Graded graphs and operads

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Outline

Graded graphs

Graded graphs of syntax trees

Graded graphs from operads
Outline

Graded graphs
The **Young lattice** is a lattice on the set of all integer partitions.

<table>
<thead>
<tr>
<th>— Example —</th>
</tr>
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<tbody>
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</table>

Its Hasse diagram is

![Hasse diagram](image)
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Paths connecting 0 with a partition \( \lambda \) are in one-to-one correspondence with the set of **standard Young tableaux** of shape \( \lambda \).
The Young lattice is a lattice on the set of all integer partitions.

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The path

\[
0 \rightarrow \square \rightarrow \square \rightarrow \bigcirc \rightarrow \bigcirc \rightarrow \bigcirc
\]

is in correspondence with the standard Young tableau

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & \\
\end{array}
\]
Sum of squares formula

By denoting by $f_{\lambda}$ the number of standard Young tableaux of shape $\lambda$, for any $n \geq 0$, one has the famous identity

$$\sum_{\lambda \vdash n} f_{\lambda}^2 = n!.$$
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This identity admits a proof [Stanley, 1988], [Sagan, 2001] based on up and down operators of the Young lattice, defined respectively by

$$U(\lambda) := \sum_{\lambda \rightarrow \lambda'} \lambda'$$

and

$$U^*(\lambda) := \sum_{\lambda' \rightarrow \lambda} \lambda'.$$

These operators are linear maps on the linear span of all integer partitions (over a field $\mathbb{K}$ of characteristic 0) and are adjoint.
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$$U\left(\begin{array}{c}
\circ \\
\bullet
\end{array}\right) = \begin{array}{c}
\begin{array}{c}
\circ \\
\bullet
\end{array} \\
\begin{array}{c}
\bullet \\
\circ
\end{array} \\
\end{array} + \begin{array}{c}
\begin{array}{c}
\circ \\
\bullet
\end{array} \\
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\circ
\end{array} \\
\end{array}$$

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Proof of the sum of squares formula

The first tool consists in observing that

\[ U^*U - UU^* = I \]

where \( I \) is the identity map, so that the Young lattice admits the structure of a differential poset [Stanley, 1988].
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for any $n \geq 1$ (provable by recurrence on $n$).
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Now, since

\[ U^n(0) = \sum_{\lambda \vdash n} f_{\lambda} \lambda \quad \text{and} \quad U^*n(\lambda) = f_{\lambda} 0, \]

for any \( \lambda \vdash n \),
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\[ U^*nU^n(0) = U^*n \left( \sum_{\lambda \vdash n} f_{\lambda \lambda} \right) = \sum_{\lambda \vdash n} f_{\lambda \lambda} U^*n(\lambda) = \sum_{\lambda \vdash n} f_{\lambda \lambda}^2 \quad 0. \]
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On the other hand, by using the differential poset structure of the Young lattice, we show the relation

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by induction on \( n \geq 1 \).
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$$U^* U^1(0) = 1! \cdot 0.$$
Proof of the sum of squares formula

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First,

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Moreover, by induction hypothesis,

$$U^n U^n(0) = U^{n-1} U^n(0)$$

$$= U^{n-1} \left( n U^{n-1} + U^n U^* \right)(0)$$

$$= \left( n U^{n-1} U^{n-1} \right)(0) + \left( U^{n-1} U^n U^* \right)(0)$$

$$= n U^{n-1} U^{n-1}(0)$$

$$= n(n - 1)! \, 0$$

$$= n! \, 0.$$
Graded graphs

It is natural to search for analogous formulas.

— Example —

Let $G$ be the set of all words on \{0, 1\} starting with 0 and graded by the length.

The map $U$ satisfying $U(u) := \sum_{i \in \{\text{length}(u)\}} u_1 \ldots u_i - 1 \theta(u_i) u_i + 1 \ldots u_{\text{length}(u)}$, where $\theta$ satisfies $\theta(0) = 00 + 01$ and $\theta(1) = 11 + 10$, defines the graded graph...
Graded graphs

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For this, we start with a graded set $G := \bigsqcup_{d \geq 0} G(d)$ and a linear map

$$U : \mathbb{K} \langle G(d) \rangle \to \mathbb{K} \langle G(d + 1) \rangle, \quad d \geq 0.$$

This pair $(G, U)$ defines a graded multigraph.
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where $\theta$ satisfies $\theta(0) = 00 + 01$ and $\theta(1) = 11 + 10$, defines the graded graph

![Diagram](image-url)
Paths in graded graphs

The weight of an edge between \( x \in G(d) \) and \( y \in G(d + 1) \) is the coefficient \( \omega_{x,y} \) of \( y \) in \( U(x) \).
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A path is a sequence \((x_1, \ldots, x_\ell)\) of vertices such that all \((x_i, x_{i+1})\) are edges. The weight of a path is the product of the weights of its edges.
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The poset of \((G, U)\) is the poset \((G, \leq)\) wherein \( x \leq y \) if there is a path from \( x \) to \( y \).
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From now, \( G(0) = \{0\} \). In this case, any path from 0 to \( x \in G \) is to \( x \) what a standard Young tableau of shape \( \lambda \) is to the integer partition \( \lambda \) in the Young lattice.
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— Lemma —

When \((G, U)\) is a multigraph (all weights are in \( \mathbb{N} \)),

\[
(I - U)^{-1}(0) = \left( \sum_{n \geq 0} U^n \right)(0) = \sum_{x \in G} h_U(x) \cdot x,
\]

where \( h_U(x) \) is the number of paths from \( 0 \) to \( x \).
Duality of graded graphs

Let \((G, U)\) and \((G, V)\) be two graded graphs on the same graded set \(G\).
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Returning paths in graded graphs

— Proposition —

Let \((G, U, V)\) be a pair of \(\phi\)-diagonal dual graded graphs. For any \(n \geq 0\),

\[
V^* U^n = U^n V^* + \sum_{k_1, k_2 \geq 0 \atop k_1 + k_2 = n - 1} U^{k_1} \phi U^{k_2}.
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When there is an \(r \in \mathbb{K}\) such that, for all \(x \in G\), \(\phi(x) = rx\), this gives

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A returning path is a pair \((p_1, p_2)\) s.t. \(p_1\) is a path in \((G, U)\) from \(0\) to \(x\) and \(p_2\) is a path in \(V^*\) from \(x\) to \(0\). It is to \(x\) what a pair of standard Young tableaux of shape \(\lambda\) is to the integer partition \(\lambda\) in the Young lattice.
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— Lemma —

When \((G, U, V)\) is a connected multigraph, for any \(n \geq 0\),

\[
V^*U^n(0) = \sum_{x \in G(n)} h_U(x)h_V(x) 0.
\]
Outline

Graded graphs of syntax trees
Syntax trees

An alphabet is a finite graded set

\[ \mathcal{G} := \bigsqcup_{n \geq 1} \mathcal{G}(n). \]

— Example —
Let \( \mathcal{G} := \mathcal{G}(2) \sqcup \mathcal{G}(3) \) such that \( \mathcal{G}(2) = \{ a, b \} \) and \( \mathcal{G}(3) = \{ c \} \).

Here is a \( \mathcal{G} \)-tree having degree 8 and arity 12:

\[
\begin{array}{c}
\text{c} \\
\text{b} \\
\text{c} \\
\text{b} \\
\text{b} \\
\text{a} \\
\text{c} \\
\end{array}
\]
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A syntax tree on \( \mathcal{G} \) (also called \( \mathcal{G} \)-tree) is a planar rooted tree \( t \) such that each internal node of arity \( n \) is labeled by a letter of \( \mathcal{G}(n) \).

Let \( F(\mathcal{G}) \) be the set of all \( \mathcal{G} \)-trees.
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Compositions of syntax trees

Let \( t, s \in \mathbf{F}(\mathcal{G}) \). For each \( i \in [|t|] \), the partial composition \( t \circ_i s \) is the tree obtained by grafting the root of \( s \) onto the \( i \)th leaf of \( t \).

— Example —

\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua...
Compositions of syntax trees

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--- Example ---

![Diagram showing partial composition]

Let $t, s_1, \ldots, s_{|t|}$ be $\mathcal{G}$-trees. The full composition $t \circ [s_1, \ldots, s_{|t|}]$ is obtained by grafting simultaneously the roots of each $s_i$ onto the $i$th leaf of $t$.

--- Example ---

![Diagram showing full composition]
A first graded graph on syntax trees

For any alphabet $\mathcal{G}$, let $(F(\mathcal{G}), U)$ be the graded graph where

$$U(t) := \sum_{a \in \mathcal{G} \atop i \in [|t|]} t \circ_i a.$$
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— Example —

For $\mathcal{G} = \{a\}$ with $|a| = 2$, the graph $(F(\mathcal{G}), U)$ is

![Diagram](image-url)
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Some properties

The rank in $(\mathbf{F}(\mathcal{G}), \mathbf{U})$ of a tree $t$ is its degree.
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Let \(N_{\text{max}}(t)\) be the set of all internal nodes of the tree \(t\) having only leaves as children.
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Let \(\mathcal{N}_{\max}(t)\) be the set of all internal nodes of the tree \(t\) having only leaves as children.

— Proposition —

If \(\mathcal{G} = \{a\}\) with \(|a| \geq 2\), the graded graph \((F(\mathcal{G}), U)\) is \(\phi\)-diagonal self-dual for the linear map \(\phi : K\langle F(\mathcal{G}) \rangle \rightarrow K\langle F(\mathcal{G}) \rangle\) satisfying

\[
\phi(t) = (|t| - \#\mathcal{N}_{\max}(t)) \cdot t
\]

for any \(\mathcal{G}\)-tree \(t\).
Some properties

The rank in \((F(G), U)\) of a tree \(t\) is its degree.

Let \(N_{\text{max}}(t)\) be the set of all internal nodes of the tree \(t\) having only leaves as children.

— Proposition —

If \(G = \{a\}\) with \(|a| \geq 2\), the graded graph \((F(G), U)\) is \(\phi\)-diagonal self-dual for the linear map \(\phi : K\langle F(G)\rangle \rightarrow K\langle F(G)\rangle\) satisfying

\[
\phi(t) = (|t| - \#N_{\text{max}}(t)) \cdot t
\]

for any \(G\)-tree \(t\).

— Example —

For \(G = \{c\}\) with \(|c| = 3\),

\[
(U^*U - UU^*) \left( \begin{array}{c} c \\ c \end{array} \right) = \left( 5 \cdot c + \frac{c}{c} + \frac{c}{c} \right) - \left( \frac{c}{c} + \frac{c}{c} + \frac{c}{c} \right) = 4 \cdot c.
\]
Prefix poset

The $\mathcal{G}$-prefix poset is the poset $(F(\mathcal{G}), \preceq)$ of $(F(\mathcal{G}), U)$. 

— Proposition —

For any $\mathcal{G}$-trees $s$ and $t$, $s \preceq t$ iff $s$ is a prefix of $t$.

Let $s \wedge s'$ be the tree being the largest common part between $s$ and $s'$.

Let $s \vee s'$ be the tree obtained by superimposing (if possible) $s$ and $s'$.

— Example —

$a \wedge e = c a$

— Example —

$a \vee c a a = c a a a$

— Proposition —

All $\mathcal{G}$-prefix posets are meet-semilattices for $\wedge$.

All intervals $[s, t]$ of these posets are distributive lattices for $\wedge$ and $\vee$. 


Prefix poset

The $\mathcal{G}$-prefix poset is the poset $(\mathcal{F}(\mathcal{G}), \preceq)$ of $(\mathcal{F}(\mathcal{G}), U)$.

A $\mathcal{G}$-tree $s$ is a prefix of a $\mathcal{G}$-tree $t$ if there exist some $\mathcal{G}$-trees $r_1, \ldots, r_{|s|}$ such that $t = s \circ [r_1, \ldots, r_{|s|}]$. 

**Proposition**

For any $\mathcal{G}$-trees $s$ and $t$, $s \preceq t$ iff $s$ is a prefix of $t$.

Let $s \wedge s'$ be the tree being the largest common part between $s$ and $s'$.

Let $s \vee s'$ be the tree obtained by superimposing (if possible) $s$ and $s'$.

**Example**

\[ a \land e = c a \]

\[ a \lor e = c a a \]

**Proposition**

All $\mathcal{G}$-prefix posets are meet-semilattices for $\wedge$.

All intervals $[s, t]$ of these posets are distributive lattices for $\wedge$ and $\lor$. 
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— Proposition —

For any $\mathcal{G}$-trees $s$ and $t$, $s \preceq t$ iff $s$ is a prefix of $t$.

Let $s \wedge s'$ be the tree being the largest common part between $s$ and $s'$.

— Example —

\[
\begin{array}{c}
  \wedge \\
  \quad a \quad a \\
  \quad c \\
  \quad e \\
  \end{array} 
\quad = \quad 
\begin{array}{c}
  \quad c \quad a \\
  \quad a \\
  \end{array}
\]
Prefix poset

The $G$-prefix poset is the poset $(F(G), \leq)$ of $(F(G), U)$.

A $G$-tree $s$ is a prefix of a $G$-tree $t$ if there exist some $G$-trees $r_1, \ldots, r_{|s|}$ such that $t = s \circ [r_1, \ldots, r_{|s|}]$.

— Proposition —

For any $G$-trees $s$ and $t$, $s \leq t$ iff $s$ is a prefix of $t$.

Let $s \wedge s'$ be the tree being the largest common part between $s$ and $s'$. Let $s \vee s'$ be the tree obtained by superimposing (if possible) $s$ and $s'$.

— Example —

\[
\begin{align*}
\wedge & \quad = \\
\end{align*}
\]

— Example —

\[
\begin{align*}
\vee & \quad = \\
\end{align*}
\]
Prefix poset

The $\mathcal{G}$-prefix poset is the poset $(F(\mathcal{G}), \preceq)$ of $(F(\mathcal{G}), U)$.

A $\mathcal{G}$-tree $s$ is a prefix of a $\mathcal{G}$-tree $t$ if there exist some $\mathcal{G}$-trees $r_1, \ldots, r_{|s|}$ such that $t = s \circ [r_1, \ldots, r_{|s|}]$.

— Proposition —

For any $\mathcal{G}$-trees $s$ and $t$, $s \preceq t$ iff $s$ is a prefix of $t$.

Let $s \land s'$ be the tree being the largest common part between $s$ and $s'$. Let $s \lor s'$ be the tree obtained by superimposing (if possible) $s$ and $s'$.

— Example —

\[
\begin{array}{c}
\text{a} & \text{c} & \text{e} \\
\text{i} & \text{a} & \text{c}
\end{array}
\quad \land \quad
\begin{array}{c}
\text{i} & \text{a} & \text{c} \\
\text{a} & \text{c} & \text{e}
\end{array}
\quad = \quad
\begin{array}{c}
\text{c} \quad \text{a}
\end{array}
\]

— Example —

\[
\begin{array}{c}
\text{a} & \text{a} & \text{c} \\
\text{i} & \text{a} & \text{c}
\end{array}
\quad \lor \quad
\begin{array}{c}
\text{c} \\
\text{a}
\end{array}
\quad = \quad
\begin{array}{c}
\text{c} \quad \text{a} \quad \text{a} \\
\text{c} \quad \text{a} \quad \text{a}
\end{array}
\]

— Proposition —

All $\mathcal{G}$-prefix posets are meet-semilattices for $\land$.

All intervals $[s, t]$ of these posets are distributive lattices for $\land$ and $\lor$. 
Prefix poset intervals

Let $s$ and $t$ be two $G$-trees such that $s \preceq t$. The difference $t \setminus s$ is the ordered forest $(r_1, \ldots, r_{|s|})$ such that the $r_i$ are the unique trees satisfying $t = s \circ [r_1, \ldots, r_{|s|}]$.

--- Example ---

```
            /\             /\      /
           | |           | |     | |
          a b           b a     a
         /   \         /   \    /   \```

$\quad \quad \quad \quad \quad \quad \quad = \quad \left( \begin{array}{c} a \\ a \\ b \\ , \\ c \end{array} \right)$
Prefix poset intervals

Let $s$ and $t$ be two $G$-trees such that $s \subseteq t$. The difference $t \setminus s$ is the ordered forest $(r_1, \ldots, r_{|s|})$ such that the $r_i$ are the unique trees satisfying $t = s \circ [r_1, \ldots, r_{|s|}]$.

— Example —

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array}
\quad \setminus \quad
\begin{array}{c}
\begin{array}{c}
\text{b} \\
\text{a} \\
\text{a}
\end{array}
\end{array}
= \quad \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{a} \\
\text{b}
\end{array}
\end{array}
\quad , \quad
\begin{array}{c}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\end{array}
\]

— Proposition —

Any interval $[s, t]$ is isomorphic as a poset to a Cartesian product of initial intervals. More precisely,

$$[s, t] \cong [l, r_1] \times \cdots \times [l, r_{|s|}]$$

where $(r_1, \ldots, r_{|s|})$ is the forest $t \setminus s$. 
Prefix poset intervals

The shadow $\text{sh}(t)$ of a $\mathcal{G}$-tree is the unordered rooted tree obtained by keeping only the internal nodes of $t$.

\[ \text{sh} \left( \begin{array}{c}
  \text{c} \\
  \text{a} \\
  \text{c} \\
  \text{e} \\
  \text{a} \\
\end{array} \right) = \begin{array}{c}
  \text{c} \\
  \text{a} \\
\end{array} = \begin{array}{c}
  \text{c} \\
  \text{a} \\
\end{array} \]
Prefix poset intervals

The shadow \( \text{sh}(t) \) of a \( \mathcal{G} \)-tree is the unordered rooted tree obtained by keeping only the internal nodes of \( t \).

\[
\text{sh} \left( \begin{array} {c}
 \begin{array} {c}
 c \\
 \downarrow \\
 a \\
 \downarrow \\
 e \\
 \downarrow \\
 a \\
 \end{array}
 \end{array} \right) = \begin{array} {c}
 \begin{array} {c}
 \circ \\
 \downarrow \\
 \circ \\
 \downarrow \\
 \circ \\
 \downarrow \\
 \circ \\
 \end{array}
 \end{array} = \begin{array} {c}
 \begin{array} {c}
 \circ \\
 \downarrow \\
 \circ \\
 \downarrow \\
 \circ \\
 \downarrow \\
 \circ \\
 \end{array}
 \end{array}
\]

\[\text{— Example —}\]

\[\text{— Proposition —}\]

The intervals \([s, t]\) and \([s', t']\) are isomorphic as posets iff

\[\text{sh} (\Diamond \circ (t \setminus s)) = \text{sh} (\Diamond \circ (t' \setminus s')).\]
Prefix poset intervals

The shadow \( \text{sh}(t) \) of a \( \mathcal{G} \)-tree is the unordered rooted tree obtained by keeping only the internal nodes of \( t \).

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
</table>
| \( \text{sh} \left( \begin{array}{c}
              a \\
              c \\
              e \\
              l \\
            \end{array} \right) \) = \[
\begin{array}{c}
  \circ \\
  \circ \\
  \circ \\
  \circ \\
\end{array}
\] \quad = \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad \xrightarrow{\text{ld}} \quad \begin{array}{c}
20 \\
4 \\
1 \\
1 \\
\end{array} \] |

<table>
<thead>
<tr>
<th>Proposition</th>
</tr>
</thead>
</table>
| The intervals \([s, t]\) and \([s', t']\) are isomorphic as posets iff
| \[ \text{sh} \left( \Diamond \circ (t \setminus s) \right) = \text{sh} \left( \Diamond \circ (t' \setminus s') \right) \] |
| The load \( \text{ld}(s) \) of a shadow \( s \) is \( \prod_i (1 + \text{ld}(s_i)) \) otherwise, where the \( s_i \) are the children of the root of \( s \). |
Prefix poset intervals

The shadow $\text{sh}(t)$ of a $\mathcal{G}$-tree is the unordered rooted tree obtained by keeping only the internal nodes of $t$.

<table>
<thead>
<tr>
<th>— Example —</th>
</tr>
</thead>
</table>

\[
\text{sh} \left( \begin{array}{c}
\text{a} \\
\text{c} \\
\text{a} \\
\text{e} \\
\text{a} \\
\end{array} \right) = \begin{array}{c}
\text{a} \\
\text{c} \\
\text{a} \\
\end{array} = \begin{array}{c}
\text{a} \\
\text{c} \\
\text{a} \\
\end{array} \rightarrow \text{ld} \begin{array}{c}
\text{a} \\
\text{c} \\
\text{a} \\
\end{array} \begin{array}{c}
\text{20} \\
\text{1} \\
\text{1} \\
\end{array}
\]

| — Proposition — |

The intervals $[s, t]$ and $[s', t']$ are isomorphic as posets iff

\[
\text{sh} (\diamond \circ (t \setminus s)) = \text{sh} (\diamond \circ (t' \setminus s')).
\]

The load $\text{ld}(s)$ of a shadow $s$ is $\prod_i (1 + \text{ld}(s_i))$ otherwise, where the $s_i$ are the children of the root of $s$.

| — Proposition — |

Any interval $[s, t]$ has cardinality $\text{ld} (\text{sh} (\diamond \circ (t \setminus s))).$
Prefix poset intervals

— Proposition —

The generating series $\mathcal{I}_{F(\mathcal{G})}(q, t)$ of all the intervals $[s, t]$ enumerated with respect to $\deg(s)$ (parameter $q$) and $\deg(t)$ (parameter $t$) satisfies

$$\mathcal{I}_{F(\mathcal{G})}(q, t) = 1 + t \, \mathcal{R}_{\mathcal{G}} \left( \mathcal{I}_{F(\mathcal{G})}(q, t) \right) + qt \, \mathcal{R}_{\mathcal{G}} \left( \mathcal{R}_{F(\mathcal{G})}(qt) \right),$$

where $\mathcal{R}_{\mathcal{G}}(t)$ is the generating series of $\mathcal{G}$ and $\mathcal{R}_{F(\mathcal{G})}(t)$ is the generating series of $F(\mathcal{G})$. 
Prefix poset intervals

— Proposition —

The generating series $\mathcal{I}_{F(\mathfrak{G})}(q, t)$ of all the intervals $[s, t]$ enumerated with respect to $\deg(s)$ (parameter $q$) and $\deg(t)$ (parameter $t$) satisfies

$$
\mathcal{I}_{F(\mathfrak{G})}(q, t) = 1 + t \mathcal{R}_{\mathfrak{G}} (\mathcal{I}_{F(\mathfrak{G})}(q, t)) + qt \mathcal{R}_{\mathfrak{G}} (\mathcal{R}_{F(\mathfrak{G})}(qt)) ,
$$

where $\mathcal{R}_{\mathfrak{G}}(t)$ is the generating series of $\mathfrak{G}$ and $\mathcal{R}_{F(\mathfrak{G})}(t)$ is the generating series of $F(\mathfrak{G})$.

— Example —

For $\mathfrak{G} = \{a\}$ with $|a| = 2$,

$$
\mathcal{I}_{F(\mathfrak{G})}(q, t) = 1 + (1 + q)t + 2 \left(1 + q + q^2\right)t^2 + \left(5 + 6q + 5q^2 + 5q^3\right)t^3
$$

$$
+ 2 \left(7 + 10q + 9q^2 + 7q^3 + 7q^4\right)t^4 + 14 \left(3 + 5q + 5q^2 + 4q^3 + 3q^4 + 3q^5\right)t^5 + \cdots .
$$
Prefix poset intervals

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The generating series $\mathcal{I}_{F(\mathcal{G})}(q, t)$ of all the intervals $[s, t]$ enumerated with respect to $\deg(s)$ (parameter $q$) and $\deg(t)$ (parameter $t$) satisfies

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For $\mathcal{G} = \{a\}$ with $|a| = 2$,

$$\mathcal{I}_{F(\mathcal{G})}(q, t) = 1 + (1 + q)t + 2 \left( 1 + q + q^2 \right) t^2 + (5 + 6q + 5q^2 + 5q^3) t^3$$
$$+ 2 \left( 7 + 10q + 9q^2 + 7q^3 + 7q^4 \right) t^4 + 14 \left( 3 + 5q + 5q^2 + 4q^3 + 3q^4 + 3q^5 \right) t^5 + \cdots.$$  

Moreover,

$$\mathcal{I}_{F(\mathcal{G})}(1, t) = 1 + 2t + 6t^2 + 21t^3 + 80t^4 + 322t^5 + 1348t^6 + \cdots$$

is the generating series of the vertices of the multiplihedra (Sequence A121988).
A **standard labeling** of a tree $t$ is a labeled version of $t$ where the internal nodes are bijectively labeled on $\{1, \ldots, \text{deg}(t)\}$ and the label of a node is smaller than the ones of its children.

**Example**

The tree admits the standard labeling $3 2 1 5 4 6$.
Enumeration of paths

A standard labeling of a tree $t$ is a labeled version of $t$ where the internal nodes are bijectively labeled on $\{1, \ldots, \deg(t)\}$ and the label of a node is smaller than the ones of its children.

--- Example ---

The tree
\[
\begin{array}{c}
  a \\
  \downarrow \\
  c \\
  \downarrow \\
  e \\
  \downarrow \\
  a
\end{array}
\quad \text{admits the standard labeling} \quad
\begin{array}{c}
  3 \\
  \downarrow \\
  1 \\
  \downarrow \\
  5 \\
  \downarrow \\
  6
\end{array}
\]

The hook-length formula for trees [Knuth, 2005]

\[
h(t) := \frac{\deg(t)!}{\prod_{u \in \mathcal{N}(t)} \deg(t(u))}
\]

counts the standard labelings of a tree $t$. 

Enumeration of paths

A standard labeling of a tree $t$ is a labeled version of $t$ where the internal nodes are bijectively labeled on $\{1, \ldots, \deg(t)\}$ and the label of a node is smaller than the ones of its children.

--- Example ---

The tree

```
   a
  c /|
 e  c
  a
```

admits the standard labeling

```
   3
  2 | 1
  5 6
```

The hook-length formula for trees [Knuth, 2005]

$$h(t) := \frac{\deg(t)!}{\prod_{u \in N(t)} \deg(t(u))}$$

counts the standard labelings of a tree $t$.

--- Proposition ---

For any alphabet $\mathcal{G}$,

$$\left(I - U\right)^{-1}(i) = \sum_{t \in F(\mathcal{G})} h(t) \cdot t.$$
A second graded graph on syntax trees

Let \((\mathbf{F}(\mathfrak{G}), \mathbf{V})\) be the graded graph defined from the adjoint \(\mathbf{V}^*\) of \(\mathbf{V}\) by

\[
\mathbf{V}^*(i) := 0,
\]
A second graded graph on syntax trees
Let $(F(\mathcal{G}), V)$ be the graded graph defined from the adjoint $V^*$ of $V$ by

$$V^* (i) := 0,$$

$$V^* (a \circ [s, i, \ldots, i]) := s,$$
A second graded graph on syntax trees

Let \((F(\mathcal{G}), V)\) be the graded graph defined from the adjoint \(V^*\) of \(V\) by

\[
V^* (I) := 0,
\]

\[
V^* (a_\circ [s, I, \ldots, I]) := s,
\]

\[
V^* (a_\circ [s_1, \ldots, s_{|a|}]) := \sum_{2 \leq j \leq |a|} a_\circ [s_1, \ldots, s_{j-1}, V^* (s_j), s_{j+1}, \ldots, s_{|a|}] .
\]
A second graded graph on syntax trees

Let \((\mathbf{F}(\mathcal{G}), \mathcal{V})\) be the graded graph defined from the adjoint \(\mathcal{V}^*\) of \(\mathcal{V}\) by

\[ \mathcal{V}^* (\mathrm{a}) := 0, \]

\[ \mathcal{V}^* (\mathrm{a} \circ [\mathcal{S}, \mathrm{i}, \ldots, \mathrm{i}]) := \mathcal{S}, \]

\[ \mathcal{V}^* (\mathrm{a} \circ [\mathcal{S}_1, \ldots, \mathcal{S}_{|a|}]) := \sum_{2 \leq j \leq |a|} \mathrm{a} \circ [\mathcal{S}_1, \ldots, \mathcal{S}_{j-1}, \mathcal{V}^* (\mathcal{S}_j), \mathcal{S}_{j+1}, \ldots, \mathcal{S}_{|a|}]. \]

— Example —

For \(\mathcal{G} = \{\mathrm{e}, \mathrm{a}, \mathrm{c}\}\) with \(|\mathrm{e}| = 1\), \(|\mathrm{a}| = 2\), and \(|\mathrm{c}| = 3\),

\[ \mathcal{V}^* \left( \begin{array}{c} \mathrm{c} \\ \mathrm{a} \\ \mathrm{e} \\ \mathrm{c} \\ \mathrm{a} \\ \mathrm{c} \\ \end{array} \right) = \begin{array}{c} \mathrm{c} \\ \mathrm{a} \\ \mathrm{e} \\ \mathrm{c} \\ \mathrm{a} \\ \mathrm{c} \\ \end{array} + \begin{array}{c} \mathrm{c} \\ \mathrm{a} \\ \mathrm{e} \\ \mathrm{c} \\ \mathrm{a} \\ \mathrm{c} \\ \end{array} + \begin{array}{c} \mathrm{c} \\ \mathrm{a} \\ \mathrm{e} \\ \mathrm{c} \\ \mathrm{a} \\ \mathrm{c} \\ \end{array}, \]
A second graded graph on syntax trees

Let \((F(\mathcal{G}), V)\) be the graded graph defined from the adjoint \(V^*\) of \(V\) by

\[
V^* (i) := 0,
\]

\[
V^* (a \circ [s, 1, \ldots, i]) := s,
\]

\[
V^* (a \circ [s_1, \ldots, s_{|a|}]) := \sum_{2 \leq j \leq |a|} a \circ [s_1, \ldots, s_{j-1}, V^* (s_j), s_{j+1}, \ldots, s_{|a|}] .
\]

— Example —

For \(\mathcal{G} = \{e, a, c\}\) with \(|e| = 1, |a| = 2, \text{ and } |c| = 3\),

\[
V^* \left( \begin{array}{c}
\text{c} \\
\text{a} \\
\text{e} \\
\end{array} \right) = \text{a} \circ \left( \begin{array}{c}
\text{c} \\
\text{a} \\
\text{e} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{c} \\
\text{a} \\
\text{e} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{c} \\
\text{a} \\
\text{e} \\
\end{array} \right) ,
\]

\[
V \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) = \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right) + \text{a} \circ \left( \begin{array}{c}
\text{a} \\
\text{a} \\
\end{array} \right).
A second graded graph on syntax trees

— Example —

For $\mathcal{G} = \{a\}$ with $|a| = 2$, the graph $(F(\mathcal{G}), V)$ is the bracket tree [Fomin, 1994].

It appears in dual pairs of graded graphs constructed from the Hopf bialgebra of binary search trees [Hivert, Novelli, Thibon, 2005].
A second graded graph on syntax trees

— Example —

For \( G = \{a\} \) with \( |a| = 2 \), the graph \( (F(G), V) \) is the bracket tree [Fomin, 1994].

It appears in dual pairs of graded graphs constructed from the Hopf bialgebra of binary search trees [Hivert, Novelli, Thibon, 2005].

— Example —

For \( G = \{a, c\} \) with \( |a| = 2 \) and \( |c| = 3 \), the graph \( (F(G), V) \) is not a tree.
For any $t \in F(\mathcal{G})$, let $\alpha(t)$ be the number of leaves of $t$ that are not in a first subtree of any internal node of $t$.

--- Example ---

```
      c
     / \  \
   a   c  c
  / \  /  \
 e  a c  .
```

$\alpha \mapsto 5$
Dual graded graphs from free operads

For any $t \in F(\mathcal{G})$, let $\alpha(t)$ be the number of leaves of $t$ that are not in a first subtree of any internal node of $t$.

**Example**

\[
\begin{array}{c}
\alpha \rightarrow 5 \\
\end{array}
\]

**Theorem [G., 2018]**

When $\mathcal{G}(1) = \emptyset$, $(F(\mathcal{G}), U, V)$ is $\phi$-diagonal dual for the linear map satisfying

\[
\phi(t) := (\#\mathcal{G}) \alpha(t) t
\]

for any $\mathcal{G}$-tree $t$. 
Outline

Graded graphs from operads
Abstract operators

An abstract operator is a device

having $n = |x|$ inputs and 1 output.
Abstract operators

An abstract operator is a device having \( n = |x| \) inputs and 1 output.

Such operators can be composed. If \( x \) and \( y \) are two operators, the composition \( x \circ_i y \) is the operator obtained by grafting the output of \( y \) onto the \( i \)th input of \( x \):

\[
\begin{align*}
\text{Composition:} & & \quad x \circ_i y = 1 \ldots i \ldots |y| \\
\quad & & = 1 \ldots i \ldots |x| + |y| - 1.
\end{align*}
\]
Operads

Operads are algebraic structures formalizing the notion of abstract operators and their composition.
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A (nonsymmetric set-theoretic) operad is a triple \((\mathcal{O}, \circ_i, 1)\) where

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\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \quad 1 \leq i \leq n, \ 1 \leq m;
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Operad axioms

The associativity relation

\[(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)\]

\[1 \leq i \leq |x|, 1 \leq j \leq |y|\]

says that the pictured operator can be constructed from top to bottom or from bottom to top.

\[1 \leq i \leq |x|, 1 \leq j \leq |y|\]
Operad axioms

The **associativity** relation

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The **commutativity** relation

\[
(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y
\]

\[
1 \leq i < j \leq |x|
\]

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The unitality relation
\[1 \circ_1 x = x = x \circ_i 1\]
\[1 \leq i \leq |x|\]

says that 1 is the identity map.
Some operads

— Example —

The free operad on an alphabet $\mathcal{G}$ is the set $F(\mathcal{G})$ of all $\mathcal{G}$-trees endowed with the partial composition $\circ_i$. 
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The associative operad $\text{As}$ is the operad wherein for any $n \geq 1$, $\text{As}(n) := \{\star n\}$, and

$$\star n \circ_i \star m := \star n + m - 1.$$
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— Example —

The diassociative operad $\text{Dias}$ is the operad wherein for any $n \geq 1$

$$\text{Dias}(n) := \{ 1^{\alpha_1} 0 1^{\alpha_2} : \alpha_1 + 1 + \alpha_2 = n \} ,$$

and

$$1^{\alpha_1} 0 1^{\alpha_2} \circ_i 1^{\beta_1} 0 1^{\beta_2} := \begin{cases} 1^{\beta_1+\beta_2} 1^{\alpha_1} 0 1^{\alpha_2} & \text{if } i \leq \alpha_1, \\ 1^{\alpha_1} 1^{\beta_1} 0 1^{\beta_2} 1^{\alpha_2} & \text{if } i = \alpha_1 + 1, \\ 1^{\alpha_1} 0 1^{\alpha_2} 1^{\beta_1+\beta_2} & \text{otherwise}. \end{cases}$$
An operad on Motzkin paths

Let $\text{Motz}$ be an operad wherein:

- $\text{Motz}(n)$ is the set of all Motzkin paths with $n$ points.

---

**Example**

![Motzkin path](image)

is a Motzkin path of arity 16.
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\begin{itemize}
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\end{itemize}

▶ The partial composition in \text{Motz} is a substitution in paths:

— Example —

\begin{itemize}
  \item The unit is \( o \).
\end{itemize}
Presentation of Motz

A presentation of an operad $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ such that

$$\mathcal{O} \simeq \frac{\mathbf{F}(\mathcal{G})}{\equiv}.$$
Presentation of Motz

A presentation of an operad $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ such that

$$\mathcal{O} \simeq \mathbf{F}(\mathcal{G})/\equiv.$$

— Proposition [G., 2015] —

The operad $\text{Motz}$ admits the presentation $(\mathcal{G}_{\text{Motz}}, \equiv_{\text{Motz}})$ where

$$\mathcal{G}_{\text{Motz}} := \{\text{\includegraphics[width=0.05\textwidth]{figures/diagram1.png}}, \text{\includegraphics[width=0.05\textwidth]{figures/diagram2.png}}\}$$

and $\equiv$ is the smallest operad congruence satisfying

- $\equiv_{\text{Motz}}(\text{\includegraphics[width=0.05\textwidth]{figures/diagram1.png}}, \text{\includegraphics[width=0.05\textwidth]{figures/diagram2.png}})$,
- $\equiv_{\text{Motz}}(\text{\includegraphics[width=0.05\textwidth]{figures/diagram3.png}}, \text{\includegraphics[width=0.05\textwidth]{figures/diagram4.png}})$,
- $\equiv_{\text{Motz}}(\text{\includegraphics[width=0.05\textwidth]{figures/diagram1.png}}, \text{\includegraphics[width=0.05\textwidth]{figures/diagram5.png}})$,
- $\equiv_{\text{Motz}}(\text{\includegraphics[width=0.05\textwidth]{figures/diagram1.png}}, \text{\includegraphics[width=0.05\textwidth]{figures/diagram6.png}})$.
An operad $\mathcal{O}$ is homogeneous if $\mathcal{O}(1) = \{1\}$, all $\mathcal{O}(n)$, $n \geq 1$, are finite, and $\mathcal{O}$ admits the presentation $(\mathcal{G}, \equiv)$ wherein $\mathcal{G}$ is finite and $t \equiv t'$ implies $\deg(t) = \deg(t')$.

Let $(\mathcal{O}, \mathcal{U})$ be the graded graph where $\mathcal{U}(x) := \sum_{a \in \mathcal{G}} \left| x \right| \circ_i a$. This graph has multiplicities and is graded by the degrees of the elements. — Example —

The graded graph $(\text{Motz}, \mathcal{U})$ is $
\begin{array}{c}
\star \quad 1 \quad \star \\
\star \quad 2 \quad \star \\
\star \quad 3 \\
\end{array}$

— Example —

The graded graph $(\text{As}, \mathcal{U})$ is $
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A first graded graph from operads

An operad $\mathcal{O}$ is homogeneous if $\mathcal{O}(1) = \{1\}$, all $\mathcal{O}(n), n \geq 1$, are finite, and $\mathcal{O}$ admits the presentation $(\mathcal{G}, \equiv)$ wherein $\mathcal{G}$ is finite and $t \equiv t'$ implies $\deg(t) = \deg(t')$.

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An operad $O$ is homogeneous if $O(1) = \{1\}$, all $O(n), n \geq 1$, are finite, and $O$ admits the presentation $(G, \equiv)$ wherein $G$ is finite and $t \equiv t'$ implies $\deg(t) = \deg(t')$.

Let $(O, U)$ be the graded graph where

$$U(x) := \sum_{a \in G} \sum_{i \in [|x|]} x \circ_i a.$$ 

This graph has multiplicities and is graded by the degrees of the elements.

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The graded graph $(\text{Motz}, U)$ is
A first graded graph from operads

An operad $\mathcal{O}$ is homogeneous if $\mathcal{O}(1) = \{1\}$, all $\mathcal{O}(n)$, $n \geq 1$, are finite, and $\mathcal{O}$ admits the presentation $(\mathcal{G}, \equiv)$ wherein $\mathcal{G}$ is finite and $t \equiv t'$ implies $\deg(t) = \deg(t')$.

Let $(\mathcal{O}, U)$ be the graded graph where

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— Example —

The graded graph $(\text{Motz, } U)$ is

— Example —

The graded graph $(\text{As, } U)$ is
A second graded graph from operads

Let $\mathcal{O}$ be an homogeneous operad.
A second graded graph from operads

Let $\mathcal{O}$ be an homogeneous operad.

Let $(\mathcal{O}, V)$ be the graded graph where

\[ V(x) := \sum_{y \in \mathcal{O}} y. \]

\[ \exists (s, t) \in \text{ev}^{-1}(x) \times \text{ev}^{-1}(y) \]
\[ (t, V(s)) \neq 0 \]
A second graded graph from operads

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where

$$\exists (s, t) \in \text{ev}^{-1}(x) \times \text{ev}^{-1}(y) \quad \langle t, V(s) \rangle \neq 0$$

This graph has no multiplicities and is graded by the degrees of the elements.
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Let $\mathcal{O}$ be an homogeneous operad.

Let $(\mathcal{O}, V)$ be the graded graph where

$$V(x) := \sum_{y \in \mathcal{O}} \sum_{\exists (s, t) \in \text{ev}^{-1}(x) \times \text{ev}^{-1}(y)} \langle t, V(s) \rangle \neq 0$$

This graph has no multiplicities and is graded by the degrees of the elements.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
</table>

In $(\text{Motz}, V)$,

$$V(\text{motz}) = \text{motz} + \text{motz} + \text{motz}.$$  

Indeed, among others,

- admits the factorization $\text{motz} \circ_1 \text{motz}$;
- admits the factorization $(\text{motz} \circ_1 \text{motz}) \circ_4 \text{motz}$;
- in $(\text{F(Motz)}, V)$, the tree $(\text{motz} \circ_1 \text{motz}) \circ_4 \text{motz}$ appears in $V(\text{motz} \circ_1 \text{motz})$.  

Some pairs of graded graphs from operads

--- Example ---

The pair \((\text{Comp}, U, V)\) is 2-dual.

The graded graph \((\text{Comp}, U)\) is the composition poset [Björner, Stanley, 2005].
Some pairs of graded graphs from operads

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— Example —

The pair \((\text{Motz}, U, V)\) is \(\phi\)-diagonal dual.
Some pairs of graded graphs from operads

— Example —

The pair \((\text{Comp}, U, V)\) is 2-dual.

The graded graph \((\text{Comp}, U)\) is the composition poset \([\text{Bjöner, Stanley, 2005}]\).

— Example —

The pair \((\text{Motz}, U, V)\) is \(\phi\)-diagonal dual.

— Example —

The pair \((\text{Dias}, U, V)\) is not \(\phi\)-diagonal dual.
### Experimental data

---

**Example: operad As**

<table>
<thead>
<tr>
<th>Degree</th>
<th>Number of Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The pair \((\text{As, } U, V)\) is dual.

**\(U\)-hook series:**

\[
(I - U)^{-1}(\ast_1) = \ast_1 + 2 \ast_3 + 6 \ast_4 + 24 \ast_5 + 120 \ast_6 + \cdots
\]

Number of initial \(U\)-paths by length: 1, 1, 2, 6, 24, 120, ...

**\(V\)-hook series:**

\[
(I - V)^{-1}(\ast_1) = \ast_1 + 2 \ast_2 + 3 \ast_4 + 4 \ast_5 + 5 \ast_6 + \cdots
\]

Number of initial \(V\)-paths by length: 1, 1, 1, 1, 1, ..., 1

Number of returning \(UV\)-paths by length: 1, 1, 2, 6, 24, 120, ...
Experimental data

— Example: operad Comp —

Number of elements by degree: 1, 2, 4, 8, 16, 32, ...

The pair (Comp, U, V) is 2-dual.

**U-hook series:**

\[(I - U)^{-1} = 1 + \frac{q}{1} + \frac{q^2}{2} + \frac{q^3}{6} + \frac{q^4}{24} + \cdots\]

Number of initial U-paths by length: 1, 2, 8, 48, 384, 3840, ... (Sequence A000165)

**V-hook series:**

\[(I - V)^{-1} = 1 + \frac{q}{1} + \frac{q^2}{2} + \frac{q^3}{6} + \frac{q^4}{24} + \cdots\]

Number of initial V-paths by length: 1, 2, 4, 8, 16, 32, ...

Number of returning UV-paths by length: 1, 2, 8, 48, 384, 3840, ... (Sequence A000165)
Experimental data

— Example: operad Motz —

Number of elements by degree: 1, 2, 6, 22, 90, 394, ... (Sequence A006318)

The pair \((\text{Motz}, U, V)\) is \(\phi\)-diagonal dual.

\(U\)-hook series:

\[(I - U)^{-1}(\circ) = \circ + \circ\circ + 2 \circ\circ\circ + \circ\circ\circ\circ + 6 \circ\circ\circ\circ\circ + 2 \circ\circ\circ\circ\circ\circ + 2 \circ\circ\circ\circ\circ\circ\circ + 24 \circ\circ\circ\circ\circ\circ\circ\circ + \cdots\]

Number of initial \(U\)-paths by length: 1, 2, 10, 82, 938, 13778, ... (Sequence A112487)

\(V\)-hook series:

\[(I - V)^{-1}(\circ) = \circ + \circ\circ + \circ\circ\circ + \circ\circ\circ\circ + \circ\circ\circ\circ\circ + \circ\circ\circ\circ\circ\circ + \circ\circ\circ\circ\circ\circ\circ + 2 \circ\circ\circ\circ\circ\circ\circ\circ + \cdots\]

Number of initial \(V\)-paths by length: 1, 2, 6, 26, ...

Number of returning \(UV\)-paths by length: 1, 2, 10, 94, 1446, ...
Questions and perspectives

▶ Necessarily condition on $\mathcal{O}$ for the $\phi$-diagonal duality of $(\mathcal{O}, U, V)$?
Questions and perspectives

▶ Necessarily condition on $O$ for the $\phi$-diagonal duality of $(O, U, V)$?

▶ Unimodality of the intervals of the posets $(O, \preceq)$ of $(F(\mathcal{G}), U)$?
Questions and perspectives

- Necessarily condition on $\mathcal{O}$ for the $\phi$-diagonal duality of $(\mathcal{O}, U, V)$?

- Unimodality of the intervals of the posets $(\mathcal{O}, \preceq)$ of $(F(\mathcal{S}), U)$?

The intervals of the Young lattice are non-unimodal. For instance, the numbers of elements smaller than the integer partition $8844$ are, degree by degree,

$$1, 2, 3, 5, 6, 9, 11, 15, 17, 21, 23, 27, 28, 31, 30, 31, 27, 24, 18, 14, 8, 5, 2, 1.$$
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▶ Formulas to count initial paths in $(O, U)$, in $(O, V)$ and returning paths in $(O, U, V)$?
Questions and perspectives

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▶ Formulas to count initial paths in $(\mathcal{O}, U)$, in $(\mathcal{O}, V)$ and returning paths in $(\mathcal{O}, U, V)$?

▶ Colored versions of the graph $(\mathcal{O}, U)$ to turn it into a tree and use it for (uniform) random generation.