Construction of combinatorial operads

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Operads (1/2)

A (non-symmetric set-)operad is a triple $(\mathcal{P}, \circ_i, \mathbf{1})$ where

- $\mathcal{P}$ is a set of the form
$$\mathcal{P} := \biguplus_{n \geqslant 1} \mathcal{P}(n);$$
- $\circ_i$ is a grafting application
$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n + m - 1),$$
defined for all $n, m \geqslant 1$ and $i \in [n]$;
- $\mathbf{1}$ is an element of $\mathcal{P}(1)$, called unit.

These data has to satisfy some associativity, commutativity, and unitarity relations.

Operads (2/2)

For all $x \in \mathcal{P}(n)$, $y \in \mathcal{P}(m)$, and $z \in \mathcal{P}(k)$, following relations must be satisfied.

1. Associativity relation:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z),$$

   for all $i \in [n]$ and $j \in [m]$.

2. Commutativity relation:

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y,$$

   for all $1 \leqslant i < j \leqslant n$.

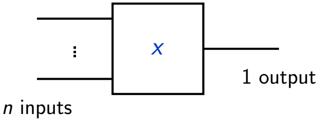
3. Unitarity relation:

$$\mathbf{1} \circ_1 x = x = x \circ_i \mathbf{1},$$

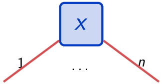
   for all $i \in [n]$.

Intuition (1/2)

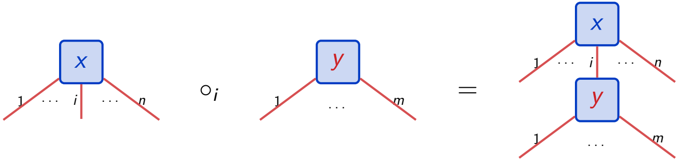
Element of $\mathcal{P}(n) \rightsquigarrow$ operator of arity n:



Operator of arity $n \rightsquigarrow$ planar rooted tree (parse trees) with n leaves:

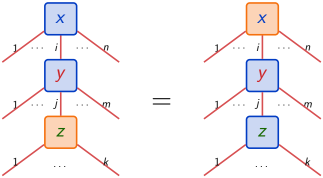


Grafting application $\rightsquigarrow$ grafting of trees:

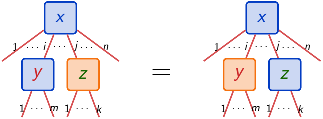


Intuition (2/2)

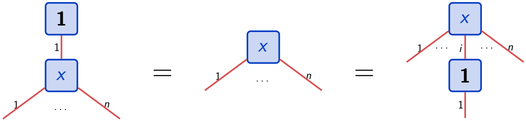
1. Associativity relation:



2. Commutativity relation:



3. Unitarity relation:



Some definitions

Let $\left(\mathcal{P}, \circ_i^{\mathcal{P}}, \mathbf{1}^{\mathcal{P}}\right)$ and $\left(\mathcal{Q}, \circ_i^{\mathcal{Q}}, \mathbf{1}^{\mathcal{Q}}\right)$ be two operads.

The arity $|x|$ of an element x of $\mathcal{P}$ is n if $x \in \mathcal{P}(n)$.

An operad morphism is a map $\phi : \mathcal{P} \to \mathcal{Q}$ mapping elements of arity n of $\mathcal{P}$ to elements of arity n of $\mathcal{Q}$, and such that, for all $x, y \in \mathcal{P}$ and $i \in [|x|]$,

$$\phi\left(x \circ_i^{\mathcal{P}} y\right) = \phi(x) \circ_i^{\mathcal{Q}} \phi(y).$$

The operad $\mathcal{Q}$ is a suboperad of $\mathcal{P}$ if for all $n \geqslant 1$, $\mathcal{Q}(n) \subseteq \mathcal{P}(n)$.

Let G be a set of elements of $\mathcal{P}$. The operad generated by G is the smallest suboperad of $\mathcal{P}$ which contains G.

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Example: The associative operad

Let $(\mathsf{Assoc}, \circ_i, \mathtt{a}_1)$ be the operad defined for all $n \geqslant 1$ by

$$\mathsf{Assoc}(n) := \{\mathtt{a}_n\},$$

and for all $n, m \geqslant 1$ and $i \in [n]$ by

$$\mathtt{a}_n \circ_i \mathtt{a}_m := \mathtt{a}_{n+m-1}.$$

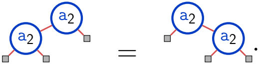
- Dimensions: 1, 1, 1, 1, 1, 1, ...
- Assoc is generated by $\mathtt{a}_2$:

$$\mathtt{a}_1, \quad \mathtt{a}_2, \quad \mathtt{a}_3 = \mathtt{a}_2 \circ_1 \mathtt{a}_2, \quad \mathtt{a}_4 = \mathtt{a}_3 \circ_1 \mathtt{a}_2, \quad \mathtt{a}_5 = \mathtt{a}_4 \circ_1 \mathtt{a}_2, \quad \ldots$$

- The generator $\mathtt{a}_2$ is subject to the relation

$$\mathtt{a}_2 \circ_1 \mathtt{a}_2 = \mathtt{a}_2 \circ_2 \mathtt{a}_2,$$

which translates into parse trees by



- Presentation by generators and relations:

$$\mathsf{Assoc} = \langle \mathtt{a}_2 \quad | \quad \mathtt{a}_2 \circ_1 \mathtt{a}_2 = \mathtt{a}_2 \circ_2 \mathtt{a}_2 \rangle .$$

Example: The magmatic operad

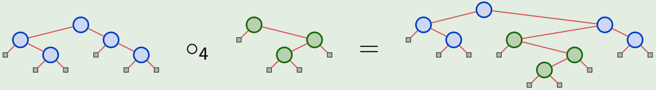
Let (Mag, $\circ_i$, ) be the operad defined for all $n \geqslant 1$ by

$$\mathrm{Mag}(n) := \{T : T \text{ binary tree with } n \text{ leaves}\},$$

and for all $n, m \geqslant 1$ and $i \in [n]$ by

$$S \circ_i T := \text{tree obtained by grafting } T \text{ on the } i\text{-th leaf of } S.$$

Example



- Dimensions : 1, 1, 2, 5, 14, 42, ... (Catalan numbers).
- Mag is generated by  (proof by induction on the arities).
- Presentation by generators and relations:

Mag = $\langle$  $\mid$ $\rangle$.

$\rightsquigarrow$ Mag is the free operad on one generator of arity 2.

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The T construction (1/2)

Let us start with a monoid $(M, \bullet, 1)$.

Let $\mathsf{T}M$ be the set $\mathsf{T}M := \uplus_{n \geqslant 1} \mathsf{T}M(n)$ where

$$\mathsf{T}M(n) := \{(x_1, \ldots, x_n) : x_i \in M \text{ for all } i \in [n]\}.$$

Let $\circ_i$ be a grafting application

$$\circ_i : \mathsf{T}M(n) \times \mathsf{T}M(m) \to \mathsf{T}M(n + m - 1),$$

defined for all $x \in \mathsf{T}M(n)$, $y \in \mathsf{T}M(m)$, and $i \in [n]$ by

$$x \circ_i y := (x_1, \ldots, x_{i-1}, x_i \bullet y_1, \ldots, x_i \bullet y_m, x_{i+1}, \ldots, x_n).$$

The T construction (2/2)

Let M and N be two monoids and $\theta : M \to N$ be a monoid morphism.

Let $\mathsf{T}\theta$ be the application

$$\mathsf{T}\theta : \mathsf{T}M \to \mathsf{T}N,$$

defined for all $(x_1, \dots, x_n) \in \mathsf{T}M(n)$ by

$$\mathsf{T}\theta\,(x_1, \dots, x_n) := (\theta(x_1), \dots, \theta(x_n))\,.$$

Some examples

$M := (\mathbb{N}, +)$. Elements of $\mathsf{T}M$: words on the alphabet $\mathbb{N}$.

Example

$$2\mathbf{1}23 \circ_2 30313 = 24142423$$

$N := \{\mathtt{a}, \mathtt{b}\}^*$. Elements of $\mathsf{T}N$: multiwords on the alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Example

$$\begin{matrix} b & a & \mathbf{a} & \epsilon & b \\ & b & & & b \\ & a & & & \end{matrix} \circ_3 \begin{matrix} \epsilon & a & \epsilon & b \\ & & & b \end{matrix} = \begin{matrix} b & a & \mathbf{a} & \mathbf{a} & \mathbf{a} & \mathbf{a} & \epsilon & b \\ & b & & a & & b & & b \\ & a & & & & b & & \end{matrix}$$

Let $\theta : N \to M$ be the monoid morphism defined by $\theta(u) := |u|$.

Example

$$\mathsf{T}\theta \begin{pmatrix} b & a & a & \epsilon & a & a \\ & b & & & a & \\ & a & & & & \end{pmatrix} = 131021$$

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Properties of the T construction (1/2)

Theorem

If M is a monoid, $\mathsf{T}M$ is an operad.

If $\theta : M \to N$ is a monoid morphism, $\mathsf{T}\theta$ is an operad morphism.

Moreover, T preserves injections and surjections.

Hence, T is an exact functor from the category of monoids with monoid morphisms to the category of operads with operad morphisms.

If N is a quotient monoid of M, then $\mathsf{T}N$ is an quotient operad of $\mathsf{T}M$.

Properties of the T construction (2/2)

The sets $\mathsf{T}M(n)$ are finite if and only if M is finite. In this case, the dimensions of $\mathsf{T}M$ are

$$m, m^2, m^3, m^4, \dots$$

where $m := \#M$.

$\mathsf{T}M$ is generated by the family

$$\{(1,1)\} \uplus \{(g) : g \in G(M)\},$$

where $G(M)$ is a set of generators of M.

The T construction coincides in some cases with a former and different construction [**Berger, Moerdijk**, 2003] which associates to any commutative bialgebra a cooperad.

Objectives and goals

Main motivations:

1. Give alternative constructions of some well-known operads;
2. Construct new operads.

The general line is as following:

We pick a monoid M, a subset G of generators of $\mathsf{T}M$, and we consider the suboperad $\mathcal{P}$ of $\mathsf{T}M$ generated by G. Typical questions:

1. Give a description of the elements of $\mathcal{P}$;
2. Find the dimensions of $\mathcal{P}$;
3. Find a bijection between elements of $\mathcal{P}$ and combinatorial objects;
4. Give an interpretation of the grafting application of $\mathcal{P}$ in terms of operations on combinatorial objects;
5. Give a presentation of $\mathcal{P}$ by generators and relations.

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Survey of some obtained operads (1/2)

Here is a list of the obtained suboperads or quotients of $\mathsf{T}(\mathbb{N}, +)$:

Operad	Generators	First dimensions	Combinatorial objects
End	—	$1, 4, 27, 256, 3125$	Endofunctions
PF	—	$1, 3, 16, 125, 1296$	Parking functions
PW	—	$1, 3, 13, 75, 541$	Packed words
Per	—	$1, 2, 6, 24, 120$	Permutations
PRT	01	$1, 1, 2, 5, 14, 42$	Planar rooted trees
$\mathsf{FCat}^{(k)}$	00, ..., 0k	Fuß-Catalan numbers	Trees of arity k
Schr	00, 01, 10	$1, 3, 11, 45, 197$	Schröder trees
Motz	00, 010	$1, 1, 2, 4, 9, 21, 51$	Motzkin paths
Comp	00, 01	$1, 2, 4, 8, 16, 32$	Int. compo.
DA	00, 01	$1, 2, 5, 13, 35, 96$	Directed animals
SComp	00, 01, 02	$1, 3, 27, 81, 243$	Segmented int. compo.

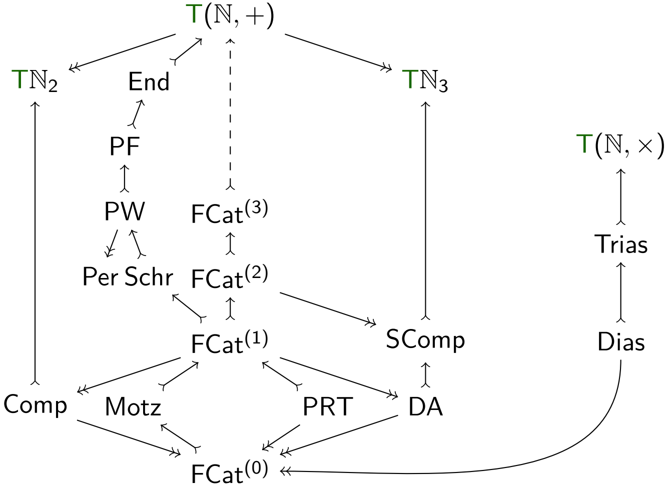
We also obtain already known operads as suboperads of $\mathsf{T}(\mathbb{N}, \times)$:

Operad	Generators	First dimensions	Combinatorial objects
Dias	01, 10	$1, 2, 3, 4, 5, 6$	Bin. words with exactly one 1
Trias	01, 10, 11	$1, 3, 7, 15, 31, 63$	Bin. words with at least one 1

Survey of some obtained operads (2/2)

These operads fit into following diagram.

$\rightarrowtail$ (resp. $\twoheadrightarrow$) stands for an injective (resp. surjective) operad morphism.



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Experimenting with Sage

Let PRT be the suboperad of $\mathsf{T}(\mathbb{N}, +)$ generated by 01.

```
sage:  M = AdditiveMonoid()
sage:  P = TConstruction(M)
sage:  G = [Word(M, [0, 1])]
sage:  PRT = SubOperad(P, G)
sage:  print [PRT.dimension(n) for n in xrange(1, 10)]
[1, 1, 2, 5, 14, 42, 132, 429, 1430]

sage:  print PRT.elements(5)
[01111, 01112, 01121, 01122, 01123, 01211, 01212, 01221,
01222, 01223, 01231, 01232, 01233, 01234]
```

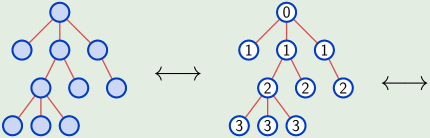
Elements and dimensions of PRT

Proposition

The elements of PRT *are exactly the words* x *of the alphabet* $\mathbb{N}$ *satisfying* $x_1 = 0$ *and* $1 \leqslant x_{i+1} \leqslant x_i + 1$ *for all* $i \in [|x| - 1]$.

The bijection between elements of PRT and planar rooted trees is computed by a depth-first traversal reading depths of nodes.

Example



$\longleftrightarrow$ 0112333212

Thus, PRT defines an operad structure on planar rooted trees, and

$$\dim \mathrm{PRT}(n) = \frac{1}{n}\binom{2n-2}{n-1}.$$

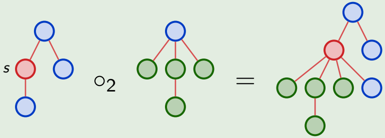
Interpretation of the grafting of PRT

Thanks to the bijection between elements of PRT and planar rooted trees, we obtain the following grafting operation on planar rooted trees:

Proposition

Let S and T be two planar rooted trees and s be the i-th node of S (for the depth-first traversal). The grafting $S \circ_i T$ in PRT *returns to graft the subtrees of the root of T as leftmost sons of s.*

Example



Presentation of PRT

Proposition

The operad PRT *is isomorphic to the magmatic operad through the operad isomorphism* ϕ : Mag $\to$ PRT *defined by*

$\phi\Big($  $\Big) :=$ .

Hence,

PRT $= \Big\langle$  $\mid$ $\Big\rangle$.

The operad PRT can be thought as a planar version of the Non Associative Permutative operad NAP [**Livernet**, 2006].

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Experimenting with Sage

Let Comp be the suboperad of $\mathsf{T}\mathbb{N}_2$ generated by 00 and 01.

```
sage:  M = CyclicMonoid(2)
sage:  P = TConstruction(M)
sage:  G = [Word(M, [0, 0]), Word(M, [0, 1])]
sage:  Comp = SubOperad(P, G)
sage:  print [Comp.dimension(n) for n in xrange(1, 10)]
[1, 2, 4, 8, 16, 32, 64, 128, 256]

sage:  print Comp.elements(5)
[00000, 00001, 00010, 00011, 00100, 00101, 00110, 00111,
01000, 01001, 01010, 01011, 01100, 01101, 01110, 01111]
```

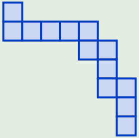
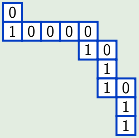
Elements and dimensions of Comp

Proposition

The elements of Comp *are exactly the words on the alphabet* $\{0, 1\}$ *which begin by* 0*.*

There is a classical bijection between such words and ribbon diagrams.

Example

 $\longleftrightarrow$ 

$\longleftrightarrow$ 0100001011011

Thus, Comp defines an operad structure on ribbon diagrams, and

$$\dim \mathrm{Comp}(n) = 2^{n-1}.$$

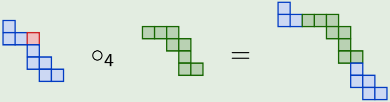
Interpretation of the grafting of Comp

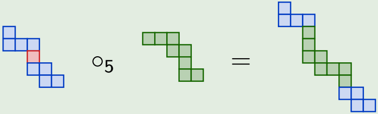
Thanks to the bijection between elements of Comp and ribbon diagrams, we obtain the following grafting operation on ribbon diagrams:

Proposition

Let C and D be two ribbon diagrams and c be the i-th box of C. The grafting $C \circ_i D$ in Comp *returns to replace c by D if c is the upper box of its column, or to replace c by the transpose of D otherwise.*

Example





Presentation of Comp

Theorem

The operad Comp *admits the following presentation:*

$$\mathrm{Comp} = \langle \boxbar, \boxminus \mid \boxbar \circ_1 \boxbar = \boxbar \circ_2 \boxbar, \quad \boxminus \circ_1 \boxbar = \boxbar \circ_2 \boxminus,$$
$$\boxminus \circ_1 \boxminus = \boxminus \circ_2 \boxbar, \quad \boxbar \circ_1 \boxminus = \boxminus \circ_2 \boxminus \rangle.$$

This implies that any algebra over the operad Comp is a set S with two applications $\boxbar, \boxminus : S \times S \to S$ which satisfy, for all $x, y, z \in S$, the relations

$$(x \boxbar y) \boxbar z = x \boxbar (y \boxbar z), \qquad (x \boxminus y) \boxminus z = x \boxminus (y \boxbar z),$$

$$(x \boxbar y) \boxminus z = x \boxbar (y \boxminus z), \qquad (x \boxminus y) \boxbar z = x \boxminus (y \boxminus z).$$

The free Comp-algebra on one generator is the set $\mathcal{C}$ of ribbon diagrams endowed with the applications $\boxbar, \boxminus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, where $x \boxbar y$ is the concatenation of y to right of x, and $x \boxminus y$ is the concatenation of the transpose of y below x.

Proof of the presentation of Comp (1/2)

Let $\mathcal{P}$ be the operad defined by

$$\mathcal{P} := \langle \mathtt{a}, \mathtt{b} \quad | \quad \mathtt{a} \circ_1 \mathtt{a} = \mathtt{a} \circ_2 \mathtt{a}, \quad \mathtt{b} \circ_1 \mathtt{a} = \mathtt{a} \circ_2 \mathtt{b}, \quad \mathtt{b} \circ_1 \mathtt{b} = \mathtt{b} \circ_2 \mathtt{a}, \quad \mathtt{a} \circ_1 \mathtt{b} = \mathtt{b} \circ_2 \mathtt{b} \rangle.$$

Let the application $\phi : \mathcal{P} \to \mathrm{Comp}$ defined by

$\phi(\mathtt{a}) :=$  and $\phi(\mathtt{b}) :=$ .

Since the relations between generators of $\mathcal{P}$ also hold in Comp by replacing $\mathtt{a}$ by  and $\mathtt{b}$ by , ϕ is an operad morphism and is surjective.

The idea is now to show that for all $n \geqslant 1$,

$$\#\mathcal{P}(n) \leqslant \#\mathrm{Comp}(n),$$

so that ϕ will turn to be an operad isomorphism.

Proof of the presentation of Comp (2/2)

Let us orient the relations of $\mathcal{P}$ in the following way:

$$\mathtt{a} \circ_1 \mathtt{a} \to \mathtt{a} \circ_2 \mathtt{a}, \qquad \mathtt{b} \circ_1 \mathtt{b} \to \mathtt{b} \circ_2 \mathtt{a},$$

$$\mathtt{b} \circ_1 \mathtt{a} \to \mathtt{a} \circ_2 \mathtt{b}, \qquad \mathtt{a} \circ_1 \mathtt{b} \to \mathtt{b} \circ_2 \mathtt{b}.$$

$\to$ is a rewriting rule on the elements of the free operad generated by $\mathtt{a}$ and $\mathtt{b}$. It exchanges in the parse trees a left oriented edge into a right oriented one, with a relabeling.

Moreover, $\to$ is terminating and its normal forms are all right comb binary trees, where each node is labeled by $\mathtt{a}$ or $\mathtt{b}$.

Hence, there are 2^{n-1} normal forms of arity n for $\to$. That implies that there are at most 2^{n-1} elements of arity n in $\mathcal{P}$.

Finally, since there are 2^{n-1} elements of arity n in Comp and ϕ is surjective, ϕ also is an isomorphism.

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Experimenting with Sage

Let DA be the suboperad of $T\mathbb{N}_3$ generated by 00 and 01.

```
sage:  M = CyclicMonoid(3)
sage:  P = TConstruction(M)
sage:  G = [Word(M, [0, 0]), Word(M, [0, 1])]
sage:  DA = SubOperad(P, G)
sage:  print [DA.dimension(n) for n in xrange(1, 10)]
[1, 2, 5, 13, 35, 96, 267, 750, 2123]
```

```
sage:  print DA.elements(5)
[00000, 00001, 00010, 00011, 00012, 00100, 00101, 00110,
00111, 00112, 00120, 00121, 00122, 01000, 01001, 01010,
01011, 01012, 01100, 01101, 01110, 01111, 01112, 01120,
01121, 01122, 01200, 01201, 01202, 01210, 01211, 01212,
01220, 01221, 01222]
```

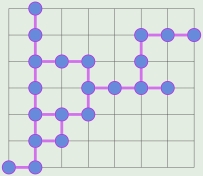
Directed animals

Directed animal: subset A of $\mathbb{N}^2$ such that $(0,0) \in A$ and

$$(i,j) \in A \text{ with } i \geqslant 1 \text{ or } j \geqslant 1 \quad \text{implies} \quad (i-1,j) \in A \text{ or } (i,j-1) \in A.$$

Example

A directed animal of size 21 (size is the number of points):



Prefixes of Motzkin paths

Prefix of Motzkin path: word x on $\{\bar{1} := -1, 0, 1\}$ such that for all $k \in [|x|]$, $x_1 + \cdots + x_k \geqslant 0$.

Example

$01\bar{1}10\bar{1}1$ is a prefix of Motzkin path;

$10\bar{1}\bar{1}1111$ is not a prefix of Motzkin path.

Theorem ([**Gouyou-Beauchamps**, **Viennot**, 1988])

Prefixes of Motzkin paths of length $n-1$ are in bijection with directed animals of size n.

Prefixes of Motzkin and elements of DA

Proposition

The application $\phi : \mathrm{DA}(n) \to \{\bar{1}, 0, 1\}^{n-1}$ *defined for all* $x \in \mathrm{DA}(n)$ *by*

$$\phi(x) := \begin{cases} \epsilon & \text{if } |x| = 1, \\ u_1 \dots u_{n-1} & \text{otherwise,} \end{cases}$$

where for all $i \in [n-1]$,

$$u_i := \begin{cases} x_{i+1} - x_i & \text{if } |x_{i+1} - x_i| \leqslant 1, \\ 1 & \text{if } x_i x_{i+1} = 20, \\ \bar{1} & \text{otherwise } (x_i x_{i+1} = 02), \end{cases}$$

is a bijection between the elements of arity n of DA *and prefixes of Motzkin paths of length* $n-1$.

Example

$$\phi(011220201) = 10101\bar{1}11$$

Elements, dimensions, and grafting of DA

Thus, DA defines an operad structure on prefixes of Motzkin paths.

By composing our bijection with the one of Gouyou-Beauchamps and Viennot, we can see elements of $\mathrm{DA}(n)$ as directed animals of size n.

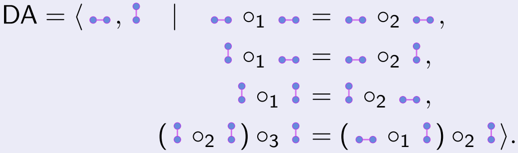
Question

What is the interpretation of the grafting in DA in terms of directed animals?

Presentation of DA

Theorem

The operad DA *admits the following presentation:*



The proof of this presentation follows same pattern as the one of Comp.

DA is binary (its generators are of arity 2) but is not quadratic (there is a nontrivial relation involving more than two generators).

Contents

Experimenting with Sage

Let D be the suboperad of $\mathsf{T}(\mathbb{N}, \times)$ generated by 01 and 10.

```
sage:  M = MultiplicativeMonoid()
sage:  P = TConstruction(M)
sage:  G = [Word(M, [0, 1]), Word(M, [1, 0])]
sage:  D = SubOperad(P, G)
sage:  print [D.dimension(n) for n in xrange(1, 10)]
[1, 2, 3, 4, 5, 6, 7, 8, 9]

sage:  print D.elements(8)
[00000001, 00000010, 00000100, 00001000, 00010000,
00100000, 01000000, 10000000]
```

Elements and dimensions of D

Proposition

The elements of D *are exactly the words on the alphabet* $\{0, 1\}$ *which have exactly one occurrence of* 1*.*

Thus,

$$\dim \mathrm{D}(n) = n.$$

The diassociative operad

The diassociative operad Dias [Loday, 2001] is defined by

$$\mathsf{Dias} := \langle \dashv, \vdash \quad | \quad \begin{aligned} \dashv \circ_1 \vdash \; &= \vdash \circ_2 \dashv, \\ \dashv \circ_1 \dashv \; &= \dashv \circ_2 \dashv \; = \; \dashv \circ_2 \vdash, \\ \vdash \circ_2 \vdash \; &= \vdash \circ_1 \vdash \; = \; \vdash \circ_1 \dashv \rangle. \end{aligned}$$

Proposition

The operad D *is isomorphic to the operad* Dias *through the operad isomorphism* $\phi : \mathsf{Dias} \to \mathsf{D}$ *defined by*

$$\phi(\dashv) := 10 \quad \textit{and} \quad \phi(\vdash) := 01.$$

Hence, D is a realization of Dias.

The triassociative operad

The triassociative operad Trias [Loday, Ronco, 2004] is defined by

$$
\begin{aligned}
\mathsf{Trias} := \langle \dashv, \perp, \vdash \quad | \quad \dashv \circ_1 \vdash &= \vdash \circ_2 \dashv, \\
\perp \circ_1 \perp &= \perp \circ_2 \perp, \\
\dashv \circ_1 \perp &= \perp \circ_2 \dashv, \\
\perp \circ_1 \dashv &= \perp \circ_2 \vdash, \\
\perp \circ_1 \vdash &= \vdash \circ_2 \perp, \\
\dashv \circ_1 \dashv = \dashv \circ_2 \dashv &= \dashv \circ_2 \vdash = \dashv \circ_2 \perp, \\
\vdash \circ_2 \vdash = \vdash \circ_1 \vdash &= \vdash \circ_1 \dashv = \vdash \circ_1 \perp \rangle.
\end{aligned}
$$

Proposition

The suboperad $\mathcal{P}$ *of* $\mathsf{T}(\mathbb{N}, \times)$ *generated by the elements* 01, 10, *and* 11 *is isomorphic to the operad* Trias *through the operad isomorphism* $\phi : \mathsf{Trias} \to \mathcal{P}$ *defined by*

$$\phi(\dashv) := 10, \quad \phi(\vdash) := 01, \quad \textit{and} \quad \phi(\perp) := 11.$$

Hence, $\mathcal{P}$ is a realization of Trias.