Some combinatorial aspects of combinatory logic

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Séminaire LIX
Palaiseau

March 31, 2021
1. Terms and rewrite systems

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1. Terms and rewrite systems
A signature is a graded set \( \mathcal{G} := \bigcup_{n \geq 0} \mathcal{G}(n) \) wherein each \( a \in \mathcal{G}(n) \) is an constant of arity \( n \).

A \( \mathcal{G} \)-term is

- either a variable \( x \) from the set \( \mathbb{X} := \{ x_1, x_2, \ldots \} \);
- either a pair \((a, (t_1, \ldots, t_n))\) where \( a \in \mathcal{G}(n) \) and each \( t_i \) is a \( \mathcal{G} \)-term.

The set of all \( \mathcal{G} \)-terms is denoted by \( \mathcal{T}(\mathcal{G}) \).

**Example**

This is the tree representation of the \( \mathcal{G} \)-term

\[
(a, ((b, (x_1, x_2)), (b, ((a, (x_1, x_1)), x_3))))
\]

where \( \mathcal{G} := \mathcal{G}(2) := \{a, b\} \).
The **frontier** of a \( \mathcal{G} \)-term \( t \) is the sequence of the variables appearing in \( t \).

The **ground arity** of \( t \) is the greatest integer \( n \) such that \( x_n \) is a variable appearing in \( t \), its **arity** is the length of its frontier, and its **degree** is its number of its internal nodes.

---

**Example**

The term \( t \) is

- **planar** if its frontier is \((x_1, \ldots, x_n)\);
- **standard** if its frontier is a permutation of \((x_1, \ldots, x_n)\);
- **linear** if there are no multiple occurrences of the same variable in the frontier of \( t \);
- **closed** if its arity is 0.
A rewrite relation on $\mathcal{L}(G)$ is a binary relation $\rightarrow$ on $\mathcal{L}(G)$ such that if $s \rightarrow s'$, then all variables of $s'$ appear in $s$.

The context closure of $\rightarrow$ is the binary relation $\Rightarrow$ satisfying $t \Rightarrow t'$ whenever $t'$ is obtained by replacing in $t$ a factor $s$ by $s'$ provided that $s \rightarrow s'$.

**Example**

For $G := G(2) := \{a\}$, let the rewrite relation $\rightarrow$ defined by

\[
\begin{align*}
    x_1 & \rightarrow x_1 \\
    x_2 x_3 & \rightarrow x_2 x_3
\end{align*}
\]

We have

\[
\begin{align*}
    & \Rightarrow \\
\end{align*}
\]

A term rewrite system (TRS) is such a pair $(G, \rightarrow)$. 
More on rewrite systems

Let \( S := (G, \rightarrow) \) be a TRS.

We define

- \( \preceq \) as the reflexive and transitive closure of \( \Rightarrow \);
- \( \equiv \) as the reflexive, symmetric, and transitive closure of \( \Rightarrow \);
- \( G(T) \) as the digraph on the set \( \{ t' \in T(G) : t \preceq t' \text{ for a } t \in T \} \) of vertices and the set \( \Rightarrow \) of edges, where \( T \) is any subset of \( T(G) \).

A term \( t \in T(G) \) is

- a **normal form** for \( S \) if there is no edge of source \( t \) in \( G(\{t\}) \);
- **weakly normalizing** in \( S \) if there is at least one normal form in \( G(\{t\}) \);
- is **strongly normalizing** in \( S \) if \( G(\{t\}) \) is finite and acyclic.

When all \( G \)-terms are strongly normalizing, \( S \) is **terminating**.

If for any \( G \)-term \( t, t \preceq s_1 \) and \( t \preceq s_2 \) implies the existence of \( t' \) such that \( s_1 \preceq t' \) and \( s_2 \preceq t' \), then \( S \) is **confluent**.
Properties of rewrite systems

Let $S := (\mathcal{G}, \rightarrow)$ be a TRS.

One has the following properties (see for instance [Baader, Nipkow, 1998]).

- Proposition –

If $S$ is terminating and confluent, then the set of normal forms of $S$ is a set of representatives of $\mathcal{T}(\mathcal{G})/\equiv$.

- Proposition –

If $S$ is terminating and confluent, then to decide if $t \equiv t'$ in $S$, compute $s$ and $s'$ as, respectively, the unique formal forms in the $\equiv$-equivalence classes of $t$ and $t'$ and check if $s = s'$.

- Proposition –

The set of normal forms of $S$ is the set of all $\mathcal{G}$-terms avoiding each $\mathcal{G}$-term $t$ where $t$ is on the left of a rule of $S$. 
The Tamari rewrite system

Let $G := G(2) := \{a\}$ and $\rightarrow$ be the rewrite relation on $\mathcal{T}(G)$ defined by

Here are the first graphs $T(k)$ of planar $G$-terms of degrees $k \geq 0$:
The Tamari rewrite system

- The binary relation $\rightarrow$ is the right rotation operation, an important operation on binary search trees appearing in algorithms handling balanced binary trees [Adelson-Velsky, Landis, 1962].

- This TRS is terminating and confluent. As consequence, $\preceq$ is a partial order relation.

  This order endows each set $T(k)$ with the structure of a lattice, known as Tamari lattice [Huang, Tamari, 1962].

- The set of normal forms of this TRS contains all right comb trees, that are the trees avoiding

  \[
  \begin{array}{c}
  \text{a} \\
  \text{x}_1 \\
  \text{x}_2 \\
  \text{x}_3 \\
  \end{array}
  \]

  There is exactly one planar normal form of degree $k$ for any $k \geq 0$. 
A variant of the Tamari rewrite system

Let \( \mathcal{G} := \mathcal{G}(2) := \{a\} \) and \( \rightarrow \) be the rewrite relation on \( \mathcal{T}(\mathcal{G}) \) defined by

\[
\begin{align*}
\begin{array}{c}
\text{a} & x_4 \\
\text{a} & x_3 \\
\text{a} & x_2 \\
\text{a} & x_1
\end{array}
\rightarrow
\begin{array}{c}
\text{a} & x_1 \\
\text{a} & x_2 \\
\text{a} & x_3 \\
\text{a} & x_4
\end{array}
\end{align*}
\]

Here are the first graphs \( T_3(k) \) of planar \( \mathcal{G} \)-terms of degrees \( k \geq 0 \):

<table>
<thead>
<tr>
<th></th>
<th>T_3(0)</th>
<th>T_3(1)</th>
<th>T_3(2)</th>
<th>T_3(3)</th>
<th>T_3(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>3</td>
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<td>4</td>
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</tbody>
</table>
A variant of the Tamari rewrite system

This TRS is terminating but **not confluent**.

The **Buchberger completion algorithm** [Knuth, Bendix, 1970], [Dotsenko, Khoroshkin, 2010] is a semi-algorithm taking as input a terminating but not confluent TRS \((\mathcal{G}, \to)\) and outputting a new rewrite relation \(\to'\) such that

- \((\mathcal{G}, \to')\) is terminating **and confluent**;
- the preorders \(\preceq\) and \(\preceq'\) are equal.

**Theorem** [Chenavier, Cordero, G., 2018] –

The normal forms of a completion of the previous TRS can be described as the \(\mathcal{G}\)-terms avoiding 11 planar \(\mathcal{G}\)-terms of degrees from 3 to 7.

The generating series of the planar normal forms of this TRS, enumerated w.r.t. their arities, is

\[
F(t) = \frac{t}{(1 - t)^2} \left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right).
\]

The first coefficients of this generating series are

1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 14, 15, 16, 17, 18, 19, 20.
Enumeration of planar terms avoiding factors

**Theorem** [G., 2020]

Let $\mathcal{G}$ be a signature, and $\mathcal{P}$ be a set of planar $\mathcal{G}$-terms. The generating series enumerating the planar $\mathcal{G}$-terms avoiding $\mathcal{P}$ w.r.t. the arity (parameter $t$) and the degree (parameter $q$) is $F(\mathcal{P}, \emptyset)$ where, for any set $\mathcal{Q}$ of planar $\mathcal{G}$-terms,

$$
F(\mathcal{P}, \mathcal{Q}) = t + q \sum_{k \geq 1} \sum_{a \in \mathcal{G}(k)} \left\{ \sum_{\ell \geq 1} (-1)^{1+\ell} \prod_{i \in [k]} F(\mathcal{P}, S_i) \right\} \subseteq M(\mathcal{P} \cup \mathcal{Q}, a) (S_1, \ldots, S_k) = \mathcal{R}(1) \oplus \cdots \oplus \mathcal{R}(\ell)
$$

**Example**

For $\mathcal{P} := \{ a \wedge b \}$, we obtain the algebraic system of equations

$$
F(\mathcal{P}, \emptyset) = t + q F(\mathcal{P}, \{a\}) F(\mathcal{P}, \emptyset) + q F(\mathcal{P}, \emptyset) F(\mathcal{P}, \{b\}) - q F(\mathcal{P}, \{a\}) F(\mathcal{P}, \{b\}) + q F(\mathcal{P}, \emptyset) F(\mathcal{P}, \emptyset),
$$

$$
F(\mathcal{P}, \{a\}) = t + q F(\mathcal{P}, \emptyset) F(\mathcal{P}, \emptyset),
$$

$$
F(\mathcal{P}, \{b\}) = t + q F(\mathcal{P}, \{a\}) F(\mathcal{P}, \emptyset) + q F(\mathcal{P}, \emptyset) F(\mathcal{P}, \{b\}) - q F(\mathcal{P}, \{a\}) F(\mathcal{P}, \{b\}).
$$
2. Combinatory logic
An **applicative signature** is a signature $\mathcal{G}$ satisfying $\mathcal{G} = \mathcal{G}(0) \sqcup \mathcal{G}(2)$ where $\mathcal{G}(2) = \{a\}$.

An **applicative term** is a term on an applicative signature.

Each applicative term can be expressed as a bracketed infix expression on $\mathcal{G}(0) \sqcup X$ wherein the symbol $a$ is made implicit and the bracketing is implicit from left to right.

**– Example –**

On the applicative signature $\mathcal{G}$ where $\mathcal{G}(0) = \{A, B, C\}$,

\[
\begin{align*}
\leftrightarrow & \quad (A \ a \ (x_1 \ a \ A)) \ a (((B \ a \ x_2) \ a \ x_1)) \\
& \leftrightarrow \quad (A \ (x_1 \ A)) (((B \ x_2) \ x_1)) \\
& \leftrightarrow \quad A(x_1 \ A)(B \ x_2x_1).
\end{align*}
\]
A combinatory logic system (CLS) is a pair \((\mathcal{C}, \rightarrow)\) where

- \(\mathcal{C}\) is a finite set;
- \(\rightarrow\) is a rewrite relation on the applicative signature \(\mathcal{G}_\mathcal{C}\) where \(\mathcal{G}_\mathcal{C}(0) = \mathcal{C}\), and for any \(X \in \mathcal{C}\), there is exactly one rule of the form

\[X \, x_1 \ldots x_n \rightarrow t\]

such that \(t\) is an \(\{a\}\)-term.

**Example**

Let the CLS \((\mathcal{C}, \rightarrow)\) [Schönfinkel, 1924] [Curry, 1930] where \(\mathcal{C} := \{I, K, S\}\) and \(\rightarrow\) satisfies

- \(I \, x_1 \rightarrow x_1\),
- \(K \, x_1 x_2 \rightarrow x_1\),
- \(S \, x_1 x_2 x_3 \rightarrow x_1 x_3 (x_2 x_3)\).

This CLS is Turing complete: there are algorithms to faithfully translate any \(\lambda\)-term into a term of this CLS. These are abstraction algorithms [Rosser, 1955], [Curry, Feys, 1958].
Some important combinators

In *To Mock a Mockingbird: and Other Logic Puzzles* [Smullyan, 1985], a large number of rules are listed. Here is a sublist:

- **Identity bird**: $I \; x_1 \rightarrow x_1$
- **Mockingbird**: $M \; x_1 \rightarrow x_1 \; x_1$
- **Kestrel**: $K \; x_1 \; x_2 \rightarrow x_1$
- **Thrush**: $T \; x_1 \; x_2 \rightarrow x_2 \; x_1$
- **Mockingbird 1**: $M_1 \; x_1 \; x_2 \rightarrow x_1 \; x_1 \; x_2$
- **Warbler**: $W \; x_1 \; x_2 \rightarrow x_1 \; x_2 \; x_2$
- **Lark**: $L \; x_1 \; x_2 \rightarrow x_1 (x_2 \; x_2)$
- **Owl**: $O \; x_1 \; x_2 \rightarrow x_2 (x_1 \; x_2)$
- **Turing bird**: $U \; x_1 \; x_2 \rightarrow x_2 (x_1 \; x_1 \; x_2)$
- **Cardinal**: $C \; x_1 \; x_2 \; x_3 \rightarrow x_1 \; x_3 \; x_2$
- **Vireo**: $V \; x_1 \; x_2 \; x_3 \rightarrow x_3 \; x_1 \; x_2$
- **Bluebird**: $B \; x_1 \; x_2 \; x_3 \rightarrow x_1 (x_2 \; x_3)$
- **Starling**: $S \; x_1 \; x_2 \; x_3 \rightarrow x_1 \; x_3 (x_2 \; x_3)$
- **Jay**: $J \; x_1 \; x_2 \; x_3 \; x_4 \rightarrow x_1 \; x_2 (x_1 \; x_4 \; x_3)$
Let $\mathcal{C} := (\mathcal{C}, \rightarrow)$ be a CLS.

### – Word problem –

Is there an algorithm to decide, given two terms $t$ and $t'$ of $\mathcal{C}$, if $t \equiv t'$? (See [Baader, Nipkow, 1998], [Statman, 2000].)

- Yes for the CLS on $L$ [Statman, 1989], [Sprenger, Wymann-Böni, 1993].
- Yes for the CLS on $W$ [Sprenger, Wymann-Böni, 1993].
- Yes for the CLS on $M_1$ [Sprenger, Wymann-Böni, 1993].
- Open for the CLS on $S$ [RTA Problem #97, 1975].

### – Strong normalization problem –

Is there an algorithm to decide, given a term $t$ of $\mathcal{C}$, if $t$ is strongly normalizing?

- Yes for the CLS on $S$ [Waldmann, 2000].
- Yes for the CLS on $J$ [Probst, Studer, 2000].
Combinatorial questions

Let $\mathcal{C} := (\mathcal{C}, \rightarrow)$ be a CLS.

- **Structure of the rewriting graphs** –
  1. Are all the connected components of the graph $G(\mathcal{G}(\mathcal{C}))$ of $\mathcal{C}$ finite?
  2. Understand when $G(\{\{t\})$ and $G(\{t'\})$ are isomorphic graphs.
  3. Is the preorder $\preceq$ an order relation on $\mathcal{G}(\mathcal{G}(\mathcal{C}))$?
  4. If so, are all intervals $[t, t']$ lattices?

- **Enumerative issues** –
  1. In the graph $G(\mathcal{G}(\mathcal{C}))$ of $\mathcal{C}$, count w.r.t. their degrees the closed terms that are
     1.1 minimal;
     1.2 maximal (which are thus normal forms);
     1.3 both minimal and maximal (which are thus isolated vertices).
  2. Count the $\equiv$-equivalence classes of closed terms w.r.t. their degrees.
  3. Count the connected components of $G(\mathcal{G}(\mathcal{C}))$ w.r.t. the minimal degrees of their closed terms.
First properties: termination

A CLS $\mathcal{C}$ can be non-terminating.

There are several possible causes:

- The set $G(\{t\})$ is infinite for a term $t$ of $\mathcal{C}$.

  - Example –

    In the CLS on $S$, the terms
    
    - $S(SS)(SS)(S(SS)(SS))$ [Zachos, 1978];
    - $SSS(SSS)(SSS)$ [Barendregt, 1984]

    have this property.

- There is a cycle, that is two terms $t$ and $t'$ of $\mathcal{C}$ such that $t \Rightarrow t' \leq t$.

  - Example –

    In the CLS on $F x_1 x_2 \rightarrow x_2 x_2$, one has $t \Rightarrow t' \Rightarrow t$ with $t := FF(FFF)$ and $t' := FFF(FFF)$. 
First properties: confluence and lattices

**– Proposition –**

Any CLS is confluent.

This is a consequence of the orthogonality \cite{Rosen, 1973} of any CLS.

A rewrite relation $\rightarrow$ is orthogonal if

- $t \rightarrow t'$ implies that $t$ is linear;
- $t \rightarrow t'$, $s \rightarrow s'$, and $t$ and $s$ overlap, implies $t = s$.

Some terminating (and confluent) CLS have intervals that are not lattices.

**– Example –**

In the CLS on $I x_1 \rightarrow x_1$, the interval $[II(III), II]$ admits the Hasse diagram
3. Mockingbird lattices
Let us focus on the CLS on $Mx_1 \rightarrow x_1x_1$.

Here is the graph $G(\mathcal{G}(\mathcal{C}))$ restrained to closed terms of degrees 3 or less:
Let \( \rightarrow' \) the rewrite relation defined by \( M(x_1x_2) \rightarrow' (x_1x_2)(x_1x_2) \).

**Lemma**
The preorders \( \preceq \) and \( \preceq' \) are equal.

**Lemma**
If \( t \Rightarrow' t' \), then \( \text{ht}(t) = \text{ht}(t') \) and \( \text{deg}(t) < \text{deg}(t') \).

As a consequence, \( \preceq' \) is an order relation.

**Proposition [G., 2021–]**
The relation \( \preceq \) is an order relation on terms on \( M \). Moreover, each connected component of the Hasse diagram of this poset is finite.
A **black and white tree (BWT)** is a planar rooted tree such that each internal node is either black or white. Let $T_{bw}$ be the set of such trees.

Let $\Rightarrow$ be the binary relation on $T_{bw}$ such that $p \Rightarrow q$ if $q$ can be obtained from $p$ by selecting a white node $u$ of $p$, by turning it into black, and by duplicating its sequence of descendants.

If $p \Rightarrow q$, then there are more black nodes in $q$ than in $p$. Hence, the reflexive and transitive closure $\ll$ of $\Rightarrow$ is a partial order.

---

**Example**

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**Lemma**

One has $p \ll p'$ iff

- either $p = \circ(p_1, \ldots, p_\ell)$, $p' = \circ(p'_1, \ldots, p'_\ell)$, and $p_i \ll p'_i$ for all $i \in [\ell]$;
- or $p = \circ(p_1, \ldots, p_\ell)$, $p' = \circ(p'_1, \ldots, p'_\ell)$, and $p_i \ll p'_i$ for all $i \in [\ell]$;
- or $p = \circ(p_1, \ldots, p_\ell)$, $p' = \circ(p'_1, \ldots, p'_\ell, p''_1, \ldots, p''_\ell)$, and $p_i \ll p'_i$ and $p_i \ll p''_i$ for all $i \in [\ell]$. 
Lattices on BWT

Let $\land$ and $\lor$ be the two partial binary, commutative, and associative operations on $T_{bw}$ defined by

\[
\bigcirc(p_1, \ldots, p_\ell) \land \bigcirc(p_1', \ldots, p_\ell') := \bigcirc(p_1 \land p_1', \ldots, p_\ell \land p_\ell'),
\]

\[
\bigcirc(p_1, \ldots, p_\ell) \land \bigcirc(p_1', \ldots, p_\ell') := \bigcirc(p_1 \land p_1', \ldots, p_\ell \land p_\ell'),
\]

\[
\bigcirc(p_1, \ldots, p_\ell) \land \bigcirc(p_1', \ldots, p_\ell', p_1'', \ldots, p_\ell'') := \bigcirc(p_1 \land p_1' \land p_1'', \ldots, p_\ell \land p_\ell' \land p_\ell''),
\]

and

\[
\bigcirc(p_1, \ldots, p_\ell) \lor \bigcirc(p_1', \ldots, p_\ell') := \bigcirc(p_1 \lor p_1', \ldots, p_\ell \lor p_\ell'),
\]

\[
\bigcirc(p_1, \ldots, p_\ell) \lor \bigcirc(p_1', \ldots, p_\ell') := \bigcirc(p_1 \lor p_1', \ldots, p_\ell \lor p_\ell'),
\]

\[
\bigcirc(p_1, \ldots, p_\ell) \lor \bigcirc(p_1', \ldots, p_\ell', p_1'', \ldots, p_\ell'') := \bigcirc(p_1 \lor p_1' \lor p_1'', \ldots, p_\ell \lor p_\ell' \lor p_\ell'').
\]

\[\text{– Proposition [G., 2021–] –}\]

Given a BTW $p$, the set $\{p' \in T_{bw} : p \ll p'\}$ is a lattice for the operations $\land$ and $\lor$. 
Lattices of closed terms on $M$

Given a closed term $t$ on $M$, let $\phi(t)$ be the BWT obtained by

1. by adding a root $\bullet$ to $t$;
2. by replacing each internal node of $t$ having a left child $M$ and a right child different from $M$ by a $\circ$;
3. by connecting these new nodes following the paths in $t$.

**– Example –**

```
M
 a
 a
 a
 a
 a
 a
 a
 M
```

**– Lemma –**

Let $t$ and $t'$ be two closed terms on $M$. One has $t \Rightarrow t'$ iff $\phi(t) \Rightarrow \phi(t')$.

**– Theorem [G., 2021–] –**

For any closed term $t$ on $M$, the poset $(G(\{t\}), \preceq)$ is a finite lattice.
Maximal and minimal terms

A closed term on $\mathbf{M}$ is maximal iff it avoids the pattern $\mathbf{M}(x_1x_2)$.

The formal sum $F$ of these trees enumerated w.r.t. their degrees satisfies the equation

$$F = \mathbf{M} + FM + (F - M)(F - M).$$

Therefore, the generating series $F$ of these terms satisfies $F = 1 + tF + (F - 1)^2$. Its first coefficients are 1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835 and are Motzkin numbers (Seq. A001006).

A closed term on $\mathbf{M}$ is minimal iff it avoids the pattern $(x_1x_2)(x_1x_2)$.

This leads to the recurrence for the number $a(d)$ of such trees of degree $d$:

$$a(0) = a(1) = 1, \quad a(d + 1) = b(d) \text{ if } d \text{ is odd}, \quad a(d + 1) = b(d) - a(d/2) \text{ otherwise},$$

where $b(d) := \sum_{0 \leq k \leq d} a(k)a(d - k)$. The first numbers are 1, 1, 2, 4, 12, 34, 108, 344, 1136, 3796, 12920.
Mockingbird lattices

The Mockingbird lattice of order $k \geq 0$ is the lattice $M(k)$ on $G(\{t\})$ where $t$ is the term $M(M(\ldots (M M) \ldots))$ of degree $k$.

Here are the Hasse diagrams of these first lattices:

<table>
<thead>
<tr>
<th>M(0)</th>
<th>M(1)</th>
<th>M(2)</th>
<th>M(3)</th>
<th>M(4)</th>
</tr>
</thead>
</table>

– Theorem [G., 2021–] –

For any BWT $p$, $\{p' \in T_{bw} : p \ll p'\}$ is isomorphic to a maximal interval of a Mockingbird lattice.
Let $p$ be a BWT.

Let $w(p)$ be the integer defined recursively as

$$w(\circ(p_1, \ldots, p_\ell)) = 1 + 2 \sum_{i \in [\ell]} w(p_i),$$

$$w(\bullet(p_1, \ldots, p_\ell)) = \sum_{i \in [\ell]} w(p_i).$$

**Example**

```
  0 15
  2  2
  0  1  1
```

**Proposition [G., 2021–]**

Let $t$ be a closed term on $M$ and $P$ be the lattice $(G(\{t\}), \preceq)$.

- A shorted path from $t$ to the maximal element of $P$ has as length the number of $\circ$ in $\phi(t)$.
- A longest path from $t$ to the maximal element of $P$ has length $w(\phi(t))$.

Therefore, a longest path in $M(k)$ for $k \geq 1$ as length $2^{k-1} - 1$. 
Number of initial intervals

Let $\mathbf{p}$ be a BWT.

Let $s(\mathbf{p})$ be the integer defined recursively as

$$s(\mathbf{p}_{1, \ldots, p_k}) = \left( \prod_{i \in [k]} s(p_i) \right) \left( 1 + \prod_{i \in [k]} s(p_i) \right),$$

$$s(\mathbf{p}_{1, \ldots, p_k}) = \prod_{i \in [k]} s(p_i).$$

**– Example –**

```
5256
1
```

```
5256
1
```

**– Proposition [G., 2021–] –**

Let $t$ be a closed term on $\mathbf{M}$ and $\mathcal{P}$ be the lattice. $(G(\{t\}), \sqsubseteq)$. Then, $\# \mathcal{P} = s(\phi(t))$.

The numbers $a(k)$ of initial intervals in $\mathbf{M}(k)$ satisfies $a(0) = a(1) = 1$, and for $k \geq 2$, $a(k) = a(k-1)(1 + a(k-1))$. The first numbers are

1, 1, 2, 6, 42, 1806, 3263442, 10650056950806, 113423713055421844361000442.
4. Conclusion and future work
When $\#C = 1$, one gets already some interesting CLS. Here are experimental data some of these.

For $L x_1 x_2 \rightarrow x_1 (x_2 x_2)$:

- Minimal terms: 1, 1, 1, 2, 5, 13, 34, 94, 265, 765, 2237, 6632;
- Maximal terms: 1, 1, …;
- Isolated terms: 1, 1, 0, 0, …;
- Equivalence classes: 1, 1, 1, 2, 5, 12, 31, 83;
- Connected components: 1, 1, 1, 2, 4, 9, 22, 60;
- $\rightarrow$ is not terminating;
- $\lessgtr$ is an order relation;
- Intervals seem lattices.
The rewriting graph of terms on \( L \) from closed terms of degrees up to 5 and up to 4 rewritings:
For $S \ x_1x_2x_3 \rightarrow \ x_1x_3(x_2x_3)$:

- Minimal terms: $1, 1, 2, 4, 10, 26, 76, 224, 690, 2158, 6882, 22208$;
- Maximal terms: $1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798$ (Motzkin numbers);
- Isolated terms: $1, 1, 2, 3, 7, 15, 37, 87, 218, 546, 1393, 3583$;
- Equivalence classes: $1, 1, 2, 4, 10, 27, 78, 234, 722, 2271$;
- Connected components: $1, 1, 2, 4, 10, 26, 74, 217, 660, 2053$;
- $\rightarrow$ is not terminating;
- $\preceq$ is an order relation [Bergstra, Klop, 1979];
- Intervals seem lattices.
Other CLS: S

The rewriting graph of terms on S from closed terms of degrees up to 6 and up to 11 rewritings:
For $T \ x_1 \ x_2 \rightarrow x_2 \ x_1$:

- Minimal terms: 1, 0, 0, …;
- Maximal terms: 1, 1, …;
- Isolated terms: 1, 0, 0, …;
- Equivalence classes: 1, 1, 2, 3, 4, 5, 6;
- Connected components: 1, 1, …;
- $\rightarrow$ is terminating;
- $\preceq$ is an order relation;
- Interval seem lattices.
The rewriting graph of terms on $T$ from closed terms of degrees up to 6 and up to 1 rewriting:
For $C x_1 x_2 x_3 \rightarrow x_1 x_3 x_2$:

- Minimal terms: $1, 1, \ldots$;
- Maximal terms: $1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798$ (Motzkin numbers);
- Isolated terms: $1, 1, \ldots$;
- Equivalence classes: $1, 1, 2, 5, 13, 33, 83, 209, 531, 1365$;
- Connected components: $1, 1, 2, 4, 9, 21, 51, 127, 323, 835$ (Motzkin numbers);
- $\rightarrow$ is terminating;
- $\preceq$ is an order relation;
- Interval seem lattices.
The rewriting graph of terms on \( C \) from closed terms of degrees up to 6 and up to 1 rewriting:
Some questions

– Connections with order theory –

Given a CLS $C$, obtain (necessary/sufficient) conditions for the fact that

1. $C$ is terminating;
2. when $\preceq$ is an order relation, all the $\preceq$-intervals of $C$ are lattices;

– Normalization –

Provide a generic way to describe all the strongly normalizing terms of a CLS.

– Connection with clone theory –

Realize a CLS as a clone.

Indeed, each CLS $C := (C, \rightarrow)$ gives rise to a clone defined as the quotient of the free clone generated by $G_C$ by the clone congruence generated by $\rightarrow$. A combinatorial realization of this algebraic structure would provide an encoding of the $\equiv$-equivalence classes of terms of $C$ compatible with term composition.