Polynomial realizations of Hopf algebras from operads

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Objectives

Present a **polynomial realization** of some Hopf algebras constructed from operads.

Main points:

1. Combinatorial Hopf algebras.
2. Polynomial realizations.
3. Nonsymmetric operads.
4. Natural Hopf algebras of nonsymmetric operads.
5. A polynomial realization of natural Hopf algebras of free operads.

Combinatorial Hopf algebras

All algebraic structures are over a field $\mathbb{K}$ of characteristic zero.

A combinatorial Hopf algebra (CHA) $\mathcal{H}$ is a graded vector space decomposing as

$$\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}(n)$$

endowed with

- an associative unital graded **product** $\star : \mathcal{H}(n_1) \otimes \mathcal{H}(n_2) \to \mathcal{H}(n_1 + n_2)$
- a coassociative counital cograded **coproduct** $\Delta : \mathcal{H}(n) \to \bigoplus_{n = n_1 + n_2} \mathcal{H}(n_1) \otimes \mathcal{H}(n_2)$

such that each $\mathcal{H}(n)$ is finite dimensional, $\dim \mathcal{H}(0) = 1$, and

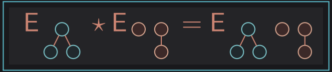
$$\Delta(x_1 \star x_2) = \Delta(x_1) \bar{\star} \Delta(x_2).$$

Let **NCK** be the space such that $\mathbf{NCK}(n)$ is the linear span of $\mathfrak{F}(n)$, the set of **ordered rooted forests** of n nodes.

The set $\{\mathsf{E}_{\mathfrak{f}} : \mathfrak{f} \in \mathfrak{F}\}$ is the elementary basis of **NCK**.

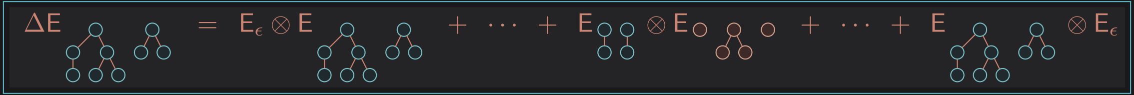
Let $\star$ be the concatenation product on **NCK**.

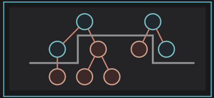
Example — Product of NCK on the E-basis



Let Δ be the pruning coproduct on **NCK**.

Example — Coproduct of NCK on the E-basis





This is the noncommutative Connes-Kreimer CHA [**Connes**, **Kreimer**, 1998] [**Foissy**, 2002].

Let **FQSym** be the space such that $\mathbf{FQSym}(n)$ is the linear span of $\mathfrak{S}(n)$, the set of **permutations** of size n.

The set $\{\mathsf{F}_\sigma : \sigma \in \mathfrak{S}\}$ is the fundamental basis of **FQSym**.

Let $\star$ be the shifted shuffle product on **FQSym**.

Example — Product of FQSym on the F-basis

$$\mathsf{F}_{231} \star \mathsf{F}_{12} = \mathsf{F}_{23145} + \mathsf{F}_{23415} + \mathsf{F}_{23451} + \mathsf{F}_{24315} + \mathsf{F}_{24351} + \mathsf{F}_{24531} + \mathsf{F}_{42315} + \mathsf{F}_{42351} + \mathsf{F}_{42531} + \mathsf{F}_{45231}$$

Let Δ be the standardized deconcatenation coproduct on **FQSym**.

Example — Coproduct of FQSym on the F-basis

$$\Delta(\mathsf{F}_{24351}) = \mathsf{F}_\epsilon \otimes \mathsf{F}_{24351} + \mathsf{F}_1 \otimes \mathsf{F}_{3241} + \mathsf{F}_{12} \otimes \mathsf{F}_{231} + \mathsf{F}_{132} \otimes \mathsf{F}_{21} + \mathsf{F}_{1324} \otimes \mathsf{F}_1 + \mathsf{F}_{24351} \otimes \mathsf{F}_\epsilon$$

This is the Malvenuto-Reutenauer CHA [**Malvenuto, Reutenauer**, 1995].

Polynomial realizations

For any alphabet A, let $\mathbb{K}\langle A\rangle$ be the space of noncommutative polynomials on A having a possibly **infinite support** but a **finite degree**.

Example — Some noncommutative polynomials

Set $A_{\mathbb{N}} := \{\mathtt{a}_0, \mathtt{a}_1, \mathtt{a}_2, \dots\}$.

- An element in $\mathbb{K}\langle A_{\mathbb{N}}\rangle$:

$$\sum_{0 \leqslant i_1 < i_2} \mathtt{a}_{i_1} \mathtt{a}_{i_2} = \mathtt{a}_0\mathtt{a}_1 + \mathtt{a}_0\mathtt{a}_2 + \cdots + \mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_1\mathtt{a}_3 + \cdots$$

- An element which is not in $\mathbb{K}\langle A_{\mathbb{N}}\rangle$:

$$\sum_{n \geqslant 0} \mathtt{a}_0^n = 1 + \mathtt{a}_0 + \mathtt{a}_0^2 + \mathtt{a}_0^3 + \cdots$$

The space $\mathbb{K}\langle A\rangle$ is endowed with the product of noncommutative polynomials.

A polynomial realization of a CHA $\mathcal{H}$ is a quadruple $(\mathcal{A}, \dotplus, \mathsf{r}_A, \mathbb{A})$ such that

1. $\mathcal{A}$ is a class of alphabets (a class of sets possibly endowed with relations).
2. $\dotplus$ is an associative operation of disjoint union on $\mathcal{A}$.
3. For any alphabet A of $\mathcal{A}$,

   $$\mathsf{r}_A : \mathcal{H} \to \mathbb{K}\langle A \rangle$$

   is an associative algebra morphism.
4. For any $x \in \mathcal{H}$ and any **mutually commuting** alphabets A' and A'' of $\mathcal{A}$ in $\mathbb{K}\langle A' \dotplus A'' \rangle$,

   $$\mathsf{r}_{A' \dotplus A''}(x) = (\mathsf{r}_{A'} \otimes \mathsf{r}_{A''}) \circ \Delta(x).$$
5. $\mathbb{A}$ is an alphabet of $\mathcal{A}$ such that $\mathsf{r}_{\mathbb{A}}$ is injective.

Point 4. offers a way to compute the coproduct of $\mathcal{H}$ by expressing the realization of x on the sum of two alphabets. This is the alphabet doubling trick.

Let A be an alphabet **endowed with a total order** $\preccurlyeq$.

The standarization of $u \in A^*$ is the word of positive integers $\mathrm{std}(u)$ such that

$$\mathrm{std}(u)_i \;=\; \#\{j : j \leqslant i \text{ and } u_j \leqslant u_i\} \;+\; \#\{j : i < j \text{ and } u_i > u_j\}.$$

Example — Standardization of a word

Let on the alphabet $A_{\mathbb{N}}$ the total order relation $\preccurlyeq$ satisfying $\mathbf{a}_{i_1} \preccurlyeq \mathbf{a}_{i_2}$ iff $i_1 \leqslant i_2$.

$$\mathrm{std}\big(\mathbf{a}_7\, \underline{\mathbf{a}_2}\, \underline{\mathbf{a}_0}\, \underline{\mathbf{a}_0}\, \underline{\mathbf{a}_2}\, \mathbf{a}_6\, \mathbf{a}_2\, \underline{\mathbf{a}_0}\, \mathbf{a}_4\big) = 941258637$$

Observations :

- std is a map from A^* to $\mathfrak{S}$;
- std is surjective iff A is infinite;
- $\mathrm{std}(u)$ is the unique permutation having the same inversion set as the one of u.

Let $\mathsf{r}_A : \mathbf{FQSym} \to \mathbb{K}\langle A \rangle$ be the map defined by

$$\mathsf{r}_A(\mathsf{F}_\sigma) := \sum_{\substack{u \in A^* \\ \mathrm{std}(u) = \sigma^{-1}}} u$$

Example — The polynomial of a basis element

$$\mathsf{r}_{A_\mathbb{N}}(\mathsf{F}_{312}) = \sum_{0 \leqslant i_1 < i_2 \leqslant i_3} \mathbf{a}_{i_2}\mathbf{a}_{i_3}\mathbf{a}_{i_1} = \mathbf{a}_1\mathbf{a}_1\mathbf{a}_0 + \mathbf{a}_1\mathbf{a}_2\mathbf{a}_0 + \cdots + \mathbf{a}_2\mathbf{a}_2\mathbf{a}_0 + \mathbf{a}_2\mathbf{a}_3\mathbf{a}_0 + \cdots + \mathbf{a}_2\mathbf{a}_2\mathbf{a}_1 + \cdots$$

The disjoint sum $A' \dotplus A''$ of the totally ordered alphabets A' and A'' is the ordinal sum of A' and A''.

Theorem [Duchamp, Hivert, Thibon, 2002]

The class of totally ordered alphabets together with the ordinal sum operation $\dotplus$, the map r_A, and the alphabet $A_\mathbb{N}$ forms a polynomial realization of $\mathbf{FQSym}$.

Example — An alphabet doubling in FQSym

$$\mathsf{r}_{A'+\!\!\!+A''}(\mathsf{F}_{312}) = \sum_{\substack{u \in (A'+\!\!\!+A'')^* \\ \mathrm{std}(u)=231}} u = \sum_{\substack{u_1,u_2,u_3 \in A'+\!\!\!+A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1$$

$$= \sum_{\substack{u_1,u_2,u_3 \in A' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1 + \sum_{\substack{u_1,u_2 \in A', u_3 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1 + \sum_{\substack{u_1,u_3 \in A', u_2 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1 + \sum_{\substack{u_2,u_3 \in A', u_1 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1$$

$$+ \sum_{\substack{u_1 \in A', u_2,u_3 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1 + \sum_{\substack{u_2 \in A', u_1,u_3 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1 + \sum_{\substack{u_3 \in A', u_1,u_2 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1 + \sum_{\substack{u_1,u_2,u_3 \in A'' \\ u_1 \preccurlyeq u_2 \preccurlyeq u_3 \\ u_1 \neq u_2}} u_2u_3u_1$$

$$= \mathsf{r}_{A'}(\mathsf{F}_{312}) \otimes \mathsf{r}_{A''}(\mathsf{F}_{\epsilon}) + \mathsf{r}_{A'}(\mathsf{F}_{21}) \otimes \mathsf{r}_{A''}(\mathsf{F}_{1}) + 0 + 0$$

$$+ \mathsf{r}_{A'}(\mathsf{F}_{1}) \otimes \mathsf{r}_{A''}(\mathsf{F}_{12}) + 0 + 0 + \mathsf{r}_{A'}(\mathsf{F}_{\epsilon}) \otimes \mathsf{r}_{A''}(\mathsf{F}_{312})$$

$$= (\mathsf{r}_{A'} \otimes \mathsf{r}_{A''}) \circ \Delta(\mathsf{F}_{312})$$

The following CHAs admit polynomial realizations:

- **NCSF**, the noncommutative symmetric functions CHA [**Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon**, 1995];
- **FQSym**, the Malvenuto-Reutenauer CHA [**Duchamp, Hivert, Thibon**, 2002];
- **PBT**, the Loday-Ronco CHA [**Hivert, Novelli, Thibon**, 2005];
- **WQSym**, the packed words CHA [**Novelli, Thibon**, 2006];
- $\mathbf{PQSym}^\star$, the dual parking functions CHA [**Novelli, Thibon**, 2007];
- **UBP**, the uniform block permutations CHA [**Maurice**, 2013];
- CK and **NCK**, the commutative and noncommutative Connes-Kreimer CHAs [**Foissy, Novelli, Thibon**, 2014];
- $\mathbf{H}_{\mathcal{FG}}$, the CHA on Feynman graphs [**Foissy**, 2020].

There are several advantages to polynomial realizations.

1. They produce from a CHA $\mathcal{H}$ a family of polynomials **generalizing symmetric functions**.

2. They lead to a **unified encoding** of the elements of a CHA $\mathcal{H}$ by polynomials, no matter how complex are the product and coproduct of $\mathcal{H}$.

3. They lead to links between a CHA $\mathcal{H}$ and other CHAs by **specializing the alphabet** on which $\mathcal{H}$ is realized.

4. Given a space $\mathcal{A}$ endowed with a product and a coproduct, the existence of a polynomial realization of $\mathcal{A}$ **proves** that $\mathcal{A}$ is a **Hopf algebra**.

Nonsymmetric operads

A nonsymmetric operad (operad) is a set

$$\mathcal{O} = \bigsqcup_{n \in \mathbb{N}} \mathcal{O}(n)$$

endowed with

- a unit $\mathbb{1} \in \mathcal{O}(1)$;
- a composition map $-[-, \dots, -] : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \dots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \dots + m_n)$

such that

$$\mathbb{1}[x] = x = x[\mathbb{1}, \dots, \mathbb{1}]$$

and

$$x[y_1, \dots, y_n][z_{1,1}, \dots, z_{1,m_1}, \dots, z_{n,1}, \dots, z_{n,m_n}] = x[y_1[z_{1,1}, \dots, z_{1,m_1}], \dots, y_n[z_{n,1}, \dots, z_{n,m_n}]].$$

The arity $\operatorname{ar}(x)$ of $x \in \mathcal{O}$ is the unique integer n such that $x \in \mathcal{O}(n)$.

Let $\mathcal{O}$ be an operad.

An element $x \in \mathcal{O}(n)$ is finitely factorizable if the set of pairs $(y, (z_1, \dots, z_n))$ satisfying

$$x = y[z_1, \dots, z_n]$$

is finite.

When all elements of $\mathcal{O}$ are finitely factorizable, by extension, $\mathcal{O}$ is finitely factorizable.

A map $\mathrm{dg} : \mathcal{O} \to \mathbb{N}$ is a grading of $\mathcal{O}$ if

- $\mathrm{dg}^{-1}(0) = \{\mathbb{1}\}$;
- for any $y \in \mathcal{O}(n)$ and $z_1, \dots, z_n \in \mathcal{O}$,

$$\mathrm{dg}(y[z_1, \dots, z_n]) = \mathrm{dg}(y) + \mathrm{dg}(z_1) + \dots + \mathrm{dg}(z_n).$$

When such a map exists, $\mathcal{O}$ is graded.

The nonsymmetric associative operad As is the operad such that

- $\mathsf{As}(0) = \emptyset$ and for any $n \geqslant 1$, $\mathsf{As}(n)$ is the set $\{\alpha_n\}$;
- the unit is α_1;
- the composition map satisfies

$$\alpha_n[\alpha_{m_1}, \dots, \alpha_{m_n}] = \alpha_{m_1 + \dots + m_n}.$$

Example — A composition in As

$$\alpha_4[\alpha_2, \alpha_2, \alpha_3, \alpha_1] = \alpha_{2+2+3+1} = \alpha_8$$

The map dg defined by $\mathrm{dg}(\alpha_n) := n - 1$ is a grading of As.

The operad As is finitely factorizable.

Natural Hopf algebras of nonsymmetric operads

Let $\mathcal{O}$ be an operad.

The reduced $\mathrm{rd}(w)$ of $w \in \mathcal{O}^*$ is the word obtained by removing the letters $\mathbb{1}$ in w.

Example — Reduced word of As^*

$$\mathrm{rd}(\alpha_2\ \alpha_2\ \alpha_1\ \alpha_4\ \alpha_1\ \alpha_1) = \alpha_2\ \alpha_2\ \alpha_4$$

The natural space $\mathbf{N}(\mathcal{O})$ of $\mathcal{O}$ is the linear span of the set of reduced elements of $\mathcal{O}^*$.

The set $\{\mathsf{E}_w : w \in \mathrm{rd}(\mathcal{O}^*)\}$ is the elementary basis of $\mathbf{N}(\mathcal{O})$.

If $\mathcal{O}$ admits a grading dg, then $\mathbf{N}(\mathcal{O})$ becomes a **graded space** by setting

$$\mathrm{dg}(\mathsf{E}_{w_1 \dots w_\ell}) := \mathrm{dg}(w_1) + \cdots + \mathrm{dg}(w_\ell).$$

Note that $\mathrm{dg}(\mathsf{E}_\epsilon) = 0$.

Let $\star$ be the **product** on $\mathbf{N}(\mathcal{O})$ defined by

$$\mathsf{E}_{w_1} \star \mathsf{E}_{w_2} := \mathsf{E}_{w_1 w_2}.$$

Let Δ be the **coproduct** on $\mathbf{N}(\mathcal{O})$ defined by

$$\Delta(\mathsf{E}_x) = \sum_{n \geqslant 0} \sum_{\substack{(y,w) \in \mathcal{O}(n) \times \mathcal{O}^n \\ x = y[w_1, \dots, w_n]}} \mathsf{E}_{\mathrm{rd}(y)} \otimes \mathsf{E}_{\mathrm{rd}(w)}.$$

Theorem [van der Laan, 2004] [Méndez, Liendo, 2014]

For any finitely factorizable operad $\mathcal{O}$, $\mathbf{N}(\mathcal{O})$ is a bialgebra.

If $\mathcal{O}$ is graded, then $\mathbf{N}(\mathcal{O})$ is a Hopf algebra.

$\mathbf{N}(\mathcal{O})$ is the natural Hopf algebra of $\mathcal{O}$.

Let us apply this construction on As endowed with the grading dg satisfying $\mathrm{dg}(\alpha_n) = n - 1$.

For any $n \geqslant 1$, $\dim \mathbf{N}(\mathrm{As})(n) = 2^{n-1}$.

Example — A product in $\mathbf{N}(\mathrm{As})$

$$\mathsf{E}_{\alpha_3\alpha_2\alpha_2\alpha_5} \star \mathsf{E}_{\alpha_4\alpha_2} = \mathsf{E}_{\alpha_3\alpha_2\alpha_2\alpha_5\alpha_4\alpha_2}$$

Example — A coproduct in $\mathbf{N}(\mathrm{As})$

$$\Delta(\mathsf{E}_{\alpha_4}) = \mathsf{E}_\epsilon \otimes \mathsf{E}_{\alpha_4} + 2\mathsf{E}_{\alpha_2} \otimes \mathsf{E}_{\alpha_3} + \mathsf{E}_{\alpha_2} \otimes \mathsf{E}_{\alpha_2\alpha_2} + 3\mathsf{E}_{\alpha_3} \otimes \mathsf{E}_{\alpha_2} + \mathsf{E}_{\alpha_4} \otimes \mathsf{E}_\epsilon.$$

Contributions to the coefficient 2 of $\mathsf{E}_{\alpha_2} \otimes \mathsf{E}_{\alpha_3}$:

$$\alpha_4 = \alpha_2[\alpha_1, \alpha_3], \quad \alpha_4 = \alpha_2[\alpha_3, \alpha_1].$$

Contributions to the coefficient 3 of $\mathsf{E}_{\alpha_3} \otimes \mathsf{E}_{\alpha_2}$:

$$\alpha_4 = \alpha_3[\alpha_1, \alpha_1, \alpha_2], \quad \alpha_4 = \alpha_3[\alpha_1, \alpha_2, \alpha_1], \quad \alpha_4 = \alpha_3[\alpha_2, \alpha_1, \alpha_1].$$

$\mathbf{N}(\mathrm{As})$ is the **noncommutative Faà di Bruno Hopf algebra $\mathbf{FdB}$** [**Figueroa, Gracia-Bondía**, 2005] [**Foissy**, 2008].

Terms and forests

A signature is a set $\mathcal{S}$ decomposing as $\mathcal{S} = \bigsqcup_{n \geqslant 0} \mathcal{S}(n)$.

An $\mathcal{S}$-term is an **ordered rooted tree** decorated on $\mathcal{S}$ such that an internal node decorated by $\mathrm{g} \in \mathcal{S}(n)$ has exactly n children.

Let $\mathfrak{T}(\mathcal{S})$ be the set of $\mathcal{S}$-terms.

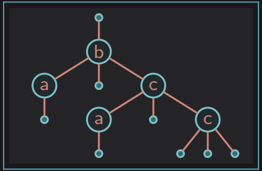
For any $\mathfrak{t} \in \mathfrak{T}(\mathcal{S})$,

- the degree $\mathrm{dg}(\mathfrak{t})$ of $\mathfrak{t}$ is the number of internal nodes of $\mathfrak{t}$;
- the arity $\mathrm{ar}(\mathfrak{t})$ of $\mathfrak{t}$ is the number of leaves of $\mathfrak{t}$.

Example — An $\mathcal{S}$-term

Let the signature $\mathcal{S} := \mathcal{S}(1) \sqcup \mathcal{S}(3)$ with $\mathcal{S}(1) := \{\mathrm{a}\}$ and $\mathcal{S}(3) := \{\mathrm{b}, \mathrm{c}\}$.

This $\mathcal{S}$-term has degree 5 and arity 7.

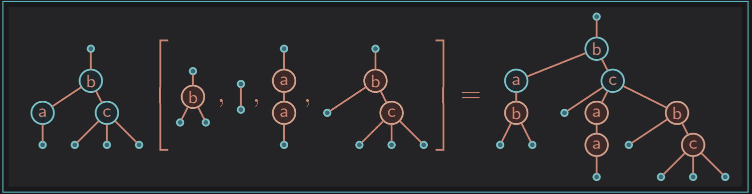


Let $\mathcal{S}$ be a signature.

The free operad on $\mathcal{S}$ is the set $\mathfrak{T}(\mathcal{S})$ such that

- $\mathfrak{T}(\mathcal{S})(n)$ is the set of $\mathcal{S}$-terms of arity n;
- the unit is the $\mathcal{S}$-term containing exactly one leaf ;
- the composition map is such that $\mathfrak{t}[\mathfrak{t}_1, \dots, \mathfrak{t}_n]$ is the $\mathcal{S}$-term obtained by grafting simultaneously each $\mathfrak{t}_i$ on the i-th leaf of $\mathfrak{t}$.

Example — A composition in a free operad



The map dg is a grading of $\mathfrak{T}(\mathcal{S})$ and this operad is finitely factorizable.

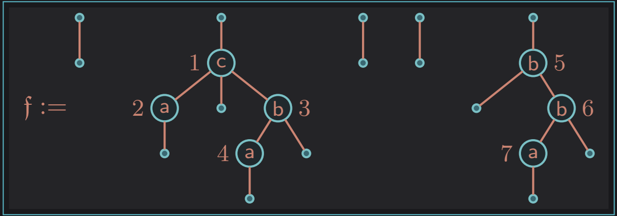
Let $\mathcal{S}$ be a signature.

An $\mathcal{S}$-forest is a word on $\mathfrak{T}(\mathcal{S})$. Let $\mathfrak{F}(\mathcal{S})$ be the set of $\mathcal{S}$-forests.

The internal nodes of an $\mathcal{S}$-forest $\mathfrak{f}$ are identified with their positions for the **preorder traversal**.

Let $\xrightarrow{\mathfrak{f}}_j$ be the binary relation on the set of internal nodes of $\mathfrak{f}$ such that $i_1 \xrightarrow{\mathfrak{f}}_j i_2$ if i_1 is the j-th child of i_2 in $\mathfrak{f}$.

Example — An $\mathcal{S}$-forest



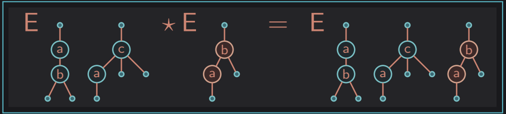
For instance, $1 \xrightarrow{\mathfrak{f}}_1 2$, $1 \xrightarrow{\mathfrak{f}}_3 3$, and $5 \xrightarrow{\mathfrak{f}}_2 6$.

Natural Hopf algebras of free operads

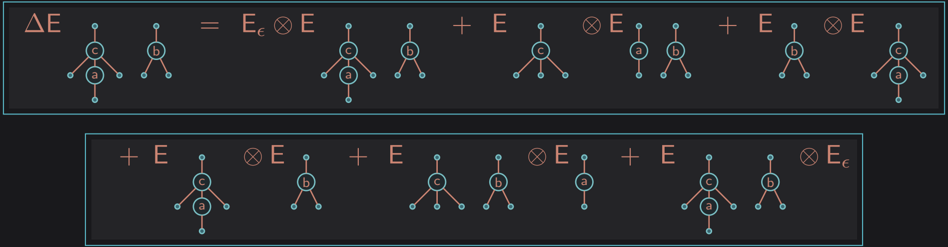
Let $\mathcal{S}$ be a signature.

The bases of $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$ are indexed by the set of **reduced $\mathcal{S}$-forests**.

Example — A product in $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$



Example — A coproduct in $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$



Let $\mathcal{S}$ be a signature such that all $\mathcal{S}(n)$, $n \geqslant 0$, are finite.

The profile of $\mathcal{S}$ is the infinite word $w_0 w_1 w_2 \dots$ such that w_i is the cardinality of $\mathcal{S}(i)$.

Example — The profile of a signature

Let the signature $\mathcal{S} := \mathcal{S}(0) \sqcup \mathcal{S}(2) \sqcup \mathcal{S}(3)$ such that $\mathcal{S}(0) = \{\mathsf{a}_1, \mathsf{a}_2\}$, $\mathcal{S}(2) = \{\mathsf{b}_1\}$, and $\mathcal{S}(3) = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$.

The profile of $\mathcal{S}$ is the infinite word 20130^ω.

Proposition [G., 2024+]

Let $\mathcal{S}$ be a signature of profile w. The Hopf algebra $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$ is

1. **commutative** iff $w = 0^\omega$ or $w = 10^\omega$;
2. **cocommutative** iff $w = k0^\omega$, $k \in \mathbb{N}$, or $w = 010^\omega$.

Polynomial realization

Let $\mathcal{S}$ be a signature.

The class of $\mathcal{S}$-forest-like alphabets is the class of alphabets A endowed with relations R, D_{g}, and $\prec_j$ such that

1. R is a unary relation called root relation;
2. for any $\mathrm{g} \in \mathcal{S}$, D_{g} is a unary relation called g-decoration relation;
3. for any $j \geqslant 1$, $\prec_j$ is a binary relation called j-edge relation.

Let $\mathcal{S}$ be a signature, and A' and A'' be to $\mathcal{S}$-forest-like alphabets.

The disjoint sum $A' + A''$ of A' and A'' is the $\mathcal{S}$-forest-like alphabet

$$A := A' \sqcup A''$$

endowed with the relations R, D_{g}, and $\prec_j$ such that

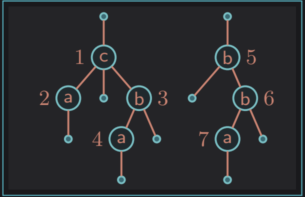
1. $\mathrm{R} := \mathrm{R}' \sqcup \mathrm{R}''$;
2. $\mathrm{D}_{\mathrm{g}} := \mathrm{D}'_{\mathrm{g}} \sqcup \mathrm{D}''_{\mathrm{g}}$;
3. $a_1 \prec_j a_2$ holds if one of the three following conditions hold:
   - $a_1 \in A'$, $a_2 \in A'$, $a_1 \prec_j' a_2$;
   - $a_1 \in A''$, $a_2 \in A''$, $a_1 \prec_j'' a_2$;
   - $a_1 \in A'$, $a_2 \in A''$, $a_2 \in \mathrm{R}''$.

Let $\mathcal{S}$ be a signature, A be an $\mathcal{S}$-forest-like alphabet, and $\mathfrak{f}$ be a reduced $\mathcal{S}$-forest.

A word $w \in A^*$ is $\mathfrak{f}$-compatible, denoted by $w \Vdash^A \mathfrak{f}$, if

1. $\ell(w) = \mathrm{dg}(\mathfrak{f})$;
2. if i is a root of $\mathfrak{f}$ then $w_i \in \mathrm{R}$;
3. if i is decorated by $\mathsf{g} \in \mathcal{S}$ in $\mathfrak{f}$ then $w_i \in \mathrm{D}_{\mathsf{g}}$;
4. if $i_1 \xrightarrow{\mathfrak{f}}_j i_2$ then $w_{i_1} \prec_j w_{i_2}$.

Example — An $\mathfrak{f}$-compatible word



Considering this reduced forest $\mathfrak{f}$, any $\mathfrak{f}$-compatible word $w \in A^*$ satisfies

- $\ell(w) = 7$;
- $w_1, w_5 \in \mathrm{R}$;
- $w_2, w_4, w_7 \in \mathrm{D}_{\mathsf{a}}$, $w_3, w_5, w_6 \in \mathrm{D}_{\mathsf{b}}$, $w_1 \in \mathrm{D}_{\mathsf{c}}$;
- $w_1 \prec_1 w_2$, $w_1 \prec_3 w_3$, $w_3 \prec_1 w_4$, $w_5 \prec_2 w_6$, $w_6 \prec_1 w_7$.

Let $\mathcal{S}$ be a signature and A be an $\mathcal{S}$-forest-like alphabet.

Let $\mathsf{r}_A : \mathbf{N}(\mathfrak{T}(\mathcal{S})) \to \mathbb{K}\langle A \rangle$ be the linear map defined for any $\mathfrak{f} \in \mathrm{rd}(\mathfrak{F}(\mathcal{S}))$ by

$$\mathsf{r}_A(\mathsf{E}_{\mathfrak{f}}) := \sum_{\substack{w \in A^* \\ w \Vdash^A \mathfrak{f}}} w.$$

This polynomial is the A-realization of $\mathfrak{f}$.

Lemma

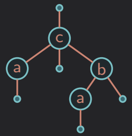
For any signature $\mathcal{S}$ and any $\mathcal{S}$-edge alphabet A, r_A is an associative algebra morphism.

Lemma

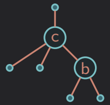
For any signature $\mathcal{S}$, any $\mathcal{S}$-edge alphabets A_1 and A_2, and any $\mathcal{S}$-term $\mathfrak{t}$ different from the leaf,

$$\mathsf{r}_{A_1 \# A_2}(\mathsf{E}_{\mathfrak{t}}) = (\mathsf{r}_{A_1} \otimes \mathsf{r}_{A_2}) \circ \Delta(\mathsf{E}_{\mathfrak{t}}).$$

Example — An alphabet doubling in $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$

$\mathsf{r}_{A' \# A''} \mathsf{E}$ 

$$= \sum_{\substack{w_1, w_2, w_3, w_4 \in A' \# A'' \\ w_1 \in \mathrm{R} \\ w_2, w_4 \in \mathrm{D}_{\mathsf{a}},\ w_1 \in \mathrm{D}_{\mathsf{c}},\ w_3 \in \mathrm{D}_{\mathsf{b}} \\ w_1 \prec_1 w_2,\ w_1 \prec_3 w_3,\ w_3 \prec_1 w_4}} w_1 w_2 w_3 w_4 = \cdots + \sum_{\substack{w_1, w_3 \in A',\ w_2, w_4 \in A'' \\ w_1, w_2, w_4 \in \mathrm{R} \\ w_2, w_4 \in \mathrm{D}_{\mathsf{a}},\ w_1 \in \mathrm{D}_{\mathsf{c}},\ w_3 \in \mathrm{D}_{\mathsf{b}} \\ w_1 \prec_3 w_3}} w_1 w_2 w_3 w_4 + \cdots$$

$= \cdots + \mathsf{r}_{A'} \mathsf{E}$ 

$\otimes\, \mathsf{r}_{A''} \mathsf{E}$ 

$+ \cdots$

Let the $\mathcal{S}$-forest-like alphabet

$$\mathbb{A}_{\mathcal{S}} := \{\mathbf{a}_{\mathrm{g},u} : \mathrm{g} \in \mathcal{S} \text{ and } u \in \mathbb{N}^*\}$$

such that

1. the root relation is defined by $\mathrm{R} := \left\{\mathbf{a}_{\mathrm{g},u} \in \mathbb{A}_{\mathcal{S}} : u = 0^{\ell}, \ell \geqslant 0\right\}$;
2. the g-decoration relation D_{g} is defined by $\mathrm{D}_{\mathrm{g}} := \{\mathbf{a}_{\mathrm{g}',u} \in \mathbb{A}_{\mathcal{S}} : \mathrm{g}' = \mathrm{g}\}$;
3. the j-edge relation $\prec_j$ is defined by $\mathbf{a}_{\mathrm{g},u} \prec_j \mathbf{a}_{\mathrm{g}',u\,j\,0^{\ell}}$.

Example — An alphabet $\mathbb{A}_{\mathcal{S}}$

Let the signature $\mathcal{S} := \mathcal{S}(1) \sqcup \mathcal{S}(3)$ such that $\mathcal{S}(1) = \{\mathrm{a}, \mathrm{b}\}$ and $\mathcal{S}(3) = \{\mathrm{c}\}$.

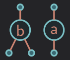
For instance,

- $\mathbf{a}_{\mathrm{a},\epsilon} \in \mathrm{R}$, $\mathbf{a}_{\mathrm{a},000} \in \mathrm{R}$, $\mathbf{a}_{\mathrm{b},00000} \in \mathrm{R}$, $\mathbf{a}_{\mathrm{b},0100} \notin \mathrm{R}$ $\mathbf{a}_{\mathrm{c},210000} \notin \mathrm{R}$;
- $\mathbf{a}_{\mathrm{c},010761} \in \mathrm{D}_{\mathrm{c}}$, $\mathbf{a}_{\mathrm{c},000} \in \mathrm{D}_{\mathrm{c}}$, $\mathbf{a}_{\mathrm{b},20101} \notin \mathrm{D}_{\mathrm{c}}$;
- $\mathbf{a}_{\mathrm{a},10212} \prec_1 \mathbf{a}_{\mathrm{b},102121000}$, $\mathbf{a}_{\mathrm{c},00200} \prec_2 \mathbf{a}_{\mathrm{a},002002}$.

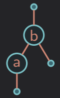
Example — Polynomial associated with some forests

$r_{\mathbb{A}_\mathcal{S}} \mathsf{E}$

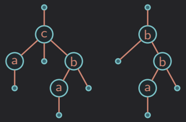

$= \sum_{\ell_1 \in \mathbb{N}} \mathsf{a}_{\mathsf{b},0^{\ell_1}}$

$r_{\mathbb{A}_\mathcal{S}} \mathsf{E}$


$= \sum_{\ell_1, \ell_2 \in \mathbb{N}} \mathsf{a}_{\mathsf{b},0^{\ell_1}} \; \mathsf{a}_{\mathsf{a},0^{\ell_2}}$

$r_{\mathbb{A}_\mathcal{S}} \mathsf{E}$


$= \sum_{\ell_1, \ell_2 \in \mathbb{N}} \mathsf{a}_{\mathsf{b},0^{\ell_1}} \; \mathsf{a}_{\mathsf{a},0^{\ell_1}10^{\ell_2}}$

$r_{\mathbb{A}_\mathcal{S}} \mathsf{E}$


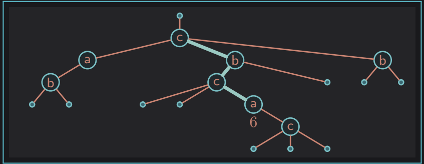
$= \sum_{\ell_1, \dots, \ell_7 \in \mathbb{N}} \mathsf{a}_{\mathsf{c},0^{\ell_1}} \; \mathsf{a}_{\mathsf{a},0^{\ell_1}10^{\ell_2}} \; \mathsf{a}_{\mathsf{b},0^{\ell_1}30^{\ell_3}} \; \mathsf{a}_{\mathsf{a},0^{\ell_1}30^{\ell_3}10^{\ell_4}} \; \mathsf{a}_{\mathsf{b},0^{\ell_5}} \; \mathsf{a}_{\mathsf{b},0^{\ell_5}20^{\ell_6}} \; \mathsf{a}_{\mathsf{b},0^{\ell_5}20^{\ell_6}10^{\ell_7}}$

Let $\mathfrak{t}$ be an $\mathcal{S}$-term.

The decoration $\mathsf{c}_i(\mathfrak{t})$ of a node i of $\mathfrak{t}$ is the element of $\mathcal{S}$ decorating it.

The address $\mathsf{p}_i(\mathfrak{t})$ of a node i of $\mathfrak{t}$ is the word specifying the positions of the edges to reach i from the root.

Example — The decoration and the address of a node of an $\mathcal{S}$-term



For this $\mathcal{S}$-term $\mathfrak{t}$, we have $\mathsf{c}_6(\mathfrak{t}) = \mathsf{a}$ and $\mathsf{p}_6(\mathfrak{t}) = 213$.

Let the associative algebra morphism $\mathsf{w}_{\mathcal{S}} : \mathbf{N}(\mathfrak{T}(\mathcal{S})) \to \mathbb{K}\langle \mathbb{A}_{\mathcal{S}} \rangle$ defined, for any $\mathcal{S}$-term $\mathfrak{t}$ of degree $d \geqslant 1$, by

$$\mathsf{w}_{\mathcal{S}}(\mathsf{E}_{\mathfrak{t}}) := \mathbf{a}_{\mathsf{c}_1(\mathfrak{t}),\mathsf{p}_1(\mathfrak{t})} \dots \mathbf{a}_{\mathsf{c}_d(\mathfrak{t}),\mathsf{p}_d(\mathfrak{t})}.$$

This is the minimal word of $\mathfrak{t}$.

Example — The minimal word of a $\mathcal{S}$-term

$$\mathsf{w}_{\mathcal{S}}(\mathsf{E}_{\mathfrak{t}}) = \mathbf{a}_{\mathsf{c},\epsilon}\ \mathbf{a}_{\mathsf{a},1}\ \mathbf{a}_{\mathsf{b},11}\ \mathbf{a}_{\mathsf{b},2}\ \mathbf{a}_{\mathsf{c},21}\ \mathbf{a}_{\mathsf{a},213}\ \mathbf{a}_{\mathsf{c},2131}\ \mathbf{a}_{\mathsf{b},3}$$

Lemma

Let $\mathcal{S}$ be a signature.

1. The map $\mathsf{w}_{\mathcal{S}}$ is injective.
2. For any reduced $\mathcal{S}$-forest $\mathfrak{f}$, the monomial $\mathsf{w}_{\mathcal{S}}(\mathsf{E}_{\mathfrak{f}})$ appears in $\mathsf{r}_{\mathbb{A}_{\mathcal{S}}}(\mathsf{E}_{\mathfrak{f}})$.
3. The map $\mathsf{r}_{\mathbb{A}_{\mathcal{S}}}$ is injective.

Theorem [G., 2024+]

For any signature $\mathcal{S}$, the class of $\mathcal{S}$-forest-like alphabets, together with the alphabet disjoint sum operation $\dotplus$, the map r_{A}, and the alphabet $\mathbb{A}_{\mathcal{S}}$, forms a polynomial realization of the Hopf algebra $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$.

Conclusion

Presented here:

- a **polynomial realization** r_A of $\mathbf{N}(\mathcal{O})$ when $\mathcal{O}$ is a **free operad**.

Not presented here:

- when $\mathcal{O}$ is **not free** and satisfies some properties, we can deduce from r_A a polynomial realization of $\mathbf{N}(\mathcal{O})$;
- by considering **alphabet specializations**, we can construct quotients or Hopf subalgebras of $\mathbf{N}(\mathfrak{T}(\mathcal{S}))$ which are isomorphic to
  - $\mathbf{NCK}$ and the noncommutative D-decorated Connes-Kreimer CHAs $\mathbf{NCK}_D$ [**Foissy**, 2002];
  - $\mathbf{FdB}$ and deformed noncommutative Faà di Bruno CHAs $\mathbf{FdB}_r$ [**Foissy**, 2008];
- fundamental $\{\mathsf{F}_{\mathfrak{f}}\}$ and homogeneous $\{\mathsf{H}_{\mathfrak{f}}\}$ bases, constructed through a new partial order on forests.

Topics to explore and open questions:

- study other alphabet specializations linking $\mathbf{N}(\mathcal{O})$ with other CHAs;
- conditions on the operad $\mathcal{O}$ for the fact that $\mathbf{N}(\mathcal{O})$ is self-dual, free, or cofree.