Some enumerative and order-theoretic properties in combinatory logic

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Outline

1. Combinatory logic
2. A combinatorial approach
3. Some results

Outline

1. Combinatory logic

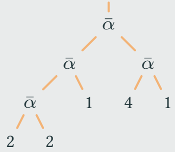
Schemes

A scheme is either

- a positive integer i, called variable;
- a pair $(\mathfrak{s}_0, \mathfrak{s}_1)$ of schemes, called application of $\mathfrak{s}_0$ on $\mathfrak{s}_1$.

Let $\bar{\alpha}$ be the binary operation defined by $\mathfrak{s}_0 \,\bar{\alpha}\, \mathfrak{s}_1 := (\mathfrak{s}_0, \mathfrak{s}_1)$.

– Example –



The tree of the left is the **tree representation** of the scheme

$$\mathfrak{s} := ((2 \,\bar{\alpha}\, 2) \,\bar{\alpha}\, 1) \,\bar{\alpha} (4 \,\bar{\alpha}\, 1).$$

The **short representation** of $\mathfrak{s}$ is obtained by considering that $\bar{\alpha}$ associates to the left. Hence,

$$\mathfrak{s} = 221(41).$$

Let $\mathfrak{S}$ be the **set of schemes** and $\mathfrak{S}(n)$, $n \geqslant 1$, be the set of schemes having only variables in $[n]$.

Terms

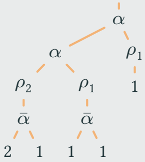
A term is either

- a pair $(n, \mathfrak{s})$ where $n \geqslant 1$ and $\mathfrak{s} \in \mathfrak{S}(n)$, called rule (or basic combinator);
- a pair $(\mathfrak{t}_0, \mathfrak{t}_1)$ of terms, called application of $\mathfrak{t}_0$ on $\mathfrak{t}_1$.

Let ρ_n, $n \geqslant 1$, be the unary operation defined by $\rho_n(\mathfrak{s}) := (n, \mathfrak{s})$ and α be the binary operation defined by $\mathfrak{t}_0 \, \alpha \, \mathfrak{t}_1 := (\mathfrak{t}_0, \mathfrak{t}_1)$.

The order of a rule $\rho_n(\mathfrak{s})$ is n.

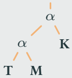
– Example –



The tree of the left is the **tree representation** of the term

$$\mathfrak{t} := (\rho_2(2 \,\bar{\alpha}\, 1) \,\alpha\, \rho_1(1 \,\bar{\alpha}\, 1)) \,\alpha\, \rho_2(1).$$

The **short representation** of terms follows the analogous conventions as the ones of schemes. Hence,

$\mathfrak{t} =$  $= \mathbf{T}\,\mathbf{M}\,\mathbf{K}$

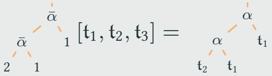
where $\mathbf{T} := \rho_2(21)$, $\mathbf{M} := \rho_1(11)$, and $\mathbf{K} := \rho_2(1)$.

Let $\mathfrak{T}$ be the **set of terms**.

Rewrite relation

Given $\mathfrak{s} \in \mathfrak{S}(n)$ and $\mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T}$, the <u>composition</u> of $\mathfrak{t}_1, \dots, \mathfrak{t}_n$ in $\mathfrak{s}$ is the term $\mathfrak{s}[\mathfrak{t}_1, \dots, \mathfrak{t}_n]$ obtained by replacing all variables i of $\mathfrak{s}$ by $\mathfrak{t}_i$.

– Example –

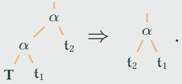


The <u>rewrite relation</u> is the binary relation $\Rightarrow$ on $\mathfrak{T}$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$ if $\mathfrak{t}'$ can be obtained from $\mathfrak{t}$ by **locally replacing** by $\mathfrak{s}[\mathfrak{t}_1, \dots, \mathfrak{t}_n]$ one **pattern** $\rho_n(\mathfrak{s})\mathfrak{t}_1 \dots \mathfrak{t}_n$.

– Example –

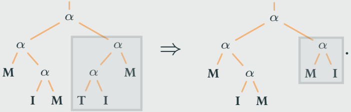
Let the rules $\mathbf{I} := \rho_1(1)$, $\mathbf{M} := \rho_1(11)$, and $\mathbf{T} := \rho_2(21)$.

The rule $\mathbf{T}$ can be seen as the rewrite rule



We have $(\mathbf{M}(\mathbf{I}\,\mathbf{M}))(\underline{\mathbf{T}\,\mathbf{I}\,\mathbf{M}}) \Rightarrow (\mathbf{M}(\mathbf{I}\,\mathbf{M}))(\underline{\mathbf{M}\,\mathbf{I}})$.

On tree representations, this expresses as



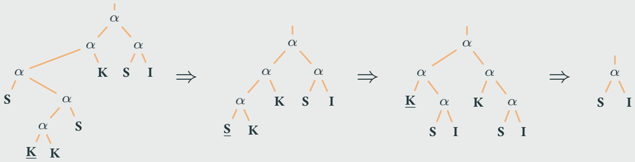
The rules S, K, I

Let the three rules [**Curry**, 1930]

$$\mathbf{S} := \rho_3(13(23)), \quad \mathbf{K} := \rho_2(1), \quad \mathbf{I} := \rho_1(1).$$

– Example –

Here is a sequence of rewrite steps, which can be seen as forming a **computation**:



The set of terms using only the rules **S**, **K**, and **I** form a **Turing-complete programming language**: any λ-term can be emulated by a term of this set through **abstraction algorithms** [**Rosser**, 1955], [**Curry, Feys**, 1958].

The Enchanted Forest of combinator birds

In *To Mock a Mockingbird: and Other Logic Puzzles* [Smullyan, 1985], a great number of rules are listed, forming the **Enchanted forest of combinator birds**.

Here is a sublist (with some others introduced since then):

- **Idiot bird**: $\mathbf{I} := \rho_1(1)$
- **Mockingbird**: $\mathbf{M} := \rho_1(11)$
- **Kestrel**: $\mathbf{K} := \rho_2(1)$
- **Kite**: $\mathbf{Ki} := \rho_2(2)$
- **Thrush**: $\mathbf{T} := \rho_2(21)$
- **Crossed Konstant Mocker**: $\mathbf{M}_{\mathrm{CK}} := \rho_2(11)$
- **Konstant Mocker**: $\mathbf{M}_{\mathrm{K}} := \rho_2(22)$
- **Mockingbird 1**: $\mathbf{M}_1 := \rho_2(112)$
- **Warbler**: $\mathbf{W} := \rho_2(122)$
- **Converse Warbler**: $\mathbf{W}^1 := \rho_2(211)$
- **Lark**: $\mathbf{L} := \rho_2(1(22))$
- **Owl**: $\mathbf{O} := \rho_2(2(12))$
- **Double Mockingbird**: $\mathbf{M}_2 := \rho_2(12(12))$
- **Turing bird**: $\mathbf{U} := \rho_2(2(112))$
- **Cardinal**: $\mathbf{C} := \rho_3(132)$
- **Robin**: $\mathbf{R} := \rho_3(231)$
- **Vireo**: $\mathbf{V} := \rho_3(312)$
- **Finch**: $\mathbf{F} := \rho_3(321)$
- **Bluebird**: $\mathbf{B} := \rho_3(1(23))$
- **Quixotic bird**: $\mathbf{Q}_1 := \rho_3(1(32))$
- **Queer bird**: $\mathbf{Q} := \rho_3(2(13))$
- **Quizzical bird**: $\mathbf{Q}_2 := \rho_3(2(31))$
- **Quirky bird**: $\mathbf{Q}_3 := \rho_3(3(12))$
- **Quarky bird**: $\mathbf{Q}_4 := \rho_3(3(21))$
- **Hummingbird**: $\mathbf{H} := \rho_3(1232)$
- **Starling**: $\mathbf{S} := \rho_3(13(23))$
- **Dove**: $\mathbf{D} := \rho_4(12(34))$
- **Goldfinch**: $\mathbf{G} := \rho_4(14(23))$
- **Blackbird**: $\mathbf{B}_1 := \rho_4(1(234))$
- **Becard**: $\mathbf{B}_3 := \rho_4(1(2(34)))$
- **Jay**: $\mathbf{J} := \rho_4(12(143))$
- **Eagle**: $\mathbf{E} := \rho_5(12(345))$
- **Bunting**: $\mathbf{B}_2 := \rho_5(1(2345))$
- **Dickcissel**: $\mathbf{D}_1 := \rho_5(123(45))$
- **Dovekies**: $\mathbf{D}_2 := \rho_5(1(23)(45))$

Rewrite graphs, prosets, and posets

- Let $\preccurlyeq$ be the **reflexive and transitive closure** of $\Rightarrow$.
- Let $\equiv$ be the **symmetric closure** of $\preccurlyeq$.
- Let $\rightleftharpoons$ be the equivalence relation on $\mathfrak{T}$ such that $\mathfrak{t} \rightleftharpoons \mathfrak{t}'$ if $\mathfrak{t} \preccurlyeq \mathfrak{t}'$ and $\mathfrak{t}' \preccurlyeq \mathfrak{t}$.

Given a term $\mathfrak{t}$,

- the set of terms accessible from $\mathfrak{t}$ is $\mathfrak{t}^\star := \{\mathfrak{t}' \in \mathfrak{T} : \mathfrak{t} \preccurlyeq \mathfrak{t}'\}$.
- the rewrite graph of $\mathfrak{t}$ is the directed multigraph $G(\mathfrak{t})$ on $\mathfrak{t}^\star$ such that there are m edges from $\mathfrak{t}'$ to $\mathfrak{t}''$ if there are exactly m ways to obtain $\mathfrak{t}''$ by a rewrite step from $\mathfrak{t}'$;
- the poset $P(\mathfrak{t})$ of $\mathfrak{t}$ is the poset $(\mathfrak{t}^\star/_{\rightleftharpoons}, \ll)$ where $\ll$ satisfies $[\mathfrak{t}']_{\rightleftharpoons} \ll [\mathfrak{t}'']_{\rightleftharpoons}$ if there are $\mathfrak{t}' \in [\mathfrak{t}']_{\rightleftharpoons}$ and $\mathfrak{t}'' \in [\mathfrak{t}'']_{\rightleftharpoons}$ such that $\mathfrak{t}' \preccurlyeq \mathfrak{t}''$.

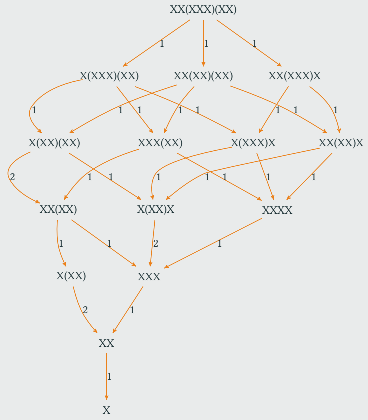
Every rewrite system $(\mathfrak{t}^\star, \Rightarrow)$ is **confluent** [**Rosen**, 1973].

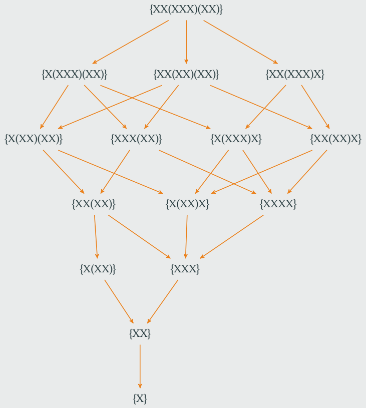
Similar structures have been considered for λ-calculus [**Barendregt**, 1981], [**Venturini-Zilli**, 1984] but $G(\mathfrak{t})$ is in general not isomorphic to the reduction graph of the natural λ-term of $\mathfrak{t}$.

Some examples — 1/6

– Example –

Let $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X})(\mathbf{X}\,\mathbf{X})$ where $\mathbf{X} := \mathbf{I} = \rho_1(1)$. Here are $\mathrm{G}(\mathfrak{t})$ and the Hasse diagram of $\mathrm{P}(\mathfrak{t})$:

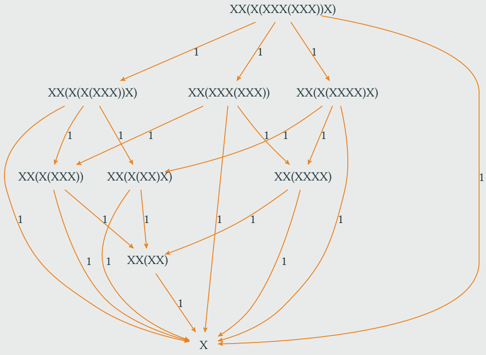


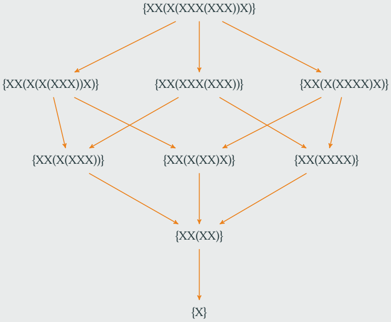


Some examples — 2/6

– Example –

Let $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}))\,\mathbf{X})$ where $\mathbf{X} := \mathbf{K} = \rho_2(1)$. Here are $\mathrm{G}(\mathfrak{t})$ and the Hasse diagram of $\mathrm{P}(\mathfrak{t})$:

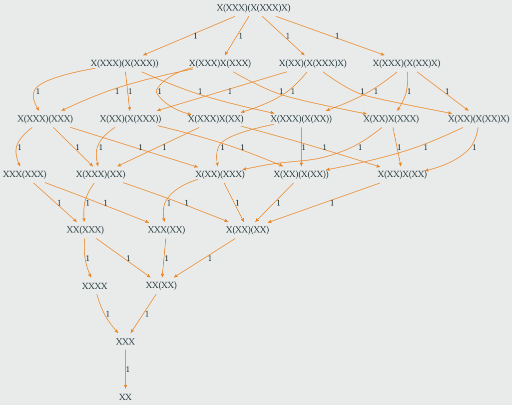


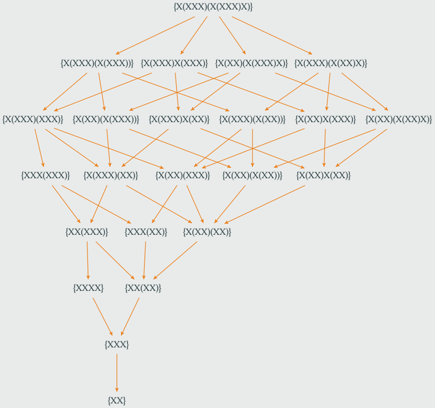


Some examples — 3/6

– Example –

Let $\mathfrak{t} := \mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X})(\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X})\,\mathbf{X})$ where $\mathbf{X} := \mathbf{T} = \rho_2(21)$. Here are $\mathrm{G}(\mathfrak{t})$ and the Hasse diagram of $\mathrm{P}(\mathfrak{t})$:

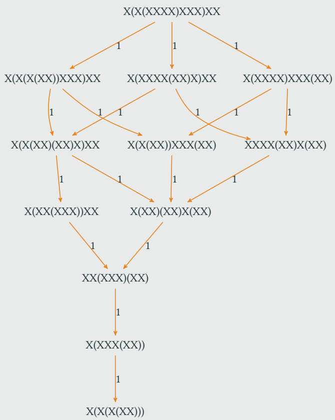


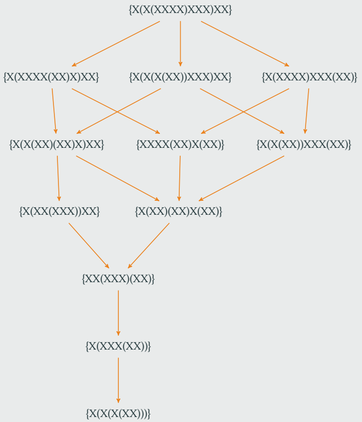


Some examples — 4/6

– Example –

Let $\mathfrak{t} := \mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\mathbf{X})\,\mathbf{X}\,\mathbf{X}\,\mathbf{X})\,\mathbf{X}\,\mathbf{X}$ where $\mathbf{X} := \mathbf{B} = \rho_3(1(23))$. Here are $\mathrm{G}(\mathfrak{t})$ and the Hasse diagram of $\mathrm{P}(\mathfrak{t})$:

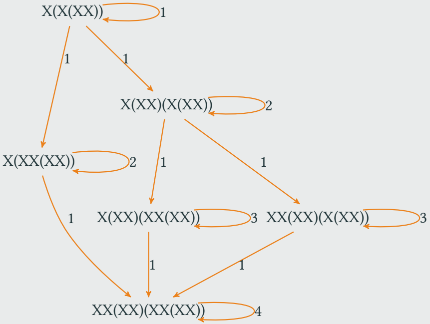


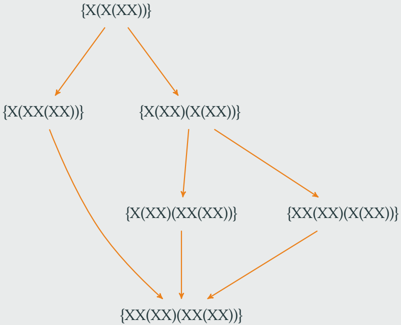


Some examples — 5/6

– Example –

Let $\mathfrak{t} := \mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X}))$ where $\mathbf{X} := \mathbf{M} = \rho_1(11)$. Here are $\mathrm{G}(\mathfrak{t})$ and the Hasse diagram of $\mathrm{P}(\mathfrak{t})$:

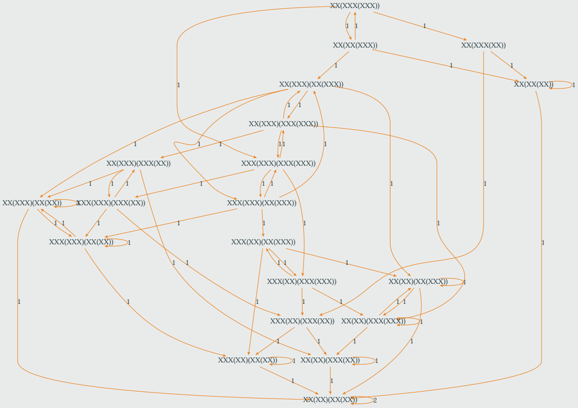


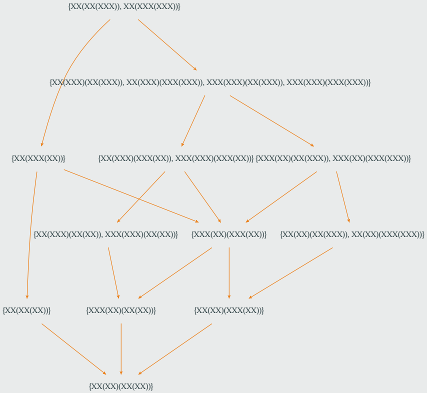


Some examples — 6/6

– Example –

Let $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}))$ where $\mathbf{X} := \rho_2(22)$. Here are $\mathrm{G}(\mathfrak{t})$ and the Hasse diagram of $\mathrm{P}(\mathfrak{t})$:





Usual questions

– Word problem –

Given a set or rules, is there an algorithm taking as input two terms t and t' on these rules and deciding if $t \equiv t'$?
See [**Baader, Nipkow**, 1998], [**Statman**, 2000].

- Yes for the terms on $\mathbf{L}$ [**Statman**, 1989], [**Sprenger, Wymann-Böni**, 1993].
- Yes for the terms on $\mathbf{W}$ [**Sprenger, Wymann-Böni**, 1993].
- Yes for the terms on $\mathbf{M_1}$ [**Sprenger, Wymann-Böni**, 1993].
- Open for the terms on $\mathbf{S}$ [**RTA Problem #97**, 1975].

– Strong normalization problem –

Given a set of rules, is there an algorithm taking as input a term t on these rules and deciding if all rewrite sequences from t are finite?

- Yes for the terms on $\mathbf{S}$ [**Waldmann**, 2000].
- Yes for the terms on $\mathbf{J}$ [**Probst, Studer**, 2000].

2. A combinatorial approach

Combinatorial questions

– On rewrite sets –

Given a term t,

- is $t^\star$ **finite**?
- If it is the case, **how many elements** it contains?

– On rewrite graphs –

Given a term t,

- is $G(t)$ a **simple** graph?
- Is $G(t)$ **acyclic**?
- Is $G(t)$ a **graded** graph?
- Is $G(t)$ **shortcutless**?

– On rewrite posets –

Given a term t,

- is the **quotient** $t^\star / \rightleftharpoons$ **trivial**?
- Is $P(t)$ a **graded** poset?
- Is $P(t)$ a **lattice**?
- If it is the case, is $P(t)$ a **distributive lattice**?

Some computer experiments

Rule	Simple graphs	Acyclic graphs	Graded graphs	Graded posets	Lattices	Max. size
$\mathbf{I} = \rho_1(1)$	$\times$	✓	✓	✓	$\times$	9
$\mathbf{K} = \rho_2(1)$	$\times$	✓	$\times$	✓	$\times$	10
$\mathbf{Ki} = \rho_2(2)$	$\times$	✓	$\times$	✓	$\times$	10
$\mathbf{T} = \rho_2(21)$	✓	✓	✓	✓	✓	11
$\mathbf{B} = \rho_3(1(23))$	✓	✓	✓	✓	✓	11
$\mathbf{C} = \rho_3(132)$	✓	✓	✓	✓	✓	11
$\rho_3(12)$	✓	✓	$\times$	✓	✓	10
$\mathbf{M} = \rho_1(11)$	$\times$	$\times$	$\times$	$\times$	✓	[G., 2022]
$\mathbf{M}_{\mathrm{K}} = \rho_2(22)$	$\times$	$\times$	$\times$	$\times$	✓	7
$\rho_3(112(22))$	$\times$	$\times$	$\times$	$\times$	$\times$	8

Any "✓" says that all terms $\mathfrak{t}$ on the specified rule and having a size less than or equal to the specified maximal size are such that $\mathrm{G}(\mathfrak{t})$ and $\mathrm{P}(\mathfrak{t})$ **have the specified property**.

Any "$\times$" says that a **counter-example** has been found.

Formal series of terms

Let $\mathbb{K}$ be any field of characteristic zero —usually $\mathbb{Q}$— and $\mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle$ be the dual space of the $\mathbb{K}$-linear span $\mathbb{K}\langle\mathfrak{T}\rangle$ of $\mathfrak{T}$.

Any $\mathbf{F} \in \mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle$ is a formal series of terms and can be expressed as a **possibly infinite formal sum**

$$\mathbf{F} = \sum_{\mathfrak{t}\in\mathfrak{T}} \langle \mathfrak{t}, \mathbf{F}\rangle\, \mathfrak{t}$$

where $\langle \mathfrak{t}, \mathbf{F}\rangle$ is the coefficient $\mathbf{F}(\mathfrak{t}) \in \mathbb{K}$ of $\mathfrak{t}$ in $\mathbf{F}$.

For any term $\mathfrak{t}$, the t-multi-application map is the linear map $\gamma_{\mathfrak{t}} : \mathrm{T}(\mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle) \to \mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle$ satisfying, for any $\mathfrak{t}_1, \dots, \mathfrak{t}_\ell \in \mathfrak{T}$, $\ell \geqslant 0$,

$$\gamma_{\mathfrak{t}}(\mathfrak{t}_1 \otimes \cdots \otimes \mathfrak{t}_\ell) = \mathfrak{t}\ \mathfrak{t}_1 \dots \mathfrak{t}_\ell.$$

– Example –

$$\gamma_{\mathbf{K}\,\mathbf{I}}(2\,\mathbf{K}\otimes\mathbf{I} \ + \ \mathbf{I}\otimes\mathbf{K}\,\mathbf{I} \ + \ \mathbf{K}\otimes\mathbf{K}\,\mathbf{I}\otimes\mathbf{K}) \quad = \quad 2\mathbf{K}\,\mathbf{I}\,\mathbf{K}\,\mathbf{I} \ + \ \mathbf{K}\,\mathbf{I}\,\mathbf{I}(\mathbf{K}\,\mathbf{I}) \ + \ \mathbf{K}\,\mathbf{I}\,\mathbf{K}(\mathbf{K}\,\mathbf{I})\,\mathbf{K}$$

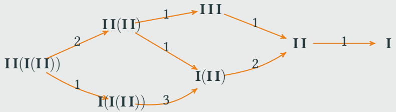
Next map

The next map is the linear map $\mathbf{nx} : \mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle \to \mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle$ satisfying, for any $\mathfrak{s} \in \mathfrak{S}(n)$, $n \geqslant 1$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_\ell \in \mathfrak{T}$, $\ell \geqslant 0$,

$$\mathbf{nx}(\rho_n(\mathfrak{s})\mathfrak{t}_1 \dots \mathfrak{t}_\ell) = [\ell \geqslant n]\mathfrak{s}[\mathfrak{t}_1, \dots, \mathfrak{t}_n]\mathfrak{t}_{n+1} \dots \mathfrak{t}_\ell + \sum_{i \in [\ell]} \gamma_{\rho_n(\mathfrak{s})}(\mathfrak{t}_1 \otimes \dots \otimes \mathbf{nx}(\mathfrak{t}_i) \otimes \dots \otimes \mathfrak{t}_\ell).$$

– Example –

$$\begin{aligned}\mathbf{nx}(\mathbf{I\,I}(\mathbf{I}(\mathbf{I\,I}))) &= \mathbf{I}(\mathbf{I}(\mathbf{I\,I})) + \gamma_{\mathbf{I}}(\mathbf{nx}(\mathbf{I}) \otimes \mathbf{I}(\mathbf{I\,I}) + \mathbf{I} \otimes \mathbf{nx}(\mathbf{I}(\mathbf{I\,I})))\\ &= \mathbf{I}(\mathbf{I}(\mathbf{I\,I})) + 2\ \mathbf{I\,I}(\mathbf{I\,I})\end{aligned}$$



– Lemma [G., 2023+] –

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be two terms.

- We have $\mathfrak{t} \Rightarrow \mathfrak{t}'$ iff $\mathfrak{t}'$ appears in $\mathbf{nx}(\mathfrak{t})$.
- The coefficient $\langle \mathfrak{t}', \mathbf{nx}(\mathfrak{t})\rangle$ is the number of ways to obtain $\mathfrak{t}'$ from $\mathfrak{t}$ by a rewrite step.

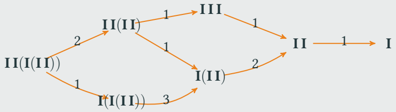
Path map

The path map is the linear map $\mathbf{ph} : \mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle \to \mathbb{K}\langle\langle\mathfrak{T}\rangle\rangle$ satisfying, for any $\mathfrak{t} \in \mathfrak{T}$,

$$\mathbf{ph}(\mathfrak{t}) = \mathfrak{t} + \mathbf{ph}(\mathbf{nx}(\mathfrak{t})).$$

– Example –

$$\begin{aligned}
\mathbf{ph}(\mathbf{II}(\mathbf{I}(\mathbf{II}))) &= \mathbf{II}(\mathbf{I}(\mathbf{II})) + \mathbf{ph}(\mathbf{nx}(\mathbf{II}(\mathbf{I}(\mathbf{II})))) \\
&= \mathbf{II}(\mathbf{I}(\mathbf{II})) + \mathbf{ph}(\mathbf{I}(\mathbf{I}(\mathbf{II})) + 2\ \mathbf{II}(\mathbf{II})) \\
&= 12\,\mathbf{I} + 12\,\mathbf{II} + 5\,\mathbf{I}(\mathbf{II}) + \mathbf{I}(\mathbf{I}(\mathbf{II})) \\
&\quad + 2\,\mathbf{III} + 2\,\mathbf{II}(\mathbf{II}) + \mathbf{II}(\mathbf{I}(\mathbf{II}))
\end{aligned}$$



– Proposition [G., 2023+] –

For any term $\mathfrak{t}$, $\mathbf{ph}(\mathfrak{t})$ is a well-defined polynomial iff the rewrite graph $G(\mathfrak{t})$ is acyclic.

When this condition holds, the coefficient $\langle \mathfrak{t}', \mathbf{ph}(\mathfrak{t})\rangle$ is the number of ways to obtain $\mathfrak{t}'$ from $\mathfrak{t}$ by a sequence of rewrite steps.

3. Some results

Properties on schemes

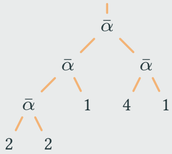
Let $\mathfrak{s}$ be a scheme.

- The frontier of $\mathfrak{s}$ is the sequence $\mathrm{fr}(\mathfrak{s})$ of the variables of $\mathfrak{s}$ read from the left to the right.
- The length $\mathrm{len}(\mathfrak{s})$ of $\mathfrak{s}$ is the length of $\mathrm{fr}(\mathfrak{s})$.
- The depth sequence $\mathrm{dep}(\mathfrak{s})$ of $\mathfrak{s}$ is the sequence of length $\mathrm{len}(\mathfrak{s})$ such that for any $j \in [\mathrm{len}(\mathfrak{s})]$, $\mathrm{dep}_j(\mathfrak{s})$ is the number of internal nodes $\bar{\alpha}$ which are ancestors of the i-th variable of $\mathfrak{s}$.

– Example –

Let the scheme

$\mathfrak{s} := 221(41) =$



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This scheme $\mathfrak{s}$ satisfies

- $\mathrm{fr}(\mathfrak{s}) = 22141$;
- $\mathrm{len}(\mathfrak{s}) = 5$;
- $\mathrm{dep}(\mathfrak{s}) = 33222$.

Properties on rules

Given $n \geqslant 1$ and $\mathfrak{s} \in \mathfrak{S}(n)$, a rule $\rho_n(\mathfrak{s})$ is

- projective if $\mathfrak{s}$ is a variable;
- linear if $\mathfrak{s}$ admits at most one occurrence of any variable;
- conservative if $\mathfrak{s}$ admits at least one occurrence of each variable of $[n]$;
- retractive if for any $j \in [\mathrm{len}(\mathfrak{s})]$, $\mathrm{dep}_j(\mathfrak{s}) \leqslant n + 1 - \mathrm{fr}_j(\mathfrak{s})$.

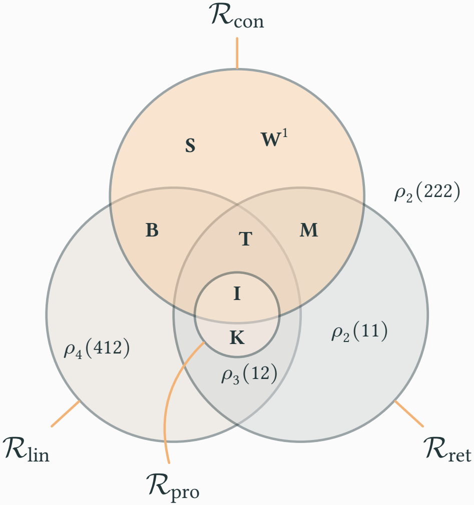
For all these properties P, a term $\mathfrak{t}$ is P (resp. anti-P) if all rules of $\mathfrak{t}$ are P (resp. are not P).

– Example –

- $\mathbf{I} = \rho_1(1)$ is projective, linear, conservative, and retractive;
- $\mathbf{K} = \rho_2(1)$ and $\mathbf{Ki} = \rho_2(2)$ are projective, linear, and retractive;
- $\mathbf{T} = \rho_2(21)$ is linear, conservative, and retractive;
- $\mathbf{B} = \rho_3(1(23))$ is linear and conservative;
- $\mathbf{M} = \rho_1(11)$ and $\mathbf{M}_1 = \rho_2(112)$ are conservative and retractive.

Classification of rules

The set of rules is structured as follows according to these properties:



where

- $\mathcal{R}_{pro}$ is the set of projective rules;
- $\mathcal{R}_{lin}$ is the set of linear rules;
- $\mathcal{R}_{con}$ is the set of conservative rules;
- $\mathcal{R}_{ret}$ is the set of retractive rules.

Conservative and linear terms

– Proposition [G., 2023+] **–**

For any $n \geqslant 1$, the number of conservative and linear rules of $\mathfrak{S}(n)$ is

$$\#(\mathfrak{S}(n) \cap \mathcal{R}_{\mathrm{con}} \cap \mathcal{R}_{\mathrm{lin}}) = \frac{(2n-2)!}{(n-1)!}.$$

The first numbers are 1, 2, 12, 120, 1680, 30240, 665280 (Sequence **A001813**).

A graph $\mathrm{G}(\mathfrak{t})$ is graded if there is a map $\phi : \mathfrak{t}^\star \to \mathbb{N}$ such that $\phi(\mathfrak{t}) = 0$ and for any terms $\mathfrak{t}'$ and $\mathfrak{t}''$ of $\mathfrak{t}^\star$ such that $\mathfrak{t}' \Rightarrow \mathfrak{t}''$, $\phi(\mathfrak{t}'') = \phi(\mathfrak{t}') + 1$.

An edge $\mathfrak{t}' \Rightarrow \mathfrak{t}''$ of $\mathrm{G}(\mathfrak{t})$ is a shortcut if there is a term $\mathfrak{t}'''$ such that $\mathfrak{t}' \neq \mathfrak{t}''' \neq \mathfrak{t}''$ and $\mathfrak{t}' \preccurlyeq \mathfrak{t}''' \preccurlyeq \mathfrak{t}''$.

– Proposition [G., 2023+] **–**

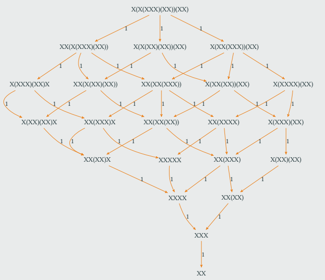
If $\mathfrak{t}$ is a conservative and linear term, then

- the graph $\mathrm{G}(\mathfrak{t})$ is graded;
- the graph $\mathrm{G}(\mathfrak{t})$ is shortcutless.

Conservative and linear terms — examples

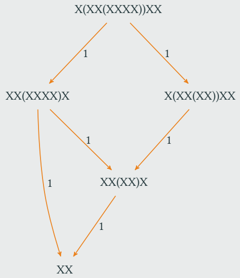
– Example –

The rewrite graph of the conservative and linear term $\mathfrak{t} := \mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X})(\mathbf{X}\,\mathbf{X}))(\mathbf{X}\,\mathbf{X})$ where $\mathbf{X} := \mathbf{T} = \rho_2(21)$ is graded and shortcutless:



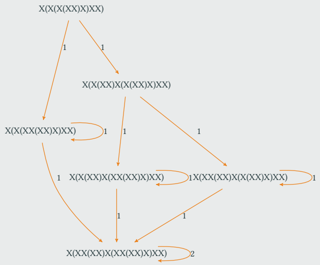
– Example –

The rewrite graph of the linear but **not conservative** term $\mathfrak{t} := \mathbf{X}(\mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\mathbf{X}))\,\mathbf{X}\,\mathbf{X}$ where $\mathbf{X} := \rho_3(13)$ is **not shortcutless**:



– Example –

The rewrite graph of the conservative but **not linear** term $\mathfrak{t} := \mathbf{X}(\mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X})\,\mathbf{X})\,\mathbf{X}\,\mathbf{X})$ where $\mathbf{X} := \mathbf{M}_1 = \rho_2(112)$ is **not graded**:



Linear terms

– Proposition [G., 2023+] –

For any $n \geqslant 1$, the number of linear rules of $\mathfrak{S}(n)$ is

$$\#(\mathfrak{S}(n) \cap \mathcal{R}_{\text{lin}}) = \sum_{k \in [n]} \binom{n}{k} \binom{2k-2}{k-1}.$$

The first numbers are 1, 4, 21, 184, 2425, 42396, 916909 (Sequence **A224500**).

A poset $\mathrm{P}(\mathfrak{t})$ is graded if its Hasse diagram is a graded graph.

– Proposition [G., 2023+] –

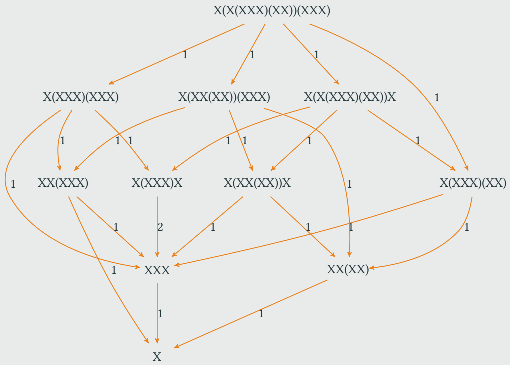
If $\mathfrak{t}$ is a linear term, then

- the set $\mathfrak{t}^\star$ is finite;
- the quotient $\mathfrak{t}^\star / _{\rightleftharpoons}$ is trivial;
- the poset $\mathrm{P}(\mathfrak{t})$ is graded.

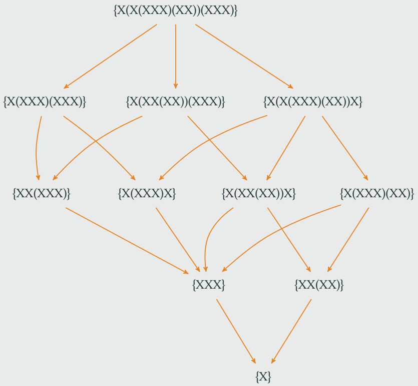
Linear terms — examples 1/2

– Example –

The rewrite graph of the linear term $\mathfrak{t} := \mathbf{X}(\mathbf{X}(\mathbf{X\,X\,X})(\mathbf{X\,X}))(\mathbf{X\,X\,X})$ where $\mathbf{X} := \mathbf{K} = \rho_2(1)$ is finite and not graded:



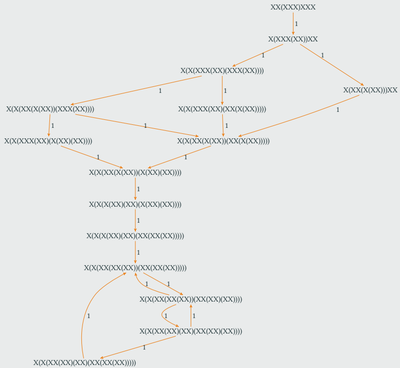
Its poset has trivial $\rightleftharpoons$-equivalence classes and is graded:



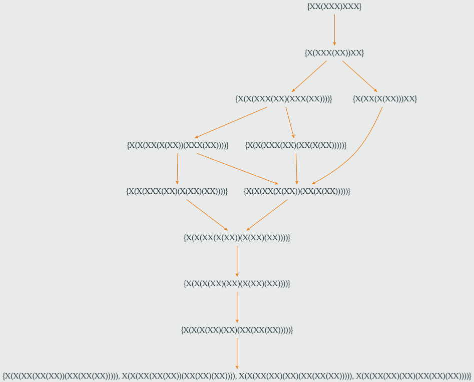
Linear terms — examples 2/2

– Example –

The rewrite graph of the non-linear term $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X})\,\mathbf{X}\,\mathbf{X}\,\mathbf{X}$ where $\mathbf{X} := \rho_3(3(2(11)))$ is finite and is **not acyclic**:



Its poset has **nontrivial $\rightleftharpoons$-equivalence classes** and is **not graded**:



Linear and anti-projective terms

A term is anti-projective if it does not have any projective rule.

The rewrite graph $G(\mathfrak{t})$ is simple if it does not have any multi-edge.

– Conjecture (work-in-progress) [G., 2023+] –

If $\mathfrak{t}$ is an anti-projective and linear term, then $G(\mathfrak{t})$ is simple.

A poset $P(\mathfrak{t})$ is a lattice if all pairs of its elements admit a greatest lower bound and a least upper bound.

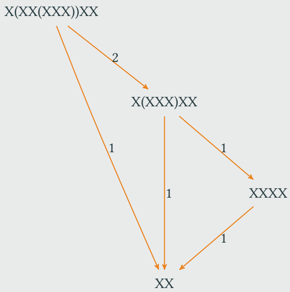
– Conjecture (work-in-progress) [G., 2023+] –

If $\mathfrak{t}$ is an anti-projective and linear term, then $P(\mathfrak{t})$ is a lattice.

Linear and anti-projective terms — examples 1/2

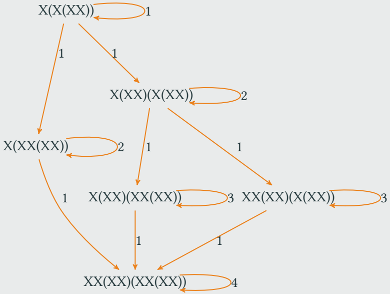
– Example –

The rewrite graph of the linear but **not anti-projective** term $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X})(\mathbf{X}\,\mathbf{X}\,\mathbf{X}))$ where $\mathbf{X} := \mathbf{Ki} = \rho_2(2)$ is **not simple**:



– Example –

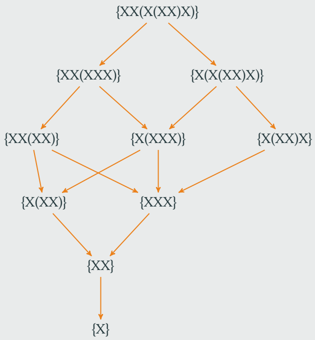
The rewrite graph of the anti-projective but **not linear** term $\mathfrak{t} := \mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X}))$ where $\mathbf{X} := \mathbf{M} = \rho_1(11)$ is **not simple**:



Linear and anti-projective terms – examples 2/2

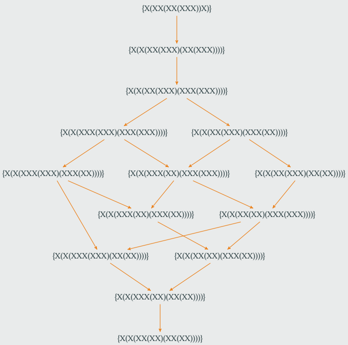
– Example –

The rewrite poset of the linear but **not anti-projective** term $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X})\,\mathbf{X})$ where $\mathbf{X} := \mathbf{I} = \rho_1(1)$ is **not a lattice**:



– Example –

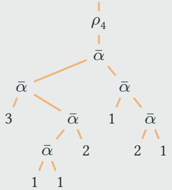
The rewrite poset of the anti-projective but **not linear** term $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}))\,\mathbf{X}$ where $\mathbf{X} := \rho_3(3(22))$ is **not a lattice**:



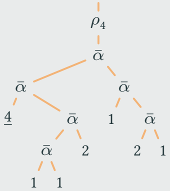
Retractive terms

If $\rho_n(\mathfrak{s})$ is a retractive rule, then in $\mathfrak{s}[\mathfrak{t}_1, \ldots, \mathfrak{t}_n]$, the respective depths of the subterms $\mathfrak{t}_1, \ldots, \mathfrak{t}_n$ are smaller than the ones they have in $\rho_n(\mathfrak{s})\mathfrak{t}_1 \ldots \mathfrak{t}_m$.

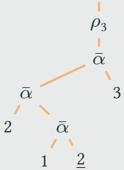
– Example –



This rule is retractive.



This rule is not retractive.



This rule is not retractive.

As a consequence, when $\mathfrak{t}$ is retractive, $\mathfrak{t} \Rightarrow \mathfrak{t}'$ implies $\mathrm{ht}(\mathfrak{t}) \geqslant \mathrm{ht}(\mathfrak{t}')$.

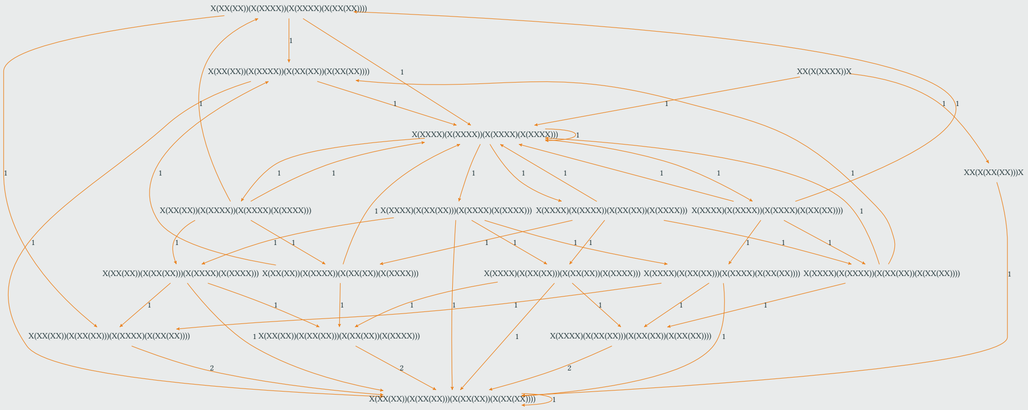
– Proposition [G., 2023+] –

If $\mathfrak{t}$ is a retractive term, then $\mathfrak{t}^\star$ is finite.

Retractive terms — example

– Example –

The rewrite graph of the retractive term $\mathfrak{t} := \mathbf{X}\,\mathbf{X}(\mathbf{X}(\mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\mathbf{X}))\,\mathbf{X}$ where $\mathbf{X} := \rho_3(22(22))$ is finite and not acyclic:



Conclusion

Rewrite graphs and rewrite posets of terms are provided with some **combinatorial properties** depending on some characteristics of the terms:

Property on t	$t^\star$ **finite**	$G(t)$ **simple**	$G(t)$ **acyc.**	$G(t)$ **grad.**	$G(t)$ **shortcutl.**	$P(t)$ **grad.**	$P(t)$ **lattice**
Lin.	✓		✓			✓	
Lin. & cons.	✓		✓	✓	✓	✓	
Lin. & anti-proj.	✓	?	✓			✓	?
Retr.	✓						

Perspectives:

- prove the conjectured properties;
- given a linear (resp. retractive) term t, describe a way to enumerate $t^\star$;
- see such rewrite graphs and rewrite posets within the framework of differential graded posets [**Stanley**, 1988] and the framework of duality of graded graphs [**Fomin**, 1994];
- describe general properties of formal series of terms w.r.t. natural operations.