Combinatorial operads, rewrite systems, and formal grammars

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Computational Logic and Applications

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Outline

Operads

Enumeration

Generation
Outline

Operads
Types of algebraic structures

Combinatorics deals with sets (or spaces) of structured objects:

- monoids;
- groups;
- lattices;
- associative alg.;
- Hopf bialg.;
- Lie alg.;
- pre-Lie alg.;
- dendriform alg.;
- duplicial alg.
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1. a collection of operations;
2. a collection of relations between operations.

— Example —

The type of monoids can be specified by

1. the operations $\star$ (binary) and $\mathbb{1}$ (nullary);
2. the relations $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$ and $x \star \mathbb{1} = x = \mathbb{1} \star x$. 
Working with operations

Strategy to study types of algebras → add a level of indirection by working with algebraic structures where

- elements are operations \( x_1 \ldots \)
- having \( n = |x| \) inputs and 1 output;
- the operation is the composition operation of operations. If \( x \) and \( y \) are two operations,
  1. by selecting an input of \( x \) specified by its position \( i \);
  2. and by grafting the output of \( y \) onto this input,
we obtain the new operation \( x_1 \mid x \mid i \ldots \ldots \circ i \mid y \mid 1 \ldots \ldots = x_1 \mid x \mid + \mid y \mid - 1 \ldots \ldots \).
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- elements are operations

\[
\begin{array}{c}
x \\
1 \ldots n
\end{array}
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\[
\begin{array}{c}
x \\
1 \ldots i \ldots |x|
\end{array} \circ_i \begin{array}{c}
y \\
1 \ldots |y|
\end{array} = \begin{array}{c}
x \\
1 \ldots \ldots |x|+|y|-1
\end{array}
\]

\[
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Operads

Operads are algebraic structures formalizing the notion of operations and their composition.
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A (nonsymmetric set-theoretic) operad is a triple \((\mathcal{O}, \circ_i, 1)\) where

1. \(\mathcal{O}\) is a graded set

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\mathcal{O} := \bigsqcup_{n \geq 1} \mathcal{O}(n);
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2. \(\circ_i\) is a map, called partial composition map,
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   \circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \quad 1 \leq i \leq n, \ 1 \leq m;
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This data has to satisfy some axioms.
Operad axioms

The **associativity** relation

\[(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)\]

\[1 \leq i \leq |x|, 1 \leq j \leq |y|\]

says that the pictured operation can be constructed from top to bottom or from bottom to top.
Operad axioms

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The commutativity relation

\[(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y\]

\[1 \leq i < j \leq |x|\]

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The **unitality** relation

\[1 \circ_1 x = x = x \circ_i 1\]

\[1 \leq i \leq |x|\]

says that 1 is the identity map.
Operad on permutations

Let $\text{Per}$ be the operad wherein:

- $\text{Per}(n)$ is the set of all permutations of size $n$, seen through their permutation matrices.

--- Example ---

The partial composition $\sigma \circ_i \nu$ is the permutation matrix obtained by replacing the $i$th point of $\sigma$ by a copy of $\nu$.

--- Example ---

The unit is the unique permutation of size 1.

--- Example ---

has arity 9 and denotes the permutation 378651294.
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Let $\text{Path}$ be the operad wherein:

- $\text{Path}(n)$ is the set of all paths with $n$ points, that are words $u_1 \ldots u_n$ of elements of $\mathbb{N}$.

— Example —

<table>
<thead>
<tr>
<th>has arity 13 and denotes the path</th>
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<tbody>
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<td>1212232100112.</td>
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**Example**

<table>
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<tr>
<th>Path (n)</th>
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<td><img src="image" alt="Path Example" /></td>
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- The partial composition \( u \circ_i v \) is the path obtained by replacing the \( i \)th point of \( u \) by a copy of \( v \).

\[ \text{Example} \]

\[ 011232101 \circ_4 11224 = 0113344632101 \]

- The unit is the unique path \( 0 \) of size 1, depicted as \( \circ \).
Some suboperads of Path

For any $m \geq 0$, an $m$-Dyck path is a path starting and ending with 0 and made of steps \( \binom{m}{0} \) and \( \circ \).

— Example —

\[ \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} 
\]

is a 2-Dyck path of size 10.
Some suboperads of Path

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For any $m \geq 0$, the set $\text{Dyck}^{(m)}$ of all $m$-Dyck paths is a suboperad of Path.
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A Motzkin path is a path starting and ending with 0 and made of steps $\begin{array}{c} 0 \\ \circ \end{array}$, $\begin{array}{c} \circ \\ 0 \end{array}$, and $\begin{array}{c} \circ \circ \end{array}$.

--- Example ---

is a Motzkin path of size 16.
Some suboperads of Path

For any $m \geq 0$, an $m$-Dyck path is a path starting and ending with 0 and made of steps $\begin{array}{c} 0 \\ \vdots \\ \end{array}$ and $\begin{array}{c} \odot \\ \vdots \\ \end{array}$.

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The set Motz of all Motzkin paths is a suboperad of Path.
Algebras over operads

Let $\mathcal{O}$ be an operad. An algebra over $\mathcal{O}$ is a space $\mathcal{V}$ equipped, for all $x \in \mathcal{O}(n)$, with linear maps

$$x : \mathcal{V} \otimes \cdots \otimes \mathcal{V} \to \mathcal{V}$$
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such that $1$ is the identity map on $\mathcal{V}$ and the compatibility relation

$$x \circ_i y = x_{v_1 \cdots v_{|x|+|y|-1}}$$

holds for any $x, y \in \mathcal{O}$, $i \in [|x|]$, and $v_1, \ldots, v_{|x|+|y|-1} \in \mathcal{V}$. 
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— Example —

Let $A_s$ be the associative operad defined by $A_s(n) := \{\ast_n\}$ for all $n \geq 1$ and $\ast_n \circ_i \ast_m := \ast_{n+m-1}$. 
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Let \( \text{As} \) be the associative operad defined by \( \text{As}(n) := \{ \star_n \} \) for all \( n \geq 1 \) and \( \star_n \circ_i \star_m := \star_{n+m-1} \). This operad is minimally generated by \( \star_2 \).
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Any algebra over $\textbf{As}$ is a space $\mathcal{V}$ endowed with linear operations $\star_n$ of arity $n \geq 1$ where $\star_2$ satisfies, for all $v_1, v_2, v_3 \in \mathcal{V}$,

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$\parallel$

$$(\star_2 \circ_2 \star_2)(v_1, v_2, v_3)$$

Using infix notation for the binary operation $\star_2$, we obtain the relation

$$(v_1 \star_2 v_2) \star_2 v_3 = v_1 \star_2 (v_2 \star_2 v_3),$$

so that algebras over $\textbf{As}$ are associative algebras.

In the same way, there are operads for $\triangleright\text{Lie alg.}; \triangleright\text{pre-Lie alg.}$ [Chapoton, Livernet, 2001]; $\triangleright\text{dendriform alg.}$ [Loday, 2001]; $\triangleright\text{duplicial alg.}$ [Loday, 2008]; $\triangleright\text{diassociative alg.}$ [Loday, 2001]; $\triangleright\text{brace alg.}$
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$$

and

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Scope of operads

As main benefits, operads

- offer a formalism to compute over operations;
- allow us to work virtually with all the structures of a type;
- lead to discover the underlying combinatorics of types of algebras.
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Endowing a set of combinatorial objects with an operad structure helps to

- highlight elementary building block for the objects;
- build combinatorial structures (graded graphs, posets, lattices, etc.);
- enumerative prospects and discovery of statistics.
Outline

 Enumeration
Syntax trees

An alphabet is a graded set $\mathcal{G} := \bigsqcup_{n \geq 1} \mathcal{G}(n)$. 
Syntax trees

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Let $S(\mathcal{G})$ be the set of $\mathcal{G}$-syntax trees, defined recursively by

- $\texttt{l} \in S(\mathcal{G})$;
- if $a \in \mathcal{G}$ and $t_1, \ldots, t_{|a|} \in S(\mathcal{G})$, then $a(t_1, \ldots, t_{|a|}) \in S(\mathcal{G})$.

---

**Example**

Let $\mathcal{G} := \mathcal{G}(2) \sqcup \mathcal{G}(3)$ such that $\mathcal{G}(2) = \{a, b\}$ and $\mathcal{G}(3) = \{c\}$.

![Syntax tree diagram](image)

denotes the $\mathcal{G}$-tree

$$c(\texttt{l}, c(a(\texttt{l}, \texttt{l}), \texttt{l}, b(a(\texttt{l}, \texttt{l}), c(\texttt{l}, \texttt{l}, \texttt{l}))), b(\texttt{l}, b(\texttt{l}, \texttt{l})))$$

having degree 8 and arity 12.
Syntax trees

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Let \( S(\mathcal{G}) \) be the set of \( \mathcal{G} \)-syntax trees, defined recursively by

1. \( \mathbf{l} \in S(\mathcal{G}) \);
2. if \( a \in \mathcal{G} \) and \( t_1, \ldots, t_{|a|} \in S(\mathcal{G}) \), then \( a(t_1, \ldots, t_{|a|}) \in S(\mathcal{G}) \).

Let \( t \in S(\mathcal{G}) \). Some definitions:

1. \( \mathbf{l} \) is the leaf;
2. the degree \( \text{deg}(t) \) of \( t \) is its number of internal nodes;
3. the arity \( |t| \) of \( t \) is its number of leaves.

— Example —

Let \( \mathcal{G} := \mathcal{G}(2) \sqcup \mathcal{G}(3) \) such that \( \mathcal{G}(2) = \{ a, b \} \) and \( \mathcal{G}(3) = \{ c \} \).

\[
c(\mathbf{l}, c(a(\mathbf{l})), b(a(\mathbf{l})), c(\mathbf{l}, \mathbf{l}, \mathbf{l})) , b(\mathbf{l}, b(\mathbf{l}, \mathbf{l}))
\]

denotes the \( \mathcal{G} \)-tree having degree 8 and arity 12.
Compositions of syntax trees

Let $t, s \in S(G)$. For each $i \in [|t|]$, the partial composition $t \circ_i s$ is the tree obtained by grafting the root of $s$ onto the $i$th leaf of $t$.

— Example —

$$
\begin{align*}
\text{c} & \quad \text{b} \\
\text{c} & \quad \text{b} \\
\text{a} & \quad \text{b} \\
\text{c} & \quad \text{b} \\
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{c} \\
\text{b} & \quad \text{b} \\
\text{a} & \quad \text{c} \\
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Compositions of syntax trees

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— Example —

Let \( t, s_1, \ldots, s_{|t|} \) be \( \mathcal{G} \)-trees. The full composition \( t \circ [s_1, \ldots, s_{|t|}] \) is obtained by grafting simultaneously the roots of each \( s_i \) onto the \( i \)th leaf of \( t \).

— Example —
Free operads

Let $\mathcal{G}$ be an alphabet.
Free operads

Let $\mathcal{E}$ be an alphabet.

The free operad on $\mathcal{E}$ is the operad on the set $S(\mathcal{E})$ wherein

- elements of arity $n$ are the $\mathcal{E}$-trees of arity $n$;
Free operads

Let $\mathcal{G}$ be an alphabet.

The free operad on $\mathcal{G}$ is the operad on the set $\mathcal{S}(\mathcal{G})$ wherein

- elements of arity $n$ are the $\mathcal{G}$-trees of arity $n$;
- the partial composition map $\circ_i$ is the one of the $\mathcal{G}$-trees;
- let $c : \mathcal{G} \to \mathcal{S}(\mathcal{G})$ be the natural injection (made implicit in the sequel).

Free operads satisfy the following universality property. For any alphabet $\mathcal{G}$, any operad $O$, and any map $f : \mathcal{G} \to O$ preserving the arities, there exists a unique operad morphism $\phi : \mathcal{S}(\mathcal{G}) \to O$ such that $f = \phi \circ c$.
Free operads

Let $\mathcal{G}$ be an alphabet.

The free operad on $\mathcal{G}$ is the operad on the set $S(\mathcal{G})$ wherein

- elements of arity $n$ are the $\mathcal{G}$-trees of arity $n$;
- the partial composition map $\circ_i$ is the one of the $\mathcal{G}$-trees;
- the unit is $I$. 

Let $c : \mathcal{G} \to S(\mathcal{G})$ be the natural injection (made implicit in the sequel).

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\[ \xymatrix@+1pc{ \mathcal{G} \ar[r]^-{c} & S(\mathcal{G}) \ar[r]^-{\phi} & O } \]
Free operads

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Factors and prefixes

Let \( t, s \in S(\mathcal{G}) \).
Factors and prefixes

Let $t, s \in S(\mathcal{G})$.

If $t$ decomposes as

$$t = r \circ_i (s \circ [r_1, \ldots, r_{|s|}])$$

for some trees $r, r_1, \ldots, r_{|s|}$, and $i \in [|r|]$, then $s$ is a factor of $t$.

This property is denoted by $s \preceq_f t$.

— Example —

[Diagram showing two trees with labels and relation $\preceq_f$.]
Factors and prefixes

Let \( t, s \in S(G) \).

If \( t \) decomposes as
\[
t = r \circ_i (s \circ [r_1, \ldots, r_{|s|}])
\]
for some trees \( r, r_1, \ldots, r_{|s|}, \) and \( i \in \|r\| \), then \( s \) is a factor of \( t \).
This property is denoted by \( s \preceq_f t \).

If in the previous decomposition \( r = t \), then \( s \) is a prefix of \( t \).
This property is denoted by \( s \preceq_p t \).
Pattern avoidance and enumeration

A $G$-tree $t$ avoids a $G$-tree $s$ if $s \preceq t$. 

Example —

$A(a a b a a b b)$ is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$.

$A(a a c a a c c)$ is enumerated by $1, 1, 2, 4, 9, 21, 51, 127, \ldots$ ($A001006$).

$A\begin{array}{c} a \hline a \end{array}$ is enumerated by $1, 2, 5, 13, 35, 96, 267, 750, \ldots$ ($A005773$).
Pattern avoidance and enumeration

A $\mathcal{G}$-tree $t$ avoids a $\mathcal{G}$-tree $s$ if $s \not\succeq t$.

For any $\mathcal{P} \subseteq S(\mathcal{G})$, let

$$A(\mathcal{P}) = \{ t \in S(\mathcal{G}) : \text{ for all } s \in \mathcal{P}, s \not\succeq t \}.$$
Pattern avoidance and enumeration

A $G$-tree $t$ avoids a $G$-tree $s$ if $s \not<_{ft} t$.

For any $\mathcal{P} \subseteq S(G)$, let

$$A(\mathcal{P}) = \{t \in S(G) : \text{for all } s \in \mathcal{P}, s \not<_{ft} t\}.$$ 

— Question —

Enumerate $A(\mathcal{P})$ w.r.t. the arities of the trees.
Pattern avoidance and enumeration

A $\mathcal{G}$-tree $t$ avoids a $\mathcal{G}$-tree $s$ if $s \nleq_f t$.

For any $\mathcal{P} \subseteq S(\mathcal{G})$, let

$$A(\mathcal{P}) = \{ t \in S(\mathcal{G}) : \text{ for all } s \in \mathcal{P}, s \nleq_f t \}.$$

— Example —

$A \left( \begin{array}{c} a \\ a \\ a \\ a \\ b \\ a \\ b \\ b \\ b \end{array} \right)$

is enumerated by 1, 2, 4, 8, 16, 32, 64, 128, . . . .

— Question —

Enumerate $A(\mathcal{P})$ w.r.t. the arities of the trees.
Pattern avoidance and enumeration

A \( G \)-tree \( t \) avoids a \( G \)-tree \( s \) if \( s \not<_{f} t \).

For any \( \mathcal{P} \subseteq S(G) \), let

\[
A(\mathcal{P}) = \{ t \in S(G) : \text{ for all } s \in \mathcal{P}, s \not<_{f} t \}.
\]

— Example —

\[
\begin{align*}
&\text{\— Example —} \\
&\downarrow A \left( \begin{array}{c} a \\ a \\ b \\ a \\ b \\ b \\ b \\ b \\ \end{array} \right) \text{ is enumerated by } 1, 2, 4, 8, 16, 32, 64, 128, \ldots.
\end{align*}
\]

\[
\downarrow A \left( \begin{array}{c} a \\ a \\ c \\ a \\ a \\ c \\ c \\ \end{array} \right) \text{ is enumerated by } 1, 1, 2, 4, 9, 21, 51, 127, \ldots. \text{(A001006)}.
\]

— Question —

Enumerate \( A(\mathcal{P}) \) w.r.t. the arities of the trees.
Pattern avoidance and enumeration

A $G$-tree $t$ avoids a $G$-tree $s$ if $s \not<_{f} t$.

For any $P \subseteq S(G)$, let

$$A(P) = \{ t \in S(G) : \text{ for all } s \in P, s \not<_{f} t \}.$$

— Example —

$A(a a b a a b b)$ is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$.

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— Question —

Enumerate $A(P)$ w.r.t. the arities of the trees.
Formal power series of trees

For any $\mathcal{P}, \mathcal{Q} \subseteq S(\mathcal{G})$, let

$$F(\mathcal{P}, \mathcal{Q}) := \sum_{\substack{t \in S(\mathcal{G}) \\ t \in A(\mathcal{P}) \\ \forall s \in \mathcal{Q}, s \not\preceq_{\mathcal{P}} t}} t.$$ 

This is the formal sum of all the $\mathcal{G}$-trees avoiding as factors all patterns of $\mathcal{P}$ and avoiding as prefixes all patterns of $\mathcal{Q}$. 
Formal power series of trees

For any \( P, Q \subseteq S(\varnothing) \), let

\[
F(P, Q) := \sum_{\substack{t \in S(\varnothing) \\
t \in A(P) \\
\forall s \in Q, s \not\approx_P t}} t.
\]

This is the formal sum of all the \( \varnothing \)-trees avoiding as factors all patterns of \( P \) and avoiding as prefixes all patterns of \( Q \).

Since

- \( F(P, \emptyset) \) is the formal sum of all the trees of \( A(P) \);
Formal power series of trees

For any $\mathcal{P}, \mathcal{Q} \subseteq S(\mathcal{G})$, let

$$F(\mathcal{P}, \mathcal{Q}) := \sum_{\substack{t \in S(\mathcal{G}) \\cap \mathcal{A}(\mathcal{P}) \\forall s \in \mathcal{Q}, s \nleq_{\mathcal{P}} t}} t.$$ 

This is the formal sum of all the $\mathcal{G}$-trees avoiding as factors all patterns of $\mathcal{P}$ and avoiding as prefixes all patterns of $\mathcal{Q}$.

Since

- $F(\mathcal{P}, \emptyset)$ is the formal sum of all the trees of $\mathcal{A}(\mathcal{P})$;
- the linear map $t \mapsto z^{|t|}$ sends $F(\mathcal{P}, \emptyset)$ to the generating series of $\mathcal{A}(\mathcal{P})$. 


Formal power series of trees

For any $\mathcal{P}, \mathcal{Q} \subseteq S(\mathfrak{G})$, let

$$F(\mathcal{P}, \mathcal{Q}) := \sum_{\begin{subarray}{c} t \in S(\mathfrak{G}) \\ t \in A(\mathcal{P}) \\ \forall s \in \mathcal{Q}, s \not\triangleleft_p t \end{subarray}} t.$$ 

This is the formal sum of all the $\mathfrak{G}$-trees avoiding as factors all patterns of $\mathcal{P}$ and avoiding as prefixes all patterns of $\mathcal{Q}$.

Since

- $F(\mathcal{P}, \emptyset)$ is the formal sum of all the trees of $A(\mathcal{P})$;
- the linear map $t \mapsto z^{|t|}$ sends $F(\mathcal{P}, \emptyset)$ to the generating series of $A(\mathcal{P})$;

the series $F(\mathcal{P}, \mathcal{Q})$ contains all the enumerative data about the trees avoiding $\mathcal{P}$. 
When \( \emptyset, \mathcal{P}, \) and \( \mathcal{Q} \) satisfy some conditions, \( F(\mathcal{P}, \mathcal{Q}) \) expresses as an inclusion-exclusion formula involving simpler terms \( F(\mathcal{P}, S_i) \).

\[ F(\mathcal{P}, \mathcal{Q}) = 1 + \sum_{k \geq 1} \sum_{\ell \geq 1} (-1)^{1+\ell} a \circ \left[ F(\mathcal{P}, S_1), \ldots, F(\mathcal{P}, S_k) \right]. \]
System of equations

When $\mathcal{G}$, $\mathcal{P}$, and $\mathcal{Q}$ satisfy some conditions, $F(\mathcal{P}, \mathcal{Q})$ expresses as an inclusion-exclusion formula involving simpler terms $F(\mathcal{P}, S_i)$.

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
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<tbody>
<tr>
<td>The series $F(\mathcal{P}, \mathcal{Q})$ satisfies</td>
</tr>
</tbody>
</table>

$$F(\mathcal{P}, \mathcal{Q}) = 1 + \sum_{k \geq 1}^{a \in \mathcal{G}(k)} \sum_{\ell \geq 1} (-1)^{1+\ell} a \overline{\circ} [F(\mathcal{P}, S_1), \ldots, F(\mathcal{P}, S_k)].$$

This leads to a system of equations for the generating series of $A(\mathcal{P})$.

Indeed, the generating series of $A(\mathcal{P})$ is the series $F(\mathcal{P}, \emptyset)$ where

$$F(\mathcal{P}, \mathcal{Q}) = z + \sum_{k \geq 1}^{a \in \mathcal{G}(k)} \sum_{\ell \geq 1} (-1)^{1+\ell} \prod_{i \in [k]} F(\mathcal{P}, S_i).$$
System of equations

— Example —

For \( \mathcal{P} := \{a, b\} \), we obtain the system of formal power series of trees

\[
\begin{align*}
F(\mathcal{P}, \emptyset) &= 1 + a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \emptyset)] + a \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \{b\})] \\
&\quad - a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \{b\})] + b \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \emptyset)], \\
F(\mathcal{P}, \{a\}) &= 1 + b \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \emptyset)], \\
F(\mathcal{P}, \{b\}) &= 1 + a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \emptyset)] + a \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \{b\})] \\
&\quad - a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \{b\})].
\end{align*}
\]
System of equations

— Example —

For $\mathcal{P} := \left\{ \begin{array}{c}
\begin{array}{c}
\vdots \\
\begin{array}{c}
\bar{a} \\
\bar{b} \\
\end{array}
\end{array}
\end{array} \right\}$, we obtain the system of formal power series of trees

$$F(\mathcal{P}, \emptyset) = 1 + a\bar{o} [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \emptyset)] + a\bar{o} [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \{b\})]$$
$$- a\bar{o} [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \{b\})] + b\bar{o} [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \emptyset)] ,$$

$$F(\mathcal{P}, \{a\}) = 1 + b\bar{o} [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \emptyset)] ,$$

$$F(\mathcal{P}, \{b\}) = 1 + a\bar{o} [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \emptyset)] + a\bar{o} [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \{b\})]$$
$$- a\bar{o} [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \{b\})] .$$

This leads to the system of generating series

$$F(\mathcal{P}, \emptyset) = z + F(\mathcal{P}, \{a\}) F(\mathcal{P}, \emptyset) + F(\mathcal{P}, \emptyset) F(\mathcal{P}, \{b\})$$
$$- F(\mathcal{P}, \{a\}) F(\mathcal{P}, \{b\}) + F(\mathcal{P}, \emptyset) F(\mathcal{P}, \emptyset) ,$$

$$F(\mathcal{P}, \{a\}) = z + F(\mathcal{P}, \emptyset) F(\mathcal{P}, \emptyset) ,$$

$$F(\mathcal{P}, \{b\}) = z + F(\mathcal{P}, \{a\}) F(\mathcal{P}, \emptyset) + F(\mathcal{P}, \emptyset) F(\mathcal{P}, \{b\})$$
$$- F(\mathcal{P}, \{a\}) F(\mathcal{P}, \{b\}) .$$
### System of equations

**— Example —**

For $\mathcal{P} := \left\{ \begin{array}{c} \begin{array}{c} a \\ b \end{array} \end{array} \right\}$, we obtain the system of formal power series of trees

\[
F(\mathcal{P}, \emptyset) = 1 + a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \emptyset)] + a \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \{b\})]
- a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \{b\})] + b \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \emptyset)],
\]

\[
F(\mathcal{P}, \{a\}) = 1 + b \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \emptyset)],
\]

\[
F(\mathcal{P}, \{b\}) = 1 + a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \emptyset)] + a \circ [F(\mathcal{P}, \emptyset), F(\mathcal{P}, \{b\})]
- a \circ [F(\mathcal{P}, \{a\}), F(\mathcal{P}, \{b\})].
\]

This leads to the system of generating series

\[
F(\mathcal{P}, \emptyset) = z + F(\mathcal{P}, \{a\})F(\mathcal{P}, \emptyset) + F(\mathcal{P}, \emptyset)F(\mathcal{P}, \{b\})
- F(\mathcal{P}, \{a\})F(\mathcal{P}, \{b\}) + F(\mathcal{P}, \emptyset)F(\mathcal{P}, \emptyset),
\]

\[
F(\mathcal{P}, \{a\}) = z + F(\mathcal{P}, \emptyset)F(\mathcal{P}, \emptyset),
\]

\[
F(\mathcal{P}, \{b\}) = z + F(\mathcal{P}, \{a\})F(\mathcal{P}, \emptyset) + F(\mathcal{P}, \emptyset)F(\mathcal{P}, \{b\})
- F(\mathcal{P}, \{a\})F(\mathcal{P}, \{b\}).
\]

As a consequence, $F(\mathcal{P}, \emptyset)$ satisfies

\[
z - F(\mathcal{P}, \emptyset) + (2 + z)F(\mathcal{P}, \emptyset)^2 - F(\mathcal{P}, \emptyset)^3 + F(\mathcal{P}, \emptyset)^4 = 0.
\]
Operads and presentations

Let $\mathcal{O}$ be an operad. A congruence of $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ preserving the arities and such that $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all $i \in [|x|]$. 

— Example —

The operad $\text{Motz}$ admits the presentation $(G, \equiv)$ where $G := \{1, 2\}$ and $\equiv$ is the smallest operad congruence satisfying $\circ_1 \equiv \circ_2$, $\circ_1 \equiv \circ_3$, $\circ_1 \equiv \circ_3$. 
Operads and presentations

Let $\mathcal{O}$ be an operad. A congruence of $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ preserving the arities and such that $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all $i \in [\lvert x \rvert]$.

A presentation of $\mathcal{O}$ is a pair $(\mathcal{G}, \equiv)$ such that $\mathcal{G}$ is an alphabet and $\equiv$ is a congruence of $\mathcal{O}$ satisfying

$$\mathcal{O} \simeq S(\mathcal{G})/\equiv.$$
Operads and presentations

Let $\mathcal{O}$ be an operad. A congruence of $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ preserving the arities and such that $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all $i \in [\lvert x \rvert]$.

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$$\mathcal{O} \simeq S(\mathcal{G})/\equiv.$$

— Example —

The operad $\textbf{Motz}$ admits the presentation $(\mathcal{G}, \equiv)$ where

$$\mathcal{G} := \{ \begin{tikzpicture}[baseline=-0.5ex]
    \draw[thick] (-0.1,0) -- (0,0);
    \draw[thick] (0,0) -- (0.1,0);
\end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    \draw[thick] (-0.1,0) -- (0,0);
    \draw[thick] (0,0) -- (0.1,0);
    \draw[thick] (0,0) -- (0,-0.2);
\end{tikzpicture} \}.$$
Operads and presentations

Let $O$ be an operad. A congruence of $O$ is an equivalence relation $\equiv$ on $O$ preserving the arities and such that $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all $i \in [|x|]$.

A presentation of $O$ is a pair $(G, \equiv)$ such that $G$ is an alphabet and $\equiv$ is a congruence of $O$ satisfying

$$O \simeq S(G)/\equiv.$$ 

— Example —

The operad $\text{Motz}$ admits the presentation $(G, \equiv)$ where

$$G := \{ \circ \circ, \circ \circ \circ \circ \}$$

and $\equiv$ is the smallest operad congruence satisfying

$$\begin{align*}
\circ \circ \circ \circ_1 \circ \circ & \equiv \circ \circ \circ \circ_2 \circ \circ, \\
\circ \circ_1 \circ \circ & \equiv \circ \circ_2 \circ \circ, \\
\circ \circ_1 \circ \circ_1 & \equiv \circ \circ_3 \circ \circ, \\
\circ \circ_1 \circ \circ_1 & \equiv \circ \circ_3 \circ \circ.
\end{align*}$$
Let $\mathcal{O}$ be an operad admitting a presentation $(\mathcal{G}, \equiv)$. 

— Example — The set $B$, described as the set of $G$-trees avoiding $P_B := \{\circ 1, \circ 1, \circ 1, \circ 1\}$, is a basis of Motz. 

Rewrite systems on $G$-trees are good tools to compute bases ($\mathcal{G}$, $\equiv$).
Operads and patterns

Let \( \mathcal{O} \) be an operad admitting a presentation \((\mathcal{G}, \equiv)\).

A basis of \( \mathcal{O} \) is a subset \( B \) of \( S(\mathcal{G}) \) such that for any \( [t]_\equiv \in S(\mathcal{G})/\equiv \), there exists a unique \( s \in [t]_\equiv \cap B \).
Operads and patterns

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathcal{G}, \equiv)$.

A basis of $\mathcal{O}$ is a subset $\mathcal{B}$ of $\mathcal{S}(\mathcal{G})$ such that for any $[t]_{\equiv} \in \mathcal{S}(\mathcal{G})/\equiv$, there exists a unique $s \in [t]_{\equiv} \cap \mathcal{B}$.

In most cases, $\mathcal{B}$ can be described as set of $\mathcal{G}$-trees avoiding a subset $\mathcal{P}_\mathcal{B}$ of $\mathcal{S}(\mathcal{G})$. 

— Example —

The set $\mathcal{B}$, described as the set of $\mathcal{G}$-trees avoiding $P_\mathcal{B} := \{ \circ_1, \circ_1, \circ_1, \circ_1 \}$, is a basis of Motz.

Rewrite systems on $\mathcal{G}$-trees are good tools to compute bases (we find terminating and confluent orientations $\Rightarrow$ of $\equiv$).
Operads and patterns

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathcal{G}, \equiv)$.

A **basis** of $\mathcal{O}$ is a subset $\mathcal{B}$ of $S(\mathcal{G})$ such that for any $[t]_\equiv \in S(\mathcal{G})/_\equiv$, there exists a unique $s \in [t]_\equiv \cap \mathcal{B}$.

In most cases, $\mathcal{B}$ can be described as set of $\mathcal{G}$-trees avoiding a subset $\mathcal{P}_\mathcal{B}$ of $S(\mathcal{G})$.

--- Example ---

The set $\mathcal{B}$, described as the set of $\mathcal{G}$-trees avoiding

$$\mathcal{P}_\mathcal{B} := \left\{ \text{\includegraphics[width=2cm]{example1.png}, \includegraphics[width=2cm]{example2.png}, \includegraphics[width=2cm]{example3.png}, \includegraphics[width=2cm]{example4.png}} \right\},$$

is a basis of $\textbf{Motz}$.
Let $\mathcal{O}$ be an operad admitting a presentation $(\mathcal{G}, \equiv)$.

A basis of $\mathcal{O}$ is a subset $\mathcal{B}$ of $\mathcal{S}(\mathcal{G})$ such that for any $[t]_\equiv \in \mathcal{S}(\mathcal{G})/\equiv$, there exists a unique $s \in [t]_\equiv \cap \mathcal{B}$.

In most cases, $\mathcal{B}$ can be described as set of $\mathcal{G}$-trees avoiding a subset $\mathcal{P}_\mathcal{B}$ of $\mathcal{S}(\mathcal{G})$.

— Example —

The set $\mathcal{B}$, described as the set of $\mathcal{G}$-trees avoiding

$$
\mathcal{P}_\mathcal{B} := \left\{ \begin{array}{c}
\circlearrowright_1 \circlearrowleft_1 \circlearrowright_1, \quad \circlearrowright_1 \circlearrowleft_1 \circlearrowright_1, \quad \circlearrowright_1 \circlearrowleft_1 \circlearrowright_1, \quad \circlearrowright_1 \circlearrowleft_1 \circlearrowright_1
\end{array} \right\},
$$

is a basis of $\textbf{Motz}$.

**Rewrite systems** on $\mathcal{G}$-trees are good tools to compute bases (we find terminating and confluent orientations $\Rightarrow$ of $\equiv$).
Operads and enumeration

Let $X$ be a family of combinatorial objects we want to enumerate.

The approach using operads consists in

1. endowing $X$ with the structure of an operad $O_X$;

---

Example

To enumerate Motzkin paths (w.r.t. their sizes), we consider their operad structure $\operatorname{Motz}$. Let $a := \emptyset$, $c := \epsilon$, and $P := \{a, a, c, a, a, c, c\}$. We have $F(P, \emptyset) = z + zF(P, \{a, c\}) + zF(P, \emptyset)^2$, so that the generating series of Motzkin paths satisfies $F(P, \emptyset) = z + zF(P, \emptyset) + zF(P, \emptyset)^2$. 

---
Operads and enumeration

Let $X$ be a family of combinatorial objects we want enumerate.

The approach using operads consists in

1. endowing $X$ with the structure of an operad $O_X$;
2. exhibiting a presentation $(\mathcal{G}, \equiv)$ of $O_X$ and a basis $B$;

— Example —

To enumerate Motzkin paths (w.r.t. their sizes), we consider their operad structure $\text{Motz}$.

Let $a := \{\}$, $c := \{\}$, and $P := \{a, ca, aca, cac\}$.

We have $F(P, \emptyset) = \bar{a} \circ [F(P, \{a, c\}), F(P, \emptyset)] + \bar{c} \circ [F(P, \{a, c\}), F(P, \emptyset), F(P, \emptyset)]$,

so that the generating series of Motzkin paths satisfies

$$F(P, \emptyset) = z + z F(P, \emptyset) + z F(P, \emptyset)^2.$$
Operads and enumeration

Let $X$ be a family of combinatorial objects we want enumerate.

The approach using operads consists in

1. endowing $X$ with the structure of an operad $O_X$;
2. exhibiting a presentation $(\mathcal{G}, \equiv)$ of $O_X$ and a basis $B$;
3. computing the series $F(P_B, \emptyset)$ where $P_B$ is a set of $\mathcal{G}$-trees satisfying $A(P_B) = B$.

— Example —

To enumerate Motzkin paths (w.r.t. their sizes), we consider their operad structure $\text{Motz}$.

Let $a := a$, $c := c$, and $P := \{a a c a a c c\}$. We have

\[ F(P_B, \emptyset) = a \bar{\phi} \circ \left[ F(P_B, \{a, c\}) , F(P_B, \emptyset) \right] + c \bar{\phi} \circ \left[ F(P_B, \{a, c\}) , F(P_B, \emptyset), F(P_B, \emptyset) \right] \]

so that, the generating series of Motzkin paths satisfies

\[ F(P_B, \emptyset) = z + z F(P_B, \emptyset) + z F(P_B, \emptyset)^2. \]
Operads and enumeration

Let $X$ be a family of combinatorial objects we want to enumerate.

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1. endowing $X$ with the structure of an operad $O_X$;
2. exhibiting a presentation $(\mathcal{G}, \equiv)$ of $O_X$ and a basis $B$;
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— Example —

To enumerate Motzkin paths (w.r.t. their sizes), we consider their operad structure $\text{Motz}$. 

Let $a := $, $c := $, and $P := \{a, c, a, c, c\}$. We have $F(P, \emptyset) = \bar{a} \cdot [F(P, \{a, c\}), F(P, \emptyset)] + \bar{c} \cdot [F(P, \{a, c\}), F(P, \emptyset), F(P, \emptyset)]$, so that the generating series of Motzkin paths satisfies $F(P, \emptyset) = \bar{z} + \bar{z} F(P, \emptyset) + \bar{z} F(P, \emptyset)^2$. 

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Operads and enumeration

Let $X$ be a family of combinatorial objects we want to enumerate.

The approach using operads consists in

1. endowing $X$ with the structure of an operad $O_X$;
2. exhibiting a presentation $(\mathcal{G}, \equiv)$ of $O_X$ and a basis $B$;
3. computing the series $F(\mathcal{P}_B, \emptyset)$ where $\mathcal{P}_B$ is a set of $\mathcal{G}$-trees satisfying $A(\mathcal{P}_B) = B$.

— Example —

To enumerate Motzkin paths (w.r.t. their sizes), we consider their operad structure $\text{Motz}$.

Let $a := \circ\circ$, $c := \circ\circ\circ$, and $\mathcal{P} := \left\{ \begin{array}{c}
\begin{array}{c}
\text{a} \quad \text{a} \\
\text{c} \quad \text{c}
\end{array}
\end{array}\right\}$. 
Operads and enumeration

Let $X$ be a family of combinatorial objects we want to enumerate.

The approach using operads consists in

1. endowing $X$ with the structure of an operad $O_X$;
2. exhibiting a presentation $(G, \equiv)$ of $O_X$ and a basis $B$;
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— Example —

To enumerate Motzkin paths (w.r.t. their sizes), we consider their operad structure $\text{Motz}$.

Let $a := \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot
\end{array}
\end{array}$, $c := \begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot
\end{array}
\end{array}$, and $P := \left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot
\end{array}
\end{array}
\end{array}\right\}$.

We have

$$F(P, \emptyset) = 1 + a \circ [F(P, \{a, c\}), F(P, \emptyset)] + c \circ [F(P, \{a, c\}), F(P, \emptyset), F(P, \emptyset)],$$

$$F(P, \{a, c\}) = 1,$$

so that, the generating series of Motzkin paths satisfies

$$F(P, \emptyset) = z + zF(P, \emptyset) + zF(P, \emptyset)^2.$$
Outline

Generation
Context-free grammars

Let $A = V \sqcup T$ be a set where $V$ is a set of variables and $T$ is a set of terminal symbols.
Context-free grammars

Let $A = V \sqcup T$ be a set where $V$ is a set of variables and $T$ is a set of terminal symbols.

A rule is a pair $(x, v) \in V \times A^*$. A set $R$ of rules specifies a rewrite rule $\rightarrow$ on $A^*$ by setting

$$u x w \rightarrow u v w$$

for any $u, w \in A^*$ provided that $(x, v) \in R$. 

Context-free grammars

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for any $u, w \in A^*$ provided that $(x, v) \in R$.

— Example —

Let $V := \{x, y\}$, $T := \{a, b, c\}$, and $R := \{(x, b), (x, xay), (y, ac)\}$.

We have

$$bxx \rightarrow bxayx \rightarrow bbayx \rightarrow bbaacx.$$
Regular tree grammars

Let $V$ be a set of variables and $T$ be an alphabet of terminal symbols.
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A $(V, T)$-tree is a planar rooted tree where internal nodes are labeled on $T$ and leaves are labeled on $V$.
Regular tree grammars

Let $V$ be a set of variables and $T$ be an alphabet of terminal symbols.

A $(V, T)$-tree is a planar rooted tree where internal nodes are labeled on $T$ and leaves are labeled on $V$.

A rule is a pair $(x, t)$ where $x \in V$ and $t$ is a $(V, T)$-tree. A set $\mathcal{R}$ of rules specifies a rewrite rule $\rightarrow$ on the set of all $(V, T)$-trees by setting

$$
\begin{array}{c}
\triangleleft \\
x
\end{array} \rightarrow
\begin{array}{c}
\triangleleft \\
t
\end{array}
$$

for any $(V, T)$-tree $s$ having a leaf labeled by $x$, provided that $(x, t) \in \mathcal{R}$. 

Regular tree grammars

Let $V$ be a set of variables and $T$ be an alphabet of terminal symbols.

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A rule is a pair $(x, t)$ where $x \in V$ and $t$ is a $(V, T)$-tree. A set $R$ of rules specifies a rewrite rule $\rightarrow$ on the set of all $(V, T)$-trees by setting

$$
\begin{array}{c}
\textcircled{s} \\
\text{x}
\end{array} 
\rightarrow 
\begin{array}{c}
\textcircled{s} \\
\text{t}
\end{array}
$$

for any $(V, T)$-tree $s$ having a leaf labeled by $x$, provided that $(x, t) \in R$.

— Example —

Let $V := \{x, y\}$, $T := \{a, b\}$ where $|a| := 1$, $|b| := 2$, and $R := \{ (x, a y), (y, b x y) \}$.

We have

$$
\begin{array}{c}
\text{x} \\
\text{a} \\
\text{x}
\end{array} 
\rightarrow 
\begin{array}{c}
\text{a} \\
\text{a} \\
\text{a}
\end{array} 
\rightarrow 
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{a}
\end{array} 
\rightarrow 
\begin{array}{c}
\text{b} \\
\text{x} \\
\text{x}
\end{array} 
\rightarrow 
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{x}
\end{array} 
\rightarrow 
\begin{array}{c}
\text{b} \\
\text{x} \\
\text{x}
\end{array} 
\rightarrow 
\begin{array}{c}
\text{a} \\
\text{a} \\
\text{x}
\end{array}.
$$
General generation

Objectives:

- Introduce generating systems for any kind of combinatorial objects;
- Retrieve the generation of words and of trees as special cases;
- Develop a toolbox for the enumeration of combinatorial objects.
General generation

Objectives:

▶ Introduce generating systems for any kind of combinatorial objects;
▶ Retrieve the generation of words and of trees as special cases;
▶ Develop a toolbox for the enumeration of combinatorial objects.

— Key idea —

Use colored operads, where

▶ colors play the role of variables and terminal symbols;
▶ Formal series on colored operad and their operations support enumeration.
Colored operads

Colored operads are algebraic structures formalizing the notion of partial operations and their composition.
Colored operads

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A **colored operad** is a quadruplet $(\mathcal{C}, C, \circ_i, 1_c)$ where

1. $\mathcal{C}$ is a finite set of **colors**;
Colored operads

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1. \(\mathcal{C}\) is a finite set of colors;

2. \(\mathcal{C}\) is a set of the form

\[
\mathcal{C} := \bigsqcup_{(a,u) \in \mathcal{C} \times \mathcal{C}^+} \mathcal{C}(a, u);
\]

where \(\mathcal{C}(a, u)\) is the set of operations obtained by replacing the \(i\)th letter of \(u\) by \(v\);
Colored operads

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2. \(C\) is a set of the form

\[
C := \bigsqcup_{(a,u) \in \mathcal{C} \times \mathcal{C}^+} C(a, u);
\]

3. \(\circ_i\) is a map, called partial composition map,

\[
\circ_i : C(a, u) \times C(u_i, v) \to C(a, u \circ_i v), \quad 1 \leq i \leq |u|,
\]

where \(u \circ_i v\) is the word obtained by replacing the \(i\)th letter of \(u\) by \(v\);
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   \]
   where \(u \circ_i v\) is the word obtained by replacing the \(i\)th letter of \(u\) by \(v\);

4. for any \(c \in \mathcal{C}\), \(1_c\) is an element of \(C(c, c)\) called \(c\)-colored unit.
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\mathcal{C} := \bigsqcup_{(a, u) \in \mathcal{C} \times \mathcal{C}^+} C(a, u);
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\circ_i : C(a, u) \times C(u_i, v) \rightarrow C(a, u \circ_i v), \quad 1 \leq i \leq |u|,
\]

where \(u \circ_i v\) is the word obtained by replacing the \(i\)th letter of \(u\) by \(v\);

4. for any \(c \in \mathcal{C}\), \(1_c\) is an element of \(C(c, c)\) called \(c\)-colored unit.

This data has to satisfy some axioms, similar to the ones of operads.
Colored operations

Any element $x$ of $C(a, u)$ can be seen as a colored operation

where a color is assigned to the output and to each input of $x$. 
Colored operations

Any element $x$ of $C(a, u)$ can be seen as a colored operation

where a color is assigned to the output and to each input of $x$.

Moreover, the partial composition map requires a condition on the colors:

$$
\circ_i \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
u_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
u|x|
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
u_i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
u|x|
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right) =
\left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
u_1
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
u_i
\end{array}
\begin{array}{c}
\begin{array}{c}
u|x|
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
v_1
\end{array}
\begin{array}{c}
\begin{array}{c}
v|y|
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
v|y|
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
i + |y| - 1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right).
$$
Bud operads

Let $O$ be an operad and $C$ be a set of colors.
Bud operads

Let $\mathcal{O}$ be an operad and $\mathcal{C}$ be a set of colors.

The $\mathcal{C}$-bud operad of $\mathcal{O}$ is the colored operad $\mathbb{B}_\mathcal{C}(\mathcal{O})$ wherein:

- $\mathbb{B}_\mathcal{C}(\mathcal{O})(a, u)$ is the set of all triples $(a, x, u)$ where $x \in \mathcal{O}$ and $(a, u) \in \mathcal{C} \times \mathcal{C}^{\mid x\mid}$.

---

Proposition

For any set of colors $\mathcal{C}$, the construction $\mathcal{O} \mapsto \mathbb{B}_\mathcal{C}(\mathcal{O})$ is a functor from the category of operads to the category of colored operads.
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- The partial composition map is defined by

  $$(a, x, u) \circ_i (u_i, y, v) := (a, x \circ_i y, u \circ_i v)$$

  where $x \circ_i y$ is the partial composition of $\mathcal{O}$.
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  where $x \circ_i y$ is the partial composition of $\mathcal{O}$.

- The colored units are the triples $(c, \mathbb{1}, c)$ where $\mathbb{1}$ is the unit of $\mathcal{O}$. 

— Proposition —

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  where $x \circ_i y$ is the partial composition of $\mathcal{O}$.

- The colored units are the triples $(c, 1, c)$ where $1$ is the unit of $\mathcal{O}$.

— Proposition —

For any set of colors $\mathcal{C}$, the construction $\mathcal{O} \mapsto \mathbf{B}_\mathcal{C}(\mathcal{O})$ is a functor from the category of operads to the category of colored operads.
Examples of bud operads

The elements of $B_C(As)$ are triples $(a, \star_\mu, u)$ where $(a, u) \in C \times C^+$. 

— Example —

In $B_{\{1,2,3\}}(As)$, $(2, \star_4, 3112) \circ_2 (1, \star_3, 233) = (2, \star_6, 323312)$. 

---

---
Examples of bud operads

The elements of $\mathcal{B}_c(\mathcal{A}s)$ are triples $(a, \star|u|, u)$ where $(a, u) \in \mathcal{C} \times \mathcal{C}^+$. 

— Example —

In $\mathcal{B}_{\{1,2,3\}}(\mathcal{A}s)$, $(2, \star_4, 3112) \circ_2 (1, \star_3, 233) = (2, \star_6, 323312)$.

The elements of $\mathcal{B}_c(\mathcal{S}(\mathcal{G}))$ are $\mathcal{C}$-typed $\mathcal{G}$-syntax trees, that are $\mathcal{G}$-trees with colors assigned with the root and with each leaf.

— Example —

$\begin{pmatrix} 2, c, a, 31122 \end{pmatrix} \in \mathcal{B}_{\{1,2,3,4\}}(\mathcal{S}(\{a, c\}))$.

This element is drawn as 

```
    a  
   / \  
  a   c 
    / \  
   311 2
```
Examples of bud operads

The elements of $B_c(\mathsf{As})$ are triples $(a, u|_u, u)$ where $(a, u) \in \mathcal{C} \times \mathcal{C}^+$.

— Example —

$$\in B_{\{1,2,3\}}(\mathsf{As}), (2, 4, 3112) \circ_2 (1, 3, 233) = (2, 6, 323312).$$

The elements of $B_c(\mathsf{S}(\mathcal{G}))$ are $\mathcal{C}$-typed $\mathcal{G}$-syntax trees, that are $\mathcal{G}$-trees with colors assigned with the root and with each leaf.

— Example —

$$\left(2, \begin{array}{c} \text{c} \\ \text{a} \end{array}, 31122 \right) \in B_{\{1,2,3,4\}}(\mathsf{S}(\{\text{a, c}\})).$$

This element is drawn as

— Example —

$$\left(1, \begin{array}{c} \text{a} \\ \text{c} \end{array}, 221222211 \right) \in B_c(\mathsf{Motz}).$$

This element is drawn as

The elements of $B_c(\mathsf{Motz})$ are Motzkin paths having a global color and a color assigned with each point.
A **bud generating system** is a quintuplet $B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$ where

1. $\mathcal{O}$ is an operad, the **ground operad**;
Bud generating systems

A bud generating system is a quintuplet \( B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T) \) where

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Bud generating systems

A bud generating system is a quintuplet $\mathcal{B} := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$ where

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Bud generating systems

A bud generating system is a quintuplet $B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$ where

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2. $\mathcal{C}$ is a set of colors;
3. $\mathcal{R} \subseteq B_\mathcal{C}(\mathcal{O})$ is a set of rules;
4. $a \in \mathcal{C}$ is the initial color;
A **bud generating system** is a quintuplet $B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$ where

1. $\mathcal{O}$ is an operad, the **ground operad**;
2. $\mathcal{C}$ is a set of **colors**;
3. $\mathcal{R} \subseteq \mathcal{B}_e(\mathcal{O})$ is a set of **rules**;
4. $a \in \mathcal{C}$ is the **initial color**;
5. $T \subseteq \mathcal{C}$ is the set of **terminal colors**.

"Bud generating systems"
A bud generating system is a quintuplet $B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$ where

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3. $\mathcal{R} \subseteq B_c(\mathcal{O})$ is a set of rules;
4. $a \in \mathcal{C}$ is the initial color;
5. $T \subseteq \mathcal{C}$ is the set of terminal colors.

Each element $(c, x, u)$ of $\mathcal{R}$ can be thought as rule having $c$ as left member and $u$ as right member.
Generation

The set $\mathcal{R}$ specifies the rewrite rule $\rightarrow$ on $B_c(O)$ by setting

$$x \rightarrow x \circ_i r$$

for any $x \in B_c(O)$, $i \in [|x|]$, and $r \in \mathcal{R}$. This is the derivation relation.
**Generation**

The set $\mathcal{R}$ specifies the rewrite rule $\rightarrow$ on $\mathbf{B}_c(\mathcal{O})$ by setting

$$x \rightarrow x \circ_i r$$

for any $x \in \mathbf{B}_c(\mathcal{O})$, $i \in [|x|]$, and $r \in \mathcal{R}$. This is the derivation relation.

An element $x$ of $\mathbf{B}_c(\mathcal{O})$ is generated by $\mathcal{B}$ if

$$1_\alpha \rightarrow \cdots \rightarrow x$$

and all input colors of $x$ are in $T$. These elements form the language of $\mathcal{B}$. 
Generation

The set $\mathcal{R}$ specifies the rewrite rule $\rightarrow$ on $\mathcal{B}_C(O)$ by setting

$$x \rightarrow x \circ_i r$$

for any $x \in \mathcal{B}_C(O)$, $i \in [|x|]$, and $r \in \mathcal{R}$. This is the derivation relation.

An element $x$ of $\mathcal{B}_C(O)$ is generated by $\mathcal{B}$ if

$$1_a \rightarrow \cdots \rightarrow x$$

and all input colors of $x$ are in $T$. These elements form the language of $\mathcal{B}$.

The set $\mathcal{R}$ specifies also the rewrite rule $\sim$ on $\mathcal{B}_C(O)$ by setting

$$x \sim x \circ [r_1, \ldots, r_{|x|}]$$

for any $x \in \mathcal{B}_C(O)$ and $r_1, \ldots, r_{|x|} \in \mathcal{R}$. This is the synchronous derivation relation.
Generation

The set $\mathcal{R}$ specifies the rewrite rule $\rightarrow$ on $\mathbb{B}_c(O)$ by setting

$$x \rightarrow x \circ_i r$$

for any $x \in \mathbb{B}_c(O)$, $i \in [|x|]$, and $r \in \mathcal{R}$. This is the derivation relation.

An element $x$ of $\mathbb{B}_c(O)$ is generated by $\mathcal{B}$ if

$$1_a \rightarrow \cdots \rightarrow x$$

and all input colors of $x$ are in $T$. These elements form the language of $\mathcal{B}$.

The set $\mathcal{R}$ specifies also the rewrite rule $\rightsquigarrow$ on $\mathbb{B}_c(O)$ by setting

$$x \rightsquigarrow x \circ [r_1, \ldots, r_{|x|}]$$

for any $x \in \mathbb{B}_c(O)$ and $r_1, \ldots, r_{|x|} \in \mathcal{R}$. This is the synchronous derivation relation.

An element $x$ of $\mathbb{B}_c(O)$ is synchronously generated by $\mathcal{B}$ if

$$1_a \rightsquigarrow \cdots \rightsquigarrow x$$

and all input colors of $x$ are in $T$. These elements form the synchronous language of $\mathcal{B}$. 
Generation of particular Motzkin paths

Let the bud generating system $\mathcal{B} := (\text{Motz}, \{1, 2\}, \mathcal{R}, 1, \{1, 2\})$ where

$$\mathcal{R} := \{(1, \circ \circ, 22), (1, \circ \circ \circ, 111)\}.$$
Generation of particular Motzkin paths

Let the bud generating system $\mathcal{B} := (\text{Motz}, \{1, 2\}, \mathcal{R}, 1, \{1, 2\})$ where

$$\mathcal{R} := \{(1, \infty, 22), (1, \circ, 111)\}.$$ 

— Example —

There are in $\mathcal{B}$ the derivations

1. $1 \rightarrow 1$
2. $1 \rightarrow 2$
3. $1 \rightarrow 2$
4. $1 \rightarrow 2$
5. $1 \rightarrow 2$

— Proposition —

There is a one-to-one correspondence between the set of Motzkin paths without consecutive steps and the language of $\mathcal{B}$. These paths are enumerated by $1, 1, 1, 3, 5, 11, 25, 55, 129, 303, 721, 1743, \ldots$ (A104545).
Generation of particular Motzkin paths

Let the bud generating system $B := (\text{Motz}, \{1, 2\}, \mathcal{R}, 1, \{1, 2\})$ where

$$\mathcal{R} := \{(1, \bigcirc, 2), (1, \bigcirc, 111)\}.$$ 

— Example —

There are in $B$ the derivations

$$1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1.$$ 

— Proposition —

There is a one-to-one correspondence between the set of Motkzin paths without consecutive $\bigcirc \bigcirc$ steps and the language of $B$.

These paths are enumerated by

$$1, 1, 1, 3, 5, 11, 25, 55, 129, 303, 721, 1743, \ldots (A104545).$$
Balanced binary trees

A balanced binary tree is a binary tree $t$ such that, for any internal node $u$ of $t$, the height of the left subtree and of the right subtree of $u$ differ by at most 1.

The first balanced binary trees are

```
   ,   a   ,   a   ,   a   ,   a   ,
   \   \   \   \   \   \   \   \   \   \
    ,   a   ,   a   ,   a   ,   a   ,   a   ,   a
    \   \   \   \   \   \   \   \   \   \   \
     ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a
     \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \
      ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a
      \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \   \
       ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a   ,   a
```

(Ans: $A006265$)
Balanced binary trees

A balanced binary tree is a binary tree $t$ such that, for any internal node $u$ of $t$, the height of the left subtree and of the right subtree of $u$ differ by at most 1.

The first balanced binary trees are

```
   ,   ,   ,   ,   ,
  / \ / \ / \ / \ / \ \
 /   /   /   /   /   \
\     \     \     \     \
|     |     |     |     |
```

These trees are enumerated by

$$1, 1, 2, 1, 4, 6, 4, 17, 32, 44, 60, 70, \ldots (A006265).$$
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The first balanced binary trees are

These trees are enumerated by

$$1, 1, 2, 1, 4, 6, 4, 17, 32, 44, 60, 70, \ldots \text{(A006265)}.$$

Their generating series is the specialization $F(x, 0)$ where

$$F(x, y) = x + F\left(x^2 + 2xy, x\right).$$
Generation of balanced binary trees

Let the bud generating system $\mathcal{B} := (\mathcal{S}(\mathcal{G}), \{1, 2\}, \mathcal{R}, 1, \{1\})$ where $\mathcal{G} := \mathcal{G}(2) := \{a\}$ and

$$\mathcal{R} := \left\{ \left(1, \begin{array}{c} a \\ a \end{array}, 11\right), \left(1, \begin{array}{c} a \\ a \end{array}, 12\right), \left(1, \begin{array}{c} a \\ a \end{array}, 21\right), (2, 1) \right\}.$$
Generation of balanced binary trees

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**— Example —**

There are in $\mathcal{B}$ the derivations

$$1 \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{1} a \xrightarrow{11}. $$

**— Proposition —**

There is a one-to-one correspondence between the set of balanced binary trees and the synchronous language of $\mathcal{B}$.
Some properties

— Proposition —

For any proper context-free grammar $G$, there exists a bud generating system $B := (A, C, R, a, T)$ such that the language generated by $G$ is in one-to-one correspondence with the language of $B$. 
Some properties

— Proposition —

For any proper context-free grammar $G$, there exists a bud generating system $\mathcal{B} := (\text{As}, \mathcal{C}, \mathcal{R}, a, T)$ such that the language generated by $G$ is in one-to-one correspondence with the language of $\mathcal{B}$.

— Proposition —

For any regular tree grammar $G$, there exists a bud generating system $\mathcal{B} := (S(\mathcal{G}), \mathcal{C}, \mathcal{R}, a, T)$ such that the language generated by $G$ is in one-to-one correspondence with the language of $\mathcal{B}$. 
Some properties

— Proposition —
For any proper context-free grammar $G$, there exists a bud generating system $B := (A, S, C, R, a, T)$ such that the language generated by $G$ is in one-to-one correspondence with the language of $B$.

— Proposition —
For any regular tree grammar $G$, there exists a bud generating system $B := (S(G), S, C, R, a, T)$ such that the language generated by $G$ is in one-to-one correspondence with the language of $B$.

— Proposition —
For any bud generating system $B$, the synchronous language of $B$ is a subset of the language of $B$. 
Random generation

For any $c \in C$, let $R_c$ be the subset of $R$ of the elements having $c$ as output color.
Random generation

For any $c \in \mathcal{C}$, let $\mathcal{R}_c$ be the subset of $\mathcal{R}$ of the elements having $c$ as output color.

Algorithm **RBS**:  

1. **Input:**
   1. a bud generating system $\mathcal{B} := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$;
   2. An integer $k \geq 0$.

2. **Output:** an element of the synchronous language of $\mathcal{B}$.

1. Let $x := 1_a$;

2. Repeat $k$ times:
   2.1 For any $i \in [|x|]$, pick $y_i$ uniformly at random in $\mathcal{R}_c$ where $c$ is the $i$th input color of $x$;
   2.2 Set $x := x \circ [y_1, \ldots, y_{|x|}]$;

3. If all input colors of $x$ belong to $T$:
   3.1 Return $x$;

4. Otherwise:
   4.1 Return failure.

**Proposition**

If $\mathcal{B} = (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T)$ is synchronously unambiguous, the RBS is a uniform random generator of the elements of the synchronous language of $\mathcal{B}$.
Random generation

For any \( c \in \mathcal{C} \), let \( \mathcal{R}_c \) be the subset of \( \mathcal{R} \) of the elements having \( c \) as output color.

**Algorithm RBS:**

- **Input:**
  1. a bud generating system \( B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T) \);
  2. An integer \( k \geq 0 \).

- **Output:** an element of the synchronous language of \( B \).

1. Let \( x := 1_a \);
2. Repeat \( k \) times:
   2.1 For any \( i \in [|x|] \), pick \( y_i \) uniformly at random in \( \mathcal{R}_c \) where \( c \) is the \( i \)th input color of \( x \);
   2.2 Set \( x := x \circ [y_1, \ldots, y_{|x|}] \);
3. If all input colors of \( x \) belong to \( T \):
   3.1 Return \( x \);
4. Otherwise:
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--- Proposition ---

If \( B = (\mathcal{O}, \mathcal{C}, \mathcal{R}, a, T) \) is synchronously unambiguous, the RBS is a uniform random random generator of the elements of the synchronous language of \( B \).