Middle orders : all distributive lattices between weak and Bruhat orders

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- A poset is a set endowed with a partial order relation $\leq$;
- A lattice is a poset such that for all pairs x, y there exists a smallest element $x \vee y$ such that $x \leq x \vee y$ and $y \leq x \vee y$, and there exists a biggest element $x \wedge y$ such that $x \wedge y \leq x$ and $x \wedge y \leq y$. In particular they have a minimal and a maximal element;
- A lattice is distributive if the operations $\vee$ and $\wedge$ distribute over each other:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

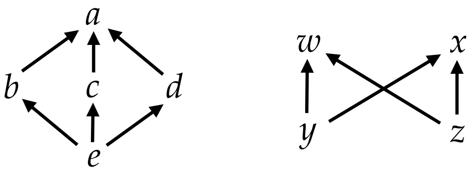


Figure: A lattice and a poset which is not a lattice.

Distributive lattices can be encoded nicely by smaller posets:

Theorem (Birkhoff 1937)

Elements of a finite distributive lattice $\mathbb{L}$ are in bijection with lower sets of $\mathrm{Irr}(\mathbb{L})$, *i.e. the poset of join-irreducible elements of* $\mathbb{L}$.

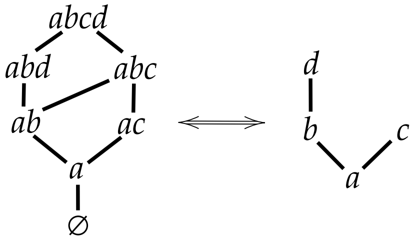


Figure: A distributive lattice $\mathbb{L}$ and its join-irreducibles poset $\mathrm{Irr}(\mathbb{L})$.

Definition (Bruhat order)

Let $\sigma, \tau \in \mathfrak{S}_n$. We let $\sigma \leq \tau$ if τ can be obtained by exchanging two values of σ which are in increasing order. We then define the Bruhat order on $\mathfrak{S}_n$ as the transitive closure of this relation, i.e. $\sigma \leq \tau$ if there exists $u_0, \ldots, u_i \in \mathfrak{S}_n$ such that $\sigma = u_0 \leq u_1 \leq \ldots \leq u_i = \tau$.

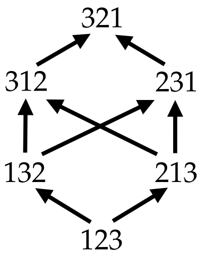


Figure: The Hasse diagram of the Bruhat order on $\mathfrak{S}_3$..

Definition (Weak order)

Let $\sigma, \tau \in \mathfrak{S}_n$. We let $\sigma \leq_R \tau$ if τ can be obtained by exchanging two adjacent values of σ which are in increasing order. We then define the (right) weak order on $\mathfrak{S}_n$ as the transitive closure of this relation.

The weak order on $\mathfrak{S}_n$ is a lattice (non distributive for $n \geq 3$).

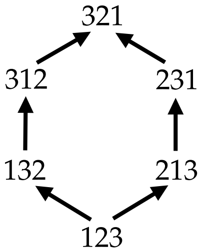


Figure: The Hasse diagram of the weak order on $\mathfrak{S}_3$.

The original middle order

Definition (Inversion sequence)

For $\sigma \in \mathfrak{S}_n$, its inversion sequence is $I(\sigma) = (i_1, \ldots, i_n)$ where

$$i_k = \#\{j < k \mid \sigma^{-1}(j) > \sigma^{-1}(k)\}.$$

For example, $I(4317256) = (0, 0, 2, 3, 0, 0, 3)$. Inversions sequences are in bijection with permutations, and they are exactly the sequences $(i_1, \ldots, i_n)$ such that for all $1 \leq k \leq n$, $0 \leq i_k < k$.

Proposition (Bouvel, Ferrari and Tenner 2024)

$\mathfrak{S}_n$ ordered by the coordinatewise comparison of inversion sequences is a distributive lattice.

The middle order is contained in the Bruhat order and contains the weak order:

$$\sigma \leq_R \tau \implies I(\sigma) \leq I(\tau) \implies \sigma \leq \tau$$

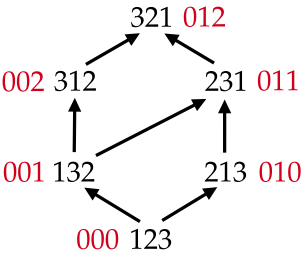
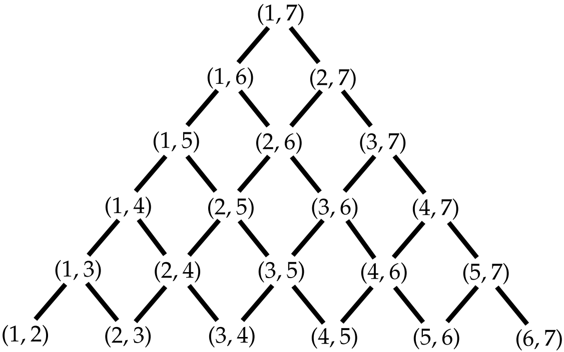


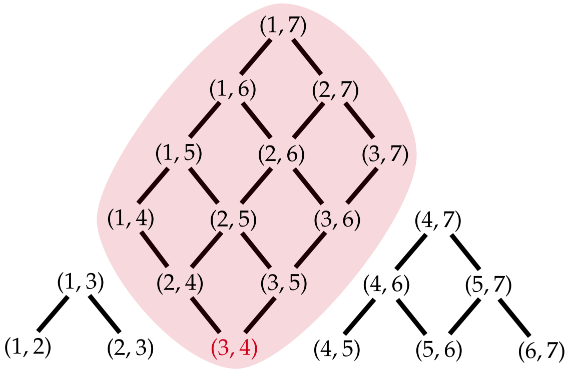
Figure: The Hasse diagram of the middle order on $\mathfrak{S}_3$, with its inversion sequences.

Our construction

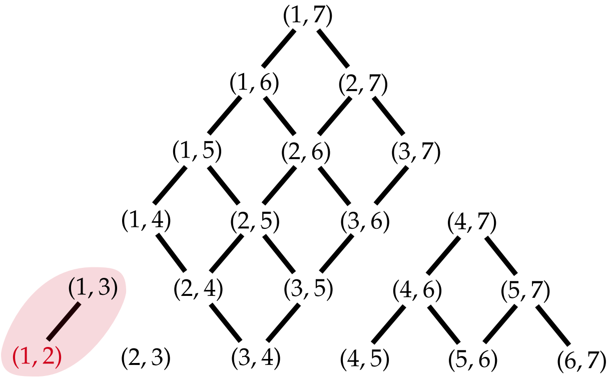
We start by constructing the posets whose ideals are in bijection with permutations. For this, consider the poset of inversions (i, j) ordered by $(i_1, j_1) \leq (i_1, j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$:



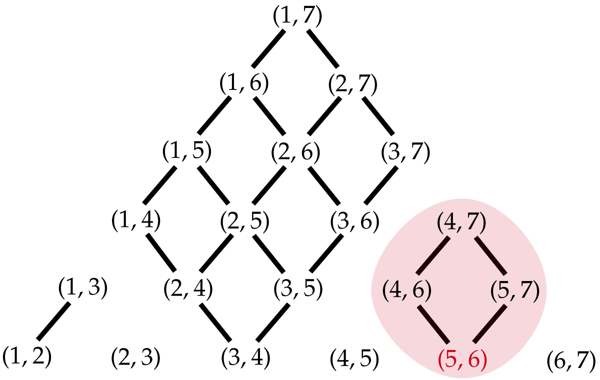
Pick any minimal element and separate the elements which are above it from the rest of the poset:



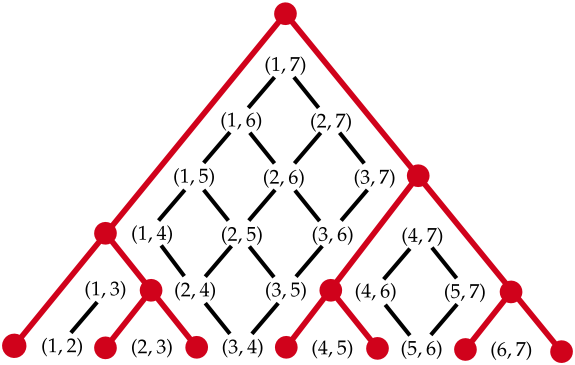
Do the same with another minimal element...



...until all minimal elements belong to different connected components.



The minimal elements can be picked in $(n-1)!$ different ways, but we get only C_{n-1} different partitions of the poset, for they are in bijection with binary trees.



Permutations are encoded by their inversion sets, which are characterized by the property that for all $i < j < k$:

- $(i, j) \in I$ and $(j, k) \in I \implies (i, k) \in I$;
- $(i, k) \in I \implies (i, j) \in I$ or $(j, k) \in I$.

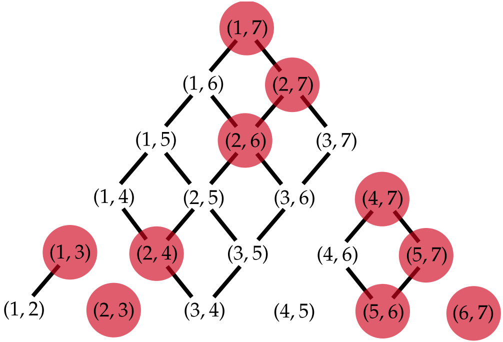
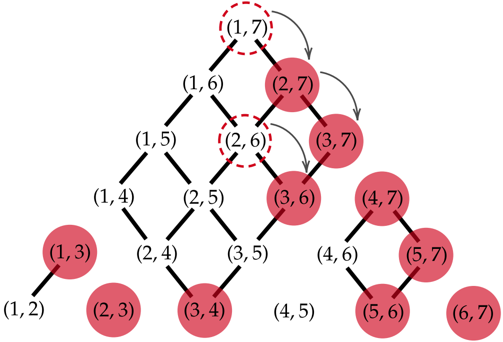
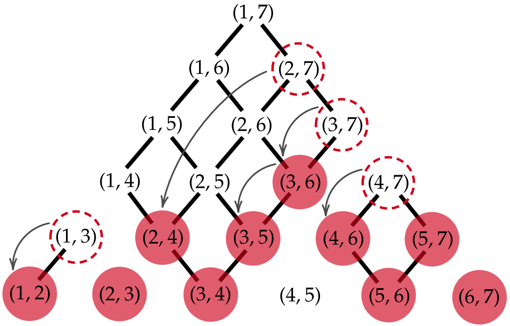


Figure: The inversion set of 3714625, inside the poset we constructed.

Let the inversions fall to the right inside each rectangular poset:

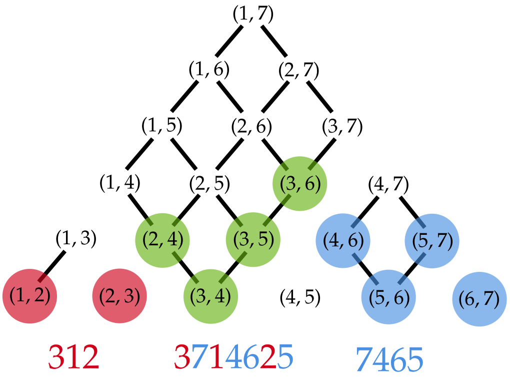


Then let the inversions fall to the left:



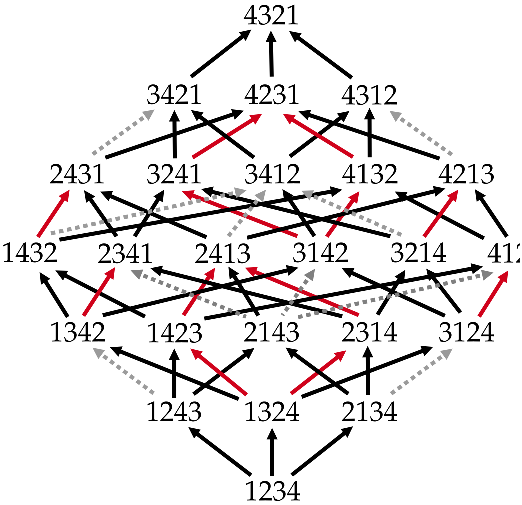
Letting the inversions fall to the left and then to the right would give the same result.

Ideals of rectangular posets encode how two subpermutations are shuffled:

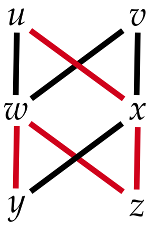


This gives us a bijection between permutations and ideals of $\mathcal{P}_T$.

When ordering permutations by the inclusion of the corresponding ideals of $\mathcal{P}_T$, we endow $\mathfrak{S}_n$ with the structure of a distributive lattice. This lattice contains the weak order and is contained in the Bruhat order.



We now want to prove that our middle orders are the only distributive lattices between the weak and Bruhat orders.
For this we consider the edges of the Bruhat order which are not in the weak order, and how we can add them to make a distributive lattice.



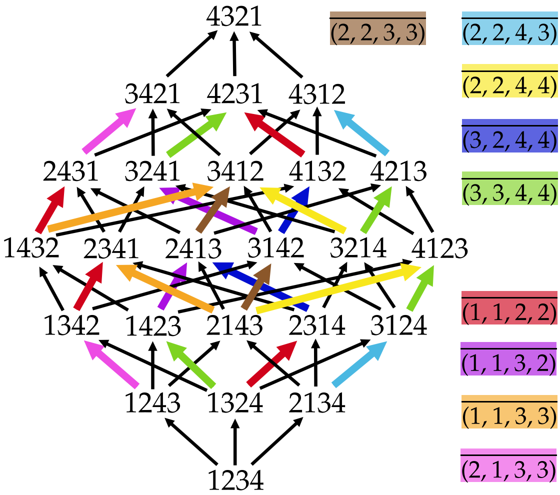
$$(x,u) \implies (z,w) \wedge (z,x) \oplus (y,w)$$

$$[(z,w) \wedge (z,x)] \oplus (y,w)$$

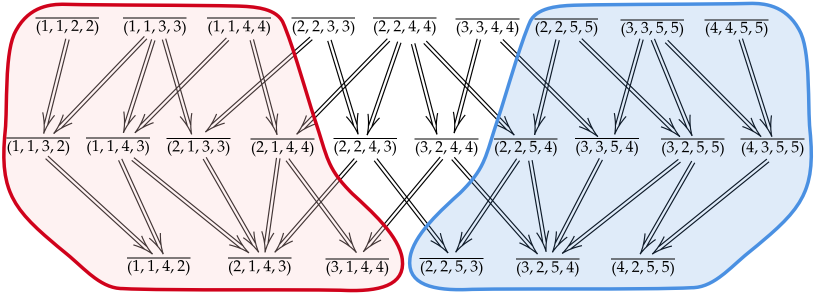
$$(y,w) \implies \neg(x,u)$$

$$(z,w) \wedge (z,x) \implies \neg(x,u)$$

Edges which can be added are grouped into equivalence classes:



These equivalence classes are ordered by implications:

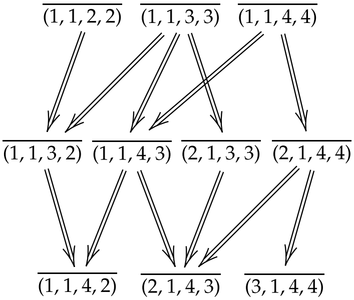


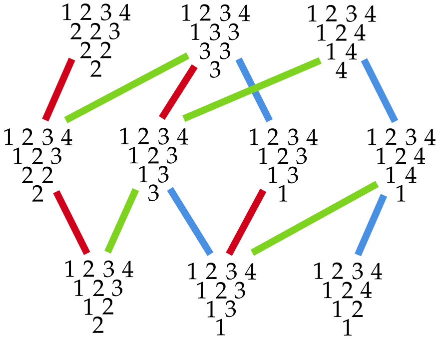
For all $i < j < k < \ell$, we have:

- $\overline{(i,1,k-1,j)} \oplus \overline{(j,i+1,n,k)}$;
- $\overline{(i,1,k-1,j)} \wedge \overline{(k,j+1,n,\ell)} \iff \overline{(i,1,\ell-1,j)} \wedge \overline{(k,i+1,n,\ell)}$.

Proof of the exhaustivity of our construction

The posets of left and right edges are isomorphic to the poset of irreducible elements of the lattice of Gelfand-Tsetlin triangles with first line 12...n:



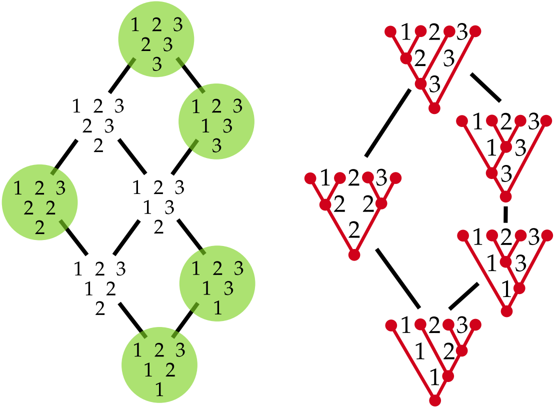


The condition

$$\overline{(i,1,k-1,j)} \wedge \overline{(k,j+1,n,\ell)} \iff \overline{(i,1,\ell-1,j)} \wedge \overline{(k,i+1,n,\ell)}$$

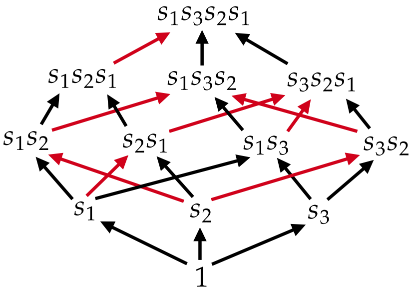
translates on Gelfand-Tsetlin triangles as

$$X_{i,j} = k \implies X_{k+i-j,k} = X_{j+n-k,j} = k.$$



The symmetric groups are generalized by Coxeter groups, *i.e.* groups with a presentation $\langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$, where $m_{ij} = m_{ji}$, $m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$. Let $\ell(u)$ be the length of any reduced expression $u = s_{i_1} \cdots s_{i_k}$. The weak and Bruhat orders are defined as the transitive closures of the following relations:

- $u \leq_R v$ if there exists $s \in S$ such that $us = v$ and $\ell(u) < \ell(v)$;
- $u \leq v$ if a reduced expression u is a subword of a reduced expression of v.



What about other Coxeter groups ?

Coxeter groups can be represented by their Coxeter diagrams, *i.e.* graphs whose vertices are generators, and whose edges give the order of the product of two generators.
Finite Coxeter groups have been classified:

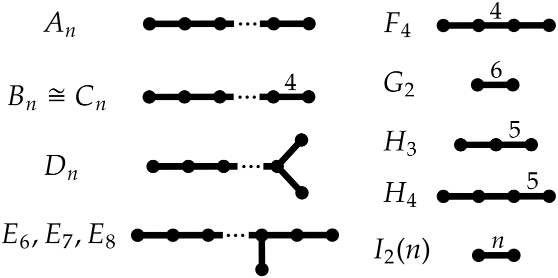


Figure: Coxeter diagrams of simple finite Coxeter groups.

Let (W, S) be a Coxeter group, and W^J the subgroup of W generated by $J \subset S$. We have the inclusions of orders

$$(W, \leq_R) \subset (W^J, \leq_R) \times W/W^J \subset (W^J, \leq) \times W/W^J \subset (W, \leq).$$

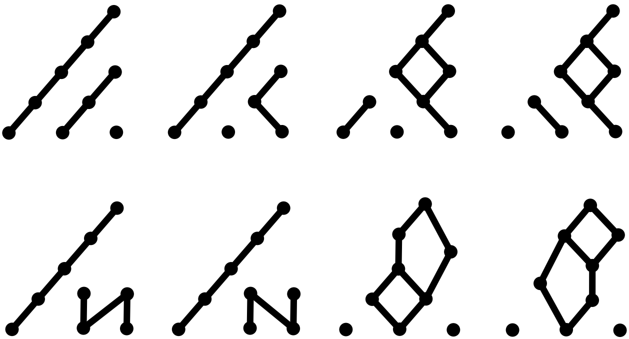
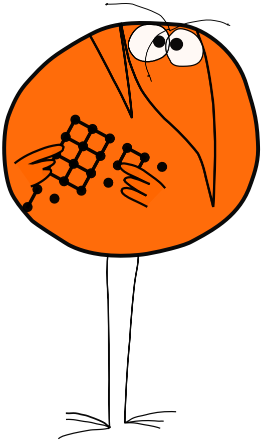


Figure: Irreducible posets of the 8 middle orders on B_3.

The end



THANKS FOR YOUR ATTENTION!