_i) & = & 1, & \text{for } i \in \mathbb{N}^*, \\ \Pi(w_1 \cdots w_n) & = & \Pi(w_1) \cdots \Pi(w_n). & \end{cases} \quad (3)$$

Then an element $\mathbf{P}$ of $\mathbb{R}<A>$ is a non-commutative analog of a polynomial $P$ if $\Pi(\mathbf{P})$ is equal to $P$. In the same way, if $L$ is a linear operator on $\mathbb{R}[x]$, we say that $\mathbf{L}$ is a non-commutative analog of $L$, if $\Pi \circ \mathbf{L}$ is equal to $L \circ \Pi$. Since the Gandhi polynomials are built from the operator $\Delta$, one way to have non-commutative Gandhi polynomials is to construct these from non-commutative analogs of $\Delta$. Since the operator $\Delta$ is related the operator consisting in shifting by 1 (denoted by $T$), it is natural to build first the non-commutative analog of $T$. Let us denote by $\mathbf{T}_i$ the algebra homomorphism such that:

$$\begin{cases} \mathbf{T}_i(a_\infty) & = & a_\infty + a_i, & \\ \mathbf{T}_i(a_j) & = & a_j, & \text{for j in } \mathbb{N}^*. \end{cases} \quad (4)$$

For each positive integer $i$, we have indeed

$$T \circ \Pi = \Pi \circ \mathbf{T}_i. \quad (5)$$

Then, we define $\mathbf{\Delta}_i$ by $\mathbf{\Delta}_i = \mathbf{T}_i - Id_{\mathbb{R}<A>}$.

**Example 3.** For example, if $w = a_2a_2a_\infty a_4a_\infty a_\infty$, then we have:

$$\begin{aligned} \mathbf{\Delta}_6(w) \quad = \quad & a_2a_2a_6a_4a_\infty a_\infty + a_2a_2a_\infty a_4a_6a_\infty + a_2a_2a_\infty a_4a_\infty a_6 \\ & + a_2a_2a_6a_4a_6a_\infty + a_2a_2a_\infty a_4a_6a_6 + a_2a_2a_6a_4a_\infty a_6 + a_2a_2a_6a_4a_6a_6. \end{aligned} \quad (6)$$

Note that for a word $w$, the terms of $\mathbf{\Delta}_i(w)$ are exactly the words obtained by replacing at least one of the $a_\infty$ of $w$ by an $a_i$. Now, let us define the non-commutative Gandhi polynomials as follows:

$$\begin{cases} \mathbf{C}_1 & = & 1 \\ \mathbf{C}_{n+1} & = & \mathbf{\Delta}_{2n}(\mathbf{C}_n a_\infty a_\infty) \quad \text{if } n > 1. \end{cases} \quad (7)$$

In particular, we have:

$$\begin{aligned} \mathbf{C}_2 \quad = \quad & a_2a_2 + a_2a_\infty + a_\infty a_2 \\ \mathbf{C}_3 \quad = \quad & a_2a_2a_4a_4 + a_2a_2a_\infty a_4 + a_2a_2a_4a_\infty + a_2a_4a_4a_4 + a_2a_\infty a_4a_4 \\ & + a_2a_4a_\infty a_4 + a_2a_4a_4a_\infty + a_2a_4a_\infty a_\infty + a_2a_\infty a_4a_\infty \\ & + a_2a_\infty a_\infty a_4 + a_4a_2a_4a_4 + a_\infty a_2a_4a_4 + a_4a_2a_\infty a_4 \\ & + a_4a_2a_4a_\infty + a_4a_2a_\infty a_\infty + a_\infty a_2a_4a_\infty + a_\infty a_2a_\infty a_4. \end{aligned} \quad (8)$$

Now, surjective pistols of positive size appear naturally through this definition. Indeed, let us define an embedding $i$ from surjective pistols of positive size onto $\mathbb{R}<A>$. If $p$ is a surjective pistol of size $2n$ (with $n \geq 1$), then $i(p)$ is equal to the word $w$ of size $2n-2$ where the $i$-th letter of $w$ is $a_\infty$ if $p(i)$ is equal to $2n$, and is $a_{p(i)}$ otherwise.

**Example 4.** If $p = 22646688$, then $i(p) = a_2a_2a_6a_4a_6a_6$. If $p = 22$, then $i(p) = \epsilon$, the empty word, and if $p = 28648688$, then $i(p) = a_2a_\infty a_6a_4a_\infty a_6$.

The map $i$ is clearly an injective map: let $w$ be a word of size $2n$ in the image of $i$. We obtain the corresponding surjective pistol by replacing the $a_\infty$ by $a_{2n+2}$, by reading the indices of the new word, and by adding twice to the right the letter $2n+2$. In the sequel, we denote by $\mathbf{p}$ the word $i(p)$. We have

**Proposition 2.1.** *Let $n \geq 1$. Then:*

$$\boldsymbol{C}_n = \sum_{p\in\mathcal{P}_{2n}} \mathbf{p}. \tag{9}$$

*Proof.* By induction on $n$. The statement is satisfied for $n = 1$. Assume that it is true for $n \geq 1$. By induction hypothesis on $\mathbf{C}_n$, we have:

$$\begin{aligned}\mathbf{C}_{n+1} &= \mathbf{\Delta}_{2n}(\mathbf{C}_n a_\infty a_\infty)\\ &= \mathbf{\Delta}_{2n}(\textstyle\sum_{p\in\mathcal{P}_{2n}} \mathbf{p}\ a_\infty a_\infty)\\ &= \textstyle\sum_{p\in\mathcal{P}_{2n}} \mathbf{\Delta}_{2n}(\mathbf{p}\ a_\infty a_\infty).\end{aligned}$$

Let $p$ be a surjective pistol of size $2n$. Then the elements of $\mathbf{\Delta}_{2n}(\mathbf{p}\ a_\infty a_\infty)$ are the words associated with surjective pistols of size $2n+2$: they are words $w = w_1 \cdots w_{2n}$ of size $2n$, such that at least one $a_\infty$ of $\mathbf{p}\ a_\infty a_\infty$ is replaced by $a_{2n}$, and such that $w_i = a_{p(i)}$ if $p(i) \leq 2n-2$, and $w_i = a_{2n}$ or $a_\infty$ otherwise. They are then associated with surjective pistols $p'$ of size $2n+2$, where for $i \leq 2n$, if $p(i) \leq 2n-2$, then $p'(i) = p(i)$, and $p'(i) = 2n$ or $2n+2$ otherwise. Conversely, if $p'$ is a surjective pistol of size $2n+2$ such that $p'(i) = p(i)$ if $p(i) < 2n$, and $p'(i) = 2n$ or $p'(i) = 2n+2$ otherwise, then $\mathbf{p'}$ is a term of $\mathbf{\Delta}_{2n}(\mathbf{p}\ a_\infty a_\infty)$.

Let us denote by $\mathcal{D}_p$ the set of pistols $p'$ such that $\mathbf{p'}$ is a term of $\mathbf{\Delta}_{2n}(\mathbf{p}\ a_\infty a_\infty)$. By the previous observation, $\mathcal{D}_p$ is in fact the following set:

$$\mathcal{D}_p = \left\{ p' \in \mathcal{P}_{2n+2} \mid \begin{array}{ll} p'(i) = p(i) & \text{if } i < 2n+1 \text{ and } p(i) < 2n, \\ p'(i) = 2n \text{ or } 2n+2 & \text{otherwise.} \end{array} \right\} \tag{10}$$

We have: $\mathcal{P}_{2n+2} = \sqcup_{p\in\mathcal{P}_{2n}} \mathcal{D}_p$. Indeed, let $p'$ be in $\mathcal{P}_{2n+2}$. By deleting the last two letters and replacing the $2n+2$ by $2n$, we obtain a surjective pistol $p$ of size $2n$, and $p'$ is in $\mathcal{D}_p$. So $\mathcal{P}_{2n+2} = \cup_{p\in\mathcal{P}_{2n}} \mathcal{D}_p$. Let now $p$ and $p'$ be two different surjective pistols of size $2n$. Then, there exists an $i$ such that $p(i)$ is different from $p'(i)$. By symmetry, we assume that $p(i) < p'(i) \leq 2n$. Then for each element $q$ in $\mathcal{D}_p$, we have $q(i) = p(i)$, and for each $q'$ in $\mathcal{D}_{p'}$, we have $q'(i) \geq p'(i)$. In particular, $q(i) < q'(i)$, so $\mathcal{D}_p \cap \mathcal{D}_{p'}$ is empty. We deduce that:

$$\begin{aligned}\mathbf{C}_{n+1} &= \textstyle\sum_{p\in\mathcal{P}_{2n}} \mathbf{\Delta}_{2n}(\mathbf{p}\ a_\infty a_\infty)\\ &= \textstyle\sum_{p\in\mathcal{P}_{2n}} \sum_{p'\in\mathcal{D}_p} \mathbf{p'}\\ &= \textstyle\sum_{p'\in\mathcal{P}_{2n+2}} \mathbf{p'}.\end{aligned} \tag{11}$$

□

## 2.2 The Dumont-Foata polynomials

Dumont and Foata defined in [DF76] a generalization of Gandhi polynomials. They proved that these polynomials count different statistics on surjective pistols. Han in [Han96] found other statistics on surjective pistols that gave back the Dumont-Foata polynomials. Moreover, with his method, he got back the result of Dumont-Foata. With the non-commutative method, we give a new direct proof of the previous combinatorial interpretations. In this section, we present the Dumont-Foata polynomials, their non-commutative analogs, and their combinatorial interpretations.

**Definition 2.2.1.** The Dumont-Foata polynomials are defined by induction as follows:

$$\begin{cases} DF_1(x,y,z) & = & 1 \\ DF_{n+1}(x,y,z) & = & DF_n(x+1,y,z)(x+z)(x+y) - DF_n(x,y,z)x^2, \text{ if } n \geq 1. \end{cases} \tag{12}$$

Since $DF_n(1,1,1) = \#\mathcal{P}_{2n}$, the different variables $x$, $y$ and $z$ record some statistics on surjective pistols.

**Definition 2.2.2.** Let $p$ be a surjective pistol of size $2n$. A position $i$ is:
- a *fixed point* if $p(i) = i$ ,
- a *surfixed point* if $p(i) = i+1$,
- a *max point* if $p(i) \geq p(j)\ \forall j \in \{1, \cdots, 2n\}$,
- a *saillance point* if for any $j < i$, then $p(j) < p(i)$, and $i$ is not a max point.

If $p$ is size of $2n$, we denote by *fix*$(p)$ (*resp.* *surfix*$(p)$, *max*$(p)$, *sai*$(p)$) the number of fixed (*resp.* surfixed, max, saillance) points of $p$ smaller than $2n-1$.

**Example 5.** If $p = 42468688$, 2 and 6 are fixed points, 3 is a surfixed point, 1 and 4 are saillance points, and 5 is a max point. And we have: *fix*$(p) = 2$, *surfix*$(p) = 1$, *sai*$(p) = 2$, and *max*$(p) = 1$.

**Theorem 2.2.** *[DF76] The Dumont-Foata polynomials have the following combinatorial interpretations:*

$$DF_n(x,y,z) = \sum_{p \in \mathcal{P}_{2n}} x^{\max(p)} y^{\mathrm{fix}(p)} z^{\mathrm{surfix}(p)} = \sum_{p \in \mathcal{P}_{2n}} x^{\max(p)} y^{\mathrm{fix}(p)} z^{\mathrm{sai}(p)}. \tag{13}$$

In order to get back Theorem 2.2, we build two non-commutative analogs of the Dumont-Foata polynomials that project naturally onto both combinatorial interpretations. In the sequel, we work with the same alphabet as before, and with the ring $R = \mathbb{Q}[y,z]$.

**Definition 2.2.3.** The non-commutative Dumont-Foata polynomials are defined as follows:

$$\begin{cases} \mathcal{DF}_1 & = 1 \\ \mathcal{DF}_{n+1} & = \mathbf{T}_{2n}(\mathcal{DF}_n)(a_\infty + z a_{2n})(a_\infty + y a_{2n}) - \mathcal{DF}_n a_\infty a_\infty, \quad \text{if } n \geq 1. \end{cases} \tag{14}$$

**Example 6.** The first polynomials are:

$$\begin{aligned} \mathcal{DF}_2 &= yz \cdot a_2 a_2 + z \cdot a_2 a_\infty + y \cdot a_\infty a_2 \\ \mathcal{DF}_3 &= y^2 z^2 \cdot a_2 a_2 a_4 a_4 + y^2 z \cdot a_2 a_2 a_\infty a_4 + yz^2 \cdot a_2 a_2 a_4 a_\infty \\ &\quad + yz^2 \cdot a_2 a_4 a_4 a_4 + yz^2 \cdot a_2 a_\infty a_4 a_4 + yz \cdot a_2 a_4 a_\infty a_4 + z^2 \cdot a_2 a_4 a_4 a_\infty \\ &\quad + z \cdot a_2 a_4 a_\infty a_\infty + z^2 \cdot a_2 a_\infty a_4 a_\infty + yz \cdot a_2 a_\infty a_\infty a_4 \\ &\quad + y^2 z \cdot a_4 a_2 a_4 a_4 + y^2 z \cdot a_\infty a_2 a_4 a_4 + y^2 \cdot a_4 a_2 a_\infty a_4 + yz \cdot a_4 a_2 a_4 a_\infty \\ &\quad + y \cdot a_4 a_2 a_\infty a_\infty + yz \cdot a_\infty a_2 a_4 a_\infty + y^2 \cdot a_\infty a_2 a_\infty a_4. \end{aligned} \tag{15}$$

**Proposition 2.3.** *Let $n \geq 1$. Then we have:*

$$\mathcal{DF}_n = \sum_{p \in \mathcal{P}_{2n}} y^{\mathrm{fix}(p)} z^{\mathrm{surfix}(p)} \mathbf{p}. \tag{16}$$

*Proof.* By induction on $n$. It is true for $n = 1$. Assume that the statement is true for a given $n \geq 1$. By definition of $\mathcal{DF}_{n+1}$, we have:

$$\begin{aligned} \mathcal{DF}_{n+1} &= \mathbf{T}_{2n}(\mathcal{DF}_n)(a_\infty + z a_{2n})(a_\infty + y a_{2n}) - \mathcal{DF}_n a_\infty a_\infty \\ &= \mathbf{\Delta}_{2n}(\mathcal{DF}_n) a_\infty a_\infty + z \mathbf{T}_{2n}(\mathcal{DF}_n) a_{2n} a_\infty \\ &\quad + y \mathbf{T}_{2n}(\mathcal{DF}_n) a_\infty a_{2n} + yz \mathbf{T}_{2n}(\mathcal{DF}_n) a_{2n} a_{2n}. \end{aligned}$$

Let $p$ be a surjective pistol of size $2n$. By using the inverse of $i$, we deduce that:

- The surjective pistols $p'$ corresponding to words in $\mathbf{\Delta}_{2n}(\mathbf{p})a_\infty a_\infty$ are the elements $p'$ of $\mathcal{D}_p$ where $p'(2n-1) = p'(2n) = 2n+2$. Since the elements $p'$ have the same fixed and surfixed points as $p$, we have:

$$\mathbf{\Delta}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_\infty a_\infty = \sum_{\substack{p' \in \mathcal{D}_p,\\ p'(2n-1) = 2n+2,\\ p'(2n) = 2n+2}} y^{\mathit{fix}(p')} z^{\mathit{surfix}(p')}\mathbf{p'}. \tag{17}$$

- The surjective pistols $p'$ associated with the words in $\mathbf{T}_{2n}(\mathbf{p})a_\infty a_{2n}$ are surjective pistols $p'$ in $\mathcal{D}_p$ where $p'(2n-1) = 2n$, and $p'(2n) = 2n+2$. In this case, the elements $p'$ have one more fixed point than $p$. So:

$$y\mathbf{T}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_\infty a_{2n} = \sum_{\substack{p' \in \mathcal{D}_p,\\ p'(2n-1) = 2n+2,\\ p'(2n) = 2n}} y^{\mathit{fix}(p')} z^{\mathit{surfix}(p')}\mathbf{p'}. \tag{18}$$

- The surjective pistols $p'$ corresponding to the words which appeared in $\mathbf{T}_{2n}(\mathbf{p})a_{2n}a_\infty$ are in $\mathcal{D}_p$, and satisfy $p'(2n-1) = 2n$ and $p'(2n) = 2n+2$. Thus, they have one more surfixed point than $p$. Then:

$$z\mathbf{T}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_{2n} a_\infty = \sum_{\substack{p' \in \mathcal{D}_p,\\ p'(2n-1) = 2n,\\ p'(2n) = 2n+2}} y^{\mathit{fix}(p')} z^{\mathit{surfix}(p')}\mathbf{p'}. \tag{19}$$

- The surjective pistols $p'$ corresponding to the words which appeared in $\mathbf{T}_{2n}(\mathbf{p})a_{2n}a_{2n}$ are in $\mathcal{D}_p$, and satisfy $p'(2n-1) = 2n$ and $p'(2n) = 2n$. Thus, they have one more fixed and surfixed point than $p$. Then:

$$yz\mathbf{T}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_{2n} a_{2n} = \sum_{\substack{p' \in \mathcal{D}_p,\\ p'(2n-1) = 2n,\\ p'(2n) = 2n}} y^{\mathit{fix}(p')} z^{\mathit{surfix}(p')}\mathbf{p'}. \tag{20}$$

By summing all Equations (17), (18), (19), (20), we get:

$$\begin{aligned}\textstyle\sum_{p'\in\mathcal{D}_p} y^{\mathit{fix}(p')} z^{\mathit{surfix}(p')}\mathbf{p'} \quad = \quad & \mathbf{\Delta}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_\infty a_\infty + y\mathbf{T}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_\infty a_{2n} \\ & + z\mathbf{T}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_{2n} a_\infty + yz\mathbf{T}_{2n}(y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p})a_{2n} a_{2n}.\end{aligned} \tag{21}$$

By using the induction hypothesis, the previous equality, and the fact that $\mathcal{P}_{2n+2}$ is equal to $\sqcup_{p\in\mathcal{P}_{2n}}\mathcal{D}_p$, we conclude that:

$$\mathcal{DF}_{n+1} = \sum_{p\in\mathcal{P}_{2n+2}} y^{\mathit{fix}(p)} z^{\mathit{surfix}(p)}\mathbf{p}. \tag{22}$$

□

Since the number of $a_\infty$ in $\mathbf{p}$ is equal to $max(p)$ we get back the first combinatorial interpretation by applying $\Pi$ to (16). Now, we build another non-commutative Dumont-Foata polynomials that projects naturally onto the second interpretation. In order to do that, we have to define a linear map on non-commutative polynomials.

**Definition 2.2.4.** Let $i$ be an integer and $w = w_1 \cdots w_n$ be a word. Let $j$ (if it exists) be the smallest value such that $w_j$ is either $a_\infty$ or $a_i$. The linear map $S_i$ on $R\langle A\rangle$ is defined as follows:

$$S_i(w) = \begin{cases} w_1 \cdots w_{j-1} w_{n-1} w_{j+1} \cdots w_{n-2} w_j w_n & \text{if } j \text{ exists,} \\ w & \text{otherwise.} \end{cases} \tag{23}$$

**Example 7.** If $w = a_2a_6a_4a_\infty a_\infty a_6$, then $S_6(w) = a_2a_\infty a_4a_\infty a_6a_6$.

**Definition 2.2.5.** The second non-commutative polynomials of Dumont-Foata are defined as follows:

$$\begin{cases} \mathcal{DF}'_1 & = 1 \\ \mathcal{DF}'_{n+1} & = S_{2n}\big(\mathbf{T}_{2n}(\mathcal{DF}'_n)(a_\infty + ya_{2n})(a_\infty + za_{2n})\big) - \mathcal{DF}'_n a_\infty a_\infty, \text{ if } n \geq 1. \end{cases} \tag{24}$$

**Example 8.** The first polynomials are:

$$\begin{array}{rcl} \mathcal{DF}'_2 & = & yz \cdot a_2a_2 + y \cdot a_2a_\infty + z \cdot a_\infty a_2 \\ \mathcal{DF}'_3 & = & y^2z^2 \cdot a_2a_2a_4a_4 + yz^2 \cdot a_2a_2a_\infty a_4 + y^2z \cdot a_2a_2a_4a_\infty \\ & & +\ y^2z \cdot a_2a_4a_4a_4 + yz \cdot a_2a_\infty a_4a_4 + y^2z \cdot a_2a_4a_\infty a_4 + y^2 \cdot a_2a_4a_4a_\infty \\ & & +\ y^2 \cdot a_2a_4a_\infty a_\infty + y \cdot a_2a_\infty a_4a_\infty + yz \cdot a_2a_\infty a_\infty a_4 \\ & & +\ yz^2 \cdot a_4a_2a_4a_4 + z^2 \cdot a_\infty a_2a_4a_4 + yz^2 \cdot a_4a_2a_\infty a_4 + yz \cdot a_4a_2a_4a_\infty \\ & & +\ yz \cdot a_4a_2a_\infty a_\infty + z \cdot a_\infty a_2a_4a_\infty + z^2 \cdot a_\infty a_2a_\infty a_4. \end{array} \tag{25}$$

**Proposition 2.4.** *For $n \geq 1$, we have the following equality:*

$$\mathcal{DF}'_n = \sum_{p \in \mathcal{P}_{2n}} y^{\mathrm{sai}(p)} z^{\mathrm{fix}(p)} \mathbf{p}. \tag{26}$$

*Proof.* By induction on $n$. The proposition is true for $n = 1$. Assume it is true for a given $n \geq 1$. By definition,

$$\mathcal{DF}'_{n+1} = S_{2n}\left(\mathbf{T}_{2n}(\mathcal{DF}'_n)(a_\infty + ya_{2n})(a_\infty + za_{2n})\right) - \mathcal{DF}'_n a_\infty a_\infty. \tag{27}$$

Since the letter $a_{2n}$ does not appear in words of $\mathcal{DF}'_n a_\infty a_\infty$, it is invariant by $S_{2n}$. So:

$$\mathcal{DF}'_{n+1} = S_{2n}\left(\mathbf{T}_{2n}(\mathcal{DF}'_n)(a_\infty + ya_{2n})(a_\infty + za_{2n}) - \mathcal{DF}'_n a_\infty a_\infty\right). \tag{28}$$

Let $p$ be a surjective pistol of size $2n$. Then define $H(\mathbf{p})$ by:

$$H(\mathbf{p}) = S_{2n}\left(y^{sai(p)} z^{fix(p)} \mathbf{T}_{2n}(\mathbf{p})(a_\infty + ya_{2n})(a_\infty + za_{2n}) - y^{sai(p)} z^{fix(p)} \mathbf{p}\ a_\infty a_\infty\right). \tag{29}$$

If $\mathbf{p} = ua_\infty v$, with $u$ without $a_\infty$ and size of $i$, then:

$$\begin{array}{rcl} H(\mathbf{p}) & = & y^{sai(p)} z^{fix(p)} u(a_\infty + ya_{2n}) \mathbf{T}_{2n}(va_\infty)(a_\infty + za_{2n}) - y^{sai(p)} z^{fix(p)} \mathbf{p}\ a_\infty a_\infty \\ & = & y^{sai(p)} z^{fix(p)} u\big(a_\infty \mathbf{\Delta}_{2n}(va_\infty)a_\infty + ya_{2n}\mathbf{T}_{2n}(va_\infty)a_\infty \\ & & +\ za_\infty \mathbf{T}_{2n}(va_\infty)a_{2n} + yza_{2n}\mathbf{T}_{2n}(va_\infty)a_{2n}\big). \end{array} \tag{30}$$

The four terms of the previous sum correspond to partition the words $w$ of $\mathcal{D}_p$ into four sets, depending whether the values of $w_{i+1}$ and $w_{2n}$ are $2n$ or $2n+2$. Moreover, since $u$ has no $a_\infty$, all values $w_j$ with $j \leq i$ are smaller than $2n$. In particular, there is an extra power of $y$ if and only if position $i+1$ is a saillance point. As before, there is an extra power of $z$ if and only if $2n$ is fixed point. Then,

$$H(\mathbf{p}) = \sum_{p' \in \mathcal{D}_p} y^{sai(p')} z^{fix(p')} \mathbf{p}'. \tag{31}$$

If $\mathbf{p}$ has no $a_\infty$, since it does not contain $a_{2n}$, we have:

$$H(\mathbf{p}) = y^{sai(p)} z^{fix(p)} \mathbf{p} \left( y \cdot a_{2n} a_\infty + z \cdot a_{2n} a_\infty + yz \cdot a_{2n} a_{2n} \right) \tag{32}$$

In particular,

$$H(\mathbf{p}) = \sum_{p' \in \mathcal{D}_p} y^{sai(p')} z^{fix(p')} \mathbf{p}'. \tag{33}$$

Then, by induction hypothesis, the previous computation, and the fact that $\sqcup_{p \in \mathcal{P}_{2n}} \mathcal{D}_p = \mathcal{P}_{2n+2}$, we conclude.

□

By applying $\Pi$ to (26), we get back the second interpretation of the Dumont-Foata polynomials.

*Remark.* In order to find the statistics associated with the parameters $x$, $y$, $z$ in the Dumont-Foata polynomials, by defining a sequence of non-commutative polynomials which projects on them, guessing the statistics in non-commutative polynomials is much easier than guessing them in the algebra of formal power series. Indeed, the words in the non-commutative polynomials appear only once with a monomial in $x$, $y$, $z$ whereas in the Dumont-Foata polynomials we have less information because of the multiplicity of terms.

## 2.3 A generalization with six parameters

In [Dum95], Dumont gave a generalization with six parameters, and some conjectures about it. Independently, Zeng in [Zen96] and Randrianarivony in [Ran94] proved these conjectures. In order to do that, they first found the induction satisfied by these polynomials. In this section, we get back the induction with the non-commutative method.

**Definition 2.3.1.** Let $p$ be a surjective pistol of size $2n$. Then $i$ is:
- a *double fixed point* if $p(i) = i$ and there exists $j \neq i$ where $p(j) = i$,
- a *non double fixed point* if $p(i) = i$ and $i$ is not a double fixed point,
- a *double surfixed point* if $p(i) = i + 1$ and there exists $j \neq i$ where $p(j) = i + 1$,
- a *non double surfixed point* if $p(i) = i$ and $i$ is not a double surfixed point,
- an *even max point* if $i$ is even and $p(i) = 2n$,
- an *odd max point* if $i$ is odd and $p(i) = 2n$.

Let us respectively denote by $dfix(p)$, $ndfix(p)$, $dsurfix(p)$, $ndsurfix(p)$, $emax(p)$ and $omax(p)$ the number of double fixed points, non double fixed points, double surfixed points, non double surfixed points, even max points, odd max points smaller than $2n-2$.

**Example 9.** If $p = 248468$ then $dfix(p) = 1$, $ndfix(p) = 1$, $dsurfix(p) = 0$, $ndsurfix(p) = 1$, $emax(p) = 1$, $omax(p) = 1$.

**Definition 2.3.2.** Let $n$ be a positive integer. The six parameters of Dumont polynomials are defined as follows:

$$\Gamma_n(x, y, z, \overline{x}, \overline{y}, \overline{z}) = \sum_{p \in \mathcal{P}_{2n}} x^{emax(p)} y^{dfix(p)} z^{dsurfix(p)} \overline{x}^{omax(p)} \overline{y}^{ndfix(p)} \overline{z}^{ndsurfix(p)}. \tag{34}$$

Zeng and Randrianarivony proved that $(\Gamma_n)$ satisfies the following induction:

$$\begin{array}{rcl} \Gamma_1 & = & 1 \\ \Gamma_{n+1}(x, y, z, \overline{x}, \overline{y}, \overline{z}) & = & \Gamma_n(x+1, y, z, \overline{x}+1, \overline{y}, \overline{z})(z+\overline{x})(y+x) \\ & & -\Gamma_n(x, y, z, \overline{x}, \overline{y}, \overline{z}) \left( \overline{x} \left( y - \overline{y} \right) + \left( z - \overline{z} \right) x + \overline{x} x \right). \end{array} \tag{35}$$

Let us adapt the non-commutative formalism to find this definition by induction. In order to do so, we change slightly the alphabet $A$, and the ring $R$. In this section, $A = \{a_i,\ i \in \mathbb{N}^*\} \cup \{a_\infty,\ b_\infty\}$, the projection $\Pi$ sends $a_i$ to 1, $a_\infty$ and $b_\infty$ respectively to $x$ and $\overline{x}$, and $R = \mathbb{Q}[y, \overline{y}, z, \overline{z}]$. We extend $\mathbf{T}_i$ by sending $b_\infty$ to $(b_\infty + a_i)$. One has to modify slightly the map $i$: an odd max point corresponds to $b_\infty$. If $p = 24846888$, then $\mathbf{p}$ is equal to $a_2a_4b_\infty a_4a_6a_\infty$. Then define the linear map $f_{2n}$ on words as follows:

$$f_{2n}(w) = \mathbf{T}_{2n}(w)(b_\infty + za_{2n})(a_\infty + ya_{2n}) - w\left((y - \overline{y})\, a_\infty a_{2n} + (z - \overline{z})b_\infty a_{2n} + b_\infty a_\infty\right). \qquad (36)$$

Now, define our non-commutative $\mathbf{\Gamma}_n$ as follows:

**Definition 2.3.3.** Let $n$ be a positive integer. Let us define the following sequence of polynomials:

$$\begin{cases} \mathbf{\Gamma}_1 &= 1 \\ \mathbf{\Gamma}_{n+1} &= f_{2n}\left(\mathbf{\Gamma}_n\right) \quad \text{if } n \geq 1. \end{cases} \qquad (37)$$

**Example 10.** The first polynomials are:

$$\begin{aligned} \mathbf{\Gamma}_2 &= yz \cdot a_2a_2 + \overline{z} \cdot a_2a_\infty + \overline{y} \cdot a_\infty a_2 \\ \mathbf{\Gamma}_3 &= y^2z^2 \cdot a_2a_2a_4a_4 + y\overline{y}z \cdot a_2a_2a_\infty a_4 + yz\overline{z} \cdot a_2a_2a_4a_\infty \\ &\quad + yz\overline{z} \cdot a_2a_4a_4a_4 + yz\overline{z} \cdot a_2a_\infty a_4a_4 + y\overline{z} \cdot a_2a_4a_\infty a_4 + yz\overline{z} \cdot a_2a_4a_4a_\infty \\ &\quad + \overline{yz} \cdot a_2a_4a_\infty a_\infty + \overline{y}^2 \cdot a_2a_\infty a_4a_\infty + \overline{yz} \cdot a_2a_\infty a_\infty a_4 \\ &\quad + y\overline{y}z \cdot a_4a_2a_4a_4 + y\overline{y}z \cdot a_\infty a_2a_4a_4 + y\overline{y} \cdot a_4a_2a_\infty a_4 + \overline{y}z \cdot a_4a_2a_4a_\infty \\ &\quad + \overline{y} \cdot a_4a_2a_\infty a_\infty + \overline{yz} \cdot a_\infty a_2a_4a_\infty + \overline{y}^2 \cdot a_\infty a_2a_\infty a_4. \end{aligned} \qquad (38)$$

**Proposition 2.5.** *We have the following identity:*

$$\mathbf{\Gamma}_n = \sum_{p \in \mathcal{P}_{2n}} y^{\mathrm{dfix}(p)}\overline{y}^{\mathrm{ndfix}(p)}z^{\mathrm{dsurfix}(p)}\overline{z}^{\mathrm{ndsurfix}(p)}\mathbf{p}. \qquad (39)$$

*Proof.* By induction on $n$. It is true for $n = 1$. Assume the statement for a given $n$. By using the fact that $\mathbf{T}_{2n} = Id + \Delta_{2n}$, we have:

$$\begin{aligned} f_{2n}(w) = \ & \mathbf{\Delta}_{2n}(w)b_\infty a_\infty + \overline{y}wb_\infty a_{2n} + \overline{z}wa_{2n}a_\infty \\ & + y\mathbf{\Delta}_{2n}(w)b_\infty a_{2n} + z\mathbf{\Delta}_{2n}(w)a_{2n}b_\infty + yz\mathbf{\Delta}_{2n}(w)a_{2n}a_{2n}. \end{aligned} \qquad (40)$$

Let $p$ be a surjective pistol of size $2n$. Then, by using the inverse of $i$, the six terms of $f_{2n}(\mathbf{p})$ correspond to a partition of $\mathcal{D}_p$ in six (possibly empty) sets, depending whether positions $2n-1$ and $2n$ are (non) double surfixed points or not, (non) double fixed points or not. If $\mathbf{p}'$ appears in $f_{2n}(\mathbf{p})$, one can check that its coefficient is equal to $y^a\overline{y}^bz^c\overline{z}^d$, with $a$, $b$, $c$, $d$ equal to 1 or 0, depending whether $2n$ is a double fixed or not, a non double fixed point or not, and $2n-1$ a double surfixed point or not, a non double surfixed point or not. So we have:

$$f_{2n}\left(y^{dfix(p)}\overline{y}^{ndfix(p)}z^{dsurfix(p)}\overline{z}^{ndsurfix(p)}\mathbf{p}\right) = \sum_{p' \in \mathcal{D}_p} y^{dfix(p')}\overline{y}^{ndfix(p')}z^{dsurfix(p')}\overline{z}^{ndsurfix(p')}\mathbf{p}'. \qquad (41)$$

Then by the same arguments as before, we deduce (39).

□

Now, by applying $\Pi$ to (39), we get back the inductive definition of the Gandhi polynomials with six parameters. For a surjective pistol $p$, it is clear that the number of occurrences of $a_\infty$ and the occurrences of $b_\infty$ in $\mathbf{p}$ are respectively equal to $emax(p)$ and $omax(p)$.

The previous generalizations correspond to recording some statistics on the induction relation. There is another way to obtain interesting generalizations of a sequence: use a $q$-analog. In [HZ99], Han and Zeng give a $q$-analog of the Gandhi polynomials and a combinatorial interpretation of them. In the next section, by using again our non-commutative method, we find another combinatorial interpretation of this $q$-analog of Gandhi polynomials. They have in fact a $q$-analog of a generalization of Gandhi polynomials. By slightly adapting the non-commutative method, we find a direct combinatorial interpretation of these polynomials.

# 3 The $q$-analog of Gandhi polynomials

In this section, the ring R is $\mathbb{Q}$, the alphabet $A$ is $\{a_i,\ i \in \mathbb{N}^*\} \cup \{a_\infty\}$, and the projection $\Pi$ sends the letters $a_i$ to one, and $a_\infty$ to $x$.

## 3.1 A $q$-analog of the finite difference operator

Let us denote by $\Delta_q$ the following $q$-analog of the operator of finite difference:

$$\Delta_q(f)(x) = \frac{f(xq+1) - f(x)}{1 + (q-1)x}. \tag{42}$$

**Example 11.** If $f(x) = x^n$, we have:

$$\begin{aligned} \Delta_q(f)(x) &= \frac{(xq+1)^n - x^n}{1+(q-1)x} = \frac{(1+qx-x)\left(\sum_{k=0}^{n-1}(1+qx)^k x^{n-1-k}\right)}{1+(q-1)x} \\ &= \sum_{k=0}^{n-1}(1+qx)^k x^{n-1-k}. \end{aligned} \tag{43}$$

By expanding $(1+qx)^k$ and by regrouping by powers of $x$, we have:

$$\Delta_q(f)(x) = \sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}\binom{k}{i} q^{k-i}\right) x^{n-1-i}. \tag{44}$$

## 3.2 The $q$-projection and the $q$-Gandhi polynomials

**Definition 3.2.1.** The $q$-Gandhi polynomials are defined as follows:

$$\begin{cases} C_1(x,q) &= 1 \\ C_{n+1}(x,q) &= \Delta_q(C_n x^2) \quad \text{for } n \geq 1. \end{cases} \tag{45}$$

**Example 12.** Here are the first terms:

$$\begin{aligned} C_2 &= (q+1)\,x + 1, \\ C_3 &= \left(q^3 + 2q^2 + 2q + 1\right) x^2 + \left(2q^2 + 4q + 2\right) x + q + 2. \end{aligned} \tag{46}$$

Since we have defined non-commutative Gandhi polynomials, in order to obtain $C_n$, we have to find a $q$-version $\Pi_q$ of the map $\Pi$, such that $\Pi_q(\mathbf{C}_n) = C_n$. By induction and linearity, it is enough to find $\Pi_q$ such that for $p$ a surjective pistol of size $2n$, we have:

$$\Pi_q \circ \mathbf{\Delta}_{2n}(\mathbf{p}\ a_\infty a_\infty) = \Delta_q\left(\Pi_q(\mathbf{p})x^2\right). \tag{47}$$

From the previous remark, it is enough to define $\Pi_q$ on words $w$ of the form $w = \mathbf{p}$ and $w = \mathbf{p}\ a_\infty a_\infty$ where $\mathbf{p}$ is associated with a surjective pistol. Equation (47) indicates that we can fix a surjective pistol $p$, and work on $\mathcal{D}_p$. Let $p$ be a surjective pistol, we search $\Pi_q$ such that:

$$\begin{array}{rcl} \Pi_q(\mathbf{p}) & = & q^{stat(p)}\Pi(\mathbf{p}) \\ \Pi_q(\mathbf{p}\ a_\infty a_\infty) & = & q^{stat(p)}\Pi(\mathbf{p})x^2. \end{array} \tag{48}$$

Now, we have to find a function *stat*. One way to proceed is to note that $\Pi(\mathbf{p}) = x^{max(p)}$, then, compute both of (47), and deduce a condition on *stat*. Note that the construction of the $n$-th polynomial only depends on the letters $a_{2n}$ and $a_\infty$. So we can search a statistic satisfying:

$$stat(p) = \sum_{i=1}^{n} stat(p, 2i). \tag{49}$$

**Example 13.** For the case $n = 2$, we have $C_2 = (q + 1)x + 1$. So the possible distribution of the parameter $q$ over $\mathcal{P}_4$ are:

| | $x$ | 1 |
|---|---|---|
| 1 | 2444 | 2244 |
| $q$ | 4244 | |

or

| | $x$ | 1 |
|---|---|---|
| 1 | 4244 | 2244 |
| $q$ | 2444 | |

.

The first case is related to the special inversion, whereas the second to the special non inversion. In another context, Novelli, Thibon and Williams in [NTW10] defined the statistic that we use. It is closely related to the statistic defined by Han and Zeng in [HZ99].

**Definition 3.2.2.** Let $w$ be a word of size $2n$ containing $a_2, \cdots, a_{2n-2}$. Let $j_2, \cdots, j_{2n-2}$ be the positions of the last occurrence of $a_2, \ldots, a_{2n-2}$. Then a *special inversion* of $w$ is a pair $(i, j)$ with $i < j$ such that $j$ is one of the $j_k$ and $w_i > w_j$. We denote by $sinv(w, a_j)$ the number of special inversions of $w$ such that the letter corresponding to the second coordinate is $a_j$, and by $sinv(w)$ the number of special inversions of $w$.

**Example 14.** If $w = a_4a_2a_6a_4a_\infty a_6$, we have $sinv(w, a_2) = 1$, $sinv(w, a_4) = 1$, $sinv(w, a_6) = 1$, and $sinv(w) = 3$.

**Definition 3.2.3.** Let $p$ be a surjective pistol. We set:

$$\begin{array}{rcccl} \Pi_q(\mathbf{p}) & = & q^{sinv(p)}\Pi(\mathbf{p}) & = & q^{sinv(p)}x^{max(p)}, \\ \Pi_q(\mathbf{p}\ a_\infty a_\infty) & = & q^{sinv(p)}\Pi(\mathbf{p}\ a_\infty a_\infty) & = & q^{sinv(p)}x^{max(p)+2}. \end{array} \tag{50}$$

**Theorem 3.1.** *Let $\Pi_q$ be defined as previously. Then we have the identity*

$$\Pi_q\left(\boldsymbol{C}_n\right) = C_n, \tag{51}$$

*for all $n$ in $\mathbb{N}$.*

The following proposition helps us to prove the theorem 3.1.

**Proposition 3.2.** *Let $p$ be a surjective pistol of size $2n$, and $l$ be the number of $a_\infty$ in $w = \mathbf{p}\ a_\infty a_\infty$. Then we have the following identities:*

$$\Pi(\mathbf{p}\ a_\infty a_\infty) = x^l, \tag{52}$$

$$\varDelta_q\left(\Pi(w)\right) = \sum_{i=0}^{l-1}\left(\sum_{k=i}^{l-1}\binom{k}{i}q^{k-i}\right)x^{l-1-i}, \tag{53}$$

*and*

$$\Pi\left(\sum_{p'\in\mathcal{D}_p} q^{\mathrm{sinv}(\mathbf{p'},a_{2n})}\mathbf{p'}\right) = \sum_{i=0}^{l-1}\left(\sum_{k=i}^{l-1}\binom{k}{i}q^{k-i}\right)x^{l-1-i} = \mathit{\Delta}_q\left(\Pi(w)\right). \tag{54}$$

*Proof.* The first identity comes from the fact that the number of $a_\infty$ in $\mathbf{p}a_\infty a_\infty$ is $l$. Then, we deduce the second one thanks to (44).

For the third one, we regroup the elements of $\mathcal{D}_p$ according to the number of occurrences of $a_\infty$, and by the value $sinv(.\ ,a_{2n})$. Thus, two surjective pistols $p'$ and $p$" are in the same class if and only if they have same number $k$ of $a_\infty$ and $sinv(\mathbf{p'},a_{2n})=sinv(\mathbf{p}\text{"},a_{2n})=j$. Since in $\mathbf{p'}$ and $\mathbf{p}$", the letter $a_\infty$ is the only letter greater than $a_{2n}$, by denoting $\{n_1,\cdots,n_l\}$ the positions of $a_\infty$ in $w$, we deduce that the position of the rightmost $a_{2n}$ of $\mathbf{p'}$ and $\mathbf{p}$" is the same and is equal to $i_{j+l-k}$, since $l-k$ is the number of $a_{2n}$ in $\mathbf{p'}$ and $\mathbf{p}$" . Note that if one $p'$ in $\mathcal{D}_p$ has $l-i-1$ letters equal to $a_\infty$, then the value $sinv(p',a_{2n})$ is between 0 and $l-i-1$. So,

$$\sum_{p'\in\mathcal{D}_p}\mathbf{p'} = \sum_{i=0}^{l-1}\sum_{k=i}^{l-1}\sum_{\substack{p'\,\in\,\mathcal{D}_p\\ oc(\mathbf{p'},a_\infty)\,=\,l-1-i\\ sinv(\mathbf{p'},a_{2n})\,=\,k-i}} q^{k-i}\mathbf{p'}, \tag{55}$$

where $oc(\mathbf{p'},a_\infty)$ is the number of $a_\infty$ in $\mathbf{p'}$. Let us enumerate one equivalence class. If $\mathbf{p'}$ has $l-i-1$ maximal letters, and $sinv(\mathbf{p'},a_{2n})$ is equal to $k-i$, then the rightmost $a_{2n}$ is at the same position $n_{k+1}$ for all elements of the class.

To the left of the position $n_{k+1}$, we know that there are $k-i$ letters equal to $a_\infty$ thanks to the equality

$$sinv\left(\mathbf{p'},a_{2n}\right)=k-i, \tag{56}$$

and $i$ letters equal to $a_{2n}$. So this class has $\binom{k}{i}$ elements. By applying $\Pi$ to (55), we find (54). □

Let us now prove Theorem 3.1.

*Proof.* By induction in $n$. If $n$ is equal to 1, Theorem 3.1 is true. Assume that is true for $n$. So we have:

$$\Pi_q\left(\mathbf{C}_n\right) = C_n. \tag{57}$$

Since

$$C_{n+1} = \mathit{\Delta}_q\left(C_n x^2\right), \tag{58}$$

we deduce that

$$C_{n+1} = \mathit{\Delta}_q\left(\Pi_q\left(\mathbf{C}_n\right)x^2\right). \tag{59}$$

By definition of $\Pi_q$, of $\mathbf{C}_n$ and linearity of $\mathit{\Delta}_q$ we have

$$\mathit{\Delta}_q\left(\Pi_q\left(\mathbf{C}_n\right)x^2\right) = \sum_{p\in\mathcal{P}_{2n}} q^{sinv(p)}\mathit{\Delta}_q\left(\Pi(\mathbf{p}a_\infty a_\infty)\right). \tag{60}$$

Then, by applying Equality (54) of Proposition 3.2, we obtain

$$\sum_{p\in\mathcal{P}_{2n}} q^{sinv(p)}\mathit{\Delta}_q\left(\Pi(\mathbf{p}a_\infty a_\infty)\right) = \sum_{p\in\mathcal{P}_{2n}} q^{sinv(p)}\Pi\left(\sum_{p'\in\mathcal{D}_p} q^{sinv(\mathbf{p'},a_{2n})}\mathbf{p'}\right). \tag{61}$$

Note that for $p'$ in $\mathcal{D}_p$ we have

$$sinv(\mathbf{p'}) = sinv(\mathbf{p}) + sinv(\mathbf{p'},a_{2n}), \tag{62}$$

and since $\mathcal{P}_{2n+2} = \sqcup_{p\in\mathcal{P}_{2n}} \mathcal{D}_p$ we deduce that

$$\sum_{p\in\mathcal{P}_{2n}} q^{sinv(p)} \Pi \left( \sum_{p'\in\mathcal{D}_p} q^{sinv(\mathbf{p}',a_{2n})} \mathbf{p}' \right) = \sum_{p'\in\mathcal{P}_{2n+2}} q^{sinv(p')} \mathbf{p}' = \Pi_q\left(\mathbf{C}_{n+1}\right), \tag{63}$$

so the theorem is true for $n+1$.

□

### 3.3 Another generalization of the $q$-Gandhi polynomials and a new $q$-projection

In [HZ99], Han and Zeng define the following generalization of Gandhi polynomials:

$$D_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0, \\ \Delta_q(D_n(x)x^2) + (y-1)D_n(x)x & \text{if } n \geq 1. \end{cases} \tag{64}$$

They also give some combinatorial interpretations of these polynomials. With our non-commutative method, we see that the parameter $y$ corresponds to non double fixed point, and another $q$-statistic arises naturally.

**Definition 3.3.1.** Let $w$ be a word containing the letters $a_1, \cdots, a_n$. A *special non inversion* is a pair $(i, j)$ such that $i < j \leq k$, and $w_i < w_j$, where $w_i = a_k$, and the factor $w_1 \cdots w_{i-1}$ does not contain the letter $a_k$. We denote by $snv(w)$ the number of special non inversions in $w$, and by $snv(w, a_k)$ the number of special non inversions where $w_i = a_k$.

**Example 15.** If $w = a_4a_2a_6a_4a_\infty a_6$, then $snv(w, a_2) = 0$, $snv(w, a_4) = 1$, $snv(w, a_6) = 1$, and $snv(w) = 2$.

Thanks to this statistic, we have a new $q$-projection.

**Definition 3.3.2.** Let $n$ be an integer, and $p$ be a surjective pistol of size $2n$. We set

$$\Pi'_q(\mathbf{p}) = q^{snv(p)}\Pi(\mathbf{p}). \tag{65}$$

In the same way as before, we have the following theorem.

**Theorem 3.3.**

$$\Pi'_q \left( \sum_{p\in\mathcal{P}_{2n}} y^{\mathrm{ndfix}(p)} \mathbf{p} \right) = D_n(x). \tag{66}$$

The proof is the same as the proof of Theorem 3.1. First, we establish a proposition which relates the special non inversions with the operator $\Delta_q$. Then, we prove by induction the theorem 3.3.

**Proposition 3.4.** *Let $p$ be a surjective pistol of size $2n$, and $w$ be equal to* $\mathbf{p}\ a_\infty a_\infty$*. Let $l$ be the number of occurrences of the letter $a_\infty$ in $w$. Then:*

$$\Pi(w) = x^l, \tag{67}$$

$$\Delta_q\left(\Pi(w)\right) = \sum_{i=0}^{l-1} \left( \sum_{k=i}^{l-1} \binom{k}{i} q^{k-i} \right) x^{l-1-i}, \tag{68}$$

*and*

$$\sum_{p'\in\mathcal{D}_p} q^{\mathrm{snv}(\mathbf{p}',a_{2n})}\Pi(\mathbf{p}') = \sum_{i=0}^{l-1} \left( \sum_{k=i}^{l-1} \binom{k}{i} q^{k-i} \right) x^{l-1-i} = \Delta_q(\Pi(w)). \tag{69}$$

*Proof.* The first identity comes from the fact that the number of $a_\infty$ in $w$ is $l$. Then, we deduce the second one thanks to (44).

For the third one, we regroup the elements of $\mathcal{D}_p$ according to the number of $a_\infty$, and to the value of $snv(.\ ,a_{2n})$. Thus, two surjective pistols $p'$ and $p$" are in the same class if and only if they have same number $k$ of $a_\infty$ and same value $j$ of $snv(.,a_{2n})$. Since in $\mathbf{p}'$ and $\mathbf{p}$", the letter $a_\infty$ is the only letter greater than $a_{2n}$, by denoting $\{n_1,\cdots,n_l\}$ the positions of $a_\infty$ in $w$, we deduce that the position of the leftmost $a_{2n}$ of $\mathbf{p}'$ and $\mathbf{p}$" is the same and is equal to $i_{k-j+1}$, because $l-k$ is the number of $a_{2n}$ in $p'$ and $p$". Note that if one $p'$ in $\mathcal{D}_p$ has $l-i-1$ letters equal to $a_\infty$, then the value $sinv(p',a_{2n})$ is between 0 and $l-i-1$. So,

$$\sum_{p'\in\mathcal{D}_p}\mathbf{p}' = \sum_{i=0}^{l-1}\sum_{k=i}^{l-1}\sum_{\substack{p'\in\mathcal{D}_p\\ oc(\mathbf{p}',a_\infty)=l-1-i\\ sinv(\mathbf{p}',a_{2n})=k-i}} q^{k-i}\mathbf{p}', \tag{70}$$

where $oc(\mathbf{p}',a_\infty)$ is the number of $a_\infty$ in $\mathbf{p}'$. Let us enumerate one equivalence class. If $\mathbf{p}'$ has $l-i-1$ maximal letters, and $snv(\mathbf{p}',a_{2n})$ is equal to $k-i$, then the leftmost $a_{2n}$ is at the same position $n_{l-k}$ for all elements of the class.

To the right of the position $n_{l-k}$, we know that there are $k-i$ letters equal to $a_\infty$ thanks to the equality

$$snv(\mathbf{p}',a_{2n})=k-i, \tag{71}$$

so there are $i$ letters equal to $a_{2n}$. Therefore, this class has $\binom{k}{i}$ elements. By applying $\Pi$ to (70), we find (69).

□

Let us now prove Theorem 3.3.

*Proof.* It is straightforward that the sequence

$$\mathbf{D}_n = \sum_{p\in\mathcal{P}_{2n}} y^{ndfix(p)}\mathbf{p} \tag{72}$$

satisfies the induction

$$\mathbf{D}_{n+1} = \begin{cases} 1 & \text{if } n=0,\\ \boldsymbol{\Delta}_{2n}(\mathbf{D}_n a_\infty a_\infty) + (y-1)\mathbf{D}_n a_\infty a_{2n} & \text{if } n\geq 1. \end{cases} \tag{73}$$

Indeed, it is a particular case of the sequence $(\boldsymbol{\Gamma}_n)_{n\geq 1}$ with $y=1$, $\overline{y}=y$, $z=1$, $\overline{z}=1$, and $b_\infty=a_\infty$. Let us now prove Theorem 3.3 by induction in $n$. For $n$ equal to 1, the theorem is true. Assume that is true for $n$. So we have:

$$\Pi'_q(\mathbf{D}_n) = D_n. \tag{74}$$

Since

$$D_{n+1} = \Delta_q\left(D_n x^2\right) + (y-1)D_n x, \tag{75}$$

we deduce that

$$D_{n+1} = \Delta_q\left(\Pi'_q(\mathbf{D}_n)\,x^2\right) + (y-1)\Pi'_q(\mathbf{D}_n)\,x. \tag{76}$$

By definition of $\Pi'_q$, of $\mathbf{D}_n$, and linearity of $\Delta_q$ we have

$$\Delta_q\left(\Pi'_q(\mathbf{D}_n)\,x^2\right) = \sum_{p\in\mathcal{P}_{2n}} q^{snv(p)}y^{ndfix(p)}\Delta_q\left(\Pi(\mathbf{p}a_\infty a_\infty)\right). \tag{77}$$

Then, by applying Equality (69) of Proposition 3.4, we obtain

$$\sum_{p\in\mathcal{P}_{2n}} q^{snv(p)}y^{ndfix(p)}\Delta_q\left(\Pi(\mathbf{p}a_\infty a_\infty)\right) = \sum_{p\in\mathcal{P}_{2n}} q^{snv(p)}y^{ndfix(p)}\Pi\left(\sum_{p'\in\mathcal{D}_p} q^{snv(\mathbf{p}',a_{2n})}\mathbf{p}'\right). \tag{78}$$

Note that for $p'$ in $\mathcal{D}_p$ we have

$$snv(\mathbf{p}') = snv(\mathbf{p}) + snv(\mathbf{p}', a_{2n}), \tag{79}$$

and since $\mathcal{P}_{2n+2} = \sqcup_{p\in\mathcal{P}_{2n}}\mathcal{D}_p$ we deduce that

$$\sum_{p\in\mathcal{P}_{2n}} q^{snv(p)}y^{ndfix(p)}\Pi\left(\sum_{p'\in\mathcal{D}_p} q^{snv(\mathbf{p}',a_{2n})}\mathbf{p}'\right) = \sum_{p\in\mathcal{P}_{2n}}\sum_{p'\in\mathcal{D}_p} q^{snv(p')}y^{ndfix(p)}\Pi\left(\mathbf{p}'\right). \tag{80}$$

All surjective pistols in $\mathcal{D}_p$ have the same number of non double fixed point as $\mathbf{p}$ but $\mathbf{p}\ a_\infty a_{2n}$ which has one more. Moreover, $snv(\mathbf{p}\ a_\infty a_{2n}, a_{2n})$ is equal to zero. Then we have

$$(y-1)\Pi'_q\left(\mathbf{D}_n\right)x = (y-1)\sum_{p\in\mathcal{P}_{2n}} y^{ndfix(p)}\Pi\left(\mathbf{p}\ a_\infty a_{2n}\right). \tag{81}$$

By adding the right members of (80) and (81), we obtain

$$\sum_{p\in\mathcal{P}_{2n}}\sum_{p'\in\mathcal{D}_p} q^{snv(p')}y^{ndfix(p)}\Pi\left(\mathbf{p}'\right) + (y-1)\sum_{p\in\mathcal{P}_{2n}} y^{ndfix(p)}\Pi\left(\mathbf{p}\ a_\infty a_{2n}\right) = \sum_{p\in\mathcal{P}_{2n+2}} q^{snv(p)}y^{ndfix(p)}\Pi\left(\mathbf{p}\right). \tag{82}$$

Since

$$\sum_{p\in\mathcal{P}_{2n+2}} q^{snv(p)}y^{ndfix(p)}\Pi\left(\mathbf{p}\right) = \Pi'_q\left(\mathbf{D}_{n+1}\right), \tag{83}$$

we deduce the statement for $n+1$.

□

# References


[Car71] L. Carlitz. A conjecture concerning Genocchi numbers. *Norske Vid. Selsk. Skr. (Trondheim)*, (9):4, 1971.

[Car80] L. Carlitz. Explicit formulas for the Dumont-Foata polynomial. *Discrete Math.*, 30:211–225, 1980.

[DF76] D. Dumont and D. Foata. Une propriété de symétrie des nombres de Genocchi. *Bull. Soc. Math. Fr.*, 104:433–451, 1976.

[Dum95] D. Dumont. Conjectures sur des symétries ternaires liées aux nombres de Genocchi. *Discrete Math.*, 139(1-3):469–472, 1995.

[Gan70] J.M. Gandhi. A conjectured representation of Genocchi numbers. *Am. Math. Mon.*, 77:505–506, 1970.

[Han96] G.-N. Han. Symétries trivariées sur les nombres de Genocchi. *Eur. J. Comb.*, 17(4):397–407, 1996.

[HZ99] G.-N. Han and J. Zeng. $q$-polynômes de Gandhi et statistique de Denert. *Discrete Math.*, 205(1-3):119–143, 1999.

[NTW10] J.-C. Novelli, J.-Y. Thibon, and L.K. Williams. Combinatorial hopf algebras, noncommutative hall–littlewood functions, and permutation tableaux. *Adv. in Math.*, 224(4):1311 – 1348, 2010.

[Ran94] A. Randrianarivony. Polynômes de Dumont-Foata généralisés. *Sémin. Lothar. Comb.*, 32:12, 1994.

[RS73] J. Riordan and P.R. Stein. Proof of a conjecture on genocchi numbers. *Discrete Math.*, 5(4):381–388, 1973.

[Vie82] G. Viennot. Interprétations combinatoires des nombres d'Euler et de Genocchi. *Sémin. Théor. Nombres, Univ. Bordeaux I*, 1981-1982:94, 1982.

[Zen96] J. Zeng. Sur quelques propriétés de symétrie des nombres de Genocchi. *Discrete Math.*, 153(1-3):319–333, 1996.