Tree rewriting and enumeration

Samuele Giraudo
LIGM, Université Paris-Est Marne-la-Vallée

GT Combinatoire Énumérative et Algébrique, LaBRI

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Outline

Trees, patterns, and rewrite systems

Tree series and pattern avoidance

Operads and enumeration
Outline

Trees, patterns, and rewrite systems
Syntax trees

A set of letters is a graded set

\[ \mathcal{G} := \bigsqcup_{n \geq 1} \mathcal{G}(n) \]

such that each \( \mathcal{G}(n) \) is finite.

Example: Let \( \mathcal{G} := \mathcal{G}(2) \sqcup \mathcal{G}(3) \) such that \( \mathcal{G}(2) = \{a, b\} \) and \( \mathcal{G}(3) = \{c\} \). Here is a \( \mathcal{G} \)-tree:

```
  c
 / \  \\
 b  c
 / \  \\
 b  a
```
Syntax trees

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such that each \( \mathcal{G}(n) \) is finite.

A syntax tree on \( \mathcal{G} \) (called \( \mathcal{G} \)-tree) is a planar rooted tree \( t \) such that each internal node of arity \( n \) is labeled by a letter of \( \mathcal{G}(n) \).
Syntax trees

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Let \( \mathcal{G} := \mathcal{G}(2) \sqcup \mathcal{G}(3) \) such that \( \mathcal{G}(2) = \{a, b\} \) and \( \mathcal{G}(3) = \{c\} \).

Here is a \( \mathcal{G} \)-tree:
Sets of syntax trees

Let $\mathcal{G}$ be a set of letters.

The set of all $\mathcal{G}$-trees is denoted by $F(\mathcal{G})$. 
Sets of syntax trees

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The set of all $\mathcal{G}$-trees is denoted by $F(\mathcal{G})$.

For any $t \in F(\mathcal{G})$, let

- $|t|$ be the arity of $t$, that is its number of leaves;
Sets of syntax trees

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The set of all $\mathcal{G}$-trees is denoted by $F(\mathcal{G})$.

For any $t \in F(\mathcal{G})$, let

- $|t|$ be the **arity** of $t$, that is its number of leaves;
- $\text{deg}(t)$ be the **degree** of $t$, that is the number of internal nodes of $t$;
Sets of syntax trees

Let \( \mathcal{G} \) be a set of letters.

The set of all \( \mathcal{G} \)-trees is denoted by \( F(\mathcal{G}) \).

For any \( t \in F(\mathcal{G}) \), let

- \( |t| \) be the arity of \( t \), that is its number of leaves;
- \( \deg(t) \) be the degree of \( t \), that is the number of internal nodes of \( t \);
- \( a(t) \) be the number of edges of \( t \) (satisfying \( a(t) = |t| + \deg(t) \)).

Therefore, \( F(\mathcal{G}) = \bigcup_{n \geq 1} F(\mathcal{G})(n) \).

Remark: since each \( \mathcal{G}(n) \) is finite, if \( \mathcal{G}(1) = \emptyset \), then all \( F(\mathcal{G})(n) \) are finite.
Sets of syntax trees

Let $\mathcal{G}$ be a set of letters.

The set of all $\mathcal{G}$-trees is denoted by $F(\mathcal{G})$.

For any $t \in F(\mathcal{G})$, let

- $|t|$ be the arity of $t$, that is its number of leaves;
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We set $F(\mathcal{G})(n)$ as the set of the $\mathcal{G}$-trees of arity $n$.

Therefore,

$$F(\mathcal{G}) = \bigsqcup_{n \geq 1} F(\mathcal{G})(n).$$
Sets of syntax trees

Let $\mathcal{G}$ be a set of letters.

The set of all $\mathcal{G}$-trees is denoted by $\mathbf{F}(\mathcal{G})$.

For any $t \in \mathbf{F}(\mathcal{G})$, let

- $|t|$ be the arity of $t$, that is its number of leaves;
- $\text{deg}(t)$ be the degree of $t$, that is the number of internal nodes of $t$;
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We set $\mathbf{F}(\mathcal{G})(n)$ as the set of the $\mathcal{G}$-trees of arity $n$.

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$$\mathbf{F}(\mathcal{G}) = \bigsqcup_{n \geq 1} \mathbf{F}(\mathcal{G})(n).$$

Remark: since each $\mathcal{G}(n)$ is finite, if $\mathcal{G}(1) = \emptyset$, then all $\mathbf{F}(\mathcal{G})(n)$ are finite.
Partial composition

Let $t, s \in F(\mathcal{G})$.

For each $i \in [|t|]$, $t \circ_i s$ is the tree obtained by grafting the root of a copy of $s$ onto the $i$th leaf of $t$.

Example
Partial composition

Let $t, s \in \mathbf{F}(G)$.

For each $i \in [|t|]$, $t \circ_i s$ is the tree obtained by grafting the root of a copy of $s$ onto the $i$th leaf of $t$.

Example

\[
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{c} \\
\text{a} \\
\text{b} \\
\text{b}
\end{array}
\end{array}
\quad \circ_5
\begin{array}{c}
\begin{array}{c}
\text{b} \\
\text{a} \\
\text{c} \\
\text{a} \\
\text{b}
\end{array}
\end{array}
= \\
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{c} \\
\text{a} \\
\text{b} \\
\text{b}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{c} \\
\text{a} \\
\text{b} \\
\text{b}
\end{array}
\end{array}
\end{array}
\]

Therefore, $\circ_i$ is a map

\[\circ_i : \mathbf{F}(G)(n) \times \mathbf{F}(G)(m) \rightarrow \mathbf{F}(G)(n + m - 1)\]

where $i \in [n]$ and $1 \leq m$, called partial composition map.
Complete composition

Let $t, s_1, \ldots, s_{|t|} \in F(\mathcal{G})$.

The $t \circ [s_1, \ldots, s_{|t|}]$ is obtained by grafting simultaneously the roots of copies of the $s_i$ onto the $i$th leaves of $t$.

Example

\[
\begin{bmatrix}
  a & a \\
  a & \, \, i, \, c
\end{bmatrix}
\]

\[
\xrightarrow{\text{}}
\]

\[
\begin{bmatrix}
  a & a \\
  a & a & a & c
\end{bmatrix}
\]
Let $t, s_1, \ldots, s_{|t|} \in F(G)$.

The $t \circ [s_1, \ldots, s_{|t|}]$ is obtained by grafting simultaneously the roots of copies of the $s_i$ onto the $i$th leaves of $t$.

**Example**

Therefore, $\circ$ is a map

$$\circ : F(G)(n) \times F(G)(m_1) \times \cdots \times F(G)(m_n) \rightarrow F(G)(m_1 + \cdots + m_n)$$

where $1 \leq n$ and $1 \leq m_1, \ldots, m_n$, called complete composition map.
Patterns and occurrences

Let $t, s \in F(G)$. A $G$-tree $t$ admits an occurrence of a $G$-tree $s$ if one can put $s$ onto $t$ by superimposing the root of $s$ and a node of $t$ and leaves of $s$ with leaves of nodes of $t$.

This property is denoted by $s \preceq t$. 

Patterns and occurrences

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Example

This relation $\preceq$ endows $F(\mathcal{G})$ with the structure of a poset.
Patterns and occurrences

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This property is denoted by $s \preceq t$.

Example

![Diagram of trees](image)

This relation $\preceq$ endows $\mathbf{F}(\mathcal{G})$ with the structure of a poset.

More formally, $s \preceq t$ holds if there exist $r, r_1, \ldots, r_{|s|} \in \mathbf{F}(\mathcal{G})$ and $i \in [|r|]$ such that

$$t = r \circ_i (s \circ [r_1, \ldots, r_{|s|}]).$$
Pattern avoidance

Given a set $\mathcal{P} \subseteq F(\mathcal{G})$, let $A(\mathcal{P})$ be the set of all $\mathcal{G}$-trees avoiding all patterns of $\mathcal{P}$.

Counting the elements of $A(\mathcal{P})$ w.r.t. the arity is a usual question.
Pattern avoidance

Given a set \( \mathcal{P} \subseteq \mathbf{F}(\mathcal{G}) \), let \( A(\mathcal{P}) \) be the set of all \( \mathcal{G} \)-trees avoiding all patterns of \( \mathcal{P} \).

Counting the elements of \( A(\mathcal{P}) \) w.r.t. the arity is a usual question.

Examples

▶ For \( \mathcal{P} := \{ \begin{array}{l} a \, a, \quad a \, b, \quad b \, a, \quad b \, b \end{array} \} \), \( A(\mathcal{P}) \) is enumerated by

\[
1, 2, 4, 8, 16, 32, 64, 128, \ldots ;
\]
Pattern avoidance

Given a set \( \mathcal{P} \subseteq \mathbf{F}(\mathcal{G}) \), let \( A(\mathcal{P}) \) be the set of all \( \mathcal{G} \)-trees avoiding all patterns of \( \mathcal{P} \).

Counting the elements of \( A(\mathcal{P}) \) w.r.t. the arity is a usual question.

Examples

- For \( \mathcal{P} := \{ a, b, a, b \} \), \( A(\mathcal{P}) \) is enumerated by
  \[
  1, 2, 4, 8, 16, 32, 64, 128, \ldots
  \]

- For \( \mathcal{P} := \{ a, c, a, c \} \), \( A(\mathcal{P}) \) is enumerated by
  \[
  1, 1, 2, 4, 9, 21, 51, 127, \ldots
  \]
Pattern avoidance

Given a set $\mathcal{P} \subseteq \mathcal{F}(\mathcal{G})$, let $A(\mathcal{P})$ be the set of all $\mathcal{G}$-trees avoiding all patterns of $\mathcal{P}$.

Counting the elements of $A(\mathcal{P})$ w.r.t. the arity is a usual question.

Examples

- For $\mathcal{P} := \begin{Bmatrix} a, & b, & a, & b, \\ \text{a} & \text{b} & \text{a} & \text{b} \end{Bmatrix}$, $A(\mathcal{P})$ is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$;

- For $\mathcal{P} := \begin{Bmatrix} a, & c, & a, & c, \\ \text{a} & \text{c} & \text{a} & \text{c} \end{Bmatrix}$, $A(\mathcal{P})$ is enumerated by $1, 1, 2, 4, 9, 21, 51, 127, \ldots$;

- For $\mathcal{P} := \begin{Bmatrix} a, & b, & b, \\ \text{a} & \text{b} & \text{a} & \text{b} \end{Bmatrix}$, $A(\mathcal{P})$ is enumerated by $1, 2, 5, 13, 35, 96, 167, 750, \ldots$. 
Rewrite rules

A rewrite rule is a binary relation $\rightarrow$ on $\mathbf{F}(\mathfrak{G})$ such that $s \rightarrow s'$ implies $|s| = |s'|$. 

Example

If $\rightarrow$ is the rewrite rule satisfying $b a a \rightarrow a b b$, one has $a a a a a a a a b \Rightarrow a b a a b a a a a a$. 


Rewrite rules

A rewrite rule is a binary relation $\to$ on $F(\mathcal{G})$ such that $s \to s'$ implies $|s| = |s'|$.

The rewrite relation induced by $\to$ is the binary relation $\Rightarrow$ on $F(\mathcal{G})$ satisfying

$$r \circ_i (s \circ [r_1, \ldots, r_{|s|}]) \Rightarrow r \circ_i (s' \circ [r_1, \ldots, r_{|s|}])$$

if $s \to s'$, where $r$ and $r_1, \ldots, r_{|s|}$ are any $\mathcal{G}$-trees.
Rewrite rules

A rewrite rule is a binary relation $\rightarrow$ on $\mathbf{F}(\mathcal{G})$ such that $s \rightarrow s'$ implies $|s| = |s'|$.

The rewrite relation induced by $\rightarrow$ is the binary relation $\Rightarrow$ on $\mathbf{F}(\mathcal{G})$ satisfying

$$r \circ_i (s \circ [r_1, \ldots, r_{|s|}]) \Rightarrow r \circ_i (s' \circ [r_1, \ldots, r_{|s|}])$$

if $s \rightarrow s'$, where $r$ and $r_1, \ldots, r_{|s|}$ are any $\mathcal{G}$-trees.

Example

If $\rightarrow$ is the rewrite rule satisfying $\begin{array}{c} \text{a} \text{b} \\ \text{a} \end{array} \rightarrow \begin{array}{c} \text{a} \\ \text{b} \end{array}$, one has

$$\begin{array}{c} \text{a} \text{b} \\ \text{a} \text{a} \\ \text{a} \end{array} \Rightarrow \begin{array}{c} \text{a} \\ \text{b} \end{array} \begin{array}{c} \text{a} \\ \text{a} \text{a} \\ \text{a} \end{array}.$$
Rewrite systems

Let $\rightarrow$ a rewrite rule on $\mathcal{G}$-trees and $\Rightarrow$ be the rewrite relation induced by $\rightarrow$.

Let us define

- $\ast \Rightarrow$ as the reflexive and transitive closure of $\Rightarrow$;
- $\ast \iff$ as the reflexive, symmetric, and transitive closure of $\Rightarrow$.

A tree $t$ rewrites into a tree $t'$ if $t \ast \Rightarrow t'$.

Two trees $t$ and $t'$ are linked if $t \ast \iff t'$.

Let $F(\mathcal{G})/\ast \iff$ be the set of all $\ast \iff$-equivalence classes.

A normal form for $\Rightarrow$ is a tree $t$ such that $t \ast \Rightarrow t'$ implies $t = t'$.

Let $N\Rightarrow$ be the set of all normal forms.
Rewrite systems

Let → a rewrite rule on $\mathcal{G}$-trees and $\Rightarrow$ be the rewrite relation induced by →.

Let us define

- $\Rightarrow^*$ as the reflexive and transitive closure of $\Rightarrow$;
- $\Rightarrow^* \Leftrightarrow$ as the reflexive, symmetric, and transitive closure of $\Rightarrow$.

A tree $t$ rewrites into a tree $t'$ if $t \Rightarrow^* t'$. 

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Rewrite systems

Let \( \rightarrow \) a rewrite rule on \( \mathcal{G} \)-trees and \( \Rightarrow \) be the rewrite relation induced by \( \rightarrow \).

Let us define

- \( \Rightarrow^* \) as the reflexive and transitive closure of \( \Rightarrow \);
- \( \Leftrightarrow^* \) as the reflexive, symmetric, and transitive closure of \( \Rightarrow \).

A tree \( t \) rewrites into a tree \( t' \) if \( t \Rightarrow^* t' \).

Two trees \( t \) and \( t' \) are linked if \( t \Leftrightarrow^* t' \). Let \( F(\mathcal{G})/\Leftrightarrow^* \) be the set of all \( \Leftrightarrow^* \)-equivalence classes.
Rewrite systems

Let $\rightarrow$ a rewrite rule on $\mathcal{G}$-trees and $\Rightarrow$ be the rewrite relation induced by $\rightarrow$.

Let us define

- $\Rightarrow^*$ as the reflexive and transitive closure of $\Rightarrow$;
- $\Rightarrow^\ast$ as the reflexive, symmetric, and transitive closure of $\Rightarrow$.

A tree $t$ rewrites into a tree $t'$ if $t \Rightarrow^* t'$.

Two trees $t$ and $t'$ are linked if $t \Rightarrow^\ast t'$. Let $\mathbf{F}(\mathcal{G})/\Rightarrow^\ast$ be the set of all $\Rightarrow^\ast$-equivalence classes.

A normal form for $\Rightarrow$ is a tree $t$ such that $t \Rightarrow^* t'$ implies $t = t'$. Let $\mathcal{N}_\Rightarrow$ be the set of all normal forms.
Termination and confluence

When there is no infinite chain $t_0 \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \cdots$, the rewrite relation $\Rightarrow$ is terminating.
Termination and confluence

When there is no infinite chain \( t_0 \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \cdots \), the rewrite relation \( \Rightarrow \) is terminating.

When \( t \Rightarrow^* s_1 \) and \( t \Rightarrow^* s_2 \) implies the existence of \( t' \) such that \( s_1 \Rightarrow t' \) and \( s_2 \Rightarrow t' \), \( \Rightarrow \) is confluent.
Termination and confluence

When there is no infinite chain $t_0 \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \cdots$, the rewrite relation $\Rightarrow$ is terminating.

When $t \Rightarrow^* s_1$ and $t \Rightarrow^* s_2$ implies the existence of $t'$ such that $s_1 \Rightarrow^* t'$ and $s_2 \Rightarrow^* t'$, $\Rightarrow$ is confluent.

When $t \Rightarrow s_1$ and $t \Rightarrow s_2$ implies the existence of $t'$ such that $s_1 \Rightarrow^* t'$ and $s_2 \Rightarrow^* t'$, $\Rightarrow$ is locally confluent.
Termination and confluence

When there is no infinite chain \( t_0 \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \cdots \), the rewrite relation \( \Rightarrow \) is terminating.

When \( t \xrightarrow{*} s_1 \) and \( t \xrightarrow{*} s_2 \) implies the existence of \( t' \) such that \( s_1 \xrightarrow{*} t' \) and \( s_2 \xrightarrow{*} t' \), \( \Rightarrow \) is confluent.

When \( t \Rightarrow s_1 \) and \( t \Rightarrow s_2 \) implies the existence of \( t' \) such that \( s_1 \Rightarrow t' \) and \( s_2 \Rightarrow t' \), \( \Rightarrow \) is locally confluent.

**Theorem (Diamond property)**

If \( \Rightarrow \) is terminating and locally confluent, then \( \Rightarrow \) is confluent.
**Termination and confluence**

When there is no infinite chain $t_0 \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \cdots$, the rewrite relation $\Rightarrow$ is terminating.

When $t^* \Rightarrow s_1$ and $t^* \Rightarrow s_2$ implies the existence of $t'$ such that $s_1 \Rightarrow t'$ and $s_2 \Rightarrow t'$, $\Rightarrow$ is confluent.

When $t \Rightarrow s_1$ and $t \Rightarrow s_2$ implies the existence of $t'$ such that $s_1 \Rightarrow t'$ and $s_2 \Rightarrow t'$, $\Rightarrow$ is locally confluent.

**Theorem (Diamond property)**

If $\Rightarrow$ is terminating and locally confluent, then $\Rightarrow$ is confluent.

**Proposition**

Let $\rightarrow$ be a rewrite rule on $\text{F}(\mathcal{G})$. If $\Rightarrow$ is terminating and confluent, then $\mathcal{N}_\Rightarrow$ is

- the set of all $\mathcal{G}$-trees avoiding the left members of $\rightarrow$;
- in a one-to-one correspondence respecting the arities with $\text{F}(\mathcal{G})/\Leftrightarrow$. 


Tamari lattices

Let $\rightarrow$ be the rewrite rule on $\mathbf{F}\{\{a\}\}$ defined by

$$a \rightarrow \overset{a}{\text{}} \overset{a}{\text{}}.$$
Tamari lattices

Let \( \rightarrow \) be the rewrite rule on \( F(\{a\}) \) defined by

\[
\begin{array}{cc}
\ \ & a \\
\rightarrow & a
\end{array}
\]

First graphs \((F(\{a\})(n), \Rightarrow)\):

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Tamari lattices

Let \( \rightarrow \) be the rewrite rule on \( \mathbb{F}(\{a\}) \) defined by

First graphs \( (\mathbb{F}(\{a\})(n), \Rightarrow) \):

Properties:

\( \Rightarrow \) is terminating and confluent;
\( \mathcal{N} \Rightarrow \) is the set of the trees avoiding \( \vdash \), that are right comb trees;
The sequence \( (\mathbb{F}(\{a\})/\cong(n))_{n\geq 1} \) is 1, 1, 1, 1, \ldots.
A variant of Tamari lattices

Let $\rightarrow$ be the rewrite rule on $F(\{a\})$ defined by $\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}.$
A variant of Tamari lattices

Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{a\})$ defined by $\rightarrow$.

First graphs $(\mathbf{F}(\{a\})(n), \Rightarrow)$:

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Theorem [Chenavier, Cordero, G., 2018] $\Rightarrow$ is terminating but not confluent; $\Rightarrow$ can be described as the set of the $\{a\}$-trees avoiding $\overline{11}$ patterns; the sequence $(\mathbf{F}(\{a\})(n)/\ast \iff (n))_{n \geq 1}$ is $1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 15, 16, 17, \ldots$ and its generating function is $t(1-t)^2(1-t+t^2+t^3+2t^4+2t^5-7t^7-2t^8+t^9+2t^{10}+t^{11})$. 

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A variant of Tamari lattices

Let $\rightarrow$ be the rewrite rule on $F(\{a\})$ defined by $\overset{a}{a} \overset{a}{a} \overset{a}{a} \rightarrow \overset{a}{a}$. 

First graphs $(F(\{a\})(n), \Rightarrow)$:

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Theorem [Chenavier, Cordero, G., 2018]

- $\Rightarrow$ is terminating but not confluent;
- $\mathcal{N} \Rightarrow$ can be described as the set of the $\{a\}$-trees avoiding 11 patterns;
A variant of Tamari lattices

Let $\rightarrow$ be the rewrite rule on $F(\{a\})$ defined by $\begin{array}{c} a \downarrow \end{array} \rightarrow \begin{array}{c} a \downarrow \end{array}$.

First graphs $(F(\{a\})(n), \Rightarrow)$:

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Theorem [Chenavier, Cordero, G., 2018]

- $\Rightarrow$ is terminating but not confluent;
- $\mathcal{N}\Rightarrow$ can be described as the set of the $\{a\}$-trees avoiding 11 patterns;
- The sequence \( (F(\{a\})/\Rightarrow(n))_{n \geq 1} \) is
  \[ 1, 1, 2, 4, 8, \ldots \]
A variant of Tamari lattices

Let $\to$ be the rewrite rule on $F(\{a\})$ defined by $\begin{array}{c} a \quad a \quad a \quad a \quad a \\ \end{array} \rightarrow \begin{array}{c} a \quad a \quad a \\ \end{array}$.

First graphs $(F(\{a\})(n), \Rightarrow)$:

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Theorem [Chenavier, Cordero, G., 2018]

- $\Rightarrow$ is terminating but not confluent;
- $\mathcal{N} \Rightarrow$ can be described as the set of the $\{a\}$-trees avoiding 11 patterns;
- The sequence $(F(\{a\})/\Rightarrow(n))_{n \geq 1}$ is $1, 1, 2, 4, 8, 14,$...
A variant of Tamari lattices

Let $\to$ be the rewrite rule on $F(\{a\})$ defined by $\quad \rightarrow \quad$.

First graphs $(F(\{a\})(n), \Rightarrow)$:

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Theorem [Chenavier, Cordero, G., 2018]

- $\Rightarrow$ is terminating but not confluent;
- $\mathcal{N} \Rightarrow$ can be described as the set of the $\{a\}$-trees avoiding 11 patterns;
- The sequence $\left( F(\{a\}) / \Rightarrow (n) \right)_{n \geq 1}$ is

$$1, 1, 2, 4, 8, 14, 20,$$
A variant of Tamari lattices

Let $\to$ be the rewrite rule on $F(\{a\})$ defined by $\begin{array}{c} \begin{array}{c} a \end{array} \end{array} \to \begin{array}{c} \begin{array}{c} a \end{array} a \end{array}$. First graphs $(F(\{a\})(n), \Rightarrow)$:

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  \[
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  \]
  
  and its generating function is
  
  \[
  \frac{t}{(1-t)^2} \left( 1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11} \right).
  \]
Outline

Tree series and pattern avoidance
Space of tree series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \ldots)$ and $\mathcal{G}$ be a set of letters.
Let $K$ be the field $\mathbb{Q}(q_0, q_1, q_2, \ldots)$ and $\mathcal{G}$ be a set of letters. A $F(\mathcal{G})$-series (tree series) is a map

$$f : F(\mathcal{G}) \to K.$$

The coefficient $f(t)$ of $t \in F(\mathcal{G})$ in $f$ is denoted by $\langle t, f \rangle$. The set of all $F(\mathcal{G})$-series is $K\langle\langle F(\mathcal{G}) \rangle\rangle$. Endowed with the pointwise addition $\langle t, f + g \rangle := \langle t, f \rangle + \langle t, g \rangle$ and the pointwise multiplication by a scalar $\langle t, \lambda f \rangle := \lambda \langle t, f \rangle$, the set $K\langle\langle F(\mathcal{G}) \rangle\rangle$ is a vector space. The sum notation of $f$ is $f = \sum_{t \in F(\mathcal{G})} \langle t, f \rangle t$. 
Space of tree series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \ldots)$ and $\mathcal{G}$ be a set of letters.

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**Space of tree series**

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \ldots)$ and $\mathcal{S}$ be a set of letters. A $\mathcal{F}(\mathcal{S})$-series (tree series) is a map

$$f : \mathcal{F}(\mathcal{S}) \to \mathbb{K}.$$ 

The coefficient $f(t)$ of $t \in \mathcal{F}(\mathcal{S})$ in $f$ is denoted by $\langle t, f \rangle$. The set of all $\mathcal{F}(\mathcal{S})$-series is $\mathbb{K}\langle \langle \mathcal{F}(\mathcal{S}) \rangle \rangle$.

Endowed with the pointwise addition

$$\langle t, f + g \rangle := \langle t, f \rangle + \langle t, g \rangle$$

and the pointwise multiplication by a scalar

$$\langle t, \lambda f \rangle := \lambda \langle t, f \rangle,$$

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The sum notation of $f$ is

$$f = \sum_{t \in F(\mathcal{G})} \langle t, f \rangle t.$$
Some tree series

Example

For \( x \in \mathcal{G} \), let \( f_x \) be the \( \mathbf{F}(\mathcal{G}) \)-series wherein \( \langle t, f_x \rangle \) is the number of occurrences of \( x \) in \( t \). For instance,

\[
f_a = \frac{1}{a} + 2 \cdot \frac{1}{a} + \frac{1}{a} + \frac{1}{b} + \frac{1}{b} + 2 \cdot \frac{1}{a} + 3 \cdot \frac{1}{a} + \cdots.
\]
**Some tree series**

**Example**

For $x \in G$, let $f_x$ be the $F(G)$-series wherein $\langle t, f_x \rangle$ is the number of occurrences of $x$ in $t$. For instance,

$$f_a = \frac{1}{a} + 2 \frac{1}{a} + \frac{1}{b} + 2 \frac{1}{a} + 3 \frac{1}{a} + \cdots.$$  

**Example**

Let $f_t$ be the $F(G)$-series wherein $\langle t, f_t \rangle := |t|$. Hence,

$$f_t = \frac{1}{a} + 2 \frac{1}{a} + 2 \frac{1}{b} + 3 \frac{1}{c} + 3 \frac{1}{a} + 3 \frac{1}{a} + 3 \frac{1}{b} + \cdots.$$
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Example

Let $f_t$ be the $\mathbf{F}(\mathcal{G})$-series wherein $\langle t, f_t \rangle := |t|$. Hence,

$$f_t = 1 + 2 \frac{1}{a} + 2 \frac{1}{b} + 3 \frac{1}{c} + 3 \frac{1}{a} + 3 \frac{1}{a} + 3 \frac{1}{b} + 3 \frac{1}{a} + \cdots.$$ 

Example

In the tree series $f_a + f_b + f_c$, the coefficient of a tree is its degree.
Some tree series

Example
For $x \in \mathcal{G}$, let $f_x$ be the $\mathbf{F}(\mathcal{G})$-series wherein $\langle t, f_x \rangle$ is the number of occurrences of $x$ in $t$. For instance,

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Example
Let $f_\mathcal{G}$ be the $\mathbf{F}(\mathcal{G})$-series wherein $\langle t, f_\mathcal{G} \rangle := |t|$. Hence,

$$f_\mathcal{G} = 1 + 2 \underbrace{\frac{a}{b}} + 2 \underbrace{\frac{b}{b}} + 3 \underbrace{\frac{c}{c}} + 3 \underbrace{\frac{a}{a}} + 3 \underbrace{\frac{a}{b}} + 3 \underbrace{\frac{b}{a}} + \cdots.$$

Example
In the tree series $f_a + f_b + f_c$, the coefficient of a tree is its degree.
In the tree series $f_\mathcal{G} + f_a + f_b + f_c$, the coefficient of a tree is its number of edges.
Evaluation and generating series

Let $S$ be a set of $G$-trees.

The characteristic series of $S$ is the $F(G)$-série

$$f_S := \sum_{t \in S} t.$$
Evaluation and generating series

Let $S$ be a set of $\mathcal{G}$-trees.

The characteristic series of $S$ is the $\mathbf{F}(\mathcal{G})$-série

$$f_S := \sum_{t \in S} t.$$

The evaluation map

$$\text{ev} : \mathbb{K} \langle \langle \mathbf{F}(\mathcal{G}) \rangle \rangle \to \mathbb{K} \langle \langle t \rangle \rangle$$

is the linear map satisfying

$$\text{ev}(t) = t^{|t|}.$$
Evaluation and generating series

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$$\text{ev}(t) = t^{|t|}.$$ 

One has

$$\text{ev}(f_S) = \sum_{t \in S} t^{|t|} = \sum_{n \geq 1} \# \{ t \in S : |t| = n \} t^n = G_S(t)$$

where $G_S(t)$ is the generating series of $S$, enumerating its elements w.r.t. the arity.
Composition of tree series

The composition of the $F(G)$-series $f$ and $g_1, \ldots, g_n$ is the series

$$f \circ [g_1, \ldots, g_n] := \sum_{t \in F(G)(n)} \left( \langle t, f \rangle \prod_{i \in [n]} \langle s_i, g_i \rangle \right) t \circ [s_1, \ldots, s_n].$$

Observe that this product is linear in all its arguments.
Composition of tree series

The composition of the \( F(G) \)-series \( f \) and \( g_1, \ldots, g_n \) is the series

\[
f \circ [g_1, \ldots, g_n] := \sum_{t \in F(G)^{(n)}} \left( \langle t, f \rangle \prod_{i \in [n]} \langle s_i, g_i \rangle \right) t \circ [s_1, \ldots, s_n].
\]

Observe that this product is linear in all its arguments.

Example

\[
\left( a + b + c \right) \circ \left[ l, c, a + b \right] = b + c + a + b + a + b
\]
Composition of tree series

The composition of the $F(\mathcal{G})$-series $f$ and $g_1, \ldots, g_n$ is the series

$$f \circ [g_1, \ldots, g_n] := \sum_{\substack{t \in F(\mathcal{G})(n) \\ \ s_1, \ldots, s_n \in F(\mathcal{G)}}} \left( \langle t, f \rangle \prod_{i \in [n]} \langle s_i, g_i \rangle \right) t \circ [s_1, \ldots, s_n].$$

Observe that this product is linear in all its arguments.

Example

$$\begin{pmatrix} a + b & b \\ \ \ \ c & b \end{pmatrix} \circ \left[ a, c, a + b \right] = \begin{pmatrix} b & b \\ \ c & b \end{pmatrix} + \begin{pmatrix} b \end{pmatrix} + \begin{pmatrix} c \end{pmatrix} + \begin{pmatrix} c \end{pmatrix}.$$

For all $t \in F(\mathcal{G})(n)$ and all $F(\mathcal{G})$-series $g_1, \ldots, g_n$,

$$\text{ev} \left( f \circ [g_1, \ldots, g_n] \right) = \prod_{i \in [n]} \text{ev} \left( g_i \right).$$
Tree series avoiding patterns

Let \( \mathcal{P} \subseteq F(\mathcal{G}) \) and set

\[
    f(\mathcal{P}) := f_{A(\mathcal{P})} = \sum_{\left.\begin{array}{c} t \in F(\mathcal{G}) \\ \forall s \in \mathcal{P}, s \not\subseteq t \end{array}\right\}} t
\]

as the series of the \( \mathcal{G} \)-trees avoiding all patterns of \( \mathcal{P} \).
Tree series avoiding patterns

Let \( \mathcal{P} \subseteq \mathcal{F}(\mathcal{G}) \) and set

\[
f(\mathcal{P}) := f_{A(\mathcal{P})} = \sum_{t \in \mathcal{F}(\mathcal{G})} t \quad \forall s \in \mathcal{P}, s \not\approx t
\]

as the series of the \( \mathcal{G} \)-trees avoiding all patterns of \( \mathcal{P} \).

When \( \mathcal{G}(1) = \emptyset \), each \( \mathcal{F}(\mathcal{G})(n) \) is finite and thus, there is a finite number of \( \mathcal{G} \)-trees of arity \( n \) avoiding \( \mathcal{P} \). Therefore, the series

\[
\text{ev}(f(\mathcal{P})) = \mathcal{G}_{A(\mathcal{P})}(t)
\]

is well-defined.
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\[
\text{ev}(f(\mathcal{P})) = G_{A(\mathcal{P})}(t)
\]

is well-defined.

**Goal**

Given \( \mathcal{G} \) and \( \mathcal{P} \subseteq \mathbf{F}(\mathcal{G}) \), provide an expression for \( f(\mathcal{P}) \).
A $\mathcal{G}$-tree $t$ admits an occurrence of a $\mathcal{G}$-tree $s$ at root if there exists $r_1, \ldots, r_{|s|} \in F(\mathcal{G})$ and $i \in [|r|]$ such that

$$t = s \circ [r_1, \ldots, r_{|s|}] .$$

This property is denoted by $s \preceq_r t$. 
Occurrences at root

A $\mathcal{G}$-tree $t$ admits an occurrence of a $\mathcal{G}$-tree $s$ at root if there exists $r_1, \ldots, r_{|s|} \in F(\mathcal{G})$ and $i \in [|r|]$ such that

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Example
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This property is denoted by $s \preceq_r t$.

Example

Assume that $t = a \circ [t_1, \ldots, t_k]$ and $s = a \circ [s_1, \ldots, s_k]$ where $a \in \mathcal{G}(k)$.
Occurrences at root

A $\mathcal{G}$-tree $t$ admits an occurrence of a $\mathcal{G}$-tree $s$ at root if there exists $r_1, \ldots, r_{|s|} \in \mathbf{F}(\mathcal{G})$ and $i \in [|r|]$ such that

$$t = s \circ [r_1, \ldots, r_{|s|}] .$$

This property is denoted by $s \preceq_r t$.

Example

Assume that $t = a \circ [t_1, \ldots, t_k]$ and $s = a \circ [s_1, \ldots, s_k]$ where $a \in \mathcal{G}(k)$. Then, $s \preceq_r t$ if and only if there exists an $i \in [k]$ such that $s_i \preceq_r t_i$. 
Admissible words

Let $\mathcal{P} \subseteq \mathcal{F}(\mathcal{G})$ and $a \in \mathcal{G}(k)$. Let

$$\mathcal{P}_a := \{s \in \mathcal{P} : a \preceq_r s\}.$$
Admissible words

Let \( \mathcal{P} \subseteq F(\mathcal{G}) \) and \( a \in \mathcal{G}(k) \). Let

\[
\mathcal{P}_a := \{ s \in \mathcal{P} : a \prec_r s \}.
\]

A \( \mathcal{G} \)-tree \( t = a \circ [t_1, \ldots, t_k] \) avoids at root all patterns of \( \mathcal{P} \) if for all patterns \( s = a \circ [s_1, \ldots, s_k] \in \mathcal{P}_a \), there is an \( i \in [k] \) such that \( s_i \not\approx_r t_i \).
Let $\mathcal{P} \subseteq F(\mathcal{G})$ and $a \in \mathcal{G}(k)$. Let
\[
\mathcal{P}_a := \{ s \in \mathcal{P} : a \preceq_r s \}.
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A $\mathcal{G}$-tree $t = a \circ [t_1, \ldots, t_k]$ avoids at root all patterns of $\mathcal{P}$ if for all patterns $s = a \circ [s_1, \ldots, s_k] \in \mathcal{P}_a$, there is an $i \in [k]$ such that $s_i \not\preceq_r t_i$.

A word $(S_1, \ldots, S_k)$ where letters are sets of $\mathcal{G}$-trees different from $\emptyset$ is $\mathcal{P}_a$-admissible if for any $s \in \mathcal{P}_a$, there is an $i \in [k]$ such that $s_i \in S_i$. 

Admissible words

Let $\mathcal{P} \subseteq \mathcal{F}(\mathcal{G})$ and $a \in \mathcal{G}(k)$. Let

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A $\mathcal{G}$-tree $t = a \circ [t_1, \ldots, t_k]$ avoids at root all patterns of $\mathcal{P}$ if for all patterns $\mathcal{s} = a \circ [\mathcal{s}_1, \ldots, \mathcal{s}_k] \in \mathcal{P}_a$, there is an $i \in [k]$ such that $\mathcal{s}_i \not\preceq_r t_i$.

A word $(S_1, \ldots, S_k)$ where letters are sets of $\mathcal{G}$-trees different from $\varnothing$ is $\mathcal{P}_a$-admissible if for any $\mathcal{s} \in \mathcal{P}_a$, there is an $i \in [k]$ such that $\mathcal{s}_i \in S_i$.

**Example**

Let $\mathcal{P} := \left\{ \begin{array}{c} \begin{array}{c} \text{a} \end{array} \begin{array}{c} \text{c} \end{array}, \\ \begin{array}{c} \text{b} \end{array} \begin{array}{c} \text{a} \end{array}, \\ \begin{array}{c} \text{c} \end{array} \begin{array}{c} \text{c} \end{array} \end{array} \right\}$. In terms of $\mathcal{P}_c$-admissibility, the word

$$\left( \{ \{ \text{a} \} \}, \varnothing, \{ \{ \text{a} \} \} \right)$$

is;
Admissible words

Let $\mathcal{P} \subseteq \mathcal{F}(\mathcal{G})$ and $a \in \mathcal{G}(k)$. Let

$$\mathcal{P}_a := \{ s \in \mathcal{P} : a \preceq_r s \} .$$

A $\mathcal{G}$-tree $t = a \circ [t_1, \ldots, t_k]$ avoids at root all patterns of $\mathcal{P}$ if for all patterns $s = a \circ [s_1, \ldots, s_k] \in \mathcal{P}_a$, there is an $i \in [k]$ such that $s_i \not\sim_r t_i$.

A word $(S_1, \ldots, S_k)$ where letters are sets of $\mathcal{G}$-trees different from $\varnothing$ is $\mathcal{P}_a$-admissible if for any $s \in \mathcal{P}_a$, there is an $i \in [k]$ such that $s_i \in S_i$.

**Example**

Let $\mathcal{P} := \{, \}$. In terms of $\mathcal{P}_c$-admissibility, the word

- $\left( \{ \frac{a}{\ }\}, \emptyset, \{ \frac{a}{\ }\} \right)$ is;
- $\left( \{ \frac{a}{\ }, \frac{c}{\ }\}, \emptyset, \{ \frac{a}{\ }\} \right)$ is;
Admissible words

Let $\mathcal{P} \subseteq F(\mathcal{G})$ and $a \in \mathcal{G}(k)$. Let

$$\mathcal{P}_a := \{ s \in \mathcal{P} : a \preceq_r s \} .$$

A $\mathcal{G}$-tree $t = a \circ [t_1, \ldots, t_k]$ avoids at root all patterns of $\mathcal{P}$ if for all patterns $s = a \circ [s_1, \ldots, s_k] \in \mathcal{P}_a$, there is an $i \in [k]$ such that $s_i \not\preceq_r t_i$.

A word $(S_1, \ldots, S_k)$ where letters are sets of $\mathcal{G}$-trees different from $\emptyset$ is $\mathcal{P}_a$-admissible if for any $s \in \mathcal{P}_a$, there is an $i \in [k]$ such that $s_i \in S_i$.

**Example**

Let $\mathcal{P} := \left\{ \begin{array}{c}
  a \xrightarrow{\text{c}} b, \\
  b \xrightarrow{\text{c}} a, \\
  a \xrightarrow{\text{c}} c
\end{array} \right\}$. In terms of $\mathcal{P}_c$-admissibility, the word

$$\left( \left\{ \frac{1}{a} \right\}, \emptyset, \left\{ \frac{1}{a} \right\} \right)$$

is;

$$\left( \left\{ \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right\}, \emptyset, \emptyset \right)$$

is;

$$\left( \left\{ \frac{1}{a}, \frac{1}{c} \right\}, \emptyset, \left\{ \frac{1}{a} \right\} \right)$$

is;
Admissible words

Let \( \mathcal{P} \subseteq \mathcal{F}(\mathcal{G}) \) and \( a \in \mathcal{G}(k) \). Let

\[ \mathcal{P}_a := \{ s \in \mathcal{P} : a \preceq r s \} . \]

A \( \mathcal{G} \)-tree \( t = a \circ [t_1, \ldots, t_k] \) avoids at root all patterns of \( \mathcal{P} \) if for all patterns \( s = a \circ [s_1, \ldots, s_k] \in \mathcal{P}_a \), there is an \( i \in [k] \) such that \( s_i \not\preceq_r t_i \).

A word \((S_1, \ldots, S_k)\) where letters are sets of \( \mathcal{G} \)-trees different from \( i \) is \( \mathcal{P}_a \)-admissible if for any \( s \in \mathcal{P}_a \), there is an \( i \in [k] \) such that \( s_i \in S_i \).

Example

Let \( \mathcal{P} := \Bigg\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \quad \text{c} \\
\text{a} \\
\text{b} \quad \text{a}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}\Bigg\} \). In terms of \( \mathcal{P}_c \)-admissibility, the word

\[ \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) , \emptyset , \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) \) is;

\[ \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) , \emptyset , \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) \) is;

\[ \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) , \emptyset , \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) \) is;

\[ \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{c}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) , \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{b} \\
\text{b}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) , \bigg( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a}
\end{array}
\end{array}
\end{array}
\end{array}\bigg) \) is not.
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[
(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).
\]
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[
(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).
\]

A \(P_a\)-admissible word \(u\) is minimal if any decomposition \(u = v \oplus v'\) where \(v\) is a \(P_a\)-admissible word and \(v'\) is a word of sets of trees implies \(u = v\).
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[
(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).
\]

A \(P_a\)-admissible word \(u\) is minimal if any decomposition \(u = v \oplus v'\) where \(v\) is a \(P_a\)-admissible word and \(v'\) is a word of sets of trees implies \(u = v\).

The set of all minimal \(P_a\)-admissible words is denoted by \(M(P_a)\).
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[
(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).
\]

A \(P_a\)-admissible word \(u\) is minimal if any decomposition \(u = v \oplus v'\) where \(v\) is a \(P_a\)-admissible word and \(v'\) is a word of sets of trees implies \(u = v\).

The set of all minimal \(P_a\)-admissible words is denoted by \(M(P_a)\).

**Example**

Let \(P := \left\{ \{a, c\}, \{b, a\}, \{c, c, a\} \right\} \). In terms of minimality, as a \(P_c\)-admissible word,

\[
\text{\textbullet } \left( \{a\}, \emptyset, \{a\} \right)
\]

is;
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[
(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).
\]

A \(\mathcal{P}_a\)-admissible word \(u\) is minimal if any decomposition \(u = v \oplus v'\) where \(v\) is a \(\mathcal{P}_a\)-admissible word and \(v'\) is a word of sets of trees implies \(u = v\).

The set of all minimal \(\mathcal{P}_a\)-admissible words is denoted by \(M(\mathcal{P}_a)\).

Example

Let \(\mathcal{P} := \left\{\begin{array}{c}
\bullet a \,
\cdot
\quad c
\end{array},
\begin{array}{c}
\cdot
\quad b
\quad a
\end{array},
\begin{array}{c}
\cdot
\quad c
\quad a
\end{array}\right\}\). In terms of minimality, as a \(\mathcal{P}_c\)-admissible word,

\[
\begin{array}{c}
\bullet a
\end{array}, \emptyset, \begin{array}{c}
\bullet a
\end{array}\]

\(\triangleright\) \(\left(\begin{array}{c}
\bullet a
\end{array}, \emptyset, \begin{array}{c}
\bullet a
\end{array}\right)\) is;

\[
\begin{array}{c}
\bullet a, \quad b, \quad c
\end{array}, \emptyset, \emptyset\]

\(\triangleright\) \(\left(\begin{array}{c}
\bullet a, \quad b, \quad c
\end{array}, \emptyset, \emptyset\right)\) is;
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).\]

A \(P_a\)-admissible word \(u\) is minimal if any decomposition \(u = v \oplus v'\) where \(v\) is a \(P_a\)-admissible word and \(v'\) is a word of sets of trees implies \(u = v\).

The set of all minimal \(P_a\)-admissible words is denoted by \(M(P_a)\).

Example

Let \(P := \left\{ \left\{ \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \right\}, \emptyset, \left\{ \begin{array}{c} \text{a} \\ \text{c} \end{array} \right\} \right\} \). In terms of minimality, as a \(P_c\)-admissible word,

\[\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array}
\end{array},
\emptyset,
\left\{ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{c}
\end{array}
\end{array} \right\}
\end{array} \text{ is;}
\end{align*}\]

\[\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a},
\text{b},
\text{c}
\end{array}
\end{array}
\end{array},
\emptyset,
\emptyset
\end{array} \text{ is;}
\end{align*}\]

\[\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array}
\end{array},
\emptyset,
\left\{ \begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array} \right\}
\end{array} \text{ is not;}
\end{align*}\]
Minimal admissible words

The union of two words \((S_1, \ldots, S_k)\) and \((S'_1, \ldots, S'_k)\) of sets of trees is defined by

\[
(S_1, \ldots, S_k) \oplus (S'_1, \ldots, S'_k) := (S_1 \cup S'_1, \ldots, S_k \cup S'_k).
\]

A \(\mathcal{P}_a\)-admissible word \(u\) is minimal if any decomposition \(u = v \oplus v'\) where \(v\) is a \(\mathcal{P}_a\)-admissible word and \(v'\) is a word of sets of trees implies \(u = v\).

The set of all minimal \(\mathcal{P}_a\)-admissible words is denoted by \(M(\mathcal{P}_a)\).

**Example**

Let \(\mathcal{P} := \left\{\begin{array}{c}
\left\{\begin{array}{c}
a \\
\end{array}\right\}, \\
\emptyset, \\
\left\{\begin{array}{c}
a \\
\end{array}\right\}
\end{array}\right\}\). In terms of minimality, as a \(\mathcal{P}_c\)-admissible word,

- \(\left(\left\{\begin{array}{c}
a \\
\end{array}\right\}, \emptyset, \left\{\begin{array}{c}
a \\
\end{array}\right\}\right)\) is;
- \(\left(\left\{\begin{array}{c}
a, b \\
\end{array}\right\}, \emptyset, \emptyset\right)\) is;
- \(\left(\left\{\begin{array}{c}
a, b, c \\
\end{array}\right\}, \emptyset, \emptyset\right)\) is not;
- \(\left(\left\{\begin{array}{c}
a, b, c \\
\end{array}\right\}, \emptyset, \left\{\begin{array}{c}
a \\
\end{array}\right\}\right)\) is not.
Back to tree series

Let $\mathcal{P}, \mathcal{R} \subseteq \mathbf{F}(\mathcal{G})$.

Let the tree series

$$ f(\mathcal{P}, \mathcal{R}) := \sum_{t \in \mathbf{F}(\mathcal{G})} \sum_{\forall s \in \mathcal{P}, s \prec t} \sum_{\forall s \in \mathcal{R}, s \prec_r t} t $$

of the $\mathcal{G}$-trees avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.
Back to tree series

Let $\mathcal{P}, \mathcal{R} \subseteq \mathcal{F}(\mathcal{G})$.

Let the tree series

$$ f(\mathcal{P}, \mathcal{R}) := \sum_{t \in \mathcal{F}(\mathcal{G})} t $$

$$ \forall s \in \mathcal{P}, s \not\sim t $$

$$ \forall s \in \mathcal{R}, s \not\sim r $$

of the $\mathcal{G}$-trees avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.

If $(S_1, \ldots, S_k)$ is a $(\mathcal{P} \cup \mathcal{R})_a$-admissible word, $a \circ [f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)]$ is the characteristic series of all the $\mathcal{G}$-trees $t = a \circ [t_1, \ldots, t_k]$ such that all $t_i$ avoid $\mathcal{P}$ and avoid $S_i$ at root.
Back to tree series

Let $\mathcal{P}, \mathcal{R} \subseteq F(\mathcal{G})$.

Let the tree series
\[
f(\mathcal{P}, \mathcal{R}) := \sum_{t \in F(\mathcal{G})} \sum_{\forall s \in \mathcal{P}, s \lessdot t} \sum_{\forall s \in \mathcal{R}, s \lessdot r} t
\]
of the $\mathcal{G}$-trees avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.

If $(S_1, \ldots, S_k)$ is a $(\mathcal{P} \cup \mathcal{R})_a$-admissible word, $a \overline{\circ} [f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)]$ is the characteristic series of all the $\mathcal{G}$-trees $t = a \circ [t_1, \ldots, t_k]$ such that all $t_i$ avoid $\mathcal{P}$ and avoid $S_i$ at root.

Moreover, the support of the tree series
\[
\sum_{(S_1, \ldots, S_k) \in M((\mathcal{P} \cup \mathcal{R})_a)} a \overline{\circ} [f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)]
\]
is the set of all $\mathcal{G}$-trees with root labeled by $a$ and avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.
System of equations

Observe that for any $\mathcal{R}, \mathcal{R}' \subseteq F(G)$, the characteristic series of the $G$-trees avoiding $\mathcal{P}$, and avoiding $\mathcal{R}$ or $\mathcal{R}'$ at root is

$$f(\mathcal{P}, \mathcal{R}) + f(\mathcal{P}, \mathcal{R}') - f(\mathcal{P}, \mathcal{R} \cup \mathcal{R}') .$$

Therefore, the description of $f(\mathcal{P}, \mathcal{R})$ uses the inclusion-exclusion principle.
System of equations

Observe that for any $\mathcal{R}, \mathcal{R}' \subseteq \mathbf{F}(\mathcal{G})$, the characteristic series of the $\mathcal{G}$-trees avoiding $\mathcal{P}$, and avoiding $\mathcal{R}$ or $\mathcal{R}'$ at root is

$$f(\mathcal{P}, \mathcal{R}) + f(\mathcal{P}, \mathcal{R}') - f(\mathcal{P}, \mathcal{R} \cup \mathcal{R}').$$

Therefore, the description of $f(\mathcal{P}, \mathcal{R})$ uses the inclusion-exclusion principle.

**Theorem** [G., 2017—]

For any set $\mathcal{G}$ of letters and $\mathcal{P}, \mathcal{R} \subseteq \mathbf{F}(\mathcal{G})$,

$$f(\mathcal{P}, \mathcal{R}) = 1 + \sum_{k \geq 1} \sum_{a \in \mathcal{G}(k)} \sum_{\ell \geq 1} (-1)^{1+\ell} a \overline{\circ} [f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)].$$
System of equations

Observe that for any $\mathcal{R}, \mathcal{R}' \subseteq F(\mathcal{G})$, the characteristic series of the $\mathcal{G}$-trees avoiding $\mathcal{P}$, and avoiding $\mathcal{R}$ or $\mathcal{R}'$ at root is

$$f(\mathcal{P}, \mathcal{R}) + f(\mathcal{P}, \mathcal{R}') - f(\mathcal{P}, \mathcal{R} \cup \mathcal{R}') .$$

Therefore, the description of $f(\mathcal{P}, \mathcal{R})$ uses the inclusion-exclusion principle.

**Theorem [G., 2017–]**

For any set $\mathcal{G}$ of letters and $\mathcal{P}, \mathcal{R} \subseteq F(\mathcal{G})$,

$$f(\mathcal{P}, \mathcal{R}) = 1 + \sum_{k \geq 1} \sum_{a \in \mathcal{G}(k)} \sum_{\ell \geq 1} (-1)^{1+\ell} a \circ [f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)] ,$$

where $a \circ [f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)]$ denotes the inner product of $a$ with the sequence $[f(\mathcal{P}, S_1), \ldots, f(\mathcal{P}, S_k)]$.

Since in particular $f(\mathcal{P}) = f(\mathcal{P}, \emptyset)$ this provides a system of equations describing $f(\mathcal{P})$. 

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Main equation for the previous example

Example

Let $\mathcal{P} := \{ \bullet \overset{c}{\rightarrow} \overset{a}{\rightarrow}, \bullet \overset{b}{\rightarrow} \overset{c}{\rightarrow} \overset{a}{\rightarrow}, \bullet \overset{c}{\rightarrow} \overset{a}{\rightarrow} \}$. One has $M (\mathcal{P}_a) = M (\mathcal{P}_b) = \{(\emptyset, \emptyset)\}$ and $M (\mathcal{P}_c) = \left\{ \left( \begin{array}{c} \{ \overset{a}{\bullet} \} \\ \overset{b}{\bullet} \overset{c}{\bullet} \overset{a}{\bullet} \end{array} \right), \emptyset, \left( \begin{array}{c} \{ \overset{a}{\bullet} \} \\ \overset{b}{\bullet} \overset{c}{\bullet} \overset{a}{\bullet} \end{array} \right) \right\}$. Therefore,
Main equation for the previous example

Example

Let $\mathcal{P} := \left\{ \begin{array}{c} \backslash \backslash a \rightarrow c \backslash \backslash \\backslash b \rightarrow c \backslash \backslash \\backslash c \rightarrow a \end{array} \right\}$.

One has $M(\mathcal{P}_a) = M(\mathcal{P}_b) = \{(\emptyset, \emptyset)\}$ and
\[
M(\mathcal{P}_c) = \left\{ \left( \begin{array}{c} \{a\}, \emptyset, \{a\} \end{array} \right), \left( \begin{array}{c} \{a\}, \{b\}, \{c\} \end{array} \right), \emptyset, \emptyset \right\}.
\]

Therefore,
\[
f(\mathcal{P}, \emptyset) = 1.
\]
Main equation for the previous example

Example

Let \( \mathcal{P} := \left\{ \begin{array}{c} a \setminus c, \ \ b \setminus c \setminus a, \ \ c \setminus a \setminus c \setminus c \end{array} \right\} \).

One has \( M (\mathcal{P}_a) = M (\mathcal{P}_b) = \{(\emptyset, \emptyset)\} \) and
\[
M (\mathcal{P}_c) = \left\{ \left( \left\{ \begin{array}{c} a \setminus b \setminus c \end{array} \right\}, \emptyset, \left\{ \begin{array}{c} a \setminus b \setminus c \end{array} \right\} \right), \left( \left\{ \begin{array}{c} a \setminus b \setminus c \end{array} \right\}, \emptyset, \emptyset \right) \right\}.
\]

Therefore,
\[
f (\mathcal{P}, \emptyset) = 1 + a \bar{a} \left[ f (\mathcal{P}, \emptyset), f (\mathcal{P}, \emptyset) \right].
\]
Main equation for the previous example

Example

Let $\mathcal{P} := \left\{ \begin{array}{c}
\begin{array}{c}
\text{a} \quad \text{c} \\
\text{b} \quad \text{a} \\
\text{c} \quad \text{a}
\end{array}
\end{array} \right\}$. 

One has $M(\mathcal{P}_{a}) = M(\mathcal{P}_{b}) = \{(\emptyset, \emptyset)\}$ and

$M(\mathcal{P}_{c}) = \left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array}, \emptyset, \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array} \right\}, \left( \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}, \emptyset, \emptyset
\end{array} \right) \right\}$. 

Therefore,

$$f(\mathcal{P}, \emptyset) = 1 + a \circ [f(\mathcal{P}, \emptyset), f(\mathcal{P}, \emptyset)] + b \circ [f(\mathcal{P}, \emptyset), f(\mathcal{P}, \emptyset)]$$
Main equation for the previous example

Example

Let $\mathcal{P} := \{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array}
\end{array} \} \}$. 

One has $M(\mathcal{P}_a) = M(\mathcal{P}_b) = \{(\emptyset, \emptyset)\}$ and 

$M(\mathcal{P}_c) = \left\{ \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right), \emptyset, \emptyset \right\}, \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right), \emptyset, \emptyset \right\} \}$. 

Therefore,

$$f(\mathcal{P}, \emptyset) = 1 + a \bar{\circ} [f(\mathcal{P}, \emptyset), f(\mathcal{P}, \emptyset)] + b \bar{\circ} [f(\mathcal{P}, \emptyset), f(\mathcal{P}, \emptyset)]$$

$$+ c \bar{\circ} \left[ f \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right), f(\mathcal{P}, \emptyset), f \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right) \right]$$

$$+ c \bar{\circ} \left[ f \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right), f(\mathcal{P}, \emptyset), f(\mathcal{P}, \emptyset) \right]$$

$$- c \bar{\circ} \left[ f \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right), f(\mathcal{P}, \emptyset), f \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \ \ \ \ \ \ \ b \ \ \ \ \ \ \ c \ \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\
\end{array}
\end{array} \right) \right] \right].$$
Example: directed animals

Example

Let \( \mathcal{P} := \begin{cases} \text{a}, & \text{a}, \text{b}, \text{b}, \text{b} \end{cases} \).
Example: directed animals

Example

Let \( \mathcal{P} := \{ a, b, b, b, a, a \} \).

One has

\[
\begin{align*}
f(\mathcal{P}, \emptyset) & = 1 + a \bar{\delta} \left[ f\left(\mathcal{P}, \{ a \} \right), f(\mathcal{P}, \emptyset) \right] \\
& + b \bar{\delta} \left[ f\left(\mathcal{P}, \{ a \} \right), f\left(\mathcal{P}, \{ a, b \} \right) \right],
\end{align*}
\]
Example: directed animals

Let $\mathcal{P} := \{ \begin{array}{l}
a \overset{1}{\rightarrow} a, \\
a \overset{1}{\rightarrow} b, \\
b \overset{1}{\rightarrow} b, \\
b \overset{1}{\rightarrow} b \
\end{array} \}$. One has

$$f(\mathcal{P}, \emptyset) = 1 + a \overset{1}{\circ} \left[ f \left( \mathcal{P}, \{ a \} \right), f(\mathcal{P}, \emptyset) \right]$$

$$+ b \overset{1}{\circ} \left[ f(\mathcal{P}, \{ a \}), f \left( \mathcal{P}, \{ a, b \} \right) \right],$$

$$f \left( \mathcal{P}, \{ a \} \right) = 1 + b \overset{1}{\circ} \left[ f \left( \mathcal{P}, \{ a \} \right), f \left( \mathcal{P}, \{ a, b \} \right) \right].$$
Example: directed animals

Example

Let \( \mathcal{P} := \left\{ \begin{array}{c} a, \\
\begin{array}{c} a \quad b \\
\hline
b \quad b \\
\end{array}
\end{array} \right\}. \)

One has

\[
f(\mathcal{P}, \emptyset) = 1 + a\bar{a} \left[ f\left(\mathcal{P}, \left\{ \begin{array}{c} a \end{array} \right\} \right), f\left(\mathcal{P}, \emptyset\right) \right] \\
+ b\bar{b} \left[ f\left(\mathcal{P}, \left\{ \begin{array}{c} a \end{array} \right\} \right), f\left(\mathcal{P}, \left\{ \begin{array}{c} a, b \\
\hline
b, b \\
\end{array} \right\} \right) \right],
\]

\[
f\left(\mathcal{P}, \left\{ \begin{array}{c} a \end{array} \right\} \right) = 1 + b\bar{b} \left[ f\left(\mathcal{P}, \left\{ \begin{array}{c} a \end{array} \right\} \right), f\left(\mathcal{P}, \left\{ \begin{array}{c} a, b \\
\hline
b, b \\
\end{array} \right\} \right) \right],
\]

\[
f\left(\mathcal{P}, \left\{ \begin{array}{c} a, b \\
\hline
b, b \\
\end{array} \right\} \right) = 1 + b\bar{b} \left[ f\left(\mathcal{P}, \left\{ \begin{array}{c} a \end{array} \right\} \right), f\left(\mathcal{P}, \left\{ \begin{array}{c} a, b, b \\
\hline
b, b, b \\
\end{array} \right\} \right) \right],
\]
Example: directed animals

Example

Let \( \mathcal{P} := \{ a, b, \overline{a}, \overline{b} \} \).

One has

\[
\begin{align*}
f(\mathcal{P}, \emptyset) &= 1 + a\overline{a} \left[ f(\mathcal{P}, \{ a \}) , f(\mathcal{P}, \emptyset) \right] \\
&\quad + b\overline{b} \left[ f(\mathcal{P}, \{ a \}) , f(\mathcal{P}, \{ a, b \}) \right], \\
f(\mathcal{P}, \{ a \}) &= 1 + b\overline{a} \left[ f(\mathcal{P}, \{ a \}) , f(\mathcal{P}, \{ a, b \}) \right], \\
f(\mathcal{P}, \{ a, b \}) &= 1 + b\overline{a} \left[ f(\mathcal{P}, \{ a \}) , f(\mathcal{P}, \{ a, b \}) \right], \\
f(\mathcal{P}, \{ a, \overline{b} \}) &= 1 \quad \text{(with \( a, \overline{b} \) swapped)}.
\end{align*}
\]
Example: directed animals

Example

By evaluating each member of the previous system, one obtains the system

\[ \mathcal{G}_S(t) = t + \mathcal{G}_{S_1}(t) \mathcal{G}_S(t) + \mathcal{G}_{S_1}(t) \mathcal{G}_{S_2}(t), \]

\[ \mathcal{G}_{S_1}(t) = t + \mathcal{G}_{S_1}(t) \mathcal{G}_{S_2}(t), \]

\[ \mathcal{G}_{S_2}(t) = t + \mathcal{G}_{S_1}(t) \mathcal{G}_{S_3}(t), \]

\[ \mathcal{G}_{S_3}(t) = t \]

for the generating series \( \mathcal{G}_S(t) \) of directed animals.

This leads to

\[ t + (3t - 1) \mathcal{G}_S(t) + (3t - 1) \mathcal{G}_S(t)^2 = 0, \]

an algebraic equation satisfied by \( \mathcal{G}_S(t) \).
Some remarks

The previous result includes, as special cases:

- pattern avoidance of factors in words [Goulden, Jackson, 1979] when $G = G(1)$;
Some remarks

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- pattern avoidance of factors in words [Goulden, Jackson, 1979] when $\mathcal{G} = \mathcal{G}(1)$;

- pattern avoidance of edges in trees [Parker, 1993], [Loday, 2005] when $\mathcal{P}$ contains only trees of degree 2;
Some remarks

The previous result includes, as special cases:

- pattern avoidance of factors in words [Goulden, Jackson, 1979] when \( G = G(1) \);

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- pattern avoidance in binary trees [Rowland, 2010] when \( G = G(2) = \{a\} \).
Some remarks

The previous result includes, as special cases:

- pattern avoidance of factors in words [Goulden, Jackson, 1979] when $\mathcal{S} = \mathcal{G}(1)$;
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- pattern avoidance in binary trees [Rowland, 2010] when $\mathcal{S} = \mathcal{G}(2) = \{a\}$.

Other systems of equations have been described for enumerating trees avoiding patterns in [Khoroshkin, Piontkovski, 2012].
Computing admissible words

Given a $\mathcal{G}$-tree $t = a \circ [t_1, \ldots, t_k]$, let the element

$$\phi(t) := \sum_{i \in [k], t_i \neq \emptyset} \left( \emptyset, \ldots, \emptyset, \{t_i\}, \emptyset, \ldots, \emptyset \right)$$

of the free module $\mathbb{B} \left\langle \left(2^{\mathcal{F}(\mathcal{G})}\right)^k \right\rangle$ on the Boolean semiring $\mathbb{B}$. 

Example

Let $P := \{ac, ab, acb, abc\}$. One has $e_P a = e_P b = (\emptyset, \emptyset)$ and $e_P c = (\{a\}, \emptyset, \emptyset) \oplus (\emptyset, \{a\}, \emptyset) \oplus (\emptyset, \emptyset, \{b\}) = (\{a\}, \emptyset, \emptyset) \oplus (\emptyset, \{a\}, \{b\}) \oplus (\emptyset, \emptyset, \{b\})$. 

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Computing admissible words

Given a $G$-tree $t = a \circ [t_1, \ldots, t_k]$, let the element

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Let the linear combination

$$e_{\mathcal{P}_a} := \bigoplus_{t \in \mathcal{P}_a} \phi(t),$$

containing all $\mathcal{P}_a$-admissible words (and other $\mathcal{P}_a$-admissible words).
Computing admissible words

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**Example**

Let $\mathcal{P} := \left\{ a \xrightarrow{c} a, \ a \xrightarrow{c} b \xrightarrow{b} b \right\}$. One has $e_{\mathcal{P}_a} = e_{\mathcal{P}_b} = (\emptyset, \emptyset)$.
Computing admissible words

Given a $G$-tree $t = a \circ [t_1, \ldots, t_k]$, let the element

$$
\phi(t) := \sum_{\substack{i \in [k] \\
i \neq 1}} \left( \emptyset, \ldots, \emptyset, \{t_i\}, \emptyset, \ldots, \emptyset \right)_{i-1, \ldots, k-i}
$$

of the free module $\mathbb{B} \left\langle \left( 2^{F(G)} \right)^k \right\rangle$ on the Boolean semiring $\mathbb{B}$.

Let the linear combination

$$
e_{P_a} := \bigoplus_{t \in P_a} \phi(t),
$$

containing all $P_a$-admissible words (and other $P_a$-admissible words).

**Example**

Let $P := \left\{ \begin{array}{c} \begin{array}{c} a \circ c \circ \ \ \ \ a \circ c \circ b \circ b \end{array} \end{array} \right\}$. One has $e_{P_a} = e_{P_b} = (\emptyset, \emptyset)$ and

$$
e_{P_c} = \left( \left( \left\{ \frac{a}{a} \right\}, \emptyset, \emptyset \right) + \left( \emptyset, \frac{a}{a}, \emptyset \right) \right) \oplus \left( \left( \left\{ \frac{a}{a} \right\}, \emptyset, \emptyset \right) + \left( \emptyset, \frac{b}{b}, \emptyset \right) + \left( \emptyset, \emptyset, \frac{b}{b} \right) \right)
$$
Computing admissible words

Given a \( G \)-tree \( t = a \circ [t_1, \ldots, t_k] \), let the element

\[
\phi(t) := \sum_{i \in [k]} \left( \begin{array}{c}
\emptyset, \ldots, \emptyset, \{t_i\}, \emptyset, \ldots, \emptyset \\
\emptyset, \ldots, \emptyset, \{t_i\}, \emptyset, \ldots, \emptyset
\end{array} \right)
\]

of the free module \( B \left\langle (2^F(G))^k \right\rangle \) on the Boolean semiring \( B \).

Let the linear combination

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e_{P_a} := \bigoplus_{t \in P_a} \phi(t),
\]

containing all \( P_a \)-admissible words (and other \( P_a \)-admissible words).

**Example**

Let \( P := \left\{ \begin{array}{c}
\begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ c
\end{array},
\begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ c \\
\circlearrowright \ b \\
\circlearrowright \ b
\end{array}
\end{array} \right\} \). One has \( e_{P_a} = e_{P_b} = (\emptyset, \emptyset) \) and

\[
e_{P_c} = \left( \left( \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \emptyset, \emptyset \right) + \left( \emptyset, \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \emptyset \right) \right) \oplus \left( \left( \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \emptyset, \emptyset \right) + \left( \emptyset, \left\{ \begin{array}{c}
\circlearrowright \ b \\
\circlearrowright \ b
\end{array} \right\}, \emptyset \right) + \left( \emptyset, \emptyset, \left\{ \begin{array}{c}
\circlearrowright \ b \\
\circlearrowright \ b
\end{array} \right\} \right) \right)
\]

\[
= \left( \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \emptyset, \emptyset \right) + \left( \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \left\{ \begin{array}{c}
\circlearrowright \ b \\
\circlearrowright \ b
\end{array} \right\}, \emptyset \right) + \left( \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \emptyset, \left\{ \begin{array}{c}
\circlearrowright \ b \\
\circlearrowright \ b
\end{array} \right\} \right)
\]

\[
+ \left( \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \emptyset, \emptyset \right) + \left( \emptyset, \left\{ \begin{array}{c}
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\circlearrowright \ a
\end{array} \right\}, \emptyset \right) + \left( \emptyset, \left\{ \begin{array}{c}
\circlearrowright \ a \\
\circlearrowright \ a
\end{array} \right\}, \left\{ \begin{array}{c}
\circlearrowright \ b \\
\circlearrowright \ b
\end{array} \right\} \right) \right) .
\]
Outline

Operads and enumeration
Operators

An operator is an entity having \( n \geq 1 \) inputs and a single output:

\[
\begin{array}{c}
\text{x} \\
\downarrow \\
1 \quad \cdots \quad n
\end{array}
\]

Its arity is its number \( n \) of inputs.
Operators

An operator is an entity having $n \geq 1$ inputs and a single output:

Its arity is its number $n$ of inputs.

Composing two operators $x$ and $y$ consists in

1. selecting an input of $x$ specified by its position $i$;
2. grafting the output of $y$ onto this input.
Operators

An operator is an entity having \( n \geq 1 \) inputs and a single output:

\[
x = \overbrace{x_1 \cdots x_n}^{1 \cdots n}.
\]

Its arity is its number \( n \) of inputs.

Composing two operators \( x \) and \( y \) consists in

1. selecting an input of \( x \) specified by its position \( i \);
2. grafting the output of \( y \) onto this input.

This produces a new operator \( x \circ_i y \) of arity \( n + m - 1 \):

\[
x \circ_i y = \overbrace{x_1 \cdots x_i \cdots x_n}^{1 \cdots i \cdots n} \circ_i \overbrace{y_1 \cdots y_m}^{1 \cdots m} = \overbrace{x_1 \cdots y_i \cdots x_n}^{1 \cdots i \cdots n + m - 1}.
\]
Operads

Operads are algebraic structures formalizing the notion of operators and their composition.
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A (nonsymmetric set-theoretic) operad is a triple \((\mathcal{O}, \circ_i, 1)\) where

1. \(\mathcal{O}\) is a graded set

\[
\mathcal{O} := \bigsqcup_{n \geq 1} \mathcal{O}(n);
\]
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2. \(\circ_i\) is a map, called partial composition map,

\[
\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \quad 1 \leq i \leq n, \ 1 \leq m;
\]
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A (nonsymmetric set-theoretic) operad is a triple \((O, \circ_i, 1)\) where

1. \(O\) is a graded set
   \[ O := \bigsqcup_{n \geq 1} O(n); \]

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   \[ \circ_i : O(n) \times O(m) \to O(n + m - 1), \quad 1 \leq i \leq n, \quad 1 \leq m; \]

3. \(1\) is an element of \(O(1)\) called unit.
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   \]

3. \(1\) is an element of \(\mathcal{O}(1)\) called unit.

This data has to satisfy some axioms.
Operad axioms

Associativity:

\[(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)\]

\[1 \leq i \leq |x|, 1 \leq j \leq |y|\]
Operad axioms

**Associativity:**

\[(x \circ_i y)\]

\[1 \leq i \leq |x|, 1 \leq j \leq |y|\]
Operad axioms

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$$(x \circ_i y) \circ_{i+j-1} z$$

$1 \leq i \leq |x|, 1 \leq j \leq |y|$$
Operad axioms

**Associativity:**

\[(x \circ_i y) \circ_{i+j-1} z = (y \circ_j z)\]

1 \leq i \leq |x|, 1 \leq j \leq |y|

**Commutativity:**

\[(x \circ_{i+j} y) \circ_{|i|} z = (y \circ_{|j|} z) \circ_{i+j} x\]

1 \leq i < j \leq |x|

**Unitality:**

1 \circ_i 1 \circ x = x = x \circ_i 1 

1 \leq i \leq |x|
Operad axioms

Associativity:

\[(x \circ_i y) \circ_{i+j-1} z \sim x \circ_i (y \circ_j z)\]

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\[(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y\]
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Unitality:

\[1 \circ_1 x = x = x \circ_i 1\]

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Unitality:

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Free operads

Let $\mathcal{G}$ be a set of letters.
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The free operad on $\mathcal{G}$ is the operad $F(\mathcal{G})$ wherein

- elements of arity $n$ are the $\mathcal{G}$-trees of arity $n$;
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Let $c : \mathcal{G} \to \mathbf{F}(\mathcal{G})$ be the natural injection (made implicit in the sequel).
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The free operad on $\mathcal{G}$ is the operad $F(\mathcal{G})$ wherein

- elements of arity $n$ are the $\mathcal{G}$-trees of arity $n$;
- the partial composition map $\circ_i$ is the one of the $\mathcal{G}$-trees;
- the unit is $1$.

Let $c : \mathcal{G} \rightarrow F(\mathcal{G})$ be the natural injection (made implicit in the sequel).

Free operads satisfy the following universality property.

For any set $\mathcal{G}$ of letters, any operad $\mathcal{O}$, and any map $f : \mathcal{G} \rightarrow \mathcal{O}$ respecting the arities, there exists a unique operad morphism $\phi : F(\mathcal{G}) \rightarrow \mathcal{O}$ such that $f = \phi \circ c$. 
An operad on paths

Let Paths be the operad wherein:

- $\text{Paths}(n)$ is the set of all paths with $n$ points, that are words $u_1 \ldots u_n$ of elements of $\mathbb{N}$.

Example

is the path $121232100112$ of arity 13.
An operad on paths

Let *Paths* be the operad wherein:

- *Paths*(\(n\)) is the set of all *paths* with \(n\) points, that are words \(u_1 \ldots u_n\) of elements of \(\mathbb{N}\).

**Example**

is the path 121232100112 of arity 13.

- The partial composition \(u \circ_i v\) is computed by replacing the \(i\)th point of \(u\) by a copy of \(v\).

**Example**

\[
\begin{align*}
011232101 \circ_4 11224 &= 0113344632101
\end{align*}
\]
An operad on paths

Let \textbf{Paths} be the operad wherein:

- \textbf{Paths}(n) is the set of all paths with \( n \) points, that are words \( u_1 \ldots u_n \) of elements of \( \mathbb{N} \).

**Example**

The partial composition \( u \circ_i v \) is computed by replacing the \( i \)th point of \( u \) by a copy of \( v \).

**Example**

The unit is the path 0, depicted as \( \circ \), having arity 1.
Suboperad on $m$-Dyck paths

Let for any $m \geq 0$ the suboperad $m\text{Dyck}$ of Paths generated by $\mathcal{G}_{m\text{Dyck}} := \mathcal{G}_{m\text{Dyck}}(m+2) := \{g_m\}$ where

$$g_m := 0 \ m \ (m-1) \ \ldots \ 1 \ 0 = \begin{array}{ccccccc}
0 & & & & & m \\
& & & & & & \\
& & & & & m+1
\end{array}.$$
Suboperad on $m$-Dyck paths

Let for any $m \geq 0$ the suboperad $m\text{Dyck}$ of Paths generated by $\mathcal{G}_{m\text{Dyck}} := \mathcal{G}_{m\text{Dyck}}(m + 2) := \{g_m\}$ where

$$g_m := 0 m (m - 1) \ldots 1 0 = \begin{array}{c}
\end{array}.$$  

Example

The elements of $2\text{Dyck}$ are, by definition, the paths obtained by composing $g_m$ with itself.

- $\text{2Dyck}(1) = \{ \circ \};$
- $\text{2Dyck}(2) = \text{2Dyck}(3) = \emptyset;$
- $\text{2Dyck}(4) = \{ \begin{array}{c}
\end{array} \};$
- $\text{2Dyck}(5) = \text{2Dyck}(6) = \emptyset;$
- $\text{2Dyck}(7) = \{ \begin{array}{c}
\end{array} \};$
- $\text{2Dyck}(8) = \text{2Dyck}(9) = \emptyset.$
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Let for any $m \geq 0$ the suboperad $m\text{Dyck}$ of Paths generated by $\mathcal{G}_{m\text{Dyck}} := \mathcal{G}_{m\text{Dyck}}(m + 2) := \{g_m\}$ where

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The elements of $2\text{Dyck}$ are, by definition, the paths obtained by composing $g_m$ with itself.

- $2\text{Dyck}(1) = \{\circ\};$
- $2\text{Dyck}(2) = 2\text{Dyck}(3) = \emptyset;$
- $2\text{Dyck}(4) = \{\text{\begin{tikzpicture}[scale=0.7, baseline=-0.5ex]
        \draw (0,0) -- (1,1);
        \draw (1,0) -- (0,1);
    \end{tikzpicture}}\};$
- $2\text{Dyck}(5) = 2\text{Dyck}(6) = \emptyset;$
- $2\text{Dyck}(7) = \{\text{\begin{tikzpicture}[scale=0.7, baseline=-0.5ex]
        \draw (0,0) -- (1,1);
        \draw (1,0) -- (0,1);
        \draw (0,1) -- (1,0);
    \end{tikzpicture}}, \text{\begin{tikzpicture}[scale=0.7, baseline=-0.5ex]
        \draw (0,0) -- (1,1);
        \draw (1,0) -- (0,1);
        \draw (0,1) -- (0,0);
    \end{tikzpicture}}, \text{\begin{tikzpicture}[scale=0.7, baseline=-0.5ex]
        \draw (0,0) -- (1,1);
        \draw (1,0) -- (0,1);
        \draw (0,0) -- (1,0);
    \end{tikzpicture}}\};$
- $2\text{Dyck}(8) = 2\text{Dyck}(9) = \emptyset.$

Proposition

For any $m \geq 0$ and $n \geq 1$, $m\text{Dyck}(n)$ is the set of all $m$-Dyck paths of length $n - 1$. 
Presentations

Let $\mathcal{O}$ be an operad.

A presentation of $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ where
Presentations

Let $\mathcal{O}$ be an operad.

A **presentation** of $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ where

- $\mathcal{G}$ is a set of letters, called **generating set**;
Let $\mathcal{O}$ be an operad.

A presentation of $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ where

- $\mathcal{G}$ is a set of letters, called generating set;
- $\equiv$ is an operad congruence of $F(\mathcal{G})$, that is an equivalence relation on the $\mathcal{G}$-trees such that if $t \equiv t'$ and $s \equiv s'$, then $t \circ_i s \equiv t' \circ_i s'$;
Presentations

Let \( \mathcal{O} \) be an operad.

A presentation of \( \mathcal{O} \) is a pair \((\mathcal{G}, \equiv)\) where

- \( \mathcal{G} \) is a set of letters, called generating set;
- \( \equiv \) is an operad congruence of \( F(\mathcal{G}) \), that is an equivalence relation on the \( \mathcal{G} \)-trees such that if \( t \equiv t' \) and \( s \equiv s' \), then \( t \circ_i s \equiv t' \circ_i s' \);

such that

\[
\mathcal{O} \simeq F(\mathcal{G})/\equiv.
\]
Presentations

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A presentation of $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ where

- $\mathcal{G}$ is a set of letters, called generating set;
- $\equiv$ is an operad congruence of $\mathcal{F}(\mathcal{G})$, that is an equivalence relation on the $\mathcal{G}$-trees such that if $t \equiv t'$ and $s \equiv s'$, then $t \circ_i s \equiv t' \circ_i s'$;

such that

$$\mathcal{O} \simeq \mathcal{F}(\mathcal{G})/\equiv.$$

The presentation $(\mathcal{G}, \equiv)$ is

- binary when $\mathcal{G} = \mathcal{G}(2)$;
Presentations

Let $\mathcal{O}$ be an operad.

A presentation of $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ where

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such that

$\mathcal{O} \simeq F(\mathcal{G})/\equiv$.

The presentation $(\mathcal{G}, \equiv)$ is

- binary when $\mathcal{G} = \mathcal{G}(2)$;
- quadratic when $\equiv$ is generated as an operad congruence by an equivalence relation on trees concentrated in degree 2.
Presentation of $m\text{Dyck}$

To find a presentation of $m\text{Dyck}$, we list all the nontrivial relations made of expressions involving the generator $g_m$ and the $o_i$. We find for instance in $2\text{Dyck}$,

\[
g_2 \circ_1 g_2 = g_2 \circ_4 g_2
\]

\[
\begin{array}{c}
\circ_1 \\
\circ_2 \\
\end{array}
\]

\[
\begin{array}{c}
\circ_4 \\
\circ_2 \\
\end{array}
\]

Proposition

For any $m \geq 0$, $m\text{Dyck}$ admits the presentation $(G_m\text{Dyck}, \equiv m\text{Dyck})$ where $G_m\text{Dyck} := \{g_m\}$ and $\equiv m\text{Dyck}$ is the smallest congruence of $F(G_m\text{Dyck})$ satisfying

\[
g_m \circ_1 g_m \equiv m\text{Dyck} g_m \circ_m + 2 g_m
\]

This says that all relations in higher degrees are consequence of this single one and the operad axioms.
Presentation of $m\text{Dyck}$

To find a presentation of $m\text{Dyck}$, we list all the nontrivial relations made of expressions involving the generator $g_m$ and the $\circ_i$. We find for instance in $2\text{Dyck}$,

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Proposition

For any $m \geq 0$, $m\text{Dyck}$ admits the presentation $(\mathcal{G}_{m\text{Dyck}}, \equiv_{m\text{Dyck}})$ where

$$\mathcal{G}_{m\text{Dyck}} := \{g_m\}$$

and $\equiv_{m\text{Dyck}}$ is the smallest congruence of $F(\mathcal{G}_{m\text{Dyck}})$ satisfying

$$g_m \circ_1 g_m \equiv_{m\text{Dyck}} g_m \circ_{m+2} g_m.$$
Presentation of $m$Dyck

To find a presentation of $m$Dyck, we list all the nontrivial relations made of expressions involving the generator $g_m$ and the $\circ_i$. We find for instance in 2Dyck,

$$g_2 \circ_1 g_2 = g_2 \circ_4 g_2$$

Proposition

For any $m \geq 0$, $m$Dyck admits the presentation $(\mathcal{G}_mDyck, \equiv_{mDyck})$ where

$$\mathcal{G}_{mDyck} := \{g_m\}$$

and $\equiv_{mDyck}$ is the smallest congruence of $F(\mathcal{G}_{mDyck})$ satisfying

$$g_m \circ_1 g_m \equiv_{mDyck} g_m \circ_{m+2} g_m.$$ 

This says that all relations in higher degrees are consequence of this single one and the operad axioms.
An operad on Motzkin paths

Let $\textbf{Motz}$ be an operad wherein:

- $\textbf{Motz}(n)$ is the set of all Motzkin paths with $n$ points.

**Example**

[Diagram of a Motzkin path]

is a Motzkin path of arity 16.
An operad on Motzkin paths

Let $\textbf{Motz}$ be an operad wherein:

- $\textbf{Motz}(n)$ is the set of all Motzkin paths with $n$ points.

**Example**

$\text{Motz}(16)$ is a Motzkin path of arity 16.

- The partial composition in $\textbf{Motz}$ is the one of $\textbf{Paths}$.

**Example**

$\circ_{4} = \text{Motz}(16)$
An operad on Motzkin paths

Let $\textbf{Motz}$ be an operad wherein:

▶ $\textbf{Motz}(n)$ is the set of all Motzkin paths with $n$ points.

**Example**

![Motzkin path example]

is a Motzkin path of arity 16.

▶ The partial composition in $\textbf{Motz}$ is the one of $\textbf{Paths}$.

**Example**

![Partial composition example]

▶ The unit is $\textcircled{0}$. 
<table>
<thead>
<tr>
<th>Proposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motz is a suboperad of Paths.</td>
</tr>
</tbody>
</table>
Properties of \textit{Motz}

\begin{proposition}
\textbf{Motz} is a suboperad of \textbf{Paths}.
\end{proposition}

\begin{proposition}
The operad \textbf{Motz} admits the presentation \((\mathcal{G}_{\text{Motz}}, \equiv_{\text{Motz}})\) where

\[\mathcal{G}_{\text{Motz}} := \{\begin{array}{c}
- \\
- \\
\end{array}, \quad \begin{array}{c}
- \\
- \\
\end{array}\}\]

and \(\equiv\) is the smallest operad congruence satisfying

\[\begin{array}{c}
\begin{array}{c}
- \\
- \\
\end{array} \equiv_{\text{Motz}} \begin{array}{c}
- \\
- \\
\end{array},
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
- \\
- \\
\end{array} \equiv_{\text{Motz}} \begin{array}{c}
- \\
- \\
\end{array},
\end{array}\]

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\begin{array}{c}
- \\
- \\
\end{array} \equiv_{\text{Motz}} \begin{array}{c}
- \\
- \\
\end{array},
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
- \\
- \\
\end{array} \equiv_{\text{Motz}} \begin{array}{c}
- \\
- \\
\end{array}.
\end{array}\]
Let $\mathcal{O}$ be an operad with presentation $(\mathcal{G}, \equiv)$. 

Poincaré-Birkhoff-Witt bases
Let \( \mathcal{O} \) be an operad with presentation \((\mathcal{G}, \equiv)\).

A Poincaré-Birkhoff-Witt basis (PBW basis) of \( \mathcal{O} \) w.r.t. \((\mathcal{G}, \equiv)\) is a set \( B \) of \( \mathcal{G} \)-trees such that for each \([t]_\equiv \in F(\mathcal{G})/\equiv\), there exists a unique \( s \in B \) such that \( s \in [t]_\equiv\).
Let $\mathcal{O}$ be an operad with presentation $(\mathcal{G}, \equiv)$.

A Poincaré-Birkhoff-Witt basis (PBW basis) of $\mathcal{O}$ w.r.t. $(\mathcal{G}, \equiv)$ is a set $B$ of $\mathcal{G}$-trees such that for each $[t] \equiv \in F(\mathcal{G})/\equiv$, there exists a unique $s \in B$ such that $s \in [t] \equiv$.

**Proposition**

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathcal{G}, \equiv)$. If

1. $\rightarrow$ is a rewrite rule on $F(\mathcal{G})$ generating $\equiv$ as an operad congruence;
2. the rewrite relation $\Rightarrow$ induced by $\rightarrow$ is terminating and confluent;

then the set of the normal forms for $\Rightarrow$ is a PBW basis of $\mathcal{O}$.
Let $\mathcal{O}$ be an operad with presentation $(\mathcal{G}, \equiv)$.

A Poincaré-Birkhoff-Witt basis (PBW basis) of $\mathcal{O}$ w.r.t. $(\mathcal{G}, \equiv)$ is a set $B$ of $\mathcal{G}$-trees such that for each $[t]_\equiv \in F(\mathcal{G})/\equiv$, there exists a unique $s \in B$ such that $s \in [t]_\equiv$.

**Proposition**

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathcal{G}, \equiv)$. If

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then the set of the normal forms for $\Rightarrow$ is a PBW basis of $\mathcal{O}$.

Such a PBW basis of $\mathcal{O}$ can be described as the set of the trees avoiding the trees appearing as left members for $\rightarrow$. 

**Poincaré-Birkhoff-Witt bases**
PBW basis of Motz

Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\mathcal{G}_{Motz})$ defined by

$\circ\circ\circ_1 \circ\circ \rightarrow \circ\circ\circ_2 \circ\circ,$

$\circ\circ\circ_1 \circ\circ \rightarrow \circ\circ\circ_3 \circ\circ,$

$\circ\circ\circ_1 \circ\circ \rightarrow \circ\circ\circ_2 \circ\circ,$

$\circ\circ\circ_1 \circ\circ \rightarrow \circ\circ\circ_3 \circ\circ.$

This rewrite rule can be seen as an orientation of $\equiv_{Motz}$. 
PBW basis of Motz

Let → be the rewrite rule on $F(\mathcal{G}_{\text{Motz}})$ defined by

\[
\begin{align*}
\circ \circ \circ \circ_1 \circ \circ & \rightarrow \circ \circ \circ \circ \circ_2, \\
\circ \circ \circ \circ_1 \circ \circ & \rightarrow \circ \circ \circ \circ \circ_2, \\
\circ \circ \circ \circ \circ_1 \circ \circ & \rightarrow \circ \circ \circ \circ \circ \circ_3, \\
\circ \circ \circ \circ \circ_1 \circ \circ & \rightarrow \circ \circ \circ \circ \circ \circ_3.
\end{align*}
\]

This rewrite rule can be seen as an orientation of $\equiv_{\text{Motz}}$.

The induced rewrite relation $\Rightarrow$ is terminating, confluent, and its normal forms are in one-to-one correspondence with Motzkin paths.
PBW basis of Motz

Let \( \to \) be the rewrite rule on \( \mathbb{F}(\mathcal{G}_{\text{Motz}}) \) defined by

\[
\begin{align*}
\bullet \bullet \circ_1 \bullet \bullet & \to \bullet \bullet \circ_2 \bullet \bullet, \\
\bullet \circ_1 \bullet \bullet & \to \bullet \circ_2 \bullet \bullet, \\
\circ_1 \circ \bullet & \to \circ_2 \bullet \bullet, \\
\circ_1 \circ \bullet & \to \circ_2 \circ \bullet \bullet.
\end{align*}
\]

This rewrite rule can be seen as an orientation of \( \equiv_{\text{Motz}} \).

The induced rewrite relation \( \Rightarrow \) is terminating, confluent, and its normal forms are in one-to-one correspondence with Motzkin paths.

**Example**

A normal form for \( \Rightarrow \) and the Motzkin path in correspondence with it:
An operad on cyclic paths

For any \( \ell \geq 1 \), let \( \ell \text{CPaths} \) be the operad wherein:

- \( \ell \text{CPaths}(n) \) is the set of all paths with \( n \) points having height smaller than \( \ell \), that are words \( u_1 \ldots u_n \) of elements of \( \{0, \ldots, \ell - 1\} \).

Example

| 3 = 0 | is the path 1212202100112 of 3CPaths. |
| 2    |                                    |
| 1    |                                    |
| 0    |                                    |
An operad on cyclic paths

For any $\ell \geq 1$, let $\ell \text{CPaths}$ be the operad wherein:

- $\ell \text{CPaths}(n)$ is the set of all paths with $n$ points having height smaller than $\ell$, that are words $u_1 \ldots u_n$ of elements of $\{0, \ldots, \ell - 1\}$.

Example

\[ 3 = 0 \]

\[ \begin{array}{c}
\text{is the path 1212202100112 of 3CPaths.}
\end{array} \]

- The partial composition $u \circ_i v$ is computed by replacing the $i$th point of $u$ by a copy of $v$, and by fitting the obtained path on the cylinder.

Example

In $3\text{CPaths}$,

\[ 011202101 \circ_4 10221 = 011(32443)_{3}02101 = 0110211002101 \]
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For any $\ell \geq 1$, let $\ell \text{CPaths}$ be the operad wherein:

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**Example**

$3 = 0$ is the path $1212202100112$ of $3 \text{CPaths}$.

- The partial composition $u \circ_i v$ is computed by replacing the $i$th point of $u$ by a copy of $v$, and by fitting the obtained path on the cylinder.

**Example**

In $3 \text{CPaths}$,

$011202101 \circ_4 10221 = 011(32443)\%_3 02101 = 0110211002101$

- The unit is $\circ$. 
A suboperad on directed animals

Let $\mathbf{DA}$ be the suboperad of $3\mathbf{CPaths}$ generated by

$$\mathcal{G}_{DA} := \{ \begin{array}{c} \begin{tikzpicture}[baseline=-0.65ex]
\draw[->] (0,0) -- (1,0);
\draw[->] (1,0) -- (2,0);
\end{tikzpicture} \\
\begin{tikzpicture}[baseline=-0.65ex]
\draw[->] (0,0) -- (1,0);
\draw[->] (0,0) -- (0,1);
\end{tikzpicture} \end{array} \}.$$
A suboperad on directed animals

Let \( \mathbf{DA} \) be the suboperad of \( \mathcal{3CPaths} \) generated by

\[
\mathcal{G}_{\mathbf{DA}} := \left\{ \begin{array}{c}
\end{array} \right\}.
\]

Example

The elements of \( \mathbf{DA}(4) \) are

\[
\begin{array}{c}
\end{array}.
\]
A suboperad on directed animals

Let $\mathbf{DA}$ be the suboperad of $3\mathbf{CPaths}$ generated by

$$\mathbf{G}_{\mathbf{DA}} := \{\text{, }\}.$$ 

Example

The elements of $\mathbf{DA}(4)$ are

$$\text{, , , , , , , , , , , , , , , , , , .}$$

Proposition

For any $n \geq 1$, $\mathbf{DA}(n)$ is in one-to-one correspondence with the set of directed animals of size $n$. 
Presentation and PBW basis of DA

**Proposition**

**DA** admits the presentation \((\mathcal{G}_{DA}, \equiv)\) where \(\equiv_{DA}\) is the smallest congruence of \(F(\mathcal{G}_{DA})\) satisfying

\[
\begin{align*}
\circ_1 \equiv_{DA} \circ_2, \\
\circ_1 \equiv_{DA} \circ_2, \\
\circ_1 \equiv_{DA} \circ_2
\end{align*}
\]

\[
\left(\begin{array}{c}
\circ_1 \\
\circ_2
\end{array}\right) \equiv_{DA} \left(\begin{array}{c}
\circ_2 \\
\circ_3
\end{array}\right)
\]
# Proposition

**DA** admits the presentation \((\mathcal{G}_{\text{DA}}, \equiv)\) where \(\equiv_{\text{DA}}\) is the smallest congruence of \(F(\mathcal{G}_{\text{DA}})\) satisfying

\[
\begin{align*}
&\circ_1 \equiv_{\text{DA}} \circ_2, \quad \circ_1 \equiv_{\text{DA}} \circ_2, \quad \circ_1 \equiv_{\text{DA}} \circ_2, \\
&\left(\circ_1 \circ_2\right) \circ_3 \equiv_{\text{DA}} \left(\circ_2 \circ_3\right) \circ_2.
\end{align*}
\]

Let \(\rightarrow\) be the orientation of \(\equiv_{\text{DA}}\) satisfying

\[
\begin{align*}
&\circ_1 \rightarrow \circ_2, \quad \circ_1 \rightarrow \circ_2, \quad \circ_2 \rightarrow \circ_1, \\
&\left(\circ_2 \circ_3\right) \circ_3 \rightarrow \left(\circ_1 \circ_2\right) \circ_2.
\end{align*}
\]
### Proposition

**DA** admits the presentation \((\mathcal{G}_{\text{DA}}, \equiv)\) where \(\equiv_{\text{DA}}\) is the smallest congruence of \(F(\mathcal{G}_{\text{DA}})\) satisfying

\[
\begin{align*}
\circ_1 & \equiv_{\text{DA}} \circ_2, \\
\circ_1 & \equiv_{\text{DA}} \circ_2, \\
\circ_1 & \equiv_{\text{DA}} \circ_2,
\end{align*}
\]

\[
\left( \begin{array}{c} \circ_1 \\ \circ_2 \end{array} \right) \circ_2 \equiv_{\text{DA}} \left( \begin{array}{c} \circ_2 \\ \circ_3 \end{array} \right) \circ_3.
\]

Let \(\rightarrow\) be the orientation of \(\equiv_{\text{DA}}\) satisfying

\[
\begin{align*}
\circ_1 & \rightarrow \circ_2, \\
\circ_1 & \rightarrow \circ_2, \\
\circ_2 & \rightarrow \circ_1,
\end{align*}
\]

\[
\left( \begin{array}{c} \circ_2 \\ \circ_3 \end{array} \right) \rightarrow \left( \begin{array}{c} \circ_1 \\ \circ_2 \end{array} \right) \rightarrow \circ_2.
\]

The rewrite relation \(\Rightarrow\) induced by \(\rightarrow\) is terminating and confluent. The \(\mathcal{G}_{\text{DA}}\)-trees avoiding the left members of \(\rightarrow\) form a PBW basis of \(\text{DA}\).
### Dimensions:

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<td>DA</td>
<td></td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>13</td>
<td>35</td>
<td>96</td>
<td>267</td>
<td>750</td>
<td>2123</td>
<td>6046</td>
</tr>
</tbody>
</table>
Application to enumeration

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1. endow $S$ with an operad structure $O$;

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3. deduce a PBW of $O$ w.r.t. $(\mathcal{G}, \equiv)$, described as the trees avoiding a certain set $\mathcal{P}$ of patterns;

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2. establish a presentation \((\mathcal{G}, \equiv)\) of \( O \);
3. deduce a PBW of \( O \) w.r.t. \((\mathcal{G}, \equiv)\), described as the trees avoiding a certain set \( \mathcal{P} \) of patterns;
4. compute \( f(\mathcal{P}) \) by using tree series and their operations.
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Then, we successively

1. endow $S$ with an operad structure $O$;
2. establish a presentation $(\mathcal{G}, \equiv)$ of $O$;
3. deduce a PBW of $O$ w.r.t. $(\mathcal{G}, \equiv)$, described as the trees avoiding a certain set $\mathcal{P}$ of patterns;
4. compute $f(\mathcal{P})$ by using tree series and their operations.

Finally, $\text{ev}(f(\mathcal{P}))$ is the Hilbert series of $O$ and the generating series of $S$. 
Hilbert series of $m\text{Dyck}$

The set $\mathcal{B}$ of the $G_{m\text{Dyck}}$-trees avoiding

$$\mathcal{P} := \{ g_m \circ_1 g_m \}$$

is a PBW basis of $m\text{Dyck}$.

The characteristic series of $\mathcal{B}$ is $f(\mathcal{P}, \emptyset)$ where

$$f(\mathcal{P}, \emptyset) = 1 + g_m \overline{o} \left[ f(\mathcal{P}, \{ g_m \}), f(\mathcal{P}, \emptyset), \ldots, f(\mathcal{P}, \emptyset) \right]_{m+1},$$

$$f(\mathcal{P}, \{ g_m \}) = 1.$$

By setting $\mathcal{H}(t) := \text{ev}(f(\mathcal{P}, \emptyset))$, the Hilbert series $\mathcal{H}(t)$ of $m\text{Dyck}$ satisfies

$$t - \mathcal{H}(t) + t\mathcal{H}(t)^{m+1} = 0.$$
Hilbert series of Motz

The set $\mathcal{B}$ of the $\mathcal{G}_{\text{Motz}}$-trees avoiding

$$\mathcal{P} := \{ \text{\includegraphics{tree1}}, \text{\includegraphics{tree2}}, \text{\includegraphics{tree3}}, \text{\includegraphics{tree4}} \}$$

is a PBW basis of $\text{Motz}$.

The characteristic series of $\mathcal{B}$ is $f(\mathcal{P}, \emptyset)$ where

$$f(\mathcal{P}, \emptyset) = 1 + \text{\includegraphics{tree5}} \left[ f(\mathcal{P}, \{ \text{\includegraphics{tree6}}, \text{\includegraphics{tree7}} \}) , f(\mathcal{P}, \emptyset) \right]$$

$$+ \text{\includegraphics{tree8}} \left[ f(\mathcal{P}, \{ \text{\includegraphics{tree9}}, \text{\includegraphics{tree10}} \}) , f(\mathcal{P}, \emptyset) , f(\mathcal{P}, \emptyset) \right]$$

$$f(\mathcal{P}, \{ \text{\includegraphics{tree11}}, \text{\includegraphics{tree12}} \}) = 1.$$

By setting $\mathcal{H}(t) := \text{ev}(f(\mathcal{P}, \emptyset))$, the Hilbert series $\mathcal{H}(t)$ of $\text{Motz}$ satisfies

$$t - (t - 1)\mathcal{H}(t) + t\mathcal{H}(t)^2 = 0.$$
Hilbert series of DA

The set $\mathcal{B}$ of the $\mathcal{G}_{DA}$-trees avoiding $\mathcal{P} := \{ \circ_1 \circ_1 \circ_0, \circ_1 \circ_2 \circ_0, (\circ_2 \circ_0 \circ_0) \circ_3 \circ_0 \}$ is a PBW basis of DA.

The characteristic series of $\mathcal{B}$ is $f(\mathcal{P}, \emptyset)$ where

$$f(\mathcal{P}, \emptyset) = 1 + \overline{\circ} \left[ f(\mathcal{P}, \{ \circ_1 \circ_1 \circ_0 \}) + f(\mathcal{P}, \{ \circ_1 \circ_2 \circ_0 \}) + f(\mathcal{P}, \{ (\circ_2 \circ_0 \circ_0) \circ_3 \circ_0 \}) \right]$$

$$f(\mathcal{P}, \{ \circ_1 \circ_1 \circ_0 \}) = 1 + \overline{\circ} \left[ f(\mathcal{P}, \{ \circ_1 \circ_1 \circ_0 \}) + f(\mathcal{P}, \{ \circ_1 \circ_2 \circ_0 \}) + f(\mathcal{P}, \{ (\circ_2 \circ_0 \circ_0) \circ_3 \circ_0 \}) \right]$$

$$f(\mathcal{P}, \{ \circ_1 \circ_2 \circ_0 \}) = 1 + \overline{\circ} \left[ f(\mathcal{P}, \{ \circ_1 \circ_1 \circ_0 \}) + f(\mathcal{P}, \{ \circ_1 \circ_2 \circ_0 \}) + f(\mathcal{P}, \{ (\circ_2 \circ_0 \circ_0) \circ_3 \circ_0 \}) \right]$$

$$f(\mathcal{P}, \{ (\circ_2 \circ_0 \circ_0) \circ_3 \circ_0 \}) = 1.$$

By setting $\mathcal{H}(t) := \text{ev}(f(\mathcal{P}, \emptyset))$, the Hilbert series $\mathcal{H}(t)$ of DA satisfies

$$t - (3t - 1)\mathcal{H}(t) + (3t - 1)\mathcal{H}(t)^2 = 0.$$