Computing with trees

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Combinatorics and combinatorial sets

Combinatorics is the art of studying concrete objects subject to construction rules.

Example — Fibonacci words

Let C be the set of words made of letters 1 or 2 such that there is no consecutive 2.

Some elements of C are

$$\epsilon, \quad 1, 2, \quad 11, 12, 21, \quad 111, 112, 121, 211, 212, \quad 1111, 1112, 1121, 1211, 1212, 2111, 2112, 2121.$$

Given a set C of combinatorial objects, a size function is a map $|-| : C \to \mathbb{N}$.

If $|x| = n$, then the size of x is n.

For any $n \in \mathbb{N}$, $C(n)$ is the set of objects of size n of C.

Definition — Combinatorial sets

A combinatorial set is a set C endowed with a size function such that $C(n)$ is finite for any $n \in \mathbb{N}$.

Example — Permutations

A permutation of size n is a word on $[n] := \{1, \dots, n\}$ such that each element of $[n]$ appears exactly once.
By denoting by $\mathfrak{S}$ this combinatorial set, we have

$$\mathfrak{S}(3) = \{123, 132, 213, 231, 312, 321\}.$$

Example — Integer compositions

An integer composition of size n is a word of positive integers such that the sum of its letters is n.
By denoting by $\mathfrak{C}$ this combinatorial set, we have

$$\mathfrak{C}(4) = \{1111, 112, 121, 211, 22, 13, 31, 4\}.$$

Example — Integer partitions

An integer partition of size n is a weakly decreasing integer composition.
By denoting by $\mathfrak{P}$ this combinatorial set, we have

$$\mathfrak{P}(4) = \{1111, 211, 22, 31, 4\}.$$

Main questions in combinatorics

Given a combinatorial set C, a main objective is to **enumerate** the elements of C.

We are looking for **formulas** depending on n and giving the number of elements of $C(n)$.

Example — Permutations

For any $n \in \mathbb{N}$,

$$\#\mathfrak{S}(n) = n!.$$

This is a consequence of the fact that there are n choices for the first letter, $n-1$ for the second, and so on.

Example — Integer compositions

For any $n \geqslant 1$,

$$\#\mathfrak{C}(n) = 2^{n-1}.$$

This is a consequence of the encoding of an integer composition using sticks and balls.

For instance, 21132 is encoded by •• | • | • | • • • | • • . Indeed, given a sequence of n balls, there are exactly $n-1$ places between two boxes wherein a stick is positioned **or** is absent (2 choices for each).

Another important objective about a combinatorial set C concerns the **generation** of the elements of C.

We are looking for **algorithms** that take n as input and construct the list of the elements of $C(n)$.

Example — Permutations

To generate the permutations of size n, consider the recursive algorithm $\mathsf{G}(n)$ defined as follows:

- if $n = 0$, then $\mathsf{G}(n)$ is the set $\{\epsilon\}$;
- otherwise, $\mathsf{G}(n)$ is the set obtained by considering each permutation σ of $\mathsf{G}(n-1)$ and by inserting the letter n in all possible ways.

The efficiency in time and space are crucial for such algorithms.

For the case of permutations, there is a much more efficient algorithm than this one: the Steinhaus–Johnson–Trotter algorithm [**Johnson**, 1963] [**Trotter**, 1962].

A last (for the moment) important question question about a combinatorial set C concerns the **relationships** between C and other combinatorial sets C'.

We are looking for **morphisms** that send objects of size n of C to objects of size n of C'.

Example — Permutations and integer compositions

Let $\phi : \mathfrak{S} \to \mathfrak{C}$ be the map such that for any $\sigma \in \mathfrak{S}(n)$, $\phi(\sigma)$ is the integer composition whose balls and sticks representation is such that there is a stick between the i-th and $i+1$-st balls whenever $\sigma(i) > \sigma(i+1)$.
For instance,

$$\phi(34276158) = \bullet\bullet \mid \bullet\bullet \mid \bullet \mid \bullet\bullet\bullet = 2213.$$

This morphism is surjective (easy exercise).

Bijective morphisms are very interesting since they provide equivalent ways to see a combinatorial set.

In general, nice morphisms between combinatorial sets have connections with **algebra** (they are morphisms of algebraic structures). This is the case for the previous morphism between $\mathfrak{S}$ and $\mathfrak{C}$.

Subfields of combinatorics

Combinatorics gather different domains, each approaching the previous questions with different methods:

- **enumerative combinatorics**, based on the understanding of combinatorial sets through their inherent properties and on generating series;
- **analytic combinatorics**, based on the use of analytical methods to understand the asymptotic behavior of large objects;
- **geometric combinatorics**, studying polytopes whose vertices are combinatorial objects;
- **algebraic combinatorics**, where combinatorial sets are studied through operations on objects.

Applications of combinatorics

Combinatorics intervenes in a lot of domains:

- in **algorithms**, since data structures can be seen as combinatorial objects (strings, lists, trees, *etc.*);
- in **theory of computation**, where treelike structures and substitution operations play a significant role;
- in **quantum field theory**, linked with Hopf algebras on trees [**Connes, Kreimer**, 2000];
- in **statistical physics**, and mainly in percolation theory [**Broadbent, Hammersley**, 1957];
- in **algebra**, since operations on combinatorial objects are useful to realize types of algebraic structures. There are also important connections with representation theory of finite groups.

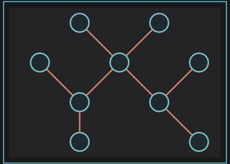
Treelike structures

Trees are known at least since Cayley [Cayley, 1857].

Definition — Trees

A tree is a connected acyclic graph.

Example — A tree



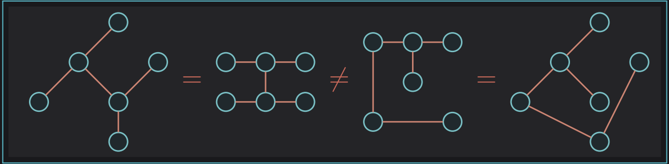
From this very general definition, many different variations exist. The main enrichments concern

- the presence of a special node (**rooted** trees);
- the presence of an ordering of the nodes adjacent to another one (**ordered** or **planar** trees);
- the presence of decorations on nodes (**decorated** trees);
- the presence of a total order on the nodes (**standard** or **labeled** trees).

Tree variations

Let $\mathfrak{T}$ be the combinatorial set of **trees** where the size of a tree is its number of nodes.

Example — Some trees



The sequence of the numbers of trees size by size is Sequence **A000055** and starts by

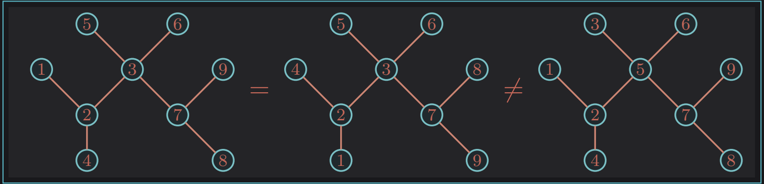
$$1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551.$$

Some important references on trees:

- *Combinatorial Species and Tree-Like Structures* [**Bergeron, Labelle, Leroux**, 1998];
- *Enumerative Combinatorics, Vol. 2* [**Stanley**, 1999].

Let $\mathfrak{LT}$ be the combinatorial set of **labeled trees** where the size of a labeled tree is its number of nodes.

Example — Some labeled trees



Theorem [Borchardt, 1860] [Cayley, 1889]

The number $\#\mathfrak{LT}(n)$ of labeled trees of size $n \geqslant 1$ satisfies

$$\#\mathfrak{RT}(n) = n^{n-2}.$$

The sequence of these numbers is Sequence **A000272** and starts by

$$1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, 100000000, 2357947691, 61917364224.$$

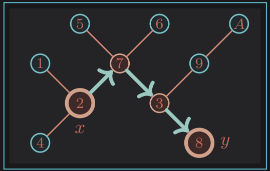
This enumeration formula admits the following very elegant interpretation.

Take a labeled tree $\mathfrak{t}$ and choose a head node x and a tail node y.

This is a vertebrate labeled tree.

Let $\mathfrak{VLT}$ be the combinatorial set of vertebrate labeled trees.

Example — A vertebrate labeled tree



Theorem [Joyal, 1981]

There is a one-to-one correspondence between vertebrate labeled trees of size n and maps $e : [n] \to [n]$.

Given this result, for any $n \geqslant 1$, observe that

- since the choice of the head and the tail are independent, $\#\mathfrak{VLT}(n) = n^2 \#\mathfrak{LT}(n)$;
- there are n^n maps $e : [n] \to [n]$.

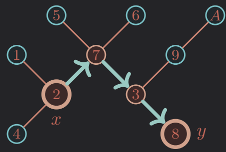
Therefore, as expected we obtain $\#\mathfrak{LT}(n) = \frac{n^n}{n^2}$.

Let $\mathfrak{t}$ be a vertebrate labeled tree. The map e is constructed as a word on $[n]$ of length n such that

- the labels of the **spine** of $\mathfrak{t}$ are put from the head to the tail as letters of indices being labels of the spine;
- the value of the letter at each remaining position z is the label of the node of $\mathfrak{t}$ which is adjacent to the node labeled by z and closest to the spine.

Example — A vertebrate labeled tree and its endofunction

$\mathfrak{t} =$



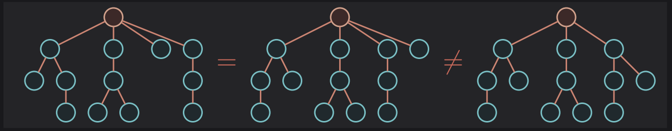
$$\longleftrightarrow \begin{pmatrix} 1 & \mathbf{2} & \mathbf{3} & 4 & 5 & 6 & \mathbf{7} & \mathbf{8} & 9 & A \\ 2 & \mathbf{2} & \mathbf{7} & 2 & 7 & 7 & \mathbf{3} & \mathbf{8} & 3 & 9 \end{pmatrix} = e$$

There are many other proofs for the enumeration of labeled trees:

- by using **species of structures** [**Joyal**, 1981] [**Bergeron, Labelle, Leroux**, 1998];
- by a **double counting method** [**Pitman**, 1999];
- by **Prüfer codes** [**Prüfer**, 1918].

Let $\mathfrak{RT}$ be the combinatorial set of **rooted trees** where the size of a rooted tree is its number of nodes.

Example — A rooted tree



Theorem [Polya, 1937] [Otter, 1948]

The number $\#\mathfrak{RT}(n)$ of rooted trees of size n satisfies $\#\mathfrak{RT}(1) = 1$ and, for any $n \geqslant 1$, the recurrence

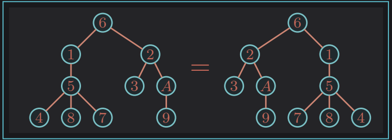
$$\#\mathfrak{RT}(n+1) = \frac{1}{n} \sum_{k \in [n]} \left(\sum_{d \mid k} d \, \#\mathfrak{RT}(d) \right) \#\mathfrak{RT}(n-k+1).$$

The sequence of these numbers is Sequence **A000081** and starts by

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766.$$

Let $\mathfrak{LRT}$ be the combinatorial set of **labeled rooted trees** where the size of a labeled rooted tree is its number of nodes.

Example — A labeled rooted tree



Theorem [Cayley, 1889]

The number $\#\mathfrak{RT}(n)$ of labeled rooted trees of size $n \geqslant 1$ satisfies

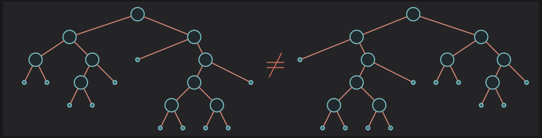
$$\#\mathfrak{RT}(n) = n^{n-1}.$$

The sequence of these numbers is Sequence **A000169** and starts by

$$1, 2, 9, 64, 625, 7776, 117649, 2097152, 43046721, 1000000000, 25937424601, 743008370688.$$

Let $\mathfrak{B}$ be the combinatorial set of **binary trees** where the size of a binary tree is its number of internal nodes.

Example — Some binary trees



Theorem

The number $\#\mathfrak{B}(n)$ of binary trees of size $n \geqslant 0$ satisfies

$$\#\mathfrak{B}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

The sequence of these numbers is Sequence **A000108** and starts by

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786.$$

These are the Catalan numbers.

A signature is a set $\mathcal{S}$ decomposing as $\mathcal{S} = \bigsqcup_{n \geqslant 0} \mathcal{S}(n)$.

An $\mathcal{S}$-planar term is defined **recursively** to be

- either the leaf $\perp$;
- or an internal node decorated by $\mathsf{g} \in \mathcal{S}(n)$ attached to n children $\mathcal{S}$-terms.

Let $\mathfrak{T}^{\perp}_{\mathrm{P}}\mathcal{S}$ (resp. $\mathfrak{T}^{\bullet}_{\mathrm{P}}\mathcal{S}$) be the set of $\mathcal{S}$-planar terms where the size of an $\mathcal{S}$-planar term is its number of leaves (resp. internal nodes).

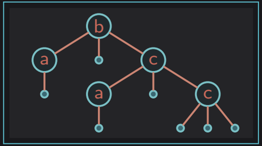
Exercise

Find a necessary and sufficient condition on $\mathcal{S}$ for the fact that $\mathfrak{T}^{\perp}_{\mathrm{P}}\mathcal{S}$ (resp. $\mathfrak{T}^{\bullet}_{\mathrm{P}}\mathcal{S}$) is combinatorial.

Example — An $\mathcal{S}$-planar term

Let the signature $\mathcal{S} := \mathcal{S}(1) \sqcup \mathcal{S}(3)$ with $\mathcal{S}(1) := \{\mathsf{a}\}$ and $\mathcal{S}(3) := \{\mathsf{b}, \mathsf{c}\}$.

This $\mathcal{S}$-planar term has size 7 in $\mathfrak{T}^{\perp}_{\mathrm{P}}\mathcal{S}$ and size 5 in $\mathfrak{T}^{\bullet}_{\mathrm{P}}\mathcal{S}$.



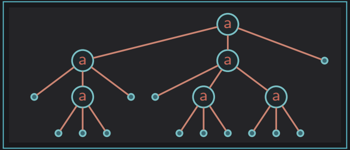
Let $m \geqslant 0$ and $\mathcal{S}_m$ be the signature such that $\mathcal{S}_m := \mathcal{S}_m(m+1) := \{\mathsf{a}\}$.

Any $\mathcal{S}_m$-planar term is a planar term such that each internal node has exactly $m+1$ children.

Such trees are m-terms.

In particular, as sets, $\mathfrak{T}^{\bullet}_{\mathrm{P}}\mathcal{S}_1 = \mathfrak{B}$.

Example — A planar 2-term



Theorem

For any $m, n \geqslant 0$,

$$\#\mathfrak{T}^{\bullet}_{\mathrm{P}}\mathcal{S}_m(n) = \frac{1}{mn+1}\binom{mn+n}{n}.$$

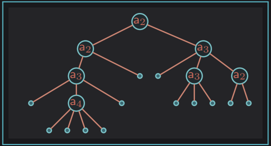
The sequence of these numbers for $m = 2$ is Sequence **A001764** and starts by

$$1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, 1430715, 8414640.$$

These are the m-Fuss-Catalan numbers.

Let $\mathcal{S}_{\mathrm{sch}}$ be the signature such that $\mathcal{S}_{\mathrm{sch}}(0) := \mathcal{S}_{\mathrm{sch}}(1) := \emptyset$ and for any $n \geqslant 2$, $\mathcal{S}_{\mathrm{sch}}(n) := \{\mathsf{a}_n\}$.

Example — A Schröder tree



Any $\mathcal{S}_{\mathrm{sch}}$-planar term is a term such that each internal node has two or more children.

Such trees are Schröder trees.

Theorem

The number $\mathfrak{T}_{\mathrm{P}}^{\perp}\mathcal{S}_{\mathrm{sch}}(n)$ of Schröder trees of size n satisfies $\mathfrak{T}_{\mathrm{P}}^{\perp}\mathcal{S}_{\mathrm{sch}}(1) = 1$ and, for any $n \geqslant 1$ the recurrence

$$\#\mathfrak{T}_{\mathrm{P}}^{\perp}\mathcal{S}_{\mathrm{sch}}(n+1) = 2 \sum_{k \in [n]} \#\mathfrak{T}_{\mathrm{P}}^{\perp}\mathcal{S}_{\mathrm{sch}}(k)\, \#\mathfrak{T}_{\mathrm{P}}^{\perp}\mathcal{S}_{\mathrm{sch}}(n+1-k) - \#\mathfrak{T}_{\mathrm{P}}^{\perp}\mathcal{S}_{\mathrm{sch}}(n).$$

The sequence of these numbers is Sequence **A001003** and starts by

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723.$$

These are the super-Catalan numbers.

Universal algebra

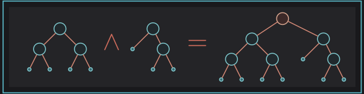
An algebraic structure is a set endowed with some operations. The arity of an operation says how many inputs this operation has.

Example — The group of integers

The set $\mathbb{Z}$ of integers endowed with the operation of addition $x_1, x_2 \mapsto x_1 + x_2$ and the operation of opposite $x_1 \mapsto -x_1$ is an algebraic structure.

Example — The magma of binary trees

The set $\mathfrak{B}$ endowed with the operation $\mathfrak{t}_1, \mathfrak{t}_2 \mapsto \mathfrak{t}_1 \wedge \mathfrak{t}_2$ such that $\mathfrak{t}_1 \wedge \mathfrak{t}_2$ is the binary tree admitting $\mathfrak{t}_1$ as left subtree and $\mathfrak{t}_2$ as right subtree is an algebraic structure.



Algebraic structures are central objects in **algebraic combinatorics**. In this field,

- algebraic structures on combinatorial sets are constructed.
  - $\rightsquigarrow$ In this direction, algebra serves as a tool to obtain **combinatorial results**;
- algebraic structures are studied by using methods coming from combinatorics.
  - $\rightsquigarrow$ In this direction, combinatorics serves as a tool to obtain **algebraic results**.

Let us introduce particular ordered rooted trees.

Definition — $\mathcal{S}$-terms

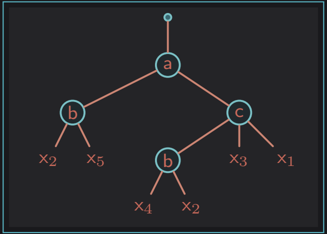
Let $\mathcal{S}$ be a signature.

An $\mathcal{S}$-term is an $\mathcal{S}$-planar term such that each leaf is decorated on the set $\mathbb{X} := \{x_1, x_2, x_3, \dots\}$ of variables.

The set of $\mathcal{S}$-terms is denoted by $\mathfrak{T}\mathcal{S}$.

Example — An $\mathcal{S}$-term

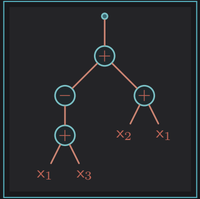
Here is an $\mathcal{S}$-term for $S := S(2) \sqcup S(3)$ with $S(2) := \{a, b\}$ and $S(3) := \{c\}$:



Terms are used to represent **compound operations**.

Example — A compound operation

On the signature $\mathcal{S} := \mathcal{S}(1) \sqcup \mathcal{S}(2)$ where $\mathcal{S}(1) := \{-\}$ and $\mathcal{S}(2) := \{+\}$, the $\mathcal{S}$-term



represents the compound operation $x_1, x_2, x_3 \mapsto -(x_1 + x_3) + (x_2 + x_1)$.

An **algebraic structure** is specified by its operations brought by a signature and by relations between some compound operations. Formally, we have the following definition.

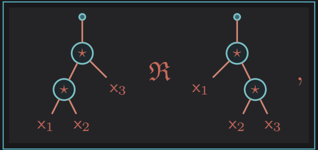
Definition — Varieties

A variety is a pair $(\mathcal{S}, \mathfrak{R})$ where $\mathcal{S}$ is a signature and $\mathfrak{R}$ is an equivalence relation on $\mathfrak{T}\mathcal{S}$.

Universal algebra studies algebraic structures through varieties.

Example — The variety of monoids

The variety of monoids is the pair $(\mathcal{S}, \mathfrak{R})$ such that $\mathcal{S} := \mathcal{S}(0) \sqcup \mathcal{S}(2)$ where $\mathcal{S}(0) := \{\mathbb{1}\}$, $\mathcal{S}(2) := \{\star\}$, and $\mathfrak{R}$ is the equivalence relation satisfying



- The first relation says that for any x_1, x_2, and x_3, the compound operations $x_1, x_2, x_3 \mapsto (x_1 \star x_2) \star x_3$ and $x_1, x_2, x_3 \mapsto x_1 \star (x_2 \star x_3)$ are the same. Hence, $\star$ is an **associative operation**.
- The second relations say that for any x_1, the compound operations $x_1 \mapsto x_1 \star \mathbb{1}$, $x_1 \mapsto x_1$, and $x_1 \mapsto \mathbb{1} \star x_1$ are the same. Hence, $\mathbb{1}$ is a **unit** w.r.t. $\star$.

Definition — Algebras over a variety

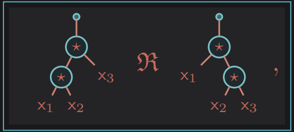
Let $\mathcal{V} := (\mathcal{S}, \mathfrak{R})$ be a variety. A $\mathcal{V}$-algebra is a set A endowed with the operations of $\mathcal{S}$ preserving the relations prescribed by $\mathfrak{R}$.

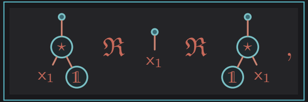
Equivalent compound operations

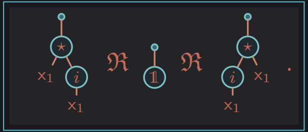
Given a variety $(\mathcal{S}, \mathfrak{R})$, it is possible that there exist compound operations $\mathfrak{t}_1$ and $\mathfrak{t}_2$ such that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ **behave in the same way**. This property is denoted by $\mathfrak{t}_1 \equiv_{\mathfrak{R}} \mathfrak{t}_2$.

Example — Two equivalent operations in the variety of groups

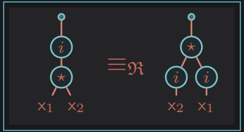
The variety of groups is the pair $(\mathcal{S}, \mathfrak{R})$ such that $\mathcal{S} := \mathcal{S}(0) \sqcup \mathcal{S}(1) \sqcup \mathcal{S}(2)$ where $\mathcal{S}(0) := \{\mathbb{1}\}$, $\mathcal{S}(1) := \{i\}$, $\mathcal{S}(2) := \{\star\}$, and $\mathfrak{R}$ is the equivalence relation satisfying







We have



This says that for any group G and any $x_1, x_2 \in G$,

$$i(x_1 \star x_2) = i(x_2) \star i(x_1).$$

Definition — Word problem

The word problem is the **decision problem** taking as input two terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of a variety $(\mathcal{S}, \mathfrak{R})$ and outputting whether $\mathfrak{t}_1 \equiv_{\mathfrak{R}} \mathfrak{t}_2$.

This problem is **undecidable** in general [**Baader, Nipkow**, 1998].

Nevertheless, for particular varieties, this problem may be decidable.

Definition — Combinatorial realizations of varieties

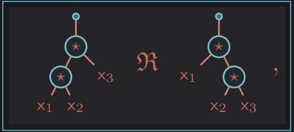
A combinatorial realization of a variety $(\mathcal{S}, \mathfrak{R})$ is a set X in one-to-one correspondence with the set of $\equiv_{\mathfrak{R}}$-equivalence classes of $\mathcal{S}$-terms together with a composition operation on X compatible with the composition of compound operations.

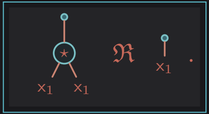
Several (possibly overlapping) tools intervene here:

- **term rewrite systems** [**Bezem, Klop, de Vrijer, Terese**, 2003];
- **operad theory** [**Loday, Vallete**, 2012] [**Méndez**, 2015] [**Giraudo**, 2018];
- **clone theory** [**Taylor**, 1993] [**Giraudo**, 2023+].

Definition — Variety of idempotent semigroups

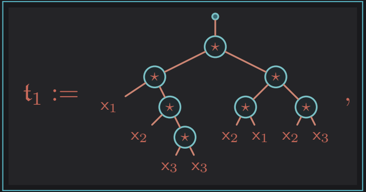
The variety of idempotent semigroups is the pair $(\mathcal{S}, \mathfrak{R})$ such that $\mathcal{S} := \mathcal{S}(2) := \{\star\}$ and $\mathfrak{R}$ is the equivalence relation satisfying

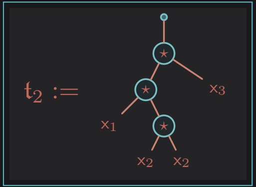




Example — Two equivalent terms of the variety of idempotent semigroups

Let the two compound operations





of the variety of idempotent semigroups. We have $t_1 \equiv_{\mathfrak{R}} t_2$.

Exercise: try to prove this by **rewriting** these two terms as a same one by using $\mathfrak{R}$.

To decide the $\equiv_{\mathfrak{R}}$-equivalence in the **variety of idempotent semigroups**, consider the following algorithm, associating with any term $\mathfrak{t}$ a word $\mathbb{P}(\mathfrak{t})$ on positive integers:

1. set u as the indexes of the variables appearing $\mathfrak{t}$, from left to right;
2. iteratively apply while possible the following transformations on u in any order:
   1. replace a factor $w \,.\, w$ by w;
   2. replace a factor $v \,.\, a \,.\, w$ by $v \,.\, w$ if $a \in \mathrm{Alph}(v)$ and $\mathrm{Alph}(v) = \mathrm{Alph}(w)$.

Theorem [Siekmann, Szabó, 1982] [Klíma, Korbelář, Polák, 2011]

Two terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are $\equiv_{\mathfrak{R}}$-equivalent in the variety of idempotent semigroups if and only if $\mathbb{P}(\mathfrak{t}_1) = \mathbb{P}(\mathfrak{t}_2)$.

Example

Consider the two terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of the previous page.

- We have $u_1 = 12332123$. Since $12\underline{3 \,.\, 3}2123 \rightsquigarrow \underline{123 \,.\, 2 \,.\, 123} \rightsquigarrow \underline{123 \,.\, 123} \rightsquigarrow 123$, we have $\mathbb{P}(\mathfrak{t}_1) = 123$.
- We have $u_2 = 1223$. Since $1\underline{2 \,.\, 2}3 \rightsquigarrow 123$, we have $\mathbb{P}(\mathfrak{t}_2) = 123$.

Therefore, $\mathfrak{t}_1 \equiv_{\mathfrak{R}} \mathfrak{t}_2$.

Algebraic structures on trees

Definition — Decorated rooted trees

Let D be a nonempty set.

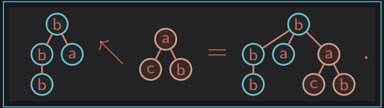
Let $\mathfrak{RT}D$ be the set of D-decorated rooted trees, which are rooted trees whose nodes are decorated on D.

Definition — Butcher product

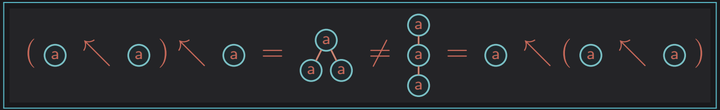
The Butcher product is the binary product $\nwarrow : \mathfrak{RT}D \times \mathfrak{RT}D \to \mathfrak{RT}D$ such that for any $\mathfrak{t}_1 \in \mathfrak{RT}D$ and $\mathfrak{t}_2 \in \mathfrak{RT}D$, $\mathfrak{t}_1 \nwarrow \mathfrak{t}_2$ is the D-decorated rooted tree obtained by putting the root of $\mathfrak{t}_2$ as a child of the root of $\mathfrak{t}_1$.

Example — A Butcher product

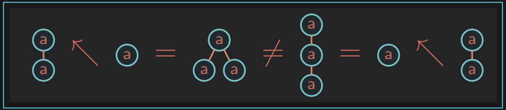
Let $D := \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$. We have the following Butcher product on D-decorated rooted trees:



The Butcher product is **not associative**:



The Butcher product is **not commutative**:



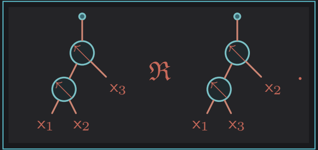
Nevertheless, this product satisfies the universal relation

$$(t_1 \nwarrow t_2) \nwarrow t_3 = (t_1 \nwarrow t_3) \nwarrow t_2.$$

This is the nonassociative permutative (NAP) relation.

Definition — Variety of NAP algebras

The variety of NAP algebras is the pair $(\mathcal{S}, \mathfrak{R})$ such that $\mathcal{S} := \mathcal{S}(2) := \{\nwarrow\}$ and $\mathfrak{R}$ is the equivalence relation satisfying



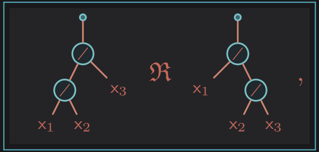
A NAP algebra is an algebra over the variety of NAP algebras.

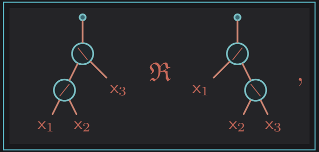
Theorem [Dzhumadil'daev, Löfwall, 2002] [Livernet, 2006]

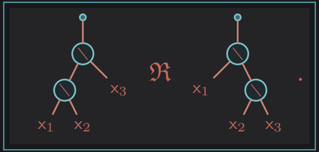
Let D be a nonempty set. The set of $\mathfrak{RT}D$ of D-decorated rooted trees endowed with the Butcher product $\nwarrow$ is the free NAP algebra generated by D.

Definition — Variety of duplicial algebras [Brouder, Frabetti, 2003]

The variety of duplicial algebras is the pair $(\mathcal{S}, \mathfrak{R})$ such that $\mathcal{S} := \mathcal{S}(2) := \{\diagup, \diagdown\}$ and $\mathfrak{R}$ is the equivalence relation satisfying







A duplicial algebra is an algebra over the variety of duplicial algebras.

By definition, this is a set A endowed with two operations $\diagup : A \times A \to A$ and $\diagdown : A \times A \to A$ such that

- $\diagup$ is associative;
- $\diagdown$ is associative;
- for any $x_1, x_2, x_3 \in A$, the relation $(x_1 \diagup x_2) \diagdown x_3 = x_1 \diagup (x_2 \diagdown x_3)$ holds.

Definition — Decorated binary trees

Let D be a nonempty set.

Let $\mathfrak{B}D$ be the set of D-decorated binary trees, which are binary trees having at least one internal node and whose internal nodes are decorated on D.

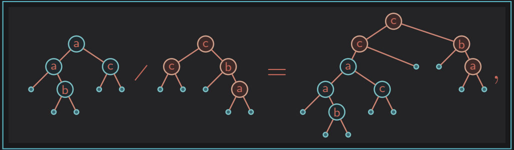
Definition — Over and under products

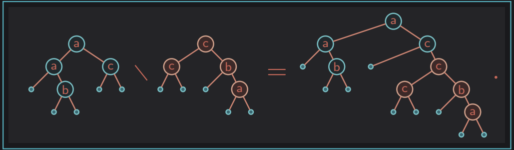
The over product is the binary product $\mathbin{/} : \mathfrak{B}D \times \mathfrak{B}D \to \mathfrak{B}D$ such that for any $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{B}D$, $\mathfrak{t}_1 \mathbin{/} \mathfrak{t}_2$ is the D-decorated binary tree obtained by grafting the root of $\mathfrak{t}_1$ on the leftmost leaf of $\mathfrak{t}_2$.

The under product $\backslash$ is defined in the same way, by considering instead the rightmost leaf.

Examples — Over and under products

Let $D := \{a, b, c\}$. We have the following over and under products on D-decorated binary trees:





Theorem [Loday, 2008]

Let D be a nonempty set. The set of $\mathfrak{B}D$ of D-decorated binary trees endowed with the over product $/$ and the under product $\backslash$ is the **free duplicial algebra** generated by D.

Exercise — A duplicial algebra on words

Let $\mathbb{N}^+$ be the set of nonempty words of nonnegative integers.

Let $\ll$ and $\gg$ be the binary products on $\mathbb{N}^+$ defined by

$$u \ll v := u(v \uparrow_{\max(u)}),$$

$$u \gg v := u(v \uparrow_{\ell(u)}).$$

For instance,

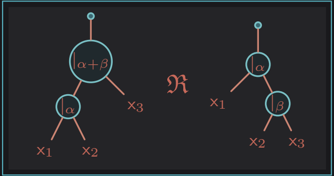
$$0211 \ll 14 = 021136,$$

$$0211 \gg 14 = 021158.$$

1. Show that $\mathbb{N}^+$ endowed with the products $\ll$ and $\gg$ is a duplicial algebra [**Novelli**, **Thibon**, 2013].
2. Describe a minimal generating set of this duplicial algebra.
3. Exhibit the nontrivial relations satisfied by these generators or prove that this duplicial algebra is free.

Definition — Variety of Fuss-Catalan algebras [Giraudo, 2015]

For any $m \in \mathbb{N}$, the variety of m-Fuss-Catalan algebras is the pair $(\mathcal{S}, \mathfrak{R})$ such that $\mathcal{S} := \mathcal{S}(2) := \{|_0, |_1, \dots, |_m\}$ and $\mathfrak{R}$ is the equivalence relation satisfying



for any $\alpha, \beta \in [\![m]\!]$ such that $\alpha + \beta \leqslant m$.

An m-Fuss-Catalan algebra is an algebra over the variety of m-Fuss-Catalan algebras.

In particular, a 1-Fuss-Catalan algebra is a set A endowed with two operations $|_0 \colon A \times A \to A$ and $|_1 \colon A \times A \to A$ such that for any $x_1, x_2, x_3 \in A$,

$$(x_1 \,|_0\, x_2) \,|_0\, x_3 = x_1 \,|_0\, (x_2 \,|_0\, x_3),$$

$$(x_1 \,|_0\, x_2) \,|_1\, x_3 = x_1 \,|_0\, (x_2 \,|_1\, x_3),$$

$$(x_1 \,|_1\, x_2) \,|_0\, x_3 = x_1 \,|_1\, (x_2 \,|_0\, x_3).$$

Definition — Decorated m-terms

Let D be a nonempty set.

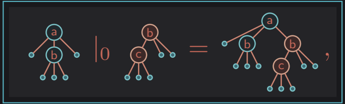
Let $\mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D$ be the set of D-decorated m-terms, which are m-terms having at least one internal nodes and whose internal nodes are decorated on D.

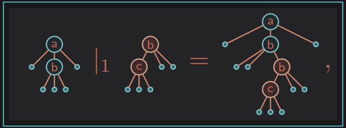
Definition — Grafting products

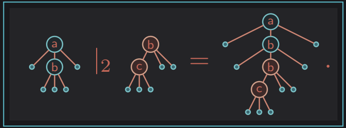
The α-th grafting product is the binary product $|_\alpha \colon \mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D \times \mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D \to \mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D$ such that for any $\mathrm{t}_1, \mathrm{t}_2 \in \mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D$, $\mathrm{t}_1 \; |_\alpha \; \mathrm{t}_2$ is the D-decorated m-term obtained by grafting t_2 on the α-th rightmost leaf of t_1.

Exemples — Grafting products

Let $D := \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. We have the following over and under products on D-decorated 2-terms:







Theorem [Giraudo, 2015]

Let $m \in \mathbb{N}$ and D be a nonempty set. The set of $\mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D$ of D-decorated m-terms endowed with the products $|_\alpha$, $\alpha \in [\![m]\!]$, is the **free m-Fuss-Catalan algebra** generated by D.

Exercises

- ☐ Interpret the operations $|_\alpha$, $\alpha \in [\![m]\!]$, on other combinatorial sets in one-to-one correspondence with $\mathfrak{T}_{\mathrm{P}}\mathcal{S}_m D$ (like dissections of polygons into $m+2$-gons or m-Dyck paths).
- ☐ Propose analog operations $|_\alpha$ on other treelike structures (rooted trees, ordered rooted trees, terms, *etc.*) and study the relations these operations satisfy.

Conclusion

We have reviewed here several aspects of **trees** and some of their variations.

They are central in **universal algebra** since compound operations can be manipulated by terms.

We have seen some **algebraic structures involving trees**.

There are a lot of other such algebraic structures:

- **pre-Lie algebras**, involving decorated rooted trees [**Chapoton, Livernet**, 2001];
- **Connes-Kreimer Hopf algebras**, involving forests of rooted trees [**Connes, Kreimer**, 1998] [**Foissy**, 2002].

There are also important links with models of computation:

- with **λ-calculus**, since expressions of this model are particular trees [**Church**, 1936];
- with **combinatory logic**, since expressions of this model are particular terms endowed with rewrite rules [**Schönfinkel**, 1924].