Combinatorics, operations, and graded graphs

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Outline

Combinatorics

Algebraic combinatorics

Operads and graded graphs

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such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

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4. Establish transformations between C and other combinatorial collections D.

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The generating series of a combinatorial collection C is

$$\mathcal{G}_C(t) := \sum_{n \in \mathbb{N}} \#C(n) t^n = \sum_{x \in C} t^{|x|}.$$

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- $\mathcal{G}_{\mathrm{BT}}(t) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots = \dfrac{1-\sqrt{1-4t}}{2t}$

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Algebraic combinatorics

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Most important algebraic structures are

- lattices;
- monoids;
- Hopf bialgebras;
- operads.

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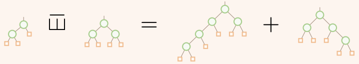
If C is endowed with operations $\star$, these operations extend as products $\bar{\star}$ on formal power series leading to expressions for $\mathbf{f}_C$.

— Example —



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Operad structures

Endowing a combinatorial collection C with the structure of an operad consists in providing a map

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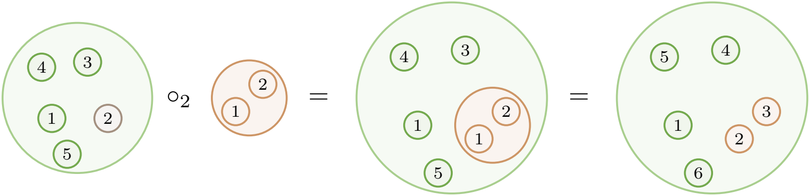
Intuition: for any $x, y \in C$ and $i \in [|x|]$, $x \circ_i y$ can be thought as the insertion of y into the ith substitution place of x. For instance,



Some operads

— Operad on words —

Let $A := \mathbb{Z}/_{\ell\mathbb{Z}}$ be an alphabet. We turn A^* into an operad where $u \circ_i v$ is obtained by replacing the ith letter of u by a copy of v obtained by incrementing $\pmod{\ell}$ its letters by u_i [**Giraudo**, 2015]. For instance, for $\ell := 3$,

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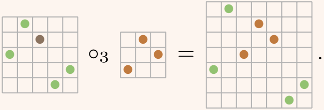
— Operad on permutations —

We turn $\mathfrak{S}$ into an operad where $\sigma \circ_i \nu$ is the permutation whose permutation matrix is obtained by replacing the ith point of the matrix of σ by a copy of the matrix of ν [**Aguiar, Livernet**, 2007]. For instance,

$$35412 \circ_3 132 = 3746512,$$



Some operads

— Operad on trees —

Let G be a set of nodes. We turn the set of trees on G into an operad $\mathbf{F}(G)$ where $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by grafting the root of a copy of $\mathfrak{s}$ onto the ith leaf of $\mathfrak{t}$. For instance, for

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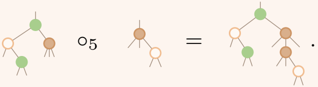

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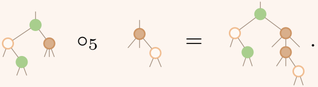

There exist many other (more or less complicated) operads involving combinatorial objects:

- on various families of trees (binary trees, m-trees, Schröder trees, rooted trees, *etc.*);
- on various families of paths (Dyck paths, Motzkin paths, *etc.*);
- on various families of graphs (cliques, drawn inside a polygon, with labeled edges, *etc.*).

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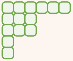
$533111 \quad \leftrightarrow$ 

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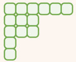
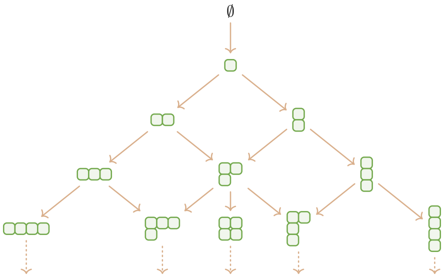
$533111 \quad \leftrightarrow$ 

The Young lattice admits as Hasse diagram the graph wherein there is an arc $\lambda \to \mu$ if μ can be obtained by adding a box from λ:



Graded graphs

A graded graph is a pair (C, U) where C is a combinatorial collection and U is a linear map

$$\mathrm{U} : \mathbb{K}\langle C(d)\rangle \to \mathbb{K}\langle C(d+1)\rangle, \qquad d \geqslant 0.$$

This map sends any $x \in C$ to its next vertices (with multiplicities).

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Classical examples include

- the Young lattice [**Stanley**, 1988];
- the bracket tree [**Fomin**, 1994];
- the composition poset [**Björner, Stanley**, 2005];
- the Fibonacci lattice [**Fomin**, 1988], [**Stanley**, 1988].

Graded graphs and duality

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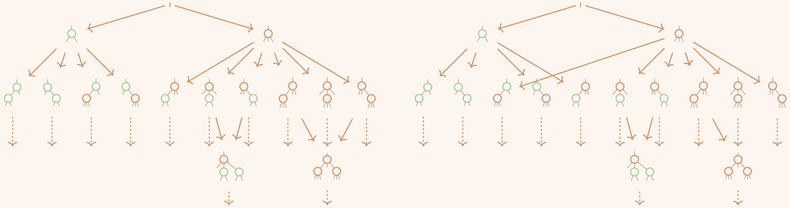
— Idea —

Use operads as a source of dual pairs of graded graphs.

Graded graphs from operads

— Example —

For $G = \Big\{$  ,  $\Big\}$, the pair $(\mathbf{F}(G), \mathrm{U}, \mathrm{V})$ is



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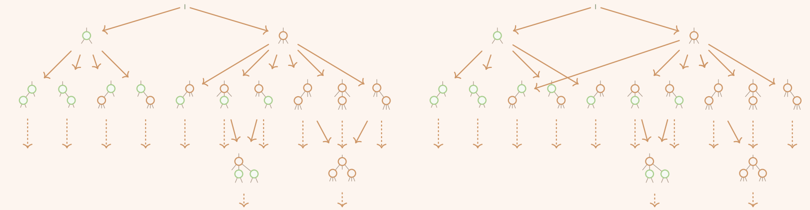

General construction: given an operad $\mathcal{O}$ (satisfying some conditions), let the graphs $(\mathcal{O}, \mathrm{U})$ and $(\mathcal{O}, \mathrm{V})$ defined by

$$\mathrm{U}(x) := \sum_{\substack{\mathsf{a} \in G \\ i \in [|x|]}} x \circ_i \mathsf{a}, \qquad \mathrm{V}(x) := \sum_{\substack{y \in \mathcal{O} \\ \exists (\mathfrak{s}, \mathfrak{t}) \in \mathrm{ev}^{-1}(x) \times \mathrm{ev}^{-1}(y) \\ \langle \mathfrak{t}, \mathrm{V}(\mathfrak{s}) \rangle \neq 0}} y.$$

Graded graphs from operads

— Theorem [Giraudo, 2018] **—**

If $\mathcal{O}$ is an homogeneous operad, then $(\mathcal{O}, \mathrm{U}, \mathrm{V})$ is a pair of graded graphs. Moreover, if $\mathcal{O}$ is a free operad, this pair is ϕ-diagonal dual.

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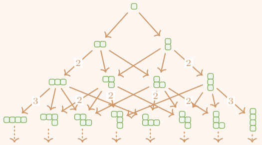
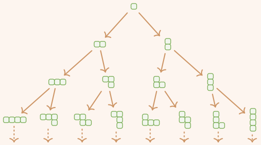
— Example —

The pair $(\mathsf{Comp}, \mathrm{U}, \mathrm{V})$ is 2-dual.

The graded graph $(\mathsf{Comp}, \mathrm{U})$ is the Hasse diagram of the composition poset [Bjöner, Stanley, 2005].





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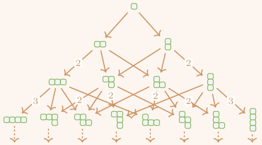

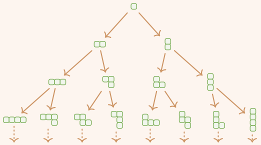

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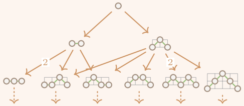
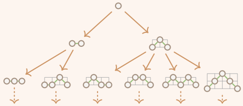
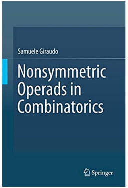
The pair $(\mathrm{Motz}, \mathrm{U}, \mathrm{V})$ is ϕ-diagonal dual

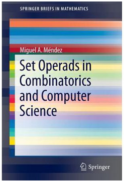




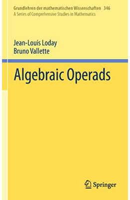
Further reading on operads



S. Giraudo. *Nonsymmetric Operads in Combinatorics*, Springer monograph, $viii+172$ pages, 2018 (Jan. 2019).



M. Méndez. *Set Operads in Combinatorics and Computer Science*, SpringerBriefs, $xv + 129$, 2015.



J.-L. Loday and B. Vallette. *Algebraic operads*, Springer, $xxiv+636$ pages, 2012.