Operads in algebra, combinatorics, and computer science

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Outline

1. Introduction

1.1 Algebraic structures

Types of algebraic structures

Combinatorics deals with sets (or spaces) of structured objects:

- monoids;
- groups;
- lattices;
- associative alg.;
- Hopf bialg.;
- Lie alg.;
- pre-Lie alg.;
- dendriform alg.;
- duplicial alg.

Such types of algebras are specified by

1. a collection of operations;
2. a collection of relations between operations.

– Example –

The type of monoids can be specified by

1. the operations $\star$ (binary) and $\mathbb{1}$ (nullary);
2. the relations $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$ and $x \star \mathbb{1} = x = \mathbb{1} \star x$.

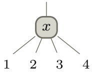
Operator structures

– Strategy to study these algebras –

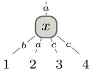
Add a level of indirection and consider algebraic structures wherein

elements are operations.

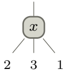
There are a lot of kind of operator structures, each dealing with a particular type of operators:



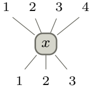
Operads



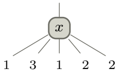
Colored operads



Symmetric operads



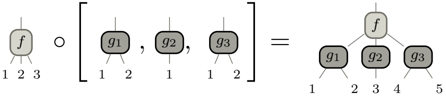
Pros



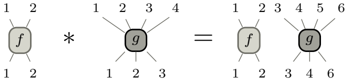
Abstract clones

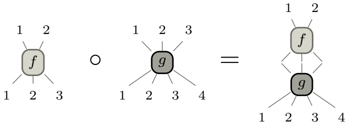
These operators can be composed in different ways.

Operads:

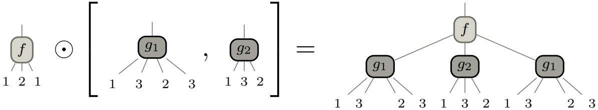


Pros:





Clones:



Operator structures and application areas

Operads are suitable to study algebraic structures where operations satisfy relations involving planar terms like

$$(x_1 \star x_2) \bullet x_3 = x_1 \bullet (x_2 \star x_3) - (x_1 \bullet x_2) \bullet x_3.$$

Symmetric operads are suitable to study algebraic structures where operations satisfy relations involving linear terms like

$$x_1 \star x_2 = x_1 \bullet x_2 + x_2 \star x_1.$$

Pros are suitable to study algebraic structures where operations can have several inputs and outputs and satisfy relations involving linear terms like

$$\Delta\left((x_1 \star x_2) \star x_3\right) = \Delta\left(x_1 \star (x_3 \star x_2)\right) + \Delta\left(x_1 \bullet (x_2 \star x_3)\right).$$

Clones are suitable to study algebraic structures where operations can satisfy any relation like

$$x_1 \star x_1 = x_1 + (x_2 \bullet x_1) \star x_2.$$

1.2 Computing over operations

Dendriform algebras

A dendriform algebra [**Loday**, 2001] is a space $\mathcal{A}$ endowed with two binary linear operations

$$\prec, \succ : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

satisfying the three relations

$$(x_1 \prec x_2) \prec x_3 = x_1 \prec (x_2 \prec x_3) + x_1 \prec (x_2 \succ x_3),$$
$$(x_1 \succ x_2) \prec x_3 = x_1 \succ (x_2 \prec x_3),$$
$$(x_1 \prec x_2) \succ x_3 + (x_1 \succ x_2) \succ x_3 = x_1 \succ (x_2 \succ x_3).$$

– Example –

On $\mathbb{K}\langle\{\mathsf{a}, \mathsf{b}\}^*\rangle$, let $\prec$ and $\succ$ be the operations defined by

$$u \prec v := u \sqcup\!\sqcup_{\leftarrow} v, \qquad u \succ v := u \sqcup\!\sqcup_{\rightarrow} v,$$

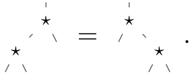
Then, for instance,

$$\mathsf{ab} \prec \mathsf{ba} = \mathsf{abab} + \mathsf{baab} + \mathsf{baab},$$
$$\mathsf{ab} \succ \mathsf{ba} = \mathsf{abba} + \mathsf{abba} + \mathsf{baba}.$$

$(\mathbb{K}\langle\{\mathsf{a}, \mathsf{b}\}^*\rangle, \prec, \succ)$ is a dendriform algebra [**Loday**, 2001].

Dendriform associative operations

A binary operation $\star$ is associative if



By using the infix notation, this says that

$$(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)\,.$$

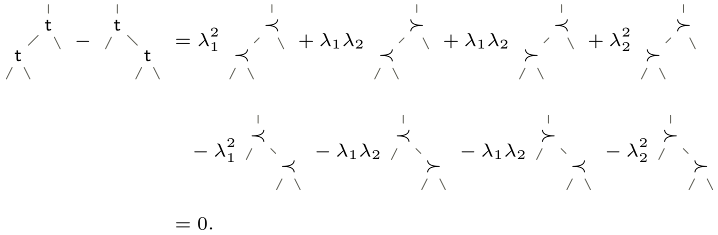
– Proposition –

In any dendriform algebra $(\mathcal{A}, \prec, \succ)$, any associative operation is proportional to the binary operation $\prec + \succ$.

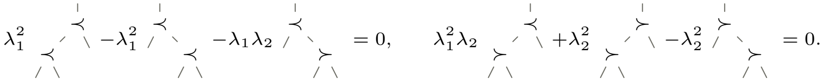
To prove this, let us consider a generic binary dendriform operation

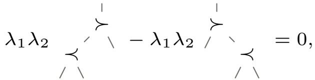
where λ_1 and λ_2 are any coefficients of $\mathbb{K}$.

This operation is associative iff



By the three dendriform relations, this is equivalent to





which is itself equivalent to $\lambda_1^2 = \lambda_1\lambda_2 = \lambda_2^2$. Therefore, $\lambda_1 = \lambda_2$.

Duplicial algebras

A duplicial algebra [**Brouder, Frabetti**, 2003] is a space $\mathcal{A}$ endowed with two binary linear operations

$$\ll, \gg : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

satisfying the three relations

$$(x_1 \ll x_2) \ll x_3 = x_1 \ll (x_2 \ll x_3)\,,$$
$$(x_1 \gg x_2) \ll x_3 = x_1 \gg (x_2 \ll x_3)\,,$$
$$(x_1 \gg x_2) \gg x_3 = x_1 \gg (x_2 \gg x_3)\,.$$

– Example –

On $\mathbb{K}\langle \mathbb{N}^* \rangle$, let $\ll$ and $\gg$ be the operations defined by

$$u \ll v := u \left(v \uparrow_{\max(u)}\right), \qquad u \gg v := u \left(v \uparrow_{|u|}\right).$$

Then, for instance,

$$0211 \ll 14 = 021136,$$
$$0211 \gg 14 = 021158.$$

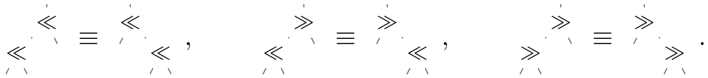
$(\mathbb{K}\langle \mathbb{N}^* \rangle, \ll, \gg)$ is a duplicial algebra [**Novelli, Thibon**, 2013].

Duplicial operations and equivalence

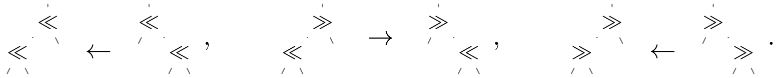
Two operations t and t′ are equivalent (written $t \equiv t'$) if they produce the same, output when evaluated with the same inputs.

Let us describe as way to test if two duplicial operations are equivalent.

By duplicial relations, we have



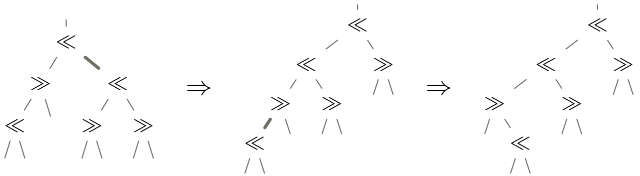
We orient them as



in order to obtain a rewrite relation $\Rightarrow$ on the set of all the duplicial operations by performing local moves.

Testing equivalence of duplicial operations

We have for instance the sequence



of rewritings.

– Proposition –

Two duplicial operations t and t′ are equivalent iff there is a duplicial operation s such that $t \overset{*}{\Rightarrow} s$ and $t' \overset{*}{\Rightarrow} s$.

To prove this, we have to establish the fact that $\Rightarrow$ is a terminating and confluent rewrite rule.

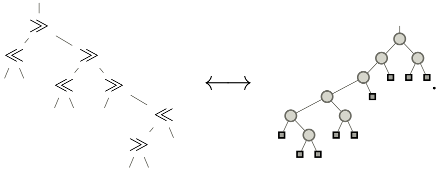
Enumerating duplicial operations

The set of all normal forms of $\Rightarrow$ contains duplicial operations that are pairwise nonequivalent.

– Proposition –

The set of normal forms of $\Rightarrow$ of operations with $n \geqslant 0$ inputs is in one-to-one correspondence with the set of all binary trees with n internal nodes.

A possible bijection puts the following two trees in correspondence:



Therefore, there are

$$\operatorname{cat}(n) := \frac{1}{n+1}\binom{2n}{n}$$

pairwise nonequivalent duplicial operations with n inputs.

Pre-Lie algebras

A pre-Lie algebra [**Vinberg**, 1963] [**Gerstenhaber**, 1963] is a space $\mathcal{A}$ endowed with a binary operation

$$\curvearrowleft : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

satisfying the relation

$$(x_1 \curvearrowleft x_2) \curvearrowleft x_3 - x_1 \curvearrowleft (x_2 \curvearrowleft x_3) = (x_1 \curvearrowleft x_3) \curvearrowleft x_2 - x_1 \curvearrowleft (x_3 \curvearrowleft x_2).$$

– Example –

On $\mathbb{K}\langle\{\mathsf{a},\mathsf{b}\}^*\rangle$, let $\curvearrowleft$ be the operation defined by

$$u \curvearrowleft v := \sum_{1 \leqslant i \leqslant |u|-1} u_1 \dots u_i \, v \, u_{i+1} \dots u_{|u|}.$$

Then, for instance,

$$\begin{aligned} \mathsf{aabab} \curvearrowleft \mathsf{bb} &= \mathsf{abbabab} + \mathsf{aabbbab} + \mathsf{aabbbab} + \mathsf{aababbb} \\ &= \mathsf{abbabab} + 2\,\mathsf{aabbbab} + \mathsf{aababbb}. \end{aligned}$$

$(\mathbb{K}\langle\{\mathsf{a},\mathsf{b}\}^*\rangle, \curvearrowleft)$ is a pre-Lie algebra.

Description of pre-Lie operations

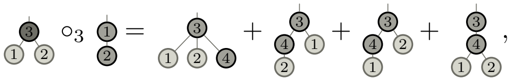
One can ask the same question as before concerning the description of equivalence classes of pre-Lie operations.

– Theorem [Chapoton, Livernet, 2001] **–**

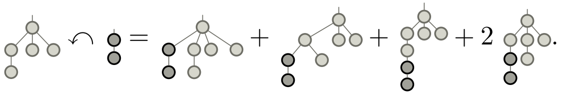
There is a one-to-one correspondence between the set of all pre-Lie operations with $n \geqslant 0$ inputs the set of all standard rooted trees with n nodes.

To prove this, one can consider a symmetric operad $\mathbf{PLie}$ on rooted trees and show that free algebras over $\mathbf{PLie}$ are free pre-Lie algebras.

Here is a composition of pre-Lie operations encoded by standard rooted trees:



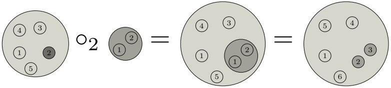
and here is a pre-Lie product in the free pre-Lie algebra over one generator:



1.3 Computing over combinatorial objects

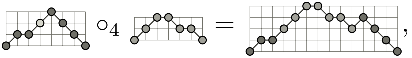
Objects as operators

By regarding objects as operators, one obtains ways to compose them. For instance,

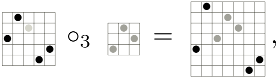


is an abstract composition of an object of size 5 with an object of size 2 at the 2-nd position.

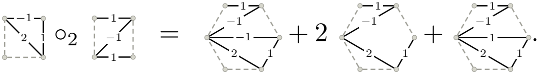
Concrete examples on Motzkin paths:



on permutations:



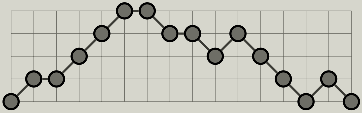
and on some labeled graphs:



Enumeration of Motzkin paths

A Motzkin path is a path in $\mathbb{N}^2$ starting from $(0,0)$ and ending at $(n,0)$, made of steps $(+1,+1)$, $(+1,-1)$, and $(+1,0)$.

– Example –



With the aid of some elementary reasoning, one can prove that the generating series $\mathcal{G}(t)$ of Motzkin paths, enumerating them w.r.t. their number of points, satisfies

$$\mathcal{G}(t) = t + t\mathcal{G}(t) + t\mathcal{G}(t)^2$$

and

$$\mathcal{G}(t) = t + t^2 + 2t^3 + 4t^4 + 9t^5 + 21t^6 + 51t^7 + 127t^8 + 323t^9 + \cdots .$$

A way to obtain this series consists in following both steps:

1. define a composition operation on the set of Motzkin paths;
2. express the infinite formal sum of all Motzkin paths.

If u and v are two Motzkin paths, the composition $u \circ_i v$ is obtained by replacing the i-th point of u by v.

The infinite formal sum of all Motzkin paths is

and one can prove that it satisfies the functional equation

This is a consequence of a property of the operad **Motz** of Motzkin paths (and more precisely, the fact that it is a Koszul operad).

1.4 Mains application fields and purposes

Purposes of the lecture

We focus in this lecture on operads.

Operads are used to study algebraic structures. They are a tool to

- compute over operations;
- describe the combinatorial heart of a type of algebraic structure;
- relate, by transformations, different types of algebraic structures.

Conversely, operads can also be used as tools to

- interpret combinatorial objects as operations;
- describe how to form objects from elementary building blocks;
- enumerate families of combinatorial objects;
- generate families of combinatorial objects.

Main topics

We shall consider and study

- collections, that are structured sets of combinatorial objects;
- treelike structure of several types (planar rooted trees and syntax trees);
- rewrite systems on syntax trees and give ways to prove termination and confluence;
- nonsymmetric operads on combinatorial objects, constructions, and study methods to prove presentations;
- some applications and open questions.

1.5 Exercises

About types of algebras

– Exercise –

Let $A := \{\mathsf{a}_0, \mathsf{a}_1, \mathsf{a}_2, \dots\}$ be an infinite alphabet totally ordered by $\mathsf{a}_i \leqslant \mathsf{a}_j$ iff $i \leqslant j$, and let $\prec$ and $\succ$ be the two binary operations defined on $\mathbb{K}\left\langle A^+\right\rangle$ by

$$u \prec v := \begin{cases} uv & \text{if } \max_{\leqslant}(u) \geqslant \max_{\leqslant}(v) \\ 0 & \text{otherwise,} \end{cases}$$

$$u \succ v := \begin{cases} uv & \text{if } \max_{\leqslant}(u) < \max_{\leqslant}(v) \\ 0 & \text{otherwise,} \end{cases}$$

Prove that $\left(\mathbb{K}\left\langle A^+\right\rangle, \prec, \succ\right)$ is a dendriform algebra.

– Exercise –

Prove that $(\mathbb{K}\langle\{\mathsf{a}, \mathsf{b}\}\rangle, ⧢_{\leftarrow}, ⧢_{\rightarrow})$ is a dendriform algebra.

– Exercise –

Let $(\mathcal{A}, \prec, \succ)$ be a dendriform algebra and set $\curvearrowleft$ as the binary operation defined by $x \curvearrowleft y := x \prec y - y \succ x$. Prove that $(\mathcal{A}, \curvearrowleft)$ is a pre-Lie algebra.

About types of algebras

– Exercise –

Let $(\mathcal{A}, \curvearrowleft)$ be a pre-Lie algebra and set $[-,-]$ as the binary operation defined by $[x, y] := x \curvearrowleft y - y \curvearrowleft x$. Prove that $(\mathcal{A}, [-,-])$ is a Lie algebra.

Recall that a Lie algebra is a space $\mathcal{A}$ endowed with a binary linear operation $[-,-]$ satisfying

$$[x, y] = -[y, x],$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

– Exercise –

Prove that each duplicial operation with $n \geqslant 0$ inputs can be encoded by a binary tree with n internal nodes.

– Exercise –

Express all associative operations in a duplicial algebra.

2. Collections

2.1 Collections and enumeration

Collections

An index set is a nonempty set I.

An I-collection is a set C decomposing as a disjoint union

$$C = \bigsqcup_{i \in I} C(i)$$

where the $C(i)$ are possibly infinite sets.

For any $x \in C(i)$, x is an object of C and $\mathrm{ind}(x) := i$ is the index of x.

When all $C(i)$ are finite, C is combinatorial.

Let C_1 and C_2 be two I-collections.

A map $\phi : C_1 \to C_2$ is an I-collection morphism if for all $x \in C_1$,

$$\mathrm{ind}(x) = \mathrm{ind}(\phi(x)).$$

If for all $i \in I$, $C_1(i) \subseteq C_2(i)$, then C_1 is a subcollection of C_2.

Graded collections

A graded collection is an $\mathbb{N}$-collection.

Let C be a graded collection. For any $x \in C$, the size $|x|$ of x is its index $\mathrm{ind}(x)$.

The map $|-| : C \to \mathbb{N}$ is the size function of C.

We say that C is

- connected if $\#C(0) = 1$;
- augmented if $C(0) = \emptyset$;
- monatomic if C is augmented and $\#C(1) = 1$;
- the unit collection if C is connected and $C = C(0)$;
- the neutral collection if C is monatomic and $C = C(1)$.

Generating series

When C is a combinatorial graded collection, the generating series of C is the formal power series

$$\mathcal{G}_C(t) := \sum_{n \geqslant 0} \#C(n)\, t^n.$$

This series satisfies

$$\mathcal{G}_C(t) = \sum_{x \in C} t^{|x|}.$$

The generating series of C encodes the integer sequence associated with C, which is the sequence

$$(\#C(0), \#C(1), \#C(2), \dots).$$

Multigraded collections

Let $k \geqslant 1$. A k-multigraded collection is an $\mathbb{N}^k$-collection.

A statistics on an I-collection C is a map $\mathrm{s} : C \to \mathbb{N}$.

Given a k-multigraded collection C, we obtain for any $j \in [k]$ the statistics s_j defined, for any $x \in C$ by

$$\mathrm{s}_j(x) := n_j$$

if $\mathrm{ind}(x) = (n_1, \ldots, n_k)$.

When C is combinatorial, the generating series of C is the multivariate formal power series

$$\mathcal{G}_C\left(t_1, \ldots, t_k\right) := \sum_{(n_1, \ldots, n_k) \in \mathbb{N}^k} \#C\left((n_1, \ldots, n_k)\right)\ t_1^{n_1} \ldots t_k^{n_k}.$$

Colored collections

Let C be a finite set, called set of colors.

A C-colored collection is an I_{C}-collection where

$$I_{\mathsf{C}} := \{(a, u) : a \in \mathsf{C} \text{ and } u \in \mathsf{C}^*\},$$

where C^* is the set of all finite words on C.

Let C be a C-colored collection, $x \in C$, and assume that $\mathrm{ind}(x) = (a, u)$. Then

- the output color $\mathrm{out}(x)$ of x is a;
- the i-th input color $\mathrm{in}_i(x)$ of x is the i-th letter u_i of u for any $i \in [|u|]$;
- the size $|x|$ of x is the length as a word of $\mathrm{in}(u)$.

In particular, any C-colored collection C gives rise to a graded collection C' where $C'(n)$ contains all elements of C of size $n \geqslant 0$.

2.2 Operations on collections

Casting

Let C be an I-collection, J be an index set, and

$$\omega : I \to J$$

be a map.

The ω-casting of C is the J-collection $\mathbf{Cast}^{\omega}(C)$ satisfying

$$\left(\mathbf{Cast}^{\omega}(C)\right)(j) = \bigcup_{\substack{i \in I \\ \omega(i) = j}} C(i)$$

for any $j \in J$.

When $J = \mathbb{N}$, one can see $\omega : I \to \mathbb{N}$ as a size function wherein the size of any $x \in C(i)$ is $\omega(i)$, and we say that $\mathbf{Cast}^{\omega}(C)$ is the ω-graduation of C.

Disjoint union

Let C_1 and C_2 be two I-collections.

The disjoint union of C_1 and C_2 is the I-collection $C_1 \sqcup C_2$ satisfying

$$(C_1 \sqcup C_2)(i) = C_1(i) \sqcup C_2(i)$$

for any $i \in I$.

– Proposition –

If C_1 and C_2 are two combinatorial I-collections, then $C_1 \sqcup C_2$ is combinatorial.

If, additionally, $I = \mathbb{N}$, then

$$\mathcal{G}_{C_1 \sqcup C_2}(t) = \mathcal{G}_{C_1}(t) + \mathcal{G}_{C_2}(t).$$

Cartesian product

Let C_1 be an I_1-collection, …, and C_p be an I_p-collection where $p \geqslant 0$.

The Cartesian product of C_1, …, C_p is the $I_1 \times \cdots \times I_p$-collection $[C_1, \ldots, C_p]_\times$ satisfying

$$[C_1, \ldots, C_p]_\times ((i_1, \ldots, i_p)) = C_1 (i_1) \times \cdots \times C_p (i_p)$$

for any $(i_1, \ldots, i_p) \in I_1 \times \cdots \times I_p$.

When $\omega : I_1 \times \cdots \times I_p \to J$ is a map, the ω-Cartesian product of C_1, …, C_p is the J-collection

$$[C_1, \ldots, C_p]_\times^\omega := \mathbf{Cast}^\omega ([C_1, \ldots, C_p]_\times) .$$

Cartesian product

When all $C_1, \ldots, C_p$ are graded collections, we denote by $+ : \mathbb{N}^p \to \mathbb{N}$ the map satisfying $+(n_1, \ldots, n_p) := n_1 + \cdots + n_p$.

In this case, $[C_1, \ldots, C_p]_\times^+$ is a graded collection.

– Proposition –

If C_1 be a combinatorial I_1-collection, ..., and C_p be a combinatorial I_p-collection, then $[C_1, \ldots, C_p]_\times$ is combinatorial.

If additionally, $I_1 = \cdots = I_p = \mathbb{N}$, then

$$\mathcal{G}_{[C_1, \ldots, C_p]_\times^+}(t) = \prod_{j \in [p]} \mathcal{G}_{C_j}(t).$$

List

Let C be an I-collection.

The list collection of C is the I^*-collection

$$\mathbf{List}(C) := \bigsqcup_{p \geqslant 0} \underbrace{[C, \dots, C]_\times}_{p \text{ terms}}.$$

When $\omega : I^* \to J$ is a map, the ω-list collection of C is the J-collection

$$\mathbf{List}^\omega(C) := \mathbf{Cast}^\omega(\mathbf{List}(C)).$$

– Proposition –

If C is a combinatorial then $\mathbf{List}(C)$ is combinatorial.

If additionally, $I = \mathbb{N}$ and C is augmented, then

$$\mathcal{G}_{\mathbf{List}^+(C)}(t) = \frac{1}{1 - \mathcal{G}_C(t)}.$$

Composition

Let C_1 and C_2 be two graded collections.

The composition of C_1 and C_2 is the graded collection $C_1 \odot C_2$ satisfying

$$(C_1 \odot C_2)(n) = \bigsqcup_{p \geqslant 0} \left[C_1(p), \underbrace{[C_2, \dots, C_2]_\times^+}_{p \text{ terms}}(n) \right]_\times^{\pi_2}$$

for any $n \in \mathbb{N}$, where $\pi_2 : \mathbb{N}^2 \to \mathbb{N}$ is the map defined by
$\pi_2(n_1, n_2) := n_2$.

– Proposition –

If C_1 and C_2 are two combinatorial and graded collections, and C_2 is augmented, then $C_1 \odot C_2$ is combinatorial.

Moreover,

$$\mathcal{G}_{C_1 \odot C_2}(t) = \mathcal{G}_{C_1}\left(\mathcal{G}_{C_2}(t)\right).$$

2.3 Some collections

Words

An alphabet is a set A wherein elements are called letters.

We can see A as a graded collection where all letters have size 1. Then $A = A(1)$.

The collections of words on A is the graded collection

$$A^* := \mathbf{List}^+(A).$$

We denote by $u_1 \dots u_n$ each element $(u_1, \dots, u_n)$ of $\mathbf{List}^+(A)(n)$. The size of a word is its length.

– Example –

Let $A := \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$.

One has $A^*(0) = \{\epsilon\}$, $A^*(1) = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, $A^*(2) = \{\mathsf{aa}, \mathsf{ab}, \mathsf{ac}, \mathsf{ba}, \mathsf{bb}, \mathsf{bc}, \mathsf{ca}, \mathsf{cb}, \mathsf{cc}\}$.

Since $\mathcal{G}_A(t) = 3t$,

$$\mathcal{G}_{A^*}(t) = \frac{1}{1 - 3t} = \sum_{n \geqslant 0} 3^n t^n.$$

Integer compositions

We can see $\mathbb{N}$ as a graded collection wherein $\mathbb{N}(n) = \{n\}$ for any $n \geqslant 0$.
The collection of integer compositions is the graded collection

$$\mathrm{Com} := \mathbf{List}^{+}(\mathbb{N} \setminus \{0\}).$$

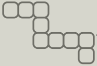
An integer composition is a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 \dots \boldsymbol{\lambda}_k$ of positive integers.
The size of $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}_1 + \cdots + \boldsymbol{\lambda}_k$. The length of $\boldsymbol{\lambda}$ is its number k of parts.
Since $\mathcal{G}_{\mathbb{N}\setminus\{0\}}(t) = \frac{t}{1-t}$,

$$\mathcal{G}_{\mathrm{Com}}(t) = \frac{1}{1 - \frac{t}{1-t}} = 1 + \sum_{n \geqslant 1} 2^{n-1} t^n.$$

The ribbon diagram of $\boldsymbol{\lambda}$ is the diagram obtained by concatenating lines of boxes for each part of $\boldsymbol{\lambda}$.

– Example –

The integer composition $\boldsymbol{\lambda} := 3141$ admits the ribbon diagram



Binary trees

The collection of binary trees is the graded collection $\mathrm{BT}_{\circ}$ defined as the one satisfying the relation

$$\mathrm{BT}_{\circ} = \{\boxdot\} \sqcup \Big[\{\circ\}, [\mathrm{BT}_{\circ}, \mathrm{BT}_{\circ}]^{+}_{\times} \Big]^{+}_{\times},$$

where $\boxdot$ is an object of size 0 called leaf and $\circ$ is an object of size 1 called internal node.

By definition, a binary tree t is

- either the leaf $\boxdot$;
- or an ordered pair $(\circ, (\mathrm{t}_1, \mathrm{t}_2))$, where t_1 and t_2 are both binary trees.

– Example –

The pair

$$(\circ, ((\circ, (\boxdot, \boxdot)), (\circ, (\boxdot, (\circ, (\boxdot, \boxdot))))))$$

is a binary tree.

Binary trees — size and representation

By definition of $\mathrm{BT}_{\circ}$, the size of a binary tree t satisfies

$$|\mathsf{t}| = \begin{cases} 0 & \text{if } \mathsf{t} = \square, \\ 1 + |\mathsf{t}_1| + |\mathsf{t}_2| & \text{otherwise, where } \mathsf{t} = (\circ, (\mathsf{t}_1, \mathsf{t}_2)) \, . \end{cases}$$

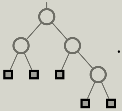
Binary trees are drawn by putting the root on the top.

– Example –

The binary tree

$$(\circ, ((\circ, (\square, \square)) , (\circ, (\square, (\circ, (\square, \square))))))$$

is drawn as



.

Binary trees – enumeration

Moreover, the generating series of $\mathrm{BT}_{\circ}$ satisfies the quadratic algebraic equation

$$\mathcal{G}_{\mathrm{BT}_{\circ}}(t) = 1 + t\mathcal{G}_{\mathrm{BT}_{\circ}}(t)^2.$$

The unique solution of this equation having a combinatorial meaning is

$$\mathcal{G}_{\mathrm{BT}_{\circ}}(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n \geqslant 0} \mathrm{cat}(n) t^n.$$

Let $\omega : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ be the map defined by $\omega(n) := n + 1$ for any $n \geqslant 0$.

By setting

$$\mathrm{BT}_{\square} := \mathbf{Cast}^{\omega}\left(\mathrm{BT}_{\circ}\right),$$

we have, for any $n \geqslant 1$.

$$\mathrm{BT}_{\square}(n) = \mathrm{BT}_{\circ}(n - 1).$$

The size of a binary tree of $\mathrm{BT}_{\square}$ is its number of leaves.

2.4 Exercises

About enumeration

– Exercise –

Show the stated formulas for the generating series for the operations of disjoint union, Cartesian product, List, and composition.

– Exercise –

Let C be a combinatorial, graded, and augmented collection. Let $\mathbf{MSet}(C)$ be the graded collection of multisets on C, that are multisets $\lbag x_1, \dots, x_p \rbag$ where $p \geqslant 0$ and $x_j \in C$ for all $j \in [p]$. The size of a multiset is the sum of the sizes of its elements w.r.t. the size function of C.

Prove that

$$\mathcal{G}_{\mathbf{MSet}(C)}(t) = \prod_{n \geqslant 1} \left(\frac{1}{1 - t^n} \right)^{\# C(n)}.$$

– Exercise –

Let C be a combinatorial and graded collection. Let $\mathbf{Set}(C)$ be the graded collection of sets on C, defined as the subcollection of $\mathbf{MSet}(C)$ restrained to multisets without repeated elements.

Prove that

$$\mathcal{G}_{\mathbf{Set}(C)}(t) = \prod_{n \geqslant 1} (1 + t^n)^{\# C(n)}.$$

Collection morphisms

– Exercise –

A Dyck path of size $n \geqslant 0$ is a path in $\mathbb{N}^2$ starting from $(0, 0)$ and ending at $(2n, 0)$, made of steps $(+1, +1)$ and $(+1, -1)$. By setting D as the graded collection of all Dyck paths, show that the collections $\mathrm{BT}_{\bigcirc}$ and D are isomorphic.

– Exercise –

Let Per be the graded collections of all permutations, that are bijections $\sigma : [n] \to [n]$, where the size of a permutation is the cardinality of its domain.

Let $\mathrm{bst} : \mathrm{Per} \to \mathrm{BT}_{\bigcirc}$ be the collection morphism wherein, for any $\sigma \in \mathrm{Per}$, $\mathrm{bst}(\sigma)$ is the binary tree obtained by inserting the letters of σ from the right to the left following the binary search tree insertion algorithm.

Prove that two permutations σ and σ' belong to the same fiber of bst iff one can transform σ into σ' through the (nonoriented) moves

$$u\,\mathsf{ac}\,v\,\mathsf{b}\,w \longleftrightarrow u\,\mathsf{ca}\,v\,\mathsf{b}\,w$$

where u, v, and w are any words of integers, and a, b, and c are letters satisfying $\mathsf{a} < \mathsf{b} < \mathsf{c}$.

This binary relation is the sylvester relation [**Hivert, Novelli, Thibon**, 2005].

3. Treelike structures

3.1 Syntax trees

Planar rooted trees

The collection of planar rooted trees is the graded collection PRT defined as the one satisfying the relation

$$\mathrm{PRT} = \{\circ\} \sqcup \left[\{\circ\}, \mathbf{List}^{+}(\mathrm{PRT}) \right]^{+}_{\times},$$

where $\circ$ is an object of size 1 called internal node.

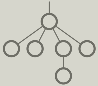
By definition, a planar rooted tree is an ordered pair $(\circ, (\mathsf{t}_1, \dots, \mathsf{t}_k))$ where $k \geqslant 0$ and all t_i, $i \in [k]$, are planar rooted trees.

The size of $\mathsf{t} = (\circ, (\mathsf{t}_1, \dots, \mathsf{t}_k))$ satisfies

$$|\mathsf{t}| = 1 + \sum_{i \in [k]} |\mathsf{t}_i| \, .$$

– Example –

The planar rooted tree $(\circ, ((\circ, \epsilon), (\circ, \epsilon), (\circ, ((\circ, \epsilon))), (\circ, \epsilon)))$ is drawn as



Planar rooted trees and binary trees

The generating series of PRT satisfies

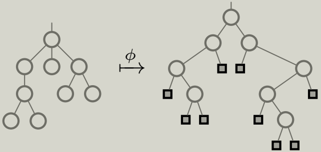
$$\mathcal{G}_{\mathrm{PRT}}(t) = t + \mathcal{G}_{\mathrm{PRT}}(t)^2,$$

so that $\mathcal{G}_{\mathrm{PRT}}(t) = \mathcal{G}_{\mathrm{BT}_{\square}}(t)$.

The Knuth rotation correspondence is the collection isomorphism $\phi : \mathrm{PRT} \to \mathrm{BT}_{\square}$ defined for any planar rooted tree $\mathsf{t} := (\circ, (\mathsf{t}_1, \ldots, \mathsf{t}_k))$ by

$$\phi(\mathsf{t}) := \begin{cases} \square & \text{if } \mathsf{t} = (\circ, \epsilon)\,, \\ (\circ, (\phi(\mathsf{t}_1)\,, \phi((\circ, (\mathsf{t}_2, \ldots, \mathsf{t}_k))))) & \text{otherwise.} \end{cases}$$

– Example –



Right partial action

Let $A := \mathbb{N} \setminus \{0\}$ and

$$\cdot : \mathrm{PRT} \times A^* \to \mathrm{PRT}$$

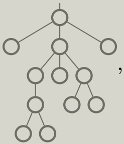
be the right partial monoid action defined for any planar rooted tree $\mathsf{t} := (\circ, (\mathsf{t}_1, \ldots, \mathsf{t}_k))$ by

$$\mathsf{t} \cdot u := \begin{cases} \mathsf{t} & \text{if } u = \epsilon, \\ \mathsf{t}_{u_1} \cdot \left(u_2 \ldots u_{|u|}\right) & \text{otherwise.} \end{cases}$$

If $u_1 \notin [k]$, then $\mathsf{t} \cdot u$ is not defined. Therefore, $\cdot$ is a partial action.

– Example –

Let

$\mathsf{t} :=$  ,

We have

$\mathsf{t} \cdot 1 = \circ$,

$\mathsf{t} \cdot 21 =$  ,

$\mathsf{t} \cdot 23 =$  .

$\mathsf{t} \cdot 231 = \circ$,

$\mathsf{t} \cdot 3 = \circ$.

Language of a tree

Let $\mathsf{t} \in \mathrm{PRT}$. Let

$$\mathcal{N}(\mathsf{t}) := \{u \in A^* : \mathsf{t} \cdot u \text{ is well-defined}\}$$

be the language of t.

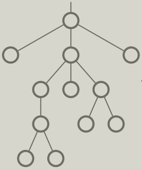
Some definitions:

- Any word of $\mathcal{N}(\mathsf{t})$ is a node;
- A maximal element of $\mathcal{N}(\mathsf{t})$ for the prefix order relation is a leaf;
- A node of t which is not a leaf is internal;
- The root of t is the node ϵ;
- A node u is an ancestor of a node v if u is a prefix of v;
- A node v is a child of u if $v = ui$ for an $i \in \mathbb{N}$;
- The depth-first order is the lexicographic total order on $\mathcal{N}(\mathsf{t})$;
- The degree $\deg(\mathsf{t})$ of t is its number of internal nodes;
- The arity $\mathrm{ari}(\mathsf{t})$ of t is its number of leaves;
- The height $\mathrm{ht}(\mathsf{t})$ of t is $\max\{|u| : u \in \mathcal{N}(\mathsf{t})\}$.

Language of a tree

– Example –

Let

$\mathsf{t} :=$  .

Then,

$$\mathcal{N}(\mathsf{t}) = \{\epsilon, 1, 2, 21, 211, 2111, 2112, 22, 23, 231, 232, 3\}.$$

The leaves of t are 1, 2111, 2112, 22, 231, 232, and 3.

The internal nodes of t are ϵ, 2, 21, 211, and 23.

The depth-first order $\preccurlyeq$ sorts the nodes of t as

$$\epsilon \preccurlyeq 1 \preccurlyeq 2 \preccurlyeq 21 \preccurlyeq 211 \preccurlyeq 2111 \preccurlyeq 2112 \preccurlyeq 22 \preccurlyeq 23 \preccurlyeq 231 \preccurlyeq 232 \preccurlyeq 3.$$

The degree of t is 5, its arity is 7, and its height is 4.

Syntax trees

Let G be an augmented graded collection.

A G-syntax tree is a planar rooted tree t together with a map

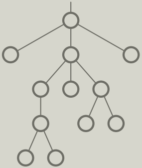
$$\lambda_{\mathsf{t}} : \mathcal{N}_{\circ}(\mathsf{t}) \to \mathsf{G},$$

where $\mathcal{N}_{\circ}(\mathsf{t})$ is the set of internal nodes of t, and such that for any $u \in \mathcal{N}_{\circ}(\mathsf{t})$, if u has $k \geqslant 1$ children, then $\lambda_{\mathsf{t}}(u) \in \mathsf{G}(k)$.

– Example –

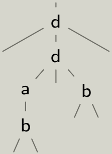
Let $\mathsf{G} := \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ such that $|\mathsf{a}| = 1$, $|\mathsf{b}| = 2$, $|\mathsf{c}| = 2$, and $|\mathsf{d}| = 3$.

Let

$\mathsf{t} :=$ 

be the G-syntax tree such that $\lambda_{\mathsf{t}}(\epsilon) = \mathsf{d}$, $\lambda_{\mathsf{t}}(2) = \mathsf{d}$, $\lambda_{\mathsf{t}}(21) = \mathsf{a}$, $\lambda_{\mathsf{t}}(211) = \mathsf{b}$, and $\lambda_{\mathsf{t}}(23) = \mathsf{b}$.

This syntax tree is depicted as



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Graded collections of syntax trees

Given an augmented graded collection G, let $\mathrm{S}(\mathsf{G})$ be the graded collection of all G-syntax trees, where for any $\mathsf{t} \in \mathrm{S}(\mathsf{G})$, $|\mathsf{t}| := \mathrm{ari}(\mathsf{t})$.

Alternatively, $\mathrm{S}(\mathsf{G})$ satisfies the relation

$$\mathrm{S}(\mathsf{G}) = \{|\} \sqcup \left[\, \{\circ\}\, , \mathsf{G} \odot \mathrm{S}(\mathsf{G})\right]_{\times}^{\pi_2},$$

where $|$ is an object of size 1 and $\circ$ is an object of size 0.

Therefore, the generating series of $\mathrm{S}(\mathsf{G})$ satisfies

$$\mathcal{G}_{\mathrm{S}(\mathsf{G})}(t) = t + \mathcal{G}_{\mathsf{G}}\left(\mathcal{G}_{\mathrm{S}(\mathsf{G})}(t)\right).$$

– Example –

Let $\mathsf{G} := \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ such that $|\mathsf{a}| = 1$, $|\mathsf{b}| = 2$, $|\mathsf{c}| = 2$, and $|\mathsf{d}| = 3$.

Since

$$\mathcal{G}_{\mathsf{G}}(t) = t + 2t^2 + t^3,$$

the generating series of $\mathrm{S}(\mathsf{G})$ satisfies

$$\mathcal{G}_{\mathrm{S}(\mathsf{G})}(t) = t + \mathcal{G}_{\mathrm{S}(\mathsf{G})}(t) + 2\mathcal{G}_{\mathrm{S}(\mathsf{G})}(t)^2 + \mathcal{G}_{\mathrm{S}(\mathsf{G})}(t)^3.$$

3.2 Operations on trees

Partial composition

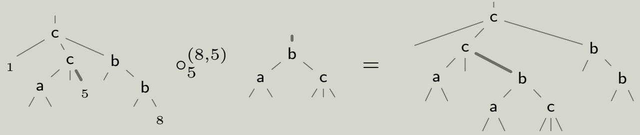
For any t $\in \mathrm{S}(\mathsf{G})(n)$ and any $i \in [n]$, the i-th leaf of t is the i-th leaf w.r.t. the depth-first order of t.

Let for $n, m \geqslant 1$ and $i \in [n]$ the product

$$\circ_i^{(n,m)} : \mathrm{S}(\mathsf{G})(n) \times \mathrm{S}(\mathsf{G})(m) \to \mathrm{S}(\mathsf{G})(n+m-1)$$

such that, for any t $\in \mathrm{S}(\mathsf{G})(n)$ and s $\in \mathrm{S}(\mathsf{G})(m)$, t $\circ_i^{(n,m)}$ s is the G-syntax tree obtained by overlying the root of s onto the i-th leaf of t.

– Example –



We shall omit the mention to n and m in $\circ_i^{(n,m)}$ in order to write simply $\circ_i$.

Full composition

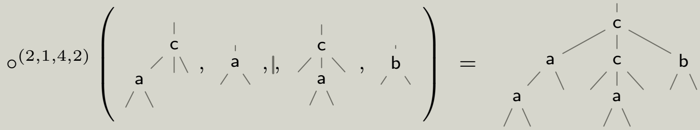
Let for any $n \geqslant 1$ and $m_1, \ldots, m_n \geqslant 0$ the product

$$\circ^{(m_1,\ldots,m_n)} : \mathrm{S}(\mathsf{G})(n) \times \mathrm{S}(\mathsf{G})(m_1) \times \cdots \times \mathrm{S}(\mathsf{G})(m_n) \to \mathrm{S}(\mathsf{G})(m_1 + \cdots + m_n)$$

defined, for where for any $\mathsf{t} \in \mathrm{S}(\mathsf{G})(n)$, $\mathsf{s}_1 \in \mathrm{S}(\mathsf{G})(m_1)$, …, $\mathsf{s}_n \in \mathrm{S}(\mathsf{G})(m_n)$, by

$$\circ^{(m_1,\ldots,m_n)}(\mathsf{t}, \mathsf{s}_1, \ldots, \mathsf{s}_n) := (\ldots((\mathsf{t} \circ_n \mathsf{s}_n) \circ_{n-1} \mathsf{s}_{n-1}) \ldots) \circ_1 \mathsf{s}_1.$$

– Example –



We shall omit the mention to m_1, …, m_n in $\circ^{(m_1,\ldots,m_n)}$ in order to write simply $\circ$.

Factors, prefixes, and suffixes

Let $\mathsf{t}, \mathsf{s} \in \mathbb{S}(\mathsf{G})$.

We say that s is a factor of t if there exist $\mathsf{r}, \mathsf{r}_1, \dots, \mathsf{r}_{|\mathsf{s}|} \in \mathbb{S}(\mathsf{G})$ and $i \in [|\mathsf{r}|]$ such that

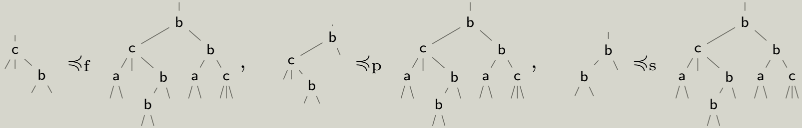
$$\mathsf{t} = \mathsf{r} \circ_i \left(\mathsf{s} \circ \left[\mathsf{r}_1, \dots, \mathsf{r}_{|\mathsf{s}|}\right]\right).$$

This property is denoted by $\mathsf{s} \preccurlyeq_{\mathrm{f}} \mathsf{t}$.

If $\mathsf{r} = |$, then s is a prefix of t. This property is denoted by $\mathsf{s} \preccurlyeq_{\mathrm{p}} \mathsf{t}$.

If all $\mathsf{r}_i = |$, then s is a suffix of t. This property is denoted by $\mathsf{s} \preccurlyeq_{\mathrm{s}} \mathsf{t}$.

– Example –



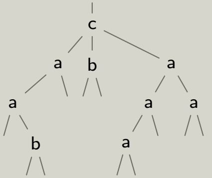
Occurrences

A node u of t is an occurrence of s in t if s is a prefix of $\mathsf{t} \cdot u$.

This node u is the position of the root of s in t.

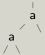
– Example –

Given the tree



,

the nodes 1, 3, and 31 are occurrences of



and the nodes 2 and 112 are occurrences of

.

By definition, a tree s admits an occurrence in t iff $\mathsf{s} \preccurlyeq_{\mathrm{f}} \mathsf{t}$.

If s admits no occurrence in t, then t avoids s.

3.3 Term rewrite systems

Rewrite relations

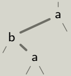
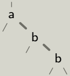
A rewrite relation on $\mathrm{S}(\mathsf{G})$ is a binary relation $\to$ on $\mathrm{S}(\mathsf{G})$ such that if $\mathsf{t} \to \mathsf{t}'$, then $|\mathsf{t}| = |\mathsf{t}'|$. Each pair $(\mathsf{t}, \mathsf{t}')$ such that $\mathsf{t} \to \mathsf{t}'$ is a rewrite rule.

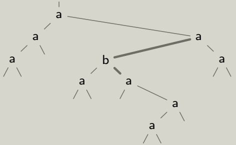
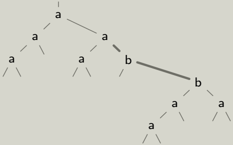
The closure of $\to$ is the binary relation $\Rightarrow$ on $\mathrm{S}(\mathsf{G})$ satisfying

$$\mathsf{r} \circ_i \left(\mathsf{s} \circ \left[\mathsf{r}_1, \ldots, \mathsf{r}_{|\mathsf{s}|}\right]\right) \Rightarrow \mathsf{r} \circ_i \left(\mathsf{s}' \circ \left[\mathsf{r}_1, \ldots, \mathsf{r}_{|\mathsf{s}|}\right]\right)$$

if $\mathsf{s} \to \mathsf{s}'$, where r and $\mathsf{r}_1, \ldots, \mathsf{r}_{|\mathsf{s}|}$ are any G-trees, and $i \in [|\mathsf{r}|]$.

– Example –

If $\to$ is the rewrite relation satisfying  $\to$ , one has

 $\Rightarrow$ .

Rewrite relations — definitions

Let $\rightarrow$ be a rewrite relation on G-trees (and $\Rightarrow$ be its closure).

Let us define

- $\overset{*}{\Rightarrow}$ as the reflexive and transitive closure of $\Rightarrow$;
- $\overset{*}{\Leftrightarrow}$ as the reflexive, symmetric, and transitive closure of $\Rightarrow$.

A tree t rewrites into a tree t' if $t \overset{*}{\Rightarrow} t'$.

Two trees t and t' are linked if $t \overset{*}{\Leftrightarrow} t'$.
Let $\mathcal{E}_{\rightarrow}$ be the graded collection of all $\overset{*}{\Leftrightarrow}$-equivalence classes.

A normal form for $\rightarrow$ is a tree t such that $t \overset{*}{\Rightarrow} t'$ implies $t = t'$.
Let $\mathcal{F}_{\rightarrow}$ be the graded collection of all normal forms for $\rightarrow$.

Termination and confluence

When there is no infinite chain $t_0 \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \cdots$, the rewrite relation $\rightarrow$ is terminating.

When $t \overset{*}{\Rightarrow} s_1$ and $t \overset{*}{\Rightarrow} s_2$ implies the existence of t' such that $s_1 \overset{*}{\Rightarrow} t'$ and $s_2 \overset{*}{\Rightarrow} t'$, $\rightarrow$ is confluent.

When $t \Rightarrow s_1$ and $t \Rightarrow s_2$ implies the existence of t' such that $s_1 \overset{*}{\Rightarrow} t'$ and $s_2 \overset{*}{\Rightarrow} t'$, $\rightarrow$ is locally confluent.

– Theorem (Diamond property) [Newman, 1942] **–**

If $\rightarrow$ is terminating and locally confluent, then $\rightarrow$ is confluent.

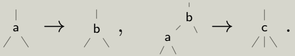
– Proposition –

- If $\rightarrow$ is terminating, then $\mathcal{F}_{\rightarrow}$ is the set of all G-trees avoiding the left members of $\rightarrow$.
- If $\rightarrow$ is terminating and confluent, then $\mathcal{F}_{\rightarrow}$ is isomorphic, as a graded collection, with $\mathcal{E}_{\rightarrow}$.

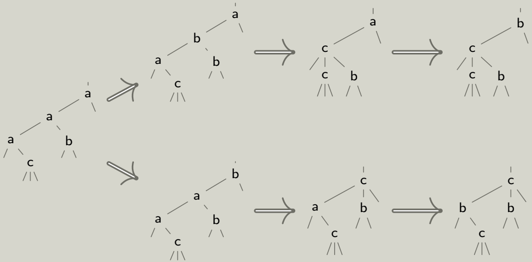
Some rewritings

– Example –

Let G $:= \{$a, b, c$\}$ with $|$a$| = |$b$| = 2$ and $|$c$| = 3$, and $\rightarrow$ be the rewrite relation satisfying



One has

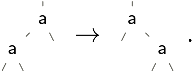


This rewrite relation $\rightarrow$ is **not confluent**.

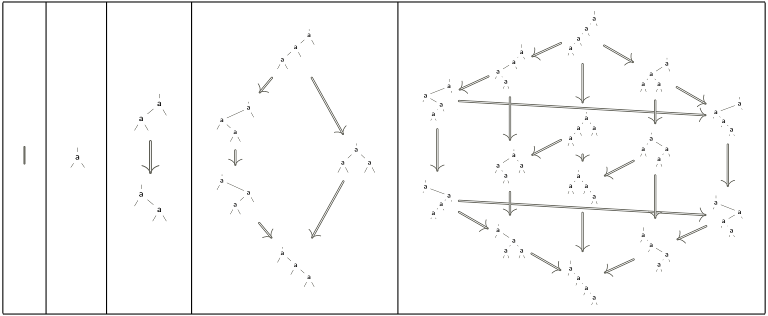
It is **terminating**. This is implied by the fact that each rewriting decreases by one the number of internal nodes labeled by a and there is a finite number of G-trees with a given arity.

An important rewrite relation: Tamari lattices

Let $\mathsf{G} := \{\mathsf{a}\}$ with $|\mathsf{a}| = 2$, and $\to$ be the rewrite relation on $\mathrm{S}(\mathsf{G})$ satisfying



First graphs $(\mathrm{S}(\mathsf{G})(n), \Rightarrow)$:



1	2	3	4	5

Tamari rewrite relation

- The binary relation $\rightarrow$ is the right rotation operation, an important operation appearing in computer science.

- This rewrite relation is terminating and confluent.

  As consequence, the binary relation $\overset{*}{\Rightarrow}$ endow each $\mathrm{S}(\mathsf{G})(n)$ with a poset structure, called Tamari order, having a least and greatest element.

- The set $\mathcal{F}_{\rightarrow}$ contains all right comb trees, that are the trees avoiding

```
      |
      a
     / \  .
    a
   / \
```

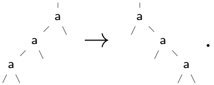
  The integer sequence associated with $\mathcal{E}_{\rightarrow}$ is $1, 1, 1, 1, \dots$.

Properties of Tamari lattices

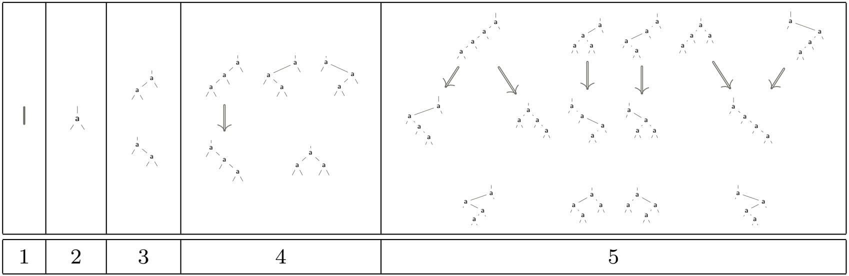
- Introduced by Tamari in order to study nonassociative operations [**Tamari**, 1962].
- The Tamari order relation $\overset{*}{\Rightarrow}$ endows each set $\mathrm{BT}(n)$ with the structure of a lattice [**Huang, Tamari**, 1972].
- Its number of intervals (thare are, pairs $(\mathsf{t}, \mathsf{t}')$ such that $\mathsf{t} \overset{*}{\Rightarrow} \mathsf{t}'$) have been enumerated in [**Chapoton**, 2006].
- These intervals can be encoded by interval-posets [**Châtel, Pons**, 2013].
- The sets of all Tamari intervals forms also lattice, the lattice of cubic coordinates, having a lot of combinatorial and geometrical properties [**Combe**, 2019].
- Some generalizations have been introduced:
  - m-Tamari lattices [**Bergeron, Préville-Ratelle**, 2012];
  - ν-Tamari lattices [**Préville-Ratelle, Viennot**, 2017];
  - δ-canyon lattices [**Combe, G.**, 2020].

A variant of Tamari lattices

Let $\mathsf{G} := \{\mathsf{a}\}$ with $|\mathsf{a}| = 2$, and $\to$ be the rewrite relation on $\mathrm{S}(\mathsf{G})$ satisfying



First graphs $(\mathrm{S}(\mathsf{G})(n), \Rightarrow)$:



Properties of a variant of Tamari lattices

This rewrite relation is terminating but **not confluent**.

The Buchberger completion algorithm [**Dotsenko, Khoroshkin**, 2010] is a semi-algorithm taking as input a terminating an not confluent rewrite relation $\rightarrow$, and outputting a new rewrite relation $\rightarrow'$ such that

- $\rightarrow'$ is terminating **and confluent**;
- $\overset{*}{\Rightarrow} = \overset{*}{\Rightarrow}'$.

– Theorem [Chenavier, Cordero, G., 2018] **–**

The collection $\mathcal{F}_{\rightarrow}$ can be described as the set of the G-trees avoiding 11 trees.

The integer sequence associated with $\mathcal{E}_{\rightarrow}$ is

$$1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 14, 15, 16, 17, \ldots$$

and its generating series satisfies

$$\mathcal{G}_{\mathcal{E}_{\rightarrow}}(t) = \frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right).$$

Proving termination

Let $\to$ be a rewrite relation on G-trees and $(\mathcal{P}, \preccurlyeq_{\mathcal{P}})$ be a well-founded poset.

A map

$$\theta : \mathrm{S}(\mathsf{G}) \to \mathcal{P}$$

is a termination invariant of $\to$ if, for any $\mathsf{t}, \mathsf{t}' \in \mathrm{S}(\mathsf{G})$,

$$\mathsf{t} \Rightarrow \mathsf{t}' \quad \text{implies} \quad \theta(\mathsf{t}') \prec_{\mathcal{P}} \theta(\mathsf{t})$$

– Proposition –

If the rewrite relation $\to$ admits a termination invariant, then $\to$ is terminating.

Indeed, assuming that there is an infinite chain $\mathsf{t}_0 \Rightarrow \mathsf{t}_1 \Rightarrow \mathsf{t}_2 \Rightarrow \cdots$, this implies that there is a infinite chain

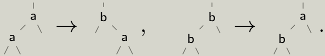
$$\cdots \prec_{\mathcal{P}} \theta(\mathsf{t}_2) \prec_{\mathcal{P}} \theta(\mathsf{t}_1) \prec_{\mathcal{P}} \theta(\mathsf{t}_0),$$

contradicting the fact that $\prec_{\mathcal{P}}$ is a well-founded relation.

Proving termination

– Example –

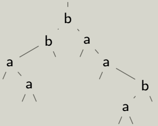
Let $\mathsf{G} := \{\mathsf{a}, \mathsf{b}\}$ with $|\mathsf{a}| = |\mathsf{b}| = 2$, and let $\to$ be the rewrite relation satisfying



Let $(\mathcal{P}, \preccurlyeq_{\mathcal{P}})$ be the poset $\mathbb{N}^2$ wherein elements are ordered lexicographically.

Let $\theta : \mathrm{S}(\mathsf{G}) \to \mathcal{P}$ be the map defined by $\theta(\mathsf{t}) := (\tau(\mathsf{t}), \#_{\mathsf{b}}\mathsf{t})$, where $\tau(\mathsf{t})$ is the sum, for all internal nodes u of t of the number of internal nodes in $\mathsf{t} \cdot u1$.

For instance,



$$\overset{\theta}{\longmapsto} \quad (3 + 2 + 0 + 0 + 0 + 0 + 1 + 0, 3) = (6, 3).$$

One has

$\theta\Big($  $\Big) = (0, 1) \prec_{\mathcal{P}} (1, 0) = \theta\Big($  $\Big)$, $\quad \theta\Big($  $\Big) = (1, 1) \prec_{\mathcal{P}} (1, 2) = \theta\Big($  $\Big)$.

Branching trees

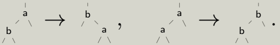
A G-tree t is branching of $\to$ if there are two different G-trees s and s′ such that $t \Rightarrow s$ and $t \Rightarrow s'$.

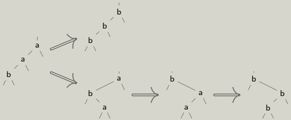
The pair $\{s, s'\}$ is a branching pair of t.

A tree r merges $\{s, s'\}$ if $s \overset{*}{\Rightarrow} r$ and $s' \overset{*}{\Rightarrow} r$.

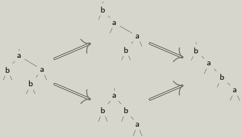
– Example –

Let $G := \{a, b\}$ with $|a| = |b| = 2$, and let $\to$ be the rewrite relation satisfying





The tree at the left is branching and the showed pair admits no merging tree.



The tree at the left is branching and the showed pair is mergeable by the rightmost tree.

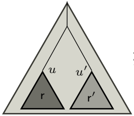
Branching trees and occurrences

Assume that $\rightarrow$ contains the two different rewrite rules $\mathsf{r} \rightarrow \mathsf{s}$ and $\mathsf{r}' \rightarrow \mathsf{s}'$.

Let t be a tree admitting an occurrence u of r and an occurrence u' of r'.

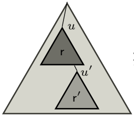
Therefore, t is branching and one has the following possible configurations:

1. The words u and u' are not prefix one of the other. Thus,

$\mathsf{t} =$ 

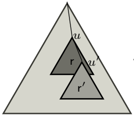
;

2. The word u is a prefix of u' and there is no overlapping internal node between the occurrences of r and r' in t. Thus,

$\mathsf{t} =$ 

;

3. The word u is a prefix of u' and there is at least one internal node shared by the occurrences of r and r' in t. Thus,

$\mathsf{t} =$ 

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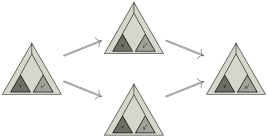
Branching trees and merging trees

Assume that t admits the branching pair $\{q, q'\}$.

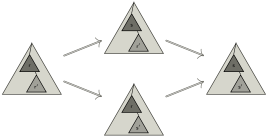
By considering the three previous configurations of occurrences, one has the following graphs of rewritings:

1.



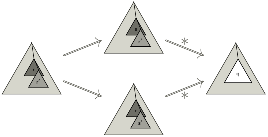
which is merging;

2.



which is merging;

3.



which is merging when the pair $\Big\{$  ,  $\Big\}$ admits a merging tree .

An algorithm to prove confluence

The degree $\deg(\rightarrow)$ of $\rightarrow$ is the maximal degree of the trees appearing as left members of $\rightarrow$.

– Proposition –

If all branching trees of $\rightarrow$ of degrees at most $2\deg(\rightarrow) - 1$ have all their branching pairs mergeable, then $\rightarrow$ is locally confluent.

For any G-tree t, let the set

$$\mathcal{R}_{\mathsf{t}} := \left\{ \mathsf{s} \in \mathrm{S}(\mathsf{G}) : \mathsf{t} \overset{*}{\Rightarrow} \mathsf{s} \right\}.$$

– Theorem –

Let $\rightarrow$ be a rewrite relation on $\mathrm{S}(\mathsf{G})$. If

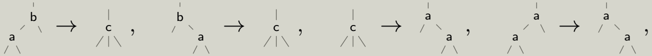
1. $\rightarrow$ admits a termination invariant;
2. for any G-tree t of degree at most $2\deg(\rightarrow) - 1$, the graph $(\mathcal{R}_{\mathsf{t}}, \Rightarrow)$ admits exactly one sink;

then $\rightarrow$ is terminating and confluent.

An algorithm to prove confluence

– Example –

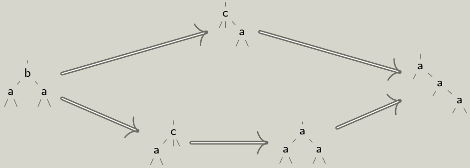
Let $\mathsf{G} := \{\mathsf{a}, \mathsf{b}\}$ with $|\mathsf{a}| = |\mathsf{b}| = 2$ and $|\mathsf{c}| = 3$, and $\to$ be the rewrite relation satisfying



This rewrite rule is terminating.

Since $\deg(\to) = 2$, to prove that $\to$ is confluent, one has to consider each graphs $\mathcal{R}_{\mathsf{t}}$ where $\mathsf{t} \in \mathrm{S}(\mathsf{G})$ and $\deg(\mathsf{t}) \leqslant 3$.

For instance, here is such a graph:



3.4 Exercises

About tree encoding

– Exercise –

Show that the Knuth rotation correspondence is a collection isomorphism between PRT and $\mathrm{BT}_{\perp}$.

Explain how to generalize this isomorphism in order to encode, by labeled binary trees, G-trees.

– Exercise –

The prefix word of a G-tree t is the word $\mathrm{p}(\mathsf{t})$ on the alphabet $A_{\mathsf{G}} := \{0\} \sqcup \mathsf{G}$ obtained by visiting the nodes of t by following the depth-first order and, for each considered node u, by writing the label of u if u is internal and by writing 0 otherwise.

1. Given the labeling graded collection G, describe a necessary and sufficient condition for a word u to be the prefix word of a G-tree.
2. Show that p is a collection isomorphism between the collection of the words on A_{G} satisfying the previous condition and $\mathrm{S}(\mathsf{G})$.
3. Given two G-trees t and s, express $\mathrm{p}(\mathsf{t} \circ_i \mathsf{s})$ in terms of $\mathrm{p}(\mathsf{t})$ and $\mathrm{p}(\mathsf{s})$.
4. Given the G-trees t and $\mathsf{s}_1, \ldots, \mathsf{s}_{|\mathsf{t}|}$, express $\mathrm{p}\left(\mathsf{t} \circ \left[\mathsf{s}_1, \ldots, \mathsf{s}_{|\mathsf{t}|}\right]\right)$ in terms of $\mathrm{p}(\mathsf{t})$ and $\mathrm{p}(\mathsf{s}_1), \ldots, \mathrm{p}\left(\mathsf{s}_{|\mathsf{t}|}\right)$.
5. Describe a necessary and sufficient condition on two prefix words u and v for the fact the tree $\mathrm{p}^{-1}(v)$ is a factor (resp. a prefix, a suffix) of the tree $\mathrm{p}^{-1}(u)$.

About rewrite relations

– Exercise –

Let G be an augmented graded collection endowed with a total order $\preccurlyeq$. This order is extended on A_{G} by setting that 0 is the least element.

Let $\to$ be a rewrite relation on G-trees such that $\mathsf{t} \to \mathsf{t}'$ implies that $\mathrm{p}(\mathsf{t})$ is greater than the prefix word of $\mathrm{p}\left(\mathsf{t}'\right)$ w.r.t. the order on A_{G} extended lexicographically on A_{G}^*.

Show that $\to$ is terminating.

– Exercise –

Prove the previous proposition stating that if $\to$ is terminating, then $\mathcal{F}_{\to}$ is the set of all trees avoiding the left members of $\to$.

– Exercise –

Let $\to$ be a rewrite relation on G-trees such that G is combinatorial and $\mathsf{G}(1) = \emptyset$. Prove that if $\to$ is not terminating, then there exist two G-trees t and t' such that $\mathsf{t} \neq \mathsf{t}'$ and $\mathsf{t} \overset{*}{\Rightarrow} \mathsf{t}' \overset{*}{\Rightarrow} \mathsf{t}$.

About rewrite relations

– Exercise –

Construct a rewrite relation which is terminating and not confluent.

– Exercise –

Construct a rewrite relation which is not terminating and not confluent.

– Exercise –

Construct a rewrite relation which is not terminating and confluent.

– Exercise –

Construct a rewrite relation $\to$ which is terminating and confluent, and such that the integer sequence associated with $\mathcal{F}_{\to}$ is $\left(2^{n-1}\right)_{n \geqslant 1}$.

A “Motzkin” rewrite relation

– Exercise –

Let the graded collection $\mathsf{G} := \{\mathsf{a}, \mathsf{c}\}$ where $|\mathsf{a}| = 2$ and $\mathsf{c}| = 3$.

Let $\to$ be the rewrite relation on $\mathrm{S}(\mathsf{G})$ satisfying

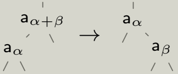
1. Show that $\to$ is terminating.
2. Show that $\to$ is confluent.
3. Describe the elements of the collection $\mathcal{F}_{\to}$.
4. Give a description of the generating series of $\mathcal{F}_{\to}$.

A "Fuss-Catalan" rewrite relation

– Exercise –

Let $m \geqslant 0$ be an integer and let the graded collection $\mathsf{G}_m := \{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_m\}$ where $|\mathsf{a}_i| = 2$ for all $i \in [0, m]$.

Let $\to$ be the rewrite relation on $\mathrm{S}(\mathsf{G}_m)$-trees satisfying



, where $\alpha, \beta \geqslant 0$ and $\alpha + \beta \leqslant m$.

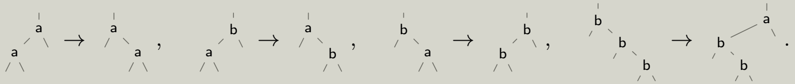
1. Show that $\to$ is terminating.
2. Show that $\to$ is confluent.
3. Describe the elements of the collection $\mathcal{F}_\to$.
4. Give a description of the generating series of $\mathcal{F}_\to$.

A “directed animal” rewrite relation

– Exercice –

Let the graded collection $\mathsf{G} := \{\mathsf{a}, \mathsf{b}\}$ where $|\mathsf{a}| = 2$ and $|\mathsf{b}| = 2$.

Let $\rightarrow$ be the rewrite relation on $\mathrm{S}(\mathsf{G})$ satisfying



1. Show that $\rightarrow$ is terminating.
2. Show that $\rightarrow$ is confluent.
3. Describe the elements of the collection $\mathcal{F}_{\rightarrow}$.
4. Give a description of the generating series of $\mathcal{F}_{\rightarrow}$.

4. Operads

4.1 Spaces and series on collections

Series on collections

Let $\mathbb{K}$ be a field of characteristic zero and C be a collection.

A C-series is a map

$$\mathbf{f} : C \to \mathbb{K}.$$

The coefficient $\mathbf{f}(x)$ of $x \in C$ in $\mathbf{f}$ is denoted by $\langle x, \mathbf{f} \rangle$.

The set of all C-series is $\mathbb{K} \langle\langle C \rangle\rangle$.

Endowed with the pointwise addition

$$\langle x, \mathbf{f} + \mathbf{g} \rangle := \langle x, \mathbf{f} \rangle + \langle x, \mathbf{g} \rangle$$

and the pointwise multiplication

$$\langle x, \lambda \mathbf{f} \rangle := \lambda \langle x, \mathbf{f} \rangle$$

for any scalar $\lambda \in \mathbb{K}$, the set $\mathbb{K} \langle\langle C \rangle\rangle$ is a vector space.

The sum notation of $\mathbf{f}$ is

$$\mathbf{f} = \sum_{x \in C} \langle x, \mathbf{f} \rangle \, x.$$

Series on collections

The scalar product of two C-series $\mathbf{f}$ and $\mathbf{g}$ is

$$\langle \mathbf{f}, \mathbf{g} \rangle := \sum_{x \in C} \langle x, \mathbf{f} \rangle \langle x, \mathbf{g} \rangle .$$

The support of $\mathbf{f}$ is the set $\mathrm{Supp}(\mathbf{f}) := \{x \in C : \langle x, \mathbf{f} \rangle \neq 0\}$.

The characteristic series of C is the series

$$\mathrm{ch}(C) := \sum_{x \in C} x.$$

We shall consider implicitly that any object x of C is the C-series defined as the characteristic series of $\{x\}$.

A C-series $\mathbf{f}$ is a C-polynomial if $\mathrm{Supp}(\mathbf{f})$ is finite.

The subspace of $\mathbb{K}\langle\langle C \rangle\rangle$ of all C-polynomials is denoted by $\mathbb{K}\langle C \rangle$.

By definition, all bases of $\mathbb{K}\langle C \rangle$ are indexed by C.

Traces of series

Let $\mathbf{f}$ be a C-series where C is a graded collection.

The trace of $\mathbf{f}$ is the generating series of $\mathbb{K}\langle\langle t \rangle\rangle$ defined by

$$\mathrm{tr}(\mathbf{f}) := \sum_{x \in C} \langle x, \mathbf{f} \rangle \, t^{|x|}.$$

Observe that

$$\mathrm{tr}\left(\mathrm{ch}\left(C\right)\right) = \mathcal{G}_C(t).$$

– Example –

Let $A := \{\mathsf{a}, \mathsf{b}\}$ and the A^*-series

$$\mathbf{f} := \sum_{\substack{u \in A^* \\ \mathsf{bb} \notin u}} u = \epsilon + \mathsf{a} + \mathsf{b} + \mathsf{aa} + \mathsf{ab} + \mathsf{ba} + \mathsf{aaa} + \mathsf{aab} + \mathsf{aba} + \mathsf{baa} + \mathsf{bab} + \cdots .$$

The trace of this series is

$$\mathrm{tr}(\mathbf{f}) = 1 + 2t + 3t^2 + 5t^3 + \cdots .$$

Formal series and algebraic structures

When C is an algebraic structure, its operations lead to operations on C-series.

Indeed, if

$$\star : C^p \to C$$

is a product of arity $p \geqslant 0$ on C, one obtains a product $\bar{\star}$ on $\mathbb{K}\langle\langle C \rangle\rangle$ defined by

$$\langle x, \bar{\star}(\mathbf{f}_1, \dots, \mathbf{f}_p) \rangle := \sum_{\substack{y_1, \dots, y_p \in C \\ x = \star(y_1, \dots, y_p)}} \prod_{i \in [p]} \langle y_i, \mathbf{f}_i \rangle .$$

– Example –

A binary product $\star : C \times C \to C$ leads to the (possibly partial) product

$$\mathbf{f}_1 \;\bar{\star}\; \mathbf{f}_2 := \sum_{y_1, y_2 \in C} \langle y_1, \mathbf{f}_1 \rangle \langle y_2, \mathbf{f}_2 \rangle (y_1 \star y_2)$$

on $\mathbb{K}\langle\langle C \rangle\rangle$.

Generalizing the product of generating series

Let C be a graded combinatorial collection.

A binary product $\star$ on C is graded if $|x_1 \star x_2| = |x_1| + |x_2|$.

– Proposition –

Let $\star$ be a graded product on C.

The map tr is an associative algebra morphism between $\mathbb{K}\left\langle\langle C \rangle\right\rangle$ endowed with the product $\bar{\star}$ and $\mathbb{K}[[t]]$ endowed with the usual generating series multiplication.

Moreover, this morphism is surjective when $C(n) \neq \emptyset$ for all $n \in \mathbb{N}$.

Therefore, $(\mathbb{K}\left\langle\langle C \rangle\right\rangle, \bar{\star})$ is a generalization of usual generating series.

Structure on C	**Sort of series**
$(\mathbb{N}, +, 0)$	Usual series $\mathbb{K}[[t]]$
Free comm. monoid	Multivariate series $\mathbb{K}[[t_1, t_2, \dots]]$
Free monoid	Noncomm. series $\mathbb{K}\left\langle\langle t_1, t_2, \dots \rangle\right\rangle$ [**Eilenberg**, 1974]
Monoid	Series on monoids [**Salomaa, Soittola**, 1978]
Operad	Series on operads [**Chapoton**, 2002, 2008]

4.2 Operads

Operads

Operads are algebraic structures formalizing the notion of some kinds of operators and their compositions.

A nonsymmetric operad is a space $\mathbb{K}\langle C\rangle$ where

- C is a graded collection;
- For any $n \geqslant 0$, $m \geqslant 0$, and $i \in [n]$,

  $$\circ_i^{(n,m)} : \mathbb{K}\langle C(n)\rangle \otimes \mathbb{K}\langle C(m)\rangle \to \mathbb{K}\langle C(n+m-1)\rangle\,,$$

  is a map, called partial composition map;
- There is a map

  $$\mathbb{1} : \mathbb{K} \to \mathbb{K}\langle C(1)\rangle\,,$$

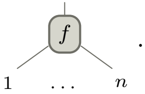
  called unit map.

When there is no ambiguity, $\circ_i^{(n,m)}$ is simply written as $\circ_i$.
Moreover, we shall write $\mathbb{1}$ for $\mathbb{1}(1)$.

This data has to satisfy some axioms, easy to understand in term of planar operators.

Planar operators

An planar operator is an entity f having $n \geqslant 0$ inputs and a single output:

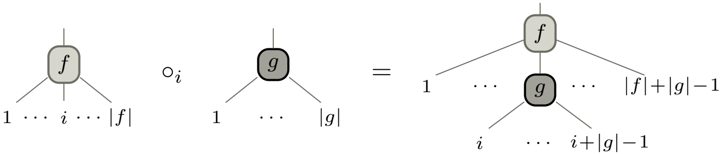


The arity $|f|$ of f is its number n of inputs, numbered from 1 to n.

Composing two planar operators f and g consists in

1. selecting an input of f specified by its position i;
2. grafting the output of g onto this input.

This produces a new operator



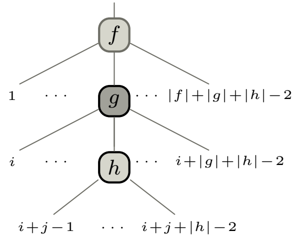
of arity $|f| + |g| - 1$.

Operad axioms

The associativity relation

$$(f \circ_i g) \circ_{i+j-1} h = f \circ_i (g \circ_j h)$$
$$1 \leqslant i \leqslant |f|, 1 \leqslant j \leqslant |g|$$

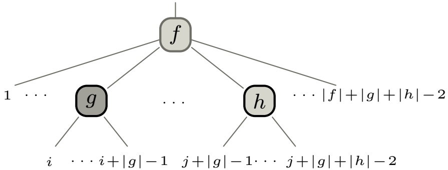
says that the pictured operation can be constructed from top to bottom or from bottom to top.



The commutativity relation

$$(f \circ_i g) \circ_{j+|g|-1} h = (f \circ_j h) \circ_i g$$
$$1 \leqslant i < j \leqslant |f|$$

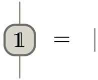
says that the pictured operation can be constructed from left to right or from right to left.



The unitality relation

$$\mathbb{1} \circ_1 f = f = f \circ_i \mathbb{1}$$
$$1 \leqslant i \leqslant |f|$$

says that $\mathbb{1}$ is the identity operation.

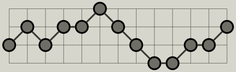


Example: Pat operad

Let $\mathbf{Pat} := \mathbb{K}\langle P \rangle$ be the operad wherein:

- P is the graded collection of all paths, that are words u on $\mathbb{N}$ and where $|u|$ is the length of u.

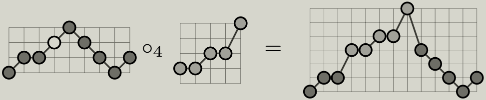
– Example –



is the path 1212232100112 and has arity 13.

- The partial composition $u \circ_i v$ is computed by replacing the i-th point of u by a copy of v.

– Example –



$011\mathbf{2}32101 \circ_4 11224 = 01133446 32101$

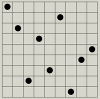
- The unit is the path 0, depicted as •, having arity 1.

Example: Per operad

Let $\mathbf{Per} := \mathbb{K}\langle \mathrm{Per} \rangle$ be the operad wherein:

- Per is the graded collection of all permutations seen as matrices and where $|\sigma|$ is the dimension of the matrix.

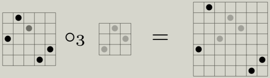
– Example –



is the permutation 972638145 and has arity 9.

- The partial composition $\sigma \circ_i \nu$ is computed by replacing the i-th point of σ by a copy of ν.

– Example –



$$35\mathbf{4}12 \circ_3 132 = 3746512$$

- The unit is the permutation 1.

Let $\mathbf{PLie} := \mathbb{K}\langle T \rangle$ [**Chapoton, Livernet**, 2001] be the operad wherein:

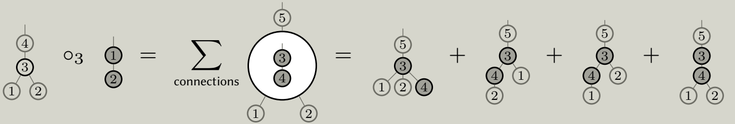
- T is the graded collection of all standard rooted trees where the size of a tree is its number of nodes.

– Example –

is a standard rooted tree of size 4.
This tree is the same as

- The partial composition $\mathsf{t} \circ_i \mathsf{s}$ is computed by replacing the node of t labeled by i by a copy of s, by relabeling the nodes in a standard way, and by summing over all possible ways to connect the children of this node on nodes of the copy.

– Example –



- The unit is the tree .

Arities, full compositions, and morphisms

Let $\mathcal{O} = \mathbb{K}\langle C \rangle$ be an operad.

For any $n \geqslant 0$, $\mathcal{O}(n) := \mathbb{K}\langle C(n) \rangle$ is the n-th homogeneous component of $\mathcal{O}$.

The arity $|f|$ of any $f \in \mathcal{O}$ is n provided that $f \in \mathcal{O}(n)$. In this case, f is homogeneous.

The full composition map of $\mathcal{O}$ is the map $\circ$ defined by

$$f \circ [g_1, \dots, g_n] := (\dots((f \circ_n g_n) \circ_{n-1} g_{n-1}) \dots) \circ_1 g_1$$

for any $f \in \mathcal{O}(n)$ and $g_1, \dots, g_n \in \mathcal{O}$.

Let $\mathcal{O}' = \mathbb{K}\langle C' \rangle$ be another operad.

A map $\phi : \mathcal{O} \to \mathcal{O}'$ is an operad morphism if

- for any $n \geqslant 0$ and any $f \in \mathcal{O}(n)$, $|\phi(f)| = n$;
- $\phi(\mathbb{1}) = \mathbb{1}'$;
- for any homogeneous elements $f, g \in \mathcal{O}$, $\phi(f \circ_i g) = \phi(f) \circ'_i \phi(g)$.

Suboperads, generating sets, and quotients

When $\mathcal{O}'$ is an operad such that for any $n \geqslant 0$, $\mathcal{O}'(n)$ is a subspace of $\mathcal{O}(n)$ and $\mathcal{O}'$ is endowed with the same partial composition maps and unit as the ones of $\mathcal{O}$, $\mathcal{O}'$ is a suboperad of $\mathcal{O}$.

The operad generated by a subset $\mathcal{G}$ of homogeneous elements of $\mathcal{O}$ is the smallest suboperad $\mathcal{O}^{\mathcal{G}}$ of $\mathcal{O}$ containing $\mathcal{G}$.

When $\mathcal{G}$ is such that $\mathcal{O}^{\mathcal{G}} = \mathcal{O}$, we say that $\mathcal{G}$ is a generating set of $\mathcal{O}$.

When moreover $\mathcal{G}$ is minimal for set inclusion among all sets satisfying this property, $\mathcal{G}$ is a minimal generating set of $\mathcal{O}$.

An operad ideal of $\mathcal{O}$ is a subspace $\mathcal{I}$ of $\mathcal{O}$ such that, for any homogeneous element f of $\mathcal{I}$ and any homogeneous element g of $\mathcal{O}$, $f \circ_i g \in \mathcal{I}$ and $g \circ_j f \in \mathcal{I}$.

In this case, $\mathcal{O}/_{\mathcal{I}}$ is the quotient operad of $\mathcal{O}$.

4.3 Free operads and presentations

Free operads

Let G be an augmented graded collection.

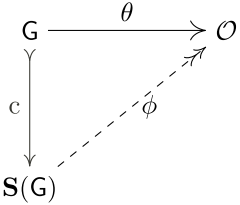
The free operad on G is the operad $\mathbf{S}(\mathsf{G}) := \mathbb{K}\langle \mathrm{S}(\mathsf{G})\rangle$ wherein:

- the partial composition maps $\circ_i$ are the ones of the G-trees;
- the unit is the tree |.

Let $\mathrm{c} : \mathsf{G} \to \mathbf{S}(\mathsf{G})$ be the natural injection, sending each label $x \in \mathsf{G}$ to the G-syntax tree of degree 1 and arity $|x|$.

Free operads satisfy the following universality property.

For any augmented graded collection G, any operad $\mathcal{O}$, and any map $\theta : \mathsf{G} \to \mathcal{O}$ preserving the arities, there exists a unique operad morphism $\phi : \mathbf{S}(\mathsf{G}) \to \mathcal{O}$ such that $\theta = \phi \circ \mathrm{c}$.



Evaluations and treelike expressions

Let $\mathcal{O} = \mathbb{K}\langle C \rangle$ be an operad.

The evaluation map of $\mathcal{O}$ is the map

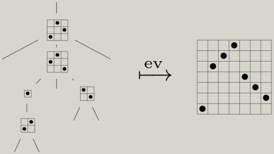
$$\mathrm{ev} : \mathbf{S}(C) \to \mathcal{O},$$

defined linearly and recursively by

$$\mathrm{ev}(\mathsf{t}) := \begin{cases} \mathbb{1} & \text{if } \mathsf{t} = |, \\ \lambda_{\mathsf{t}}(\epsilon) \circ [\mathrm{ev}(\mathsf{t} \cdot 1), \dots, \mathrm{ev}(\mathsf{t} \cdot k)] & \text{otherwise, where } \mathsf{t} \text{ has } k \text{ children.} \end{cases}$$

– Example –

In $\mathbf{Per}$, we have



Given $f \in \mathcal{O}$, if g is an element of $\mathbf{S}(C)$ such that $\mathrm{ev}(g) = f$, then g is a treelike expression of f.

Presentations by generators and relations

Let $\mathcal{O} = \mathbb{K}\langle C \rangle$ be an operad.

A presentation of $\mathcal{O}$ is a pair $(\mathcal{G}, \mathcal{R})$ such that

- $\mathcal{G}$ is a subcollection of C and is a minimal generating set of $\mathcal{O}$, called set of generators;
- $\mathcal{R}$ is a subset of $\mathbf{S}(\mathcal{G})$, called set of relations;
- by denoting by $\langle \mathcal{R} \rangle$ the operad ideal of $\mathbf{S}(C)$ generated by $\mathcal{R}$,

$$\mathcal{O} \simeq \mathbf{S}(\mathcal{G})/_{\langle \mathcal{R} \rangle}.$$

We say that a presentation $(\mathcal{G}, \mathcal{R})$ is

- quadratic if all trees appearing in $\mathcal{R}$ have degree 2;
- binary if all elements of $\mathcal{G}$ are of size 2.

By extension, $\mathcal{O}$ is quadratic (resp. binary) if $\mathcal{O}$ admits a quadratic (resp. binary) presentation.

Realizations

On the other way round, it is possible to define operads through a presentation.

In this way, a presentation specifies a quotient of a free operad.

A realization of a presentation $(\mathcal{G}, \mathcal{R})$ consists in

- a space $\mathcal{O} = \mathbb{K}\langle C \rangle$;
- an explicit description of the partial compositions maps $\circ_i$ on C;

such that $\mathcal{O} \simeq \mathbf{S}(\mathcal{G})/_{\langle \mathcal{R} \rangle}$.

Of course, there can be different realizations $\mathcal{O} = \mathbb{K}\langle C \rangle$ and $\mathcal{O}' = \mathbb{K}\langle C' \rangle$ of $(\mathcal{G}, \mathcal{R})$. We have necessarily that C and C' are isomorphic graded collections, and that $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic operads.

As we shall see, proving presentations and establishing realizations of operads use rewrite relations on syntax trees.

The duplicial operad $\mathbf{Dup}$ [**Loday**, 2008] is the operad admitting the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is the graded collection $\{\ll, \gg\}$ with $|\ll| = 2$ and $|\gg| = 2$, and $\mathcal{R}$ is the set containing the three $\mathrm{S}(\mathcal{G})$-polynomials

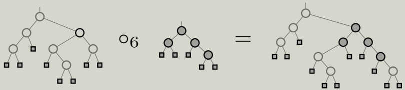
$$\mathrm{c}(\ll) \circ_1 \mathrm{c}(\ll) - \mathrm{c}(\ll) \circ_2 \mathrm{c}(\ll),$$

$$\mathrm{c}(\ll) \circ_1 \mathrm{c}(\gg) - \mathrm{c}(\gg) \circ_2 \mathrm{c}(\ll),$$

$$\mathrm{c}(\gg) \circ_1 \mathrm{c}(\gg) - \mathrm{c}(\gg) \circ_2 \mathrm{c}(\gg).$$

This operad is realized as $\mathbf{Dup} = \mathbb{K}\langle \mathrm{BT}_{\circ} - \{\square\}\rangle$ where for any binary trees t and s, t $\circ_i$ s is obtained by replacing the i-th internal node u of t by a copy of s, and by grafting the left (resp. right) subtree of u to the first (resp. last) leaf of the copy.

– Example –



The dendriform operad $\mathbf{Dendr}$ [**Loday**, 2001] is the operad admitting the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is the graded collection $\{\prec, \succ\}$ with $|\prec| = 2$ and $|\succ| = 2$, and $\mathcal{R}$ is the set containing the $\mathrm{S}(\mathcal{G})$-polynomials

$$\mathrm{c}(\prec) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\succ),$$

$$\mathrm{c}(\prec) \circ_1 \mathrm{c}(\succ) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\prec),$$

$$\mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec) + \mathrm{c}(\succ) \circ_1 \mathrm{c}(\succ) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\succ).$$

This operad is realized as $\mathbf{Dendr} = \mathbb{K} \langle \mathrm{BT}_{\circ} - \{\blacksquare\} \rangle$ where for any binary trees t and s, if the root of t is its i-th internal node, then

$$\mathsf{t} \circ_i \mathsf{s} = \sum_{\substack{\mathsf{q}, \mathsf{r} \in \mathrm{BT}_{\circ} \\ \mathsf{t} \cdot 1 / \mathsf{s} \cdot 1 \preccurlyeq \mathsf{q} \preccurlyeq \mathsf{t} \cdot 1 \backslash \mathsf{s} \cdot 1 \\ \mathsf{s} \cdot 2 / \mathsf{t} \cdot 2 \preccurlyeq \mathsf{r} \preccurlyeq \mathsf{s} \cdot 2 \backslash \mathsf{t} \cdot 2}} (\circ, (\mathsf{q}, \mathsf{r})),$$

where $\preccurlyeq$ is the Tamari order, and for binary trees p and p', p / p' (resp. $\mathsf{p} \backslash \mathsf{p}'$) is the binary tree obtained by grafting the root of p (resp. p') onto the first (resp. last) leaf of p' (resp. p).

Dendriform operad and generalizations

Dendriform operads are important devices used to **split associative operations** in two pieces.

There are in fact some rigidity theorems saying that associative operations that can be split by dendriform operations are free.

Several generalizations of the dendriform operad exist:

- tridendriform operad [**Loday, Ronco**, 2004], where associative products are split into three parts;
- quadendriform operad [**Aguiar, Loday**, 2004], where associative products are split into four parts;
- enneadendriform operad [**Leroux**, 2007], where associative products are split into nine parts;
- m-polydendriform operad [**G.**, 2016], where associative products are split into $2m$ parts, $m \geqslant 0$.
- m-polytridendriform operad [**G.**, 2016], where associative products are split into $2m+1$ parts, $m \geqslant 0$.

The diassociative operad

The diassociative operad $\mathbf{Dias}$ [**Loday**, 2001] is the operad admitting the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is the graded collection $\{\dashv, \vdash\}$ with $|\dashv| = 2$ and $|\vdash| = 2$, and $\mathcal{R}$ is the set containing the $\mathrm{S}(\mathcal{G})$-polynomials

$$\mathrm{c}(\dashv) \circ_1 \mathrm{c}(\dashv) - \mathrm{c}(\dashv) \circ_2 \mathrm{c}(\dashv), \quad \mathrm{c}(\dashv) \circ_1 \mathrm{c}(\dashv) - \mathrm{c}(\dashv) \circ_2 \mathrm{c}(\vdash),$$

$$\mathrm{c}(\dashv) \circ_1 \mathrm{c}(\vdash) - \mathrm{c}(\vdash) \circ_2 \mathrm{c}(\dashv),$$

$$\mathrm{c}(\vdash) \circ_1 \mathrm{c}(\dashv) - \mathrm{c}(\vdash) \circ_2 \mathrm{c}(\vdash), \quad \mathrm{c}(\vdash) \circ_1 \mathrm{c}(\vdash) - \mathrm{c}(\vdash) \circ_2 \mathrm{c}(\vdash).$$

This operad is realized as $\mathbf{Dias} = \mathbb{K}\langle W \rangle$ where W is the graded collection of all words on $\{0, 1\}$ having exactly one occurrence of 0, and where for any $u, v \in W$, $u \circ_i v$ is obtained by replacing the i-th letter of u by v if $u_i = 0$ and by $1^{|v|}$ otherwise.

– Examples –

$$101\mathbf{1}1 \circ_4 110 = 1011111$$

$$1\mathbf{0}111 \circ_2 110 = 1110111$$

Reduced set-operads

Let $\mathcal{O} = \mathbb{K} \langle C \rangle$ be an operad.

- If C is augmented and $\#C(1) = 1$, then $\mathcal{O}$ is reduced.
- If for any $x, y \in C$ and $i \in [|x|]$, $\mathrm{Supp}(x \circ_i y)$ is a singleton, then $\mathcal{O}$ is a set-operad.

– Example –

- $\mathbf{Pat}$ is a set-operad but is not reduced. Indeed, $\mathbf{Pat}(0) = \{\epsilon\}$ and $\mathbf{Pat}(1) = \mathbb{N}$.
- $\mathbf{Per}$ is a set-operad but is not reduced. Indeed, $\mathbf{Per}(0) = \{\epsilon\}$.
- $\mathbf{PLie}$ is a reduced operad but is not a set-operad.
- If G is augmented and $\mathsf{G}(1) = \emptyset$, then $\mathbf{S}(\mathsf{G})$ is a reduced set-operad.
- $\mathbf{Dup}$ is a reduced set-operad.
- $\mathbf{Dendr}$ is a reduced operad but not a set-operad.
- $\mathbf{Dias}$ is a reduced set-operad.

Minimal generating sets of reduced set-operads

Given a reduced set-operad $\mathcal{O} = \mathbb{K}\langle C \rangle$, $x \in C$ is indecomposable if for any $y, z \in C$ and $i \in [|y|]$, $x = y \circ_i z$ implies $(y, z) \in \{(x, \mathbb{1}), (\mathbb{1}, x)\}$.

– Example –

Let the reduced suboperad $\mathbf{Per}' := \mathbf{Per} - \mathbb{K}\langle \mathrm{Per}(0) \rangle$ of $\mathbf{Per}$.

The permutation 143562 is not indecomposable in $\mathbf{Per}'$ since $143562 = 1342 \circ_2 213$.

The permutation 4135726 is indecomposable in $\mathbf{Per}'$.

– Proposition –

If $\mathcal{O}$ is a reduced set-operad, then the set of all the indecomposable elements of $\mathcal{O}$ is a minimal generating set of $\mathcal{O}$.

– Example –

A permutation σ belongs to the minimal generating set of $\mathbf{Per}'$ iff $|\sigma| \geqslant 2$ and σ is simple: σ does not admit any factor of length between 2 and $|\sigma| - 1$ which is a segment.

The integer sequence associated with this graded collection is Sequence **A111111**, beginning by

$$0, 2, 0, 2, 6, 46, 338, 2926, 28146, 298526.$$

Proving presentations of set-operads

Our objective is, given a realization of a reduced set-operad $\mathcal{O} = \mathbb{K}\langle C \rangle$, to provide a presentation $(\mathcal{G}, \mathcal{R})$ of $\mathcal{O}$.

A tree polynomial f of the form $f = \mathsf{t} - \mathsf{t}'$ where $\mathsf{t}, \mathsf{t}' \in \mathrm{S}(C)$ is a nontrivial relation of $\mathcal{O}$ if $\mathsf{t} \neq \mathsf{t}'$ and $\mathrm{ev}(f) = 0$.

An orientation of a set $\mathcal{R}$ of nontrivial relations is a rewrite relation $\to$ such that, for any $\mathsf{t} - \mathsf{t}' \in \mathcal{R}$, then either $\mathsf{t} \to \mathsf{t}'$ or $\mathsf{t}' \to \mathsf{t}$.

– Proposition –

Let $\mathcal{R}$ be a set of C-tree polynomials of the form $\mathsf{t} - \mathsf{t}'$ with $\mathsf{t} \neq \mathsf{t}'$.

Let $\to$ be an orientation of $\mathcal{R}$.

The operad ideal $\langle \mathcal{R} \rangle$ of $\mathbf{S}(C)$ is the linear span of all the C-tree polynomials g of the form $g = \mathsf{s} - \mathsf{s}'$ such that $\mathsf{s} \overset{*}{\Leftrightarrow} \mathsf{s}'$.

Proving presentations of set-operads

– Theorem –

Let $\mathcal{O} := \mathbb{K}\langle C \rangle$ be a reduced set-operad. If

- $\mathcal{G}$ is a minimal generating set of $\mathcal{O}$;
- $\mathcal{R}$ is a set of nontrivial relations of $\mathcal{O}$;
- there exists a terminating and confluent orientation $\rightarrow$ of $\mathcal{R}$ such that $\mathcal{F}_{\rightarrow} \simeq C$;

then the pair $(\mathcal{G}, \mathcal{R})$ is a presentation of $\mathcal{O}$.

– Proposition –

Let $\mathcal{O} := \mathbb{K}\langle C \rangle$ be a reduced set-operad where C is combinatorial. If

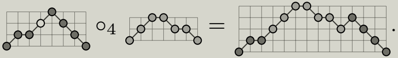
- $\mathcal{G}$ is a minimal generating set of $\mathcal{O}$;
- $\mathcal{R}$ is a set of nontrivial relations of $\mathcal{O}$;
- there exists a terminating orientation $\rightarrow$ of $\mathcal{R}$ such that $\mathcal{F}_{\rightarrow} \simeq C$;

then the pair $(\mathcal{G}, \mathcal{R})$ is a presentation of $\mathcal{O}$.

– Example 1/2 –

Let **Motz** [G., 2015] be the subcollection of **Pat** restrained on all nonnempty Motzkin paths.

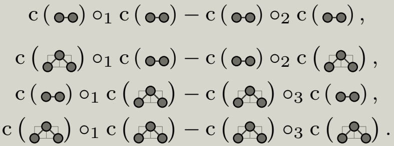
It is easy to show that **Motz** is a suboperad of **Pat**, and that it is a reduced set-operad. For instance,



By induction on the arities, one can show that

is a minimal generating set of **Motz**.

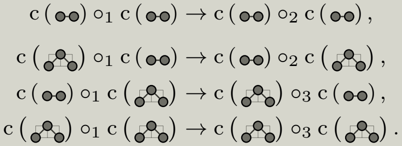
The only quadratic nontrivial relations on $\mathcal{G}$-trees are



Let $\mathcal{R}$ be the set of these four relations. The question now is to prove that **there are no further nontrivial relations** involving trees of higher degrees.

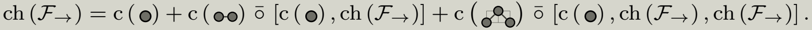
– Example 2/2 –

Let $\rightarrow$ be the rewrite relation on $\mathcal{G}$-trees, defined as the orientation of $\mathcal{R}$ satisfying



First, $\rightarrow$ is terminating since the map τ seen previously is a termination invariant.

The normal forms for $\rightarrow$ are the $\mathcal{G}$-trees avoiding the four trees appearing as left members of $\rightarrow$. By an obvious description of these trees, one obtains that the characteristic series of $\mathcal{F}_{\rightarrow}$ satisfies



The trace of this formal power series is the generating series of $\mathcal{F}_{\rightarrow}$ which satisfies

$$\mathcal{G}_{\mathcal{F}_{\rightarrow}}(t) = t + t\mathcal{G}_{\mathcal{F}_{\rightarrow}}(t) + t\mathcal{G}_{\mathcal{F}_{\rightarrow}}(t)^2$$

This is the generating series of Motzkin paths, so that $\mathcal{F}_{\rightarrow} \simeq M$, where M is the graded collection of all Motzkin paths.

This proves that $\mathbf{Motz}$ contains exactly the four previous nontrivial relations.

4.4 Algebras over operads

Algebras over an operad

Let $\mathcal{O}$ be an operad.

A space $\mathcal{A}$ is an $\mathcal{O}$-algebra if there are, for all $n \geqslant 0$, linear maps

$$\curlywedge^{(n)} : \mathcal{O}(n) \otimes \mathcal{A}^{\otimes n} \to \mathcal{A}$$

such that, by writing simply $f(a_1, \ldots, a_n)$ instead of $\curlywedge^{(n)}(f \otimes a_1 \otimes \cdots \otimes a_n)$ for any $f \in \mathcal{O}(n)$ and any $a_1, \ldots, a_n \in \mathcal{A}$, the two relations

- for any $a \in \mathcal{A}$,
$$\mathbb{1}(a) = a;$$
- for any $f \in \mathcal{O}(n)$, $g \in \mathcal{O}(m)$, and $a_1, \ldots, a_{n+m-1} \in \mathcal{A}$,
$$(f \circ_i g)(a_1, \ldots, a_{n+m-1}) \\ = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1});$$

are satisfied.

Algebras over an operad and presentations

Let $\mathcal{O}$ be an operad and $(\mathcal{G}, \mathcal{R})$ be a presentation of $\mathcal{O}$.

From the previous definition, a space $\mathcal{A}$ is an $\mathcal{O}$-algebra if and only

- $\mathcal{A}$ is endowed with operations $\left\{x : \mathcal{A}^{\otimes |x|} \to \mathcal{A} : x \in \mathcal{G}\right\}$;
- For any $f \in \mathcal{R}$ and any $a_1, \dots, a_{|f|} \in \mathcal{A}$, $(\mathrm{ev}(f))\left(a_1, \dots, a_{|f|}\right) = 0$.

– Example –

Let $\mathcal{A}$ be a $\mathbf{Motz}$-algebra.

The space $\mathcal{A}$ is endowed with two generating operations

$$\circ\!\!-\!\!\circ : \mathcal{A}^{\otimes 2} \to \mathcal{A} \quad \text{and} \quad \overset{\circ}{\circ\,\circ} : \mathcal{A}^{\otimes 3} \to \mathcal{A},$$

satisfying

$$\circ\!\!-\!\!\circ\left(\circ\!\!-\!\!\circ(a_1, a_2), a_3\right) - \circ\!\!-\!\!\circ\left(a_1, \circ\!\!-\!\!\circ(a_2, a_3)\right) = 0,$$

$$\overset{\circ}{\circ\,\circ}\left(\circ\!\!-\!\!\circ(a_1, a_2), a_3, a_4\right) - \circ\!\!-\!\!\circ\left(a_1, \overset{\circ}{\circ\,\circ}(a_2, a_3, a_4)\right) = 0,$$

$$\circ\!\!-\!\!\circ\left(\overset{\circ}{\circ\,\circ}(a_1, a_2, a_3), a_4\right) - \circ\!\!-\!\!\circ\left(a_1, \overset{\circ}{\circ\,\circ}(a_2, a_3, a_4)\right) = 0,$$

$$\overset{\circ}{\circ\,\circ}\left(\overset{\circ}{\circ\,\circ}(a_1, a_2, a_3), a_4, a_5\right) - \overset{\circ}{\circ\,\circ}\left(a_1, a_2, \overset{\circ}{\circ\,\circ}(a_3, a_4, a_5)\right) = 0.$$

Categories and constructions

Any operad $\mathcal{O}$ gives rise to a category, the category of $\mathcal{O}$-algebras, wherein

- objects are all $\mathcal{O}$-algebras;
- arrows are all linear maps $\phi : \mathcal{A} \to \mathcal{A}'$ such that, for any $f \in \mathcal{O}(n)$ and $a_1, \ldots, a_n \in \mathcal{A}$,

$$\phi\left(f\left(a_1, \ldots, a_n\right)\right) = f\left(\phi\left(a_1\right), \ldots, \phi\left(a_n\right)\right).$$

The space $\mathcal{O}$ is an $\mathcal{O}$-algebra itself by setting, for any $f \in \mathcal{O}(n)$ and $a_1, \ldots, a_n \in \mathcal{O}$, that $f\left(a_1, \ldots, a_n\right) := f \circ [a_1, \ldots, a_n]$. This is the free $\mathcal{O}$-algebra over one generator.

– Theorem –

Let $\mathcal{O}$ and $\mathcal{O}'$ be two operads and $\phi : \mathcal{O} \to \mathcal{O}'$ be an operad morphism.

If $\mathcal{A}$ is an $\mathcal{O}'$-algebra, then by defining for each $f \in \mathcal{O}(n)$ the linear map f from $\mathcal{A}^{\otimes n}$ to $\mathcal{A}$ by

$$f\left(a_1, \ldots, a_n\right) := \left(\phi(f)\right)\left(a_1, \ldots, a_n\right),$$

the space $\mathcal{A}$ becomes an $\mathcal{O}$-algebra.

From dendriform algebras to associative algebras

The associative operad $\mathbf{As}$ is the operad $\mathbb{K}\langle\mathbb{N}\rangle$ where, for any $a, b \in \mathbb{N}$, $a \circ_i b := a + b - 1$. This operad admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \{\star\}$ with $|\star| = 2$, and $\mathcal{R}$ contains the single relation

$$\mathrm{c}(\star) \circ_1 \mathrm{c}(\star) - \mathrm{c}(\star) \circ_2 \mathrm{c}(\star).$$

Let $\phi : \mathbf{As} \to \mathbf{Dendr}$ be the unique operad morphism satisfying

$$\phi(\star) := \prec + \succ .$$

Since, by using the relations of $\mathbf{Dendr}$,

$$\phi(\star) \circ_1 \phi(\star) - \phi(\star) \circ_2 \phi(\star) = 0,$$

this morphism is well-defined.

Therefore, ϕ describes a construction from $\mathbf{Dendr}$-algebras to $\mathbf{As}$-algebras: given a dendriform algebra $(\mathcal{A}, \prec, \succ)$, one obtains an $\mathbf{As}$-algebra $(\mathcal{A}, \star)$ by setting

$$f \star g := f \prec g + f \succ g$$

for any $f, g \in \mathcal{A}$.

4.5 Koszul duality and Koszulity

Koszul duality

Let $\mathcal{O}$ be a binary and quadratic operad admitting the presentation $(\mathcal{G}, \mathcal{R})$.

The Koszul dual of $\mathcal{O}$ is the operad $\mathcal{O}^!$ isomorphic to the operad admitting the presentation $\left(\mathcal{G}, \mathcal{R}^\perp\right)$ where $\mathcal{R}^\perp$ is a basis of the annihilator in $\mathbf{S}(\mathcal{G})(3)$ of the linear span of $\mathcal{R}$ w.r.t. the linear map

$$\langle -, - \rangle : \mathbf{S}(\mathcal{G})(3) \otimes \mathbf{S}(\mathcal{G})(3) \to \mathbb{K},$$

linearly defined, for any $x, x', y, y' \in \mathcal{G}$ by

$$\langle \mathrm{c}(x) \circ_i \mathrm{c}(y), \mathrm{c}\left(x'\right) \circ_{i'} \mathrm{c}\left(y'\right) \rangle := \begin{cases} 1 & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1 & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, from the knowledge of a presentation of $\mathcal{O}$, one can compute a presentation of $\mathcal{O}^!$.

The Koszul pair $(\mathbf{Dendr}, \mathbf{Dias})$

Let $(\mathcal{G}, \mathcal{R})$ be the usual presentation of $\mathbf{Dendr}$.

A tree polynomial $f := \sum_{\mathsf{t} \in \mathrm{S}(\mathcal{G})(3)} \alpha_{\mathsf{t}}\, \mathsf{t}$ belongs to the linear span of $\mathcal{R}^\perp$ iff

$$\langle f, \mathrm{c}(\prec) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\succ) \rangle = 0,$$

$$\langle f, \mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\prec) \rangle = 0,$$

$$\langle f, \mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec) + \mathrm{c}(\succ) \circ_1 \mathrm{c}(\succ) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\succ) \rangle = 0.$$

This is equivalent to

$$\alpha_{\mathrm{c}(\prec) \circ_1 \mathrm{c}(\prec)} + \alpha_{\mathrm{c}(\prec) \circ_2 \mathrm{c}(\prec)} + \alpha_{\mathrm{c}(\prec) \circ_2 \mathrm{c}(\succ)} = 0,$$

$$\alpha_{\mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec)} + \alpha_{\mathrm{c}(\succ) \circ_2 \mathrm{c}(\prec)} = 0,$$

$$\alpha_{\mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec)} + \alpha_{\mathrm{c}(\succ) \circ_1 \mathrm{c}(\succ)} + \alpha_{\mathrm{c}(\succ) \circ_2 \mathrm{c}(\succ)} = 0.$$

Therefore, a basis of $\mathcal{R}^\perp$ is

$$\{\mathrm{c}(\prec) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\prec), \qquad \mathrm{c}(\prec) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\prec) \circ_2 \mathrm{c}(\succ),$$
$$\mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\prec),$$
$$\mathrm{c}(\succ) \circ_1 \mathrm{c}(\prec) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\succ), \qquad \mathrm{c}(\succ) \circ_1 \mathrm{c}(\succ) - \mathrm{c}(\succ) \circ_2 \mathrm{c}(\succ)\}.$$

We recognize the relations of $\mathbf{Dias}$.

Koszulity

Formally, an operad $\mathcal{O}$ is a Koszul operad if the Koszul complex of $\mathcal{O}$ is acyclic. Moreover, $\mathcal{O}$ is Koszul iff $\mathcal{O}^!$ is Koszul.

There is a combinatorial criterion to prove that some operads are Koszul involving rewrite relations on trees:

– Theorem –

Let $\mathcal{O} = \mathbb{K}\langle C \rangle$ be a binary and quadratic operad. If $\mathcal{O}$ admits a presentation $(\mathcal{G}, \mathcal{R})$ and a terminating and confluent orientation $\to$, then $\mathcal{O}$ is a Koszul operad.

– Example –

The rewrite relation

$$\mathrm{c}(\dashv) \circ_2 \mathrm{c}(\dashv) \to \mathrm{c}(\dashv) \circ_1 \mathrm{c}(\dashv), \qquad \mathrm{c}(\dashv) \circ_2 \mathrm{c}(\vdash) \to \mathrm{c}(\dashv) \circ_1 \mathrm{c}(\dashv),$$

$$\mathrm{c}(\vdash) \circ_2 \mathrm{c}(\dashv) \to \mathrm{c}(\dashv) \circ_1 \mathrm{c}(\vdash),$$

$$\mathrm{c}(\vdash) \circ_1 \mathrm{c}(\dashv) \to \mathrm{c}(\vdash) \circ_2 \mathrm{c}(\vdash), \qquad \mathrm{c}(\vdash) \circ_1 \mathrm{c}(\vdash) \to \mathrm{c}(\vdash) \circ_2 \mathrm{c}(\vdash),$$

is terminating and convergent, and is an orientation of the set of nontrivial relations of $\mathbf{Dias}$. Therefore, $\mathbf{Dias}$ and $\mathbf{Dendr} \simeq \mathbf{Dias}^!$ are Koszul.

Koszulity and generating series

– Theorem –

Let $\mathcal{O} = \mathbb{K}\langle C \rangle$ be a Koszul, binary, and quadratic operad where C is combinatorial. Then,

$$\mathcal{G}_C\left(-\mathcal{G}_{C^!}(-t)\right) = t = \mathcal{G}_{C^!}\left(-\mathcal{G}_C(-t)\right)$$

where $\mathcal{O}^! = \mathbb{K}\left\langle C^! \right\rangle$.

– Example –

The generating series $\mathcal{G}(t)$ of the underlying collection of $\mathbf{Dendr}$ satisfies $\mathcal{G}(t) = t + 2t\mathcal{G}(t) + t\mathcal{G}(t)^2$ so that $t = \mathcal{G}(t)\left(1 + \mathcal{G}(t)\right)^{-2}$.

Let $\mathcal{G}'(t)$ be the generating series of the underlying collection of $\mathbf{Dias}$.

By substitution, we obtain

$$-\mathcal{G}'(-t) = \mathcal{G}\left(-\mathcal{G}'(-t)\right)\left(1 + \mathcal{G}\left(-\mathcal{G}'(-t)\right)\right)^{-2}$$

and by the previous theorem,

$$\mathcal{G}'(t) = t(1-t)^{-2} = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + \cdots .$$

Generic realizations

There is a generic way to build realizations of presentations of (in particular) Koszul operads.

Let $(\mathcal{G}, \mathcal{R})$ be an operad presentation, and $\to$ be a terminating and confluent orientation of $\to$.

Let $\mathcal{O}_{(\mathcal{G},\mathcal{R}),\to}$ be the operad on the space $\mathbb{K}\langle \mathcal{F}_{\to} \rangle$ wherein

- for any $\mathsf{t}, \mathsf{s} \in \mathcal{F}_{\to}$, the partial composition $\mathsf{t} \circ_i \mathsf{s}$ is the unique normal form r such that $\mathsf{t} \circ_i \mathsf{s} \overset{*}{\Rightarrow} \mathsf{r}$ (here, $\circ_i$ is the partial composition of syntax trees);
- the unit is the tree $|$.

– Proposition –

The operad $\mathcal{O}_{(\mathcal{G},\mathcal{R}),\to}$ admits $(\mathcal{G}, \mathcal{R})$ as presentation.

In particular, when $(\mathcal{G}, \mathcal{R})$ is binary and quadratic, this presentation is the one of a Koszul operad and one can build in this way a realization of this presentation.

Generic realizations

– Example –

Let $\mathbf{DA}$ [G., 2015] be the operad admitting the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is the graded collection $\{\mathsf{a}, \mathsf{b}\}$ whith $|\mathsf{a}| = 2$ and $|\mathsf{b}| = 2$, and $\mathcal{R}$ is the set containing the four $\mathrm{S}(\mathcal{G})$-polynomials

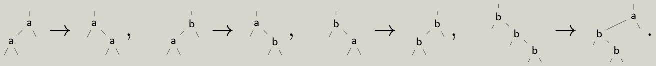
$$\mathrm{c}(\mathsf{a}) \circ_1 \mathrm{c}(\mathsf{a}) - \mathrm{c}(\mathsf{a}) \circ_2 \mathrm{c}(\mathsf{a}),$$

$$\mathrm{c}(\mathsf{b}) \circ_1 \mathrm{c}(\mathsf{a}) - \mathrm{c}(\mathsf{a}) \circ_2 \mathrm{c}(\mathsf{b}),$$

$$\mathrm{c}(\mathsf{b}) \circ_1 \mathrm{c}(\mathsf{b}) - \mathrm{c}(\mathsf{b}) \circ_2 \mathrm{c}(\mathsf{a}),$$

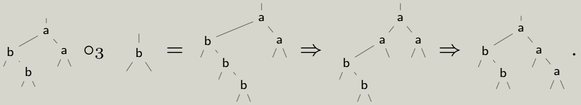
$$(\mathrm{c}(\mathsf{a}) \circ_1 \mathrm{c}(\mathsf{b})) \circ_2 \mathrm{c}(\mathsf{b}) - (\mathrm{c}(\mathsf{b}) \circ_2 \mathrm{c}(\mathsf{b})) \circ_3 \mathrm{c}(\mathsf{b}).$$

Let $\to$ be the rewrite relation on $\mathrm{S}(\mathsf{G})$ satisfying



This rewrite relation is an orientation of $\mathcal{R}$ and is terminating and convergent.

In the realization $\mathcal{O}_{(\mathcal{G},\mathcal{R}),\to}$ of $\mathbf{DA}$, one has



4.6 Exercises

About presentations

– Exercise –

Let $\mathcal{O} = \mathbb{K}\langle C \rangle$ be a reduced set operad. When C is combinatorial, propose an algorithm to compute up to a given arity $n \geqslant 1$, the minimal generating set of $\mathcal{O}$.

– Exercise –

Describe a minimal generating set of the operad $\mathbf{Pat}$.

Describe a minimal generating set of the operad $\mathbf{Pat}' := \mathbf{Pat} - \mathbb{K}\langle P(0) \rangle$.

– Exercise –

Describe a minimal generating set of the operad $\mathbf{Per}$.

About Koszul duality

– Exercise –

1. Compute a presentation for the Koszul dual of the operad $\mathbf{Dup}$.
2. Prove that $\mathbf{Dup}$ is a Koszul operad.
3. Compute the generating series of the underlying collection of $\mathbf{Dup}^!$.

– Exercise –

The operad $\mathbf{BS}$ is the operad admitting the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is the graded collection $\{\star_0, \star_1\}$ with $|\star_0| = 2$ and $|\star_1| = 2$, and $\mathcal{R}$ is the set containing the $\mathrm{S}(\mathcal{G})$-polynomials

$$\mathrm{c}\,(\star_0) \circ_1 \mathrm{c}\,(\star_0) - \mathrm{c}\,(\star_0) \circ_2 \mathrm{c}\,(\star_0)$$
$$\mathrm{c}\,(\star_1) \circ_1 \mathrm{c}\,(\star_1) - \mathrm{c}\,(\star_1) \circ_2 \mathrm{c}\,(\star_1)\,.$$

This operad is sometimes called the two-associative operad [**Loday, Ronco**, 2006].

1. Compute a presentation for the Koszul dual of the operad $\mathbf{BS}$.
2. Prove that $\mathbf{BS}$ is a Koszul operad.
3. By considering first the generic realization of $\mathbf{BS}$, construct a realization of this operad.