Clones of pigmented words and realizations of varieties of monoids

Samuele Giraudo

LIGM, Université Gustave Eiffel

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Outline

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1. Varieties, clones, and realizations

Terms

A signature is a graded set $\mathfrak{G} := \bigsqcup_{n \geqslant 0} \mathfrak{G}(n)$ wherein each $\mathsf{g} \in \mathfrak{G}(n)$ is an constant of arity n.

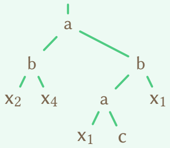
A $\mathfrak{G}$-term is either

- a variable x from the set $\mathbb{X}_k := \{\mathsf{x}_1, \dots, \mathsf{x}_k\}$ for a $k \geqslant 0$;
- or a pair $(\mathsf{g}, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ where $\mathsf{g} \in \mathfrak{G}(n)$ and each $\mathfrak{t}_i$ is a $\mathfrak{G}$-term.

  This pair is denoted by $\mathsf{g}[\mathfrak{t}_1, \dots, \mathfrak{t}_n]$.

Let $\mathfrak{T}(\mathfrak{G}) := \bigsqcup_{n \geqslant 0} \mathfrak{T}(\mathfrak{G})(n)$ where $\mathfrak{T}(\mathfrak{G})(n)$ is the set of the $\mathfrak{G}$-terms having all variables in $\mathbb{X}_n$.

– Example –



This is the tree representation of the $\mathfrak{G}$-term

$$\mathsf{a}[\mathsf{b}[\mathsf{x}_2, \mathsf{x}_4], \mathsf{b}[\mathsf{a}[\mathsf{x}_1, \mathsf{c}[]], \mathsf{x}_1]]$$

of $\mathfrak{T}(\mathfrak{G})(6)$ where

$$\mathfrak{G} := \mathfrak{G}(0) \sqcup \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$$

with $\mathfrak{G}(0) := \{\mathsf{c}\}$, $\mathfrak{G}(2) := \{\mathsf{a}, \mathsf{b}\}$, and $\mathfrak{G}(3) := \{\mathsf{d}\}$.

Varieties

A variety is a pair $(\mathfrak{G}, \mathcal{R})$ where $\mathfrak{G}$ is a signature and $\mathcal{R}$ is an equivalence relation on $\mathfrak{T}(\mathfrak{G})$.

An $\mathcal{R}$-equation is a pair $(\mathfrak{t}, \mathfrak{t}')$ of $\mathfrak{G}$-terms such that $\mathfrak{t} \; \mathcal{R} \; \mathfrak{t}'$.

– Examples –

- The variety of idempotent semigroups is the pair $(\mathfrak{G}, \mathcal{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$, and $\mathcal{R}$ is the finest equivalence relation satisfying

$$\star[\star[x_1, x_2], x_3] \; \mathcal{R} \; \star[x_1, \star[x_2, x_3]] \qquad \text{and} \qquad \star[x_1, x_1] \; \mathcal{R} \; x_1.$$

- The variety of distributive lattices is the pair $(\mathfrak{G}, \mathcal{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\wedge, \vee\}$, and $\mathcal{R}$ is the finest equivalence relation satisfying

$$\begin{aligned}
\wedge[\wedge[x_1, x_2], x_3] \; \mathcal{R} \; \wedge[x_1, \wedge[x_2, x_3]], &\qquad \vee[\vee[x_1, x_2], x_3] \; \mathcal{R} \; \vee[x_1, \vee[x_2, x_3]], \\
\wedge[x_1, x_2] \; \mathcal{R} \; \wedge[x_2, x_1], &\qquad \vee[x_1, x_2] \; \mathcal{R} \; \vee[x_2, x_1], \\
\wedge[x_1, \vee[x_1, x_2]] \; \mathcal{R} \; x_1, &\qquad \vee[x_1, \wedge[x_1, x_2]] \; \mathcal{R} \; x_1, \\
\vee[x_1, \wedge[x_2, x_3]] \; \mathcal{R} \; \wedge[\vee[x_1, x_2], \vee[x_1, x_3]], &\qquad \wedge[x_1, \vee[x_2, x_3]] \; \mathcal{R} \; \vee[\wedge[x_1, x_2], \wedge[x_1, x_3]].
\end{aligned}$$

Algebras over a variety

Let $\mathfrak{G}$ be a signature and $\mathcal{A}$ be a set. An interpretation of $\mathfrak{G}$ on $\mathcal{A}$ is a map

$$\mathfrak{op} : \mathfrak{G}(n) \to (\mathcal{A}^n \to \mathcal{A}).$$

Such map associates with each $\mathrm{g} \in \mathfrak{G}(n)$ an **operation** on $\mathcal{A}$ admitting n inputs.

– Example –

The map $\mathfrak{op}$ defined by $\mathfrak{op}(\star)(x_1, x_2) := \max\{x_1, x_2\}$ is an interpretation of the signature $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ on the set $\mathcal{A} := \mathbb{Z}$.

For any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})(n)$, $\mathfrak{op}(\mathfrak{t})$ is the operation of arity n obtained by **composing** the operations carried by the constants forming $\mathfrak{t}$.

– Example –

By considering the previous interpretation, for $\mathfrak{t} := \star[\mathrm{x}_1, \star[\star[\mathrm{x}_2, \mathrm{x}_3], \mathrm{x}_2]]$, $\mathfrak{op}(\mathfrak{t})$ is the operation of arity 3 satisfying $\mathfrak{op}(\mathfrak{t})(x_1, x_2, x_3) := \max\{x_1, \max\{\max\{x_2, x_3\}, x_2\}\}$.

An algebra over a variety $(\mathfrak{G}, \mathcal{R})$ on a set $\mathcal{A}$ is a pair $(\mathcal{A}, \mathfrak{op})$ such that $\mathfrak{op}$ is an interpretation of $\mathfrak{G}$ on $\mathcal{A}$, and, for any $\mathcal{R}$-equation $(\mathfrak{t}, \mathfrak{t}')$, $\mathfrak{op}(\mathfrak{t})$ and $\mathfrak{op}(\mathfrak{t}')$ are the **same operation**.

Equivalence of terms

Given a variety $\mathcal{V} := (\mathfrak{G}, \mathcal{R})$, let $\equiv_{\mathcal{R}}$ be the **equivalence relation** such that for any $n \geqslant 0$ and $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}(\mathfrak{G})(n)$, $\mathfrak{t} \equiv_{\mathcal{R}} \mathfrak{t}'$ if for any algebra $(\mathcal{A}, \mathfrak{op})$ over $\mathcal{V}$, $\mathfrak{op}(\mathfrak{t})$ and $\mathfrak{op}(\mathfrak{t}')$ are the same operations.

– Examples –

- In the variety of idempotent semigroups,

$$\star[x_1, \star[\star[x_2, x_1], x_2]] \equiv_{\mathcal{R}} \star[x_1, x_2].$$

- In the variety of distributive lattices,

$$\wedge[\wedge[x_1, x_1], x_2] \equiv_{\mathcal{R}} \wedge[x_2, x_1].$$

– Questions –

1. Is the $\equiv_{\mathcal{R}}$-equivalence decidable in a variety $\mathcal{V}$? This is the word problem [**Baader, Nipkow**, 1998].
2. If it is the case, design an (efficient) **algorithm** to decide if two terms are $\equiv_{\mathcal{R}}$-equivalent.

Idempotent semigroups

To decide the $\equiv_{\mathcal{R}}$-equivalence in the **variety of idempotent semigroups**, consider the following algorithm, associating with any term $\mathfrak{t}$ a word $\mathbb{P}(\mathfrak{t})$ on positive integers:

1. set u as the indexes of the variables appearing $\mathfrak{t}$, from left to right;
2. iteratively apply while possible the following transformations on u in any order:
   a. replace a factor $w.w$ by w;
   b. replace a factor $v.a.w$ by $v.w$ if $a \in \mathrm{Alph}(v)$ and $\mathrm{Alph}(v) = \mathrm{Alph}(w)$.

– Theorem [Siekmann, Szabó, 1982] [Klíma, Korbelář, Polák, 2011] –

Two terms $\mathfrak{t}$ and $\mathfrak{t}'$ are $\equiv_{\mathcal{R}}$-equivalent in the variety of the idempotent semigroups if and only if $\mathbb{P}(\mathfrak{t}) = \mathbb{P}(\mathfrak{t}')$.

– Example –

Let $\mathfrak{t} := \star[\star[\mathtt{x}_1, \star[\mathtt{x}_2, \star[\mathtt{x}_3, \mathtt{x}_3]]], \star[\star[\mathtt{x}_2, \mathtt{x}_1], \star[\mathtt{x}_2, \mathtt{x}_3]]]$.
We have $u = 12332123$. Since $12\underline{3.3}2123 \rightsquigarrow \underline{123.2.123} \rightsquigarrow \underline{123.123} \rightsquigarrow 123$, $\mathbb{P}(\mathfrak{t}) = 123$.

Let $\mathfrak{t}' := \star[\star[\mathtt{x}_1, \star[\mathtt{x}_2, \mathtt{x}_2]], \mathtt{x}_3]$.
We have and $u' = 1223$. Since $1\underline{2.2}3 \rightsquigarrow 123$, $\mathbb{P}(\mathfrak{t}') = 123$.

Therefore, $\mathfrak{t} \equiv_{\mathcal{R}} \mathfrak{t}'$.

Clones

Abstract clones provide an **algebraic and combinatorial framework** to study varieties.

An abstract clone [**Cohn**, 1965] is a triple $(\mathcal{C}, [\ ], \mathbb{1}_{i,n})$ where

- $\mathcal{C}$ is a graded set $\mathcal{C} = \bigsqcup_{n \geqslant 0} \mathcal{C}(n)$;
- $[\ ]$ is a map $-[-, \ldots, -] : \mathcal{C}(n) \times \mathcal{C}(m)^n \to \mathcal{C}(m)$ called superposition map;
- for each $n \geqslant 1$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is an element of $\mathcal{C}(n)$ called projection.

The following relations have to hold:

- for all $x_i \in \mathcal{C}(m)$,
$$\mathbb{1}_{i,n}[x_1, \ldots, x_n] = x_i;$$
- for all $x \in \mathcal{C}(n)$,
$$x[\mathbb{1}_{1,n}, \ldots, \mathbb{1}_{n,n}] = x;$$
- for all $x \in \mathcal{C}(n)$, $y_i \in \mathcal{C}(m)$, and $z_j \in \mathcal{C}(k)$,
$$(x[y_1, \ldots, y_n])[z_1, \ldots, z_m] = x[y_1[z_1, \ldots, z_m], \ldots, y_n[z_1, \ldots, z_m]].$$

Free clones

Let $\mathfrak{G}$ be a signature.

The free clone on $\mathfrak{G}$ is the clone $(\mathfrak{T}(\mathfrak{G}), [\ ], \mathbb{1}_{i,n})$ where

- for any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})(n)$, $\mathfrak{t}[\mathfrak{t}'_1, \ldots, \mathfrak{t}'_n]$ is the $\mathfrak{G}$-term obtained by replacing, for all $i \in [n]$, the occurrences of x_i in $\mathfrak{t}$ by a copy of $\mathfrak{t}'_i$;
- $\mathbb{1}_{i,n}$ is the $\mathfrak{G}$-term x_i of arity n.

– Example –

By setting $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ where $\mathfrak{G}(2) := \{\mathrm{a}, \mathrm{b}\}$ and $\mathfrak{G}(3) := \{\mathrm{d}\}$, in the free clone $\mathfrak{T}(\mathfrak{G})$, we have

$$\mathrm{d}[\mathrm{x}_3, \mathrm{x}_1, \mathrm{a}[\mathrm{x}_3, \mathrm{x}_1]] \left[\mathrm{a}[\mathrm{a}[\mathrm{x}_1, \mathrm{x}_2], \mathrm{x}_2], \quad \mathrm{b}[\mathrm{x}_2, \mathrm{x}_2], \quad \mathrm{b}[\mathrm{x}_2, \mathrm{x}_1] \right]$$
$$= \mathrm{d}[\ \mathrm{b}[\mathrm{x}_2, \mathrm{x}_1], \quad \mathrm{a}[\mathrm{a}[\mathrm{x}_1, \mathrm{x}_2], \mathrm{x}_2], \quad \mathrm{a}[\ \mathrm{b}[\mathrm{x}_2, \mathrm{x}_1], \quad \mathrm{a}[\mathrm{a}[\mathrm{x}_1, \mathrm{x}_2], \mathrm{x}_2]\ ]\ ].$$

The free clone satisfies the usual **universal property** of free structures in the category of clones.

Clone realizations of varieties

A clone congruence of a clone $\mathcal{C}$ is an equivalence relation $\equiv$ on $\mathcal{C}$ compatible with the superposition map, that is, for any $x, x' \in \mathcal{C}(n)$ and $y_1, y_1', \dots, y_n, y_n' \in \mathcal{C}(m)$, if $x \equiv x'$ and $y_1 \equiv y_1'$, $\dots$, $y_n \equiv y_n'$, then

$$x[y_1, \dots, y_n] \equiv x'[y_1', \dots, y_n'].$$

– Proposition –

For any variety $(\mathfrak{G}, \mathcal{R})$, the equivalence relation $\equiv_{\mathcal{R}}$ is a clone congruence of $\mathfrak{T}(\mathfrak{G})$.

A clone $\mathcal{C}$ is a clone realization [**Neumann**, 1970] of a variety $\mathcal{V} := (\mathfrak{G}, \mathcal{R})$ if

$$\mathcal{C} \simeq \mathfrak{T}(\mathfrak{G})/_{\equiv_{\mathcal{R}}}.$$

In this case, $\mathcal{V}$ is a presentation of $\mathcal{C}$, and a $\mathcal{C}$-algebra is an algebra over the variety $\mathcal{V}$.

– Main objective –

Build explicit clones which are clone realizations of algebraic varieties.

2. Clones of pigmented words

Pigmented words

Let $(\mathcal{M}, \cdot, \mathrm{e})$ be a monoid.

An $\mathcal{M}$-pigmented letter is a pair (i, α) denoted by i^{α} where $i \geqslant 1$ and $\alpha \in \mathcal{M}$.
An $\mathcal{M}$-pigmented word of arity $n \geqslant 0$ is a **word** on $\mathcal{M}$-pigmented letters i^{α} such that $i \in [n]$.

– Example –

Let A^* the **free monoid** $(A^*, ., \epsilon)$ on $A := \{a, b, c\}$.

$3^{aa}\, 2^{\epsilon}\, 5^{ba}\, 5^{bbaa}$ is an A^*-pigmented word of arity 5.

Let $\mathbf{P}(\mathcal{M}) := \bigsqcup_{n \geqslant 0} \mathbf{P}(\mathcal{M})(n)$ where $\mathbf{P}(\mathcal{M})(n)$ is the **set of the $\mathcal{M}$-pigmented words** of arity n.

For any $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathbf{P}(\mathcal{M})$ and $\alpha \in \mathcal{M}$, let $\alpha \,\bar{\cdot}\, i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} := i_1^{\alpha \cdot \alpha_1} \dots i_\ell^{\alpha \cdot \alpha_\ell}$.

We endow $\mathbf{P}(\mathcal{M})$ with the **superposition maps** defined by

$$i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}[\mathfrak{p}_1, \dots, \mathfrak{p}_n] := (\alpha_1 \,\bar{\cdot}\, \mathfrak{p}_{i_1}) \; . \; \dots \; . \; (\alpha_\ell \,\bar{\cdot}\, \mathfrak{p}_{i_\ell}).$$

– Example –

We have in $\mathbf{P}(A^*)$,

$$2^{ba}2^{aa}4^{baa}3^{\epsilon}\left[2^{b}1^{aa}, 1^{bbb}1^{\epsilon}2^{b}, 2^{aa}2^{a}, \epsilon\right] = 1^{babbb}1^{ba}2^{bab} \quad 1^{aabbb}1^{aa}2^{aab} \quad \epsilon \quad 2^{aa}2^{a} = 1^{babbb}1^{ba}2^{bab}1^{aabbb}1^{aa}2^{aab}2^{aa}2^{a}.$$

Clone of pigmented words

– Theorem [G., 2020–] –

For any monoid $\mathcal{M}$, $\mathbf{P}(\mathcal{M})$ is a clone.

– Theorem [G., 2020–] –

For any monoid $(\mathcal{M}, \cdot, \mathrm{e})$, $\mathbf{P}(\mathcal{M})$ admits the presentation $(\mathfrak{G}_{\mathcal{M}}, \mathcal{R}_{\mathcal{M}})$ such that

$$\mathfrak{G}_{\mathcal{M}} := \mathfrak{G}_{\mathcal{M}}(0) \sqcup \mathfrak{G}_{\mathcal{M}}(1) \sqcup \mathfrak{G}_{\mathcal{M}}(2)$$

where $\mathfrak{G}_{\mathcal{M}}(0) := \{\mathrm{u}\}$, $\mathfrak{G}_{\mathcal{M}}(1) := \{\mathrm{p}_\alpha : \alpha \in \mathcal{M}\}$, and $\mathfrak{G}_{\mathcal{M}}(2) := \{\star\}$, and $\mathcal{R}_{\mathcal{M}}$ is the finest equivalence relation on $\mathfrak{T}(\mathfrak{G}_{\mathcal{M}})$ satisfying

$$\star[\star[\mathrm{x}_1, \mathrm{x}_2], \mathrm{x}_3] \;\mathcal{R}_{\mathcal{M}}\; \star[\mathrm{x}_1, \star[\mathrm{x}_2, \mathrm{x}_3]],$$

$$\star[\mathrm{u}, \mathrm{x}_1] \;\mathcal{R}_{\mathcal{M}}\; \mathrm{x}_1 \;\mathcal{R}_{\mathcal{M}}\; \star[\mathrm{x}_1, \mathrm{u}],$$

$$\mathrm{p}_\alpha[\star[\mathrm{x}_1, \mathrm{x}_2]] \;\mathcal{R}_{\mathcal{M}}\; \star[\mathrm{p}_\alpha[\mathrm{x}_1], \mathrm{p}_\alpha[\mathrm{x}_2]],$$

$$\mathrm{p}_\alpha[\mathrm{u}] \;\mathcal{R}_{\mathcal{M}}\; \mathrm{u},$$

$$\mathrm{p}_{\alpha_1}[\mathrm{p}_{\alpha_2}[\mathrm{x}_1]] \;\mathcal{R}_{\mathcal{M}}\; \mathrm{p}_{\alpha_1 \cdot \alpha_2}[\mathrm{x}_1],$$

$$\mathrm{p}_{\mathrm{e}}[\mathrm{x}_1] \;\mathcal{R}_{\mathcal{M}}\; \mathrm{x}_1,$$

for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

Varieties of pigmented monoids

We call $\mathcal{V}_{\mathcal{M}} := (\mathfrak{G}_{\mathcal{M}}, \mathcal{R}_{\mathcal{M}})$ the variety of $\mathcal{M}$-pigmented monoids.

By the previous presentation, an algebra over $\mathcal{V}_{\mathcal{M}}$ is a set $\mathcal{A}$ endowed with a constant u, unary maps p_α, $\alpha \in \mathcal{M}$, and a binary product $\star$, such that, for any $x_1, x_2, x_3 \in \mathcal{A}$,

$$(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3),$$

$$\mathrm{u} \star x_1 = x_1 = x_1 \star \mathrm{u},$$

$$\mathrm{p}_\alpha(x_1 \star x_2) = \mathrm{p}_\alpha(x_1) \star \mathrm{p}_\alpha(x_2),$$

$$\mathrm{p}_\alpha(\mathrm{u}) = \mathrm{u},$$

$$\mathrm{p}_{\alpha_1}(\mathrm{p}_{\alpha_2}(x_1)) = \mathrm{p}_{\alpha_1 \cdot \alpha_2}(x_1),$$

$$\mathrm{p}_{\mathrm{e}}(x_1) = x_1.$$

Therefore, given an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$,

- $(\mathcal{A}, \star, \mathrm{u})$ is a **monoid**,
- each p_α is a **monoid endomorphism** of $(\mathcal{A}, \star, \mathrm{u})$,
- the map $\cdot : \mathcal{M} \times \mathcal{A} \to \mathcal{A}$ defined by $\alpha \cdot x := \mathrm{p}_\alpha(x)$ is a **left monoid action** of $\mathcal{M}$ on $\mathcal{A}$.

3. Realizations of some varieties of monoids

A quotient of $\mathbf{P}(\mathcal{M})$

An occurrence of i^α in $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ is a left witness if all letters on the left of this occurrence are of the form j^β with $j \neq i$.

Right witnesses are defined symmetrically.

– Example –

$2^a\ 2^b\ 3^a\ 1^a\ 3^a$

Let $\mathrm{left}(\mathfrak{p})$ (resp. $\mathrm{right}(\mathfrak{p})$) be the subword of $\mathfrak{p}$ of the letters which are left (resp. right) witnesses.

Let $\equiv$ be the **equivalence relation** on $\mathbf{P}(\mathcal{M})$ such that $\mathfrak{p} \equiv \mathfrak{p}'$ if $(\mathrm{left}(\mathfrak{p}), \mathrm{right}(\mathfrak{p})) = (\mathrm{left}(\mathfrak{p}'), \mathrm{right}(\mathfrak{p}'))$.

– Example –

$$2^a\ 2^b\ 3^b\ 1^a \equiv 2^a\ 3^b\ 1^a\ 2^b\ 2^b\ 3^b\ 1^a$$

$$\left(2^a 3^b 1^a,\ 2^b 3^b 1^a\right) = \left(2^a 3^b 1^a,\ 2^b 3^b 1^a\right)$$

– Proposition [G., 2020–] –

For any monoid $\mathcal{M}$, $\equiv$ is a **clone congruence** of $\mathbf{P}(\mathcal{M})$.

Let us consider and study the clone $\mathrm{Magn}(\mathcal{M}) := \mathbf{P}(\mathcal{M})/_{\equiv}$.

– Objectives –

1. Provide a **combinatorial description** of $\mathrm{Magn}(\mathcal{M})$.
2. Provide a **presentation** of $\mathrm{Magn}(\mathcal{M})$.

Pigmented magnets

An $\mathcal{M}$-pigmented magnet is an $\mathcal{M}$-pigmented word $\mathfrak{p}$ such that

1. for any occurrence i^{α} of a letter in $\mathfrak{p}$ is a left witness, a right witness, or both;
2. for any factor $i_1^{\alpha_1}\ i_2^{\alpha_2}$ of $\mathfrak{p}$, $i_1 = i_2$ and $\alpha_1 \neq \alpha_2$.

– Examples –

- $1^{b}\ 1^{a}\ 2^{ab}\ 1^{b}$ is not an $\mathcal{M}$-pigment magnet (it admits a letter which neither a left nor a right 1-witness).
- $2^{bb}\ 3^{ba}\ 2^{a}\ 1^{a}\ 3^{b}$ is not an $\mathcal{M}$-pigmented magnet (it admits the factor $3^{ba}\ 2^{a}$).
- $3^{a}\ 2^{b}\ 4^{bb}\ 1^{a}\ 2^{ba}$ is an $\mathcal{M}$-pigmented magnet.
- $2^{aa}\ 1^{a}\ 1^{b}\ 2^{ba}$ is an $\mathcal{M}$-pigmented magnet.

An algorithm to decide equivalence

Let $\mathbb{P} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map defined by the **algorithm**. Given $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$,

1. delete iteratively in $\mathfrak{p}$ the leftmost letter which is not a witness;
2. replace iteratively in $\mathfrak{p}$ the leftmost factor $\underline{i_1^{\alpha_1}}\ \underline{i_2^{\alpha_2}}$ such that $i_1 \neq i_2$ by $\underline{i_2^{\alpha_2}}\ \underline{i_1^{\alpha_1}}$;
3. replace iteratively in $\mathfrak{p}$ the leftmost factor $\underline{i^{\alpha}}\ \underline{i^{\alpha}}$ by $\underline{i^{\alpha}}$.

– Example –

For $\mathfrak{p} := 2^{\epsilon}1^{\epsilon}2^{\epsilon}3^{\epsilon}1^{\epsilon}1^{\epsilon}3^{\epsilon}$, by highlighting the left and right 1-witnesses, we have $\mathfrak{p} = \underline{2^{\epsilon}}\ \underline{1^{\epsilon}}\ \underline{2^{\epsilon}}\ \underline{3^{\epsilon}}\ 1^{\epsilon}\ \underline{1^{\epsilon}}\ \underline{3^{\epsilon}}$, and Step 1 produces $\mathfrak{p} = \underline{2^{\epsilon}}\ \underline{1^{\epsilon}}\ \underline{2^{\epsilon}}\ \underline{3^{\epsilon}}\ \underline{1^{\epsilon}}\ \underline{3^{\epsilon}}$, Step 2 produces $\mathfrak{p} = \underline{2^{\epsilon}}\ \underline{2^{\epsilon}}\ \underline{1^{\epsilon}}\ \underline{1^{\epsilon}}\ \underline{3^{\epsilon}}\ \underline{3^{\epsilon}}$, and Step 3 produces $\mathfrak{p} = \underline{2^{\epsilon}}\ \underline{1^{\epsilon}}\ \underline{3^{\epsilon}}$, so that $\mathbb{P}(\mathfrak{p}) = 2^{\epsilon}1^{\epsilon}3^{\epsilon}$.

– Proposition [G., 2020–] –

For any monoid $\mathcal{M}$ and $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$,

- $\mathfrak{p} \equiv \mathbb{P}(\mathfrak{p})$;
- $\mathfrak{p} \equiv \mathfrak{p}'$ iff $\mathbb{P}(\mathfrak{p}) = \mathbb{P}(\mathfrak{p}')$;
- $\mathbb{P}(\mathbf{P}(\mathcal{M}))$ is the set of the $\mathcal{M}$-pigmented magnets.

Realization of regular bands

The previous description of $\mathsf{Magn}(\mathcal{M})$ in terms of $\mathcal{M}$-**pigmented magnets** leads to the following result.

– Theorem [G., 2020–] –

For any monoid $\mathcal{M}$, the clone $\mathsf{Magn}(\mathcal{M})$ admits the **presentation** $\left(\mathfrak{G}_{\mathcal{M}}, \mathcal{R}'_{\mathcal{M}}\right)$ where $\mathcal{R}'_{\mathcal{M}}$ is the set $\mathcal{R}_{\mathcal{M}}$ augmented with the $\mathfrak{G}_{\mathcal{M}}$-equations

$$\star[1^{\alpha}, 1^{\alpha}] \; \mathcal{R}'_{\mathcal{M}} \; 1^{\alpha}, \quad \alpha \in \mathcal{M},$$

$$\star[1^{\alpha_1}, \star[2^{\alpha_2}, \star[1^{\alpha_3}, \star[3^{\alpha_4}, 1^{\alpha_5}]]]] \; \mathcal{R}'_{\mathcal{M}} \; \star[1^{\alpha_1}, \star[2^{\alpha_2}, \star[3^{\alpha_4}, 1^{\alpha_5}]]], \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathcal{M}.$$

Any $\mathsf{Magn}(\mathcal{M})$-algebra is an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ where $\star$ is idempotent, and $\star$ and p_α satisfy

$$\mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\alpha_2}(x_1) \star x_3 \star \mathrm{p}_{\alpha_3}(x_1) = \mathrm{p}_{\alpha_1}(x_1) \star x_2 \star x_3 \star \mathrm{p}_{\alpha_3}(x_1)$$

for any $x_1, x_2, x_3 \in \mathcal{A}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$.

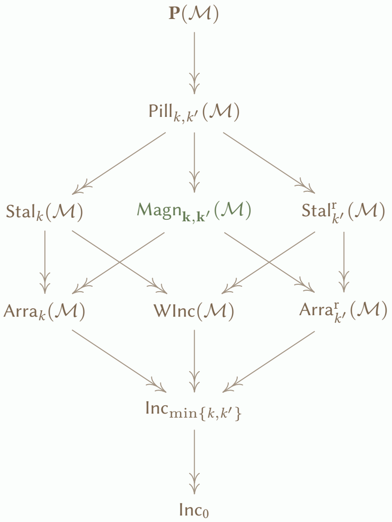
In particular, $\mathsf{Magn}(\mathcal{E})$ is a **clone realization** of the variety of **regular band monoids**, where $\mathcal{E}$ is the trivial monoid.

A hierarchy of clones

By considering quotients of $\mathbf{P}(\mathcal{M})$ by some natural congruences $\equiv_{\mathrm{sort}}$, $\equiv_{\mathrm{first}_k}$, and $\equiv^{\mathrm{r}}_{\mathrm{first}_{k'}}$, and their intersections and compositions, we obtain a **hierarchy of clones** linked by surjective clone morphisms.

For instance,

- $\mathsf{Inc}_k := \mathbf{P}(\mathcal{M})/_{\equiv_{\mathrm{sort}} \circ \equiv_{\mathrm{first}_k}}$,
- $\mathsf{WInc}(\mathcal{M}) = \mathbf{P}(\mathcal{M})/_{\equiv_{\mathrm{sort}}}$,
- $\mathsf{Arra}_k(\mathcal{M}) = \mathbf{P}(\mathcal{M})/_{\equiv_{\mathrm{first}_k}}$,
- $\mathsf{Stal}_k(\mathcal{M}) = \mathbf{P}(\mathcal{M})/_{\equiv_{\mathrm{sort}} \cap \equiv_{\mathrm{first}_k}}$,
- $\mathsf{Magn}_{\mathbf{k},\mathbf{k'}}(\mathcal{M}) = \mathbf{P}(\mathcal{M})/_{\equiv_{\mathrm{first}_k} \cap \equiv^{\mathrm{r}}_{\mathrm{first}_{k'}}}$,
- $\mathsf{Pill}_{k,k'}(\mathcal{M}) = \mathbf{P}(\mathcal{M})/_{\equiv_{\mathrm{first}_k} \cap \equiv_{\mathrm{sort}} \cap \equiv^{\mathrm{r}}_{\mathrm{first}_{k'}}}$.



Conclusion and perspectives

Given a variety $\mathcal{V} := (\mathfrak{G}, \mathcal{R})$, deciding if two terms of $\mathcal{V}$ are $\equiv_{\mathcal{R}}$-equivalent is an important question.

Here, we proposed the **construction P**, producing **clones from monoids**. The constructed clones are rich enough to contain as quotients some clones, generalizing some varieties of monoids.

In particular, we obtained a **realization of the variety of regular band monoids** in terms of magnets.

As perspectives:

- consider other clone congruences of $\mathbf{P}(\mathcal{M})$ in order to **extend** the previous **hierarchy of clones** and capture other varieties of monoids;
- consider variations of the variety of pigmented monoids, obtained by **removing some relations**. Instead of pigmented words, this could give rise to particular trees or configurations of chords in polygons.