Clone realizations of semigroup varieties

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1. Introduction and overview
**M-semigroups**

--- Definition [G., 2015] ---

Let \((M, \cdot, \epsilon)\) be a monoid. An **M-semigroup** is a set \(S\) endowed with a binary operation \(\star : S \times S \to S\) and unary operations \(\theta_\alpha : S \to S, \alpha \in M\), satisfying

\[
\begin{align*}
(x_1 \star x_2) \star x_3 &= x_1 \star (x_2 \star x_3), \\
\theta_\alpha(x_1 \star x_1) &= \theta_\alpha(x_1) \star \theta_\alpha(x_1), \\
\theta_{\alpha_1}(\theta_{\alpha_2}(x_1)) &= \theta_{\alpha_1 \cdot \alpha_2}(x_1), \\
\theta_\epsilon(x_1) &= x_1.
\end{align*}
\]

Such structures (and variations) appear quite often in combinatorics.

--- Example ---

Let \(M := (\mathbb{N}, \max, 0)\), \(S := \mathbb{N}^*\) (the set of sequences of nonnegative integers), \(\star\) be the concatenation product, and \(\theta_\alpha\) be the map sending any word to its subword made of the letters nonsmaller than \(\alpha\). For instance,

\[
\theta_2(0015213 \star 41200) = \theta_2(001521341200) = 52342.
\]

An **M-semigroup** is hence a semigroup \((S, \star)\) endowed with semigroups endomorphisms \(\theta_\alpha\) for any \(\alpha \in M\), and such that the map \((x, \alpha) \mapsto \theta_\alpha(x)\) is a monoid action of \(M\) on \(S\).
Overview

– Goals –

- Introduce a clone having, as algebras, $M$-semigroups and provide a combinatorial realization of it.
- Study this clone and understand what it contains (as quotients or sub-clones).

We will use terms, rewrite systems, and clone theory.

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2. Varieties, clones, and rewrite systems
A **signature** is a graded set \( \mathcal{G} := \bigsqcup_{n \geq 0} \mathcal{G}(n) \) wherein each \( a \in \mathcal{G}(n) \) is a **constant** of arity \( n \).

A **\( \mathcal{G} \)-term** is

- either a **variable** \( x \) from the set \( \mathbb{X}_k := \{x_1, \ldots, x_k\} \) for a \( k \geq 0 \);
- either a pair \((a, (t_1, \ldots, t_n))\) where \( a \in \mathcal{G}(n) \) and each \( t_i \) is a **\( \mathcal{G} \)-term**.

The set of all **\( \mathcal{G} \)-terms** is denoted by \( \mathcal{T}(\mathcal{G}) \).

---

**Example**

This is the tree representation of the **\( \mathcal{G} \)-term**

\[
(a, ((b, (x_2, x_4)), (b, ((a, (x_1, c)), x_1))))
\]

where \( \mathcal{G} := \mathcal{G}(0) \sqcup \mathcal{G}(2) \) with \( \mathcal{G}(0) := \{c\} \) and \( \mathcal{G}(2) := \{a, b\} \).
Varieties

A $\mathcal{G}$-equation is a pair $(t, t')$ where $t$ and $t'$ are both $\mathcal{G}$-terms.

A variety is a pair $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is a signature and $\mathcal{R}$ is a set of $\mathcal{G}$-equations. We denote by $t \mathcal{R} t'$ the fact that $(t, t') \in \mathcal{R}$.

- Example -

The variety of groups is the pair $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(0) \sqcup \mathcal{G}(1) \sqcup \mathcal{G}(2)$ with $\mathcal{G}(0) := \{1\}$, $\mathcal{G}(1) := \{i\}$, and $\mathcal{G}(2) := \{\star\}$, and $\mathcal{R}$ is the set of $\mathcal{G}$-equations satisfying

\[
\begin{align*}
\star & \mathcal{R} \star, & \star & \mathcal{R} \star, & \star & \mathcal{R} \star, & \star & \mathcal{R} \star, & \star & \mathcal{R} \star.
\end{align*}
\]

- Example -

The variety of semilattices is the pair $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\land\}$, and $\mathcal{R}$ is the set of $\mathcal{G}$-equations satisfying

\[
\begin{align*}
\land & \mathcal{R} \land, & \land & \mathcal{R} \land, & \land & \mathcal{R} \land.
\end{align*}
\]
Substitutions, interpretations, and evaluations

Let $\mathcal{A}$ be a nonempty set. An **$\mathcal{A}$-substitution** is a map $\sigma : \mathbb{X} \to \mathcal{A}$, where $\mathbb{X} := \{x_1, x_2, \ldots\}$. An **$\mathcal{A}$-interpretation** of a signature $\mathcal{G}$ is a set

$$\mathcal{G}_\mathcal{A} := \left\{ a_\mathcal{A} : \mathcal{A}^k \to \mathcal{A} : a \in \mathcal{G}(k), k \geq 0 \right\}.$$

The **evaluation** $ev^\sigma_\mathcal{A}(t)$ of a $\mathcal{G}$-term $t$ is the element of $\mathcal{A}$ defined recursively by

$$ev^\sigma_\mathcal{A}(t) := \begin{cases} 
\sigma(x) & \text{if } t = x \text{ is a variable,} \\
 a_\mathcal{A}(ev^\sigma_\mathcal{A}(t_1), \ldots, ev^\sigma_\mathcal{A}(t_k)) & \text{otherwise, where } t = (a, (t_1, \ldots, t_k)). 
\end{cases}$$

**Example**

$$t := x_2 x_3 x_3 x_1 x_1 \xrightarrow{\sigma} \begin{array}{c} x \times x_2 2 \times 0 \\
\times x_3 0 x_1 1 1 \end{array} \xrightarrow{ev^\sigma_\mathcal{A}} 2$$

With $\mathcal{A} := \mathbb{N}$, $\mathcal{G}_\mathcal{A}$ defined naturally, and $\sigma$ satisfying $\sigma(x_1) := 1$, $\sigma(x_2) := 2$, and $\sigma(x_3) := 0$, one obtains $ev^\sigma_\mathcal{A}(t) = 2$. \

---
Algebras of a variety

An algebra of a variety \((\mathcal{G}, \mathcal{R})\) is a pair \((\mathcal{A}, \mathcal{G}_A)\) such that, for any \((t, t') \in \mathcal{R}\) and any \(\mathcal{A}\)-substitution \(\sigma\), we have \(\text{ev}_{\mathcal{A}}^\sigma(t) = \text{ev}_{\mathcal{A}}^\sigma(t')\).

\[-\text{Example}\-\]

Any algebra of the variety \((\mathcal{G}, \mathcal{R})\) of groups is a set \(\mathcal{A}\) endowed with three operations \(\mathbb{1}\) (nullary), \(i\) (unary), and \(\star\) (binary), such that, for all \(x_1, x_2, x_3 \in \mathcal{A}\),

\[
(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3), \quad x_1 \star \mathbb{1} = x_1 = \mathbb{1} \star x_1, \quad i(x_1) \star x_1 = \mathbb{1} = x_1 \star i(x_1).
\]

Two \(\mathcal{G}\)-terms \(t\) and \(t'\) are \(\mathcal{R}\)-equivalent if for all algebras \((\mathcal{A}, \mathcal{G}_A)\) of \((\mathcal{G}, \mathcal{R})\) and for all \(\mathcal{A}\)-substitutions \(\sigma\), one has \(\text{ev}_{\mathcal{A}}^\sigma(t) = \text{ev}_{\mathcal{A}}^\sigma(t')\). This property is denoted by \(t \equiv_{\mathcal{R}} t'\).

\[-\text{Example}\-\]

In the variety of groups,

\[
\begin{align*}
\ast & \quad \equiv_{\mathcal{R}} & \ast \\
\ast & \quad \equiv_{\mathcal{R}} & \ast \\
\ast & \quad \equiv_{\mathcal{R}} & \ast \\
\ast & \quad \equiv_{\mathcal{R}} & \ast
\end{align*}
\]

and

\[
\begin{align*}
\ast & \quad \equiv_{\mathcal{R}} & \ast \\
\ast & \quad \equiv_{\mathcal{R}} & \ast \\
\ast & \quad \equiv_{\mathcal{R}} & \ast
\end{align*}
\]
Clones

Abstract clones [Cohn, 1965] provide a framework to study varieties.

An abstract clone is a triple \((C, \odot, 1_{i,n})\) where

- \(C\) is a graded set \(C = \bigsqcup_{n \geq 0} C(n)\);
- \(\odot\) is a map \(\odot : C(n) \times C(m)^n \to C(m)\) called superposition map;
- for each \(n \geq 0\) and \(i \in [n]\), \(1_{i,n}\) is an element of \(C(n)\) called projection.

The following relations have to hold:

- for all \(x_i \in C(m)\),
  \[
  1_{i,n} \odot [x_1, \ldots, x_n] = x_i;
  \]

- for all \(x \in C(n)\),
  \[
  x \odot [1_{1,n}, \ldots, 1_{n,n}] = x;
  \]

- for all \(x \in C(n), y_i \in C(m)\), and \(z_j \in C(k)\),
  \[
  (x \odot [y_1, \ldots, y_n]) \odot [z_1, \ldots, z_m] = x \odot [y_1 \odot [z_1, \ldots, z_m], \ldots, y_n \odot [z_1, \ldots, z_m]].
  \]
Let $\mathcal{G}$ be a signature.

The **free clone** on $\mathcal{G}$ is the clone $(\mathcal{T}(\mathcal{G}), \odot, 1_{i,n})$ where

- for any $n \geq 0$, $\mathcal{T}(\mathcal{G})(n)$ is the set of all $\mathcal{G}$-terms on $X_n$;
- $\odot$ is defined as follows. The $\mathcal{G}$-term $t \odot [s_1, \ldots, s_n]$ is obtained by replacing each occurrence of a variable $x_i$ of $t$ by the root of $s_i$;
- $1_{i,n}$ is the term $\frac{1}{x_i}$ of arity $n$.

**Example**

By setting $\mathcal{G} := \mathcal{G}(2) \sqcup \mathcal{G}(3)$ where $\mathcal{G}(2) := \{a, b\}$ and $\mathcal{G}(3) := \{c\}$, in the free clone $\mathcal{T}(\mathcal{G})$, one has
Clone realizations of varieties

A **clone congruence** of a clone $C$ is an equivalence relation $\equiv$ on $C$ compatible with the superposition map, that is, for any $x, x' \in C(n)$ and $y_1, y'_1, \ldots, y_n, y'_n \in C(m)$, if $x \equiv x'$ and $y_1 \equiv y'_1, \ldots, y_n \equiv y'_n$, then

$$x \circ [y_1, \ldots, y_n] \equiv x' \circ [y'_1, \ldots, y'_n].$$

For any variety $(\mathcal{G}, \mathcal{R})$, the $\mathcal{R}$-equivalence relation $\equiv_{\mathcal{R}}$ is a clone congruence of $\mathcal{T}(\mathcal{G})$.

A **presentation** of a clone $C$ is a variety $(\mathcal{G}, \mathcal{R})$ such that

$$C \simeq \mathcal{T}(\mathcal{G})/\equiv_{\mathcal{R}}.$$

Conversely, we say in this case that $C$ is a **clone realization** of $(\mathcal{G}, \mathcal{R})$ (see [Neumann, 1970]).

An **algebra** on $C$ is an algebra of the variety $(\mathcal{G}, \mathcal{R})$. 
Let \((\mathcal{G}, \mathcal{R})\) be a variety.

The **term realization** of \((\mathcal{G}, \mathcal{R})\) is the clone constructed from an orientation \(\rightarrow\) of \(\mathcal{R}\) such that

- \(C(n)\) is in one-to-one correspondence with the set of the normal forms on \(X_n\) for \(\rightarrow\);
- to compute \(t \odot [s_1, \ldots, s_n]\) in \(C\), compute this term in the free clone on \(\mathcal{G}\) and then consider the unique normal form for \(\rightarrow\) reachable from it;
- the projections of \(C\) are the normal forms reachable from the terms consisting in one leaf.

Term realizations allow us to decide the **word problem**: to decide if two \(\mathcal{G}\)-terms are \(\equiv_R\)-equivalent, just compare their normal forms [Baader, Nipkow, 1998]. This can be undecidable.

In this context, **completion algorithms** are important [Knuth, Bendix, 1970].

They can also be used to construct free objects of the category of the algebras of \((\mathcal{G}, \mathcal{R})\).
Rewrite systems on terms

A **rewrite relation** on $\mathcal{T}(G)$ is a binary relation $\rightarrow$ on $\mathcal{T}(G)$ such that if $s \rightarrow s'$, then $s$ and $s'$ are two terms on $\mathcal{X}_n$ for an $n \geq 0$.

The **context closure** of $\rightarrow$ is the binary relation $\Rightarrow$ satisfying $t \Rightarrow t'$ whenever $t'$ is obtained by replacing in $t$ a factor $s$ by $s'$ provided that $s \rightarrow s'$.

--- Example ---

For $G := G(2) := \{a\}$, let the rewrite relation $\rightarrow$ defined by

\[
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_1 & | & \text{x}_1 \\
\end{array} \quad \rightarrow \quad
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_1 & | & \text{x}_1 \\
\end{array} \quad \text{and} \quad
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_1 & | & \text{x}_1 \\
\end{array} \quad \rightarrow \quad
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_1 & | & \text{x}_1 \\
\end{array}.
\]

We have

\[
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_2 & | & \text{x}_2 \\
\end{array} \quad \Rightarrow \quad
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_2 & | & \text{x}_2 \\
\end{array} \quad \text{and} \quad
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_5 & | & \text{x}_5 \\
\end{array} \quad \Rightarrow \quad
\begin{array}{cccc}
| & | & | & | \\
| & \text{a} & | & \text{a} \\
| & \text{x}_2 & | & \text{x}_2 \\
\end{array}.
\]
Let $\rightarrow$ be a rewrite relation on $T(\mathcal{G})$.

Let $(\mathcal{G}, \mathcal{R})$ be a variety. A rewrite relation $\rightarrow$ on $T(\mathcal{G})$ is an orientation of $\mathcal{R}$ if for any $(t, t') \in \mathcal{R}$, we have either $t \rightarrow t'$ or $t' \rightarrow t$.

A normal form for $\rightarrow$ is a $\mathcal{G}$-term $t$ such that there is no $\mathcal{G}$-term $t'$ satisfying $t \Rightarrow t'$.

When there is no infinite chain $t_0 \Rightarrow t_1 \Rightarrow \cdots$, the rewrite relation $\rightarrow$ is terminating.

If $t \Rightarrow s_1$ and $t \Rightarrow s_2$ implies the existence of $t'$ such that $s_1 \Rightarrow t'$ and $s_2 \Rightarrow t'$, then $\rightarrow$ is confluent.

Some properties:

- For any two $\mathcal{G}$-terms $t$ and $t'$, $t \equiv_{\mathcal{R}} t'$ iff $t \Leftrightarrow t'$.
- If $\rightarrow$ is terminating and confluent, then $t \equiv_{\mathcal{R}} t'$ iff there is a normal form $s$ such that $t \Rightarrow s$ and $t' \Rightarrow s$. 
Example: duplicial algebras

Let the variety \((\mathcal{G}, \mathcal{R})\) where \(\mathcal{G} := \mathcal{G}(2) := \{\ll, \gg\}\) and \(\mathcal{R}\) is the set \(\mathcal{G}\)-equations satisfying

\[
\ll x_1 x_2 x_3 \ll R \ll x_1 x_2 x_3 \ll, \quad \ll x_1 x_2 x_3 \ll R \ll x_1 x_2 x_3 \ll,
\]

\[
\gg x_1 x_2 x_3 \gg R \gg x_1 x_2 x_3 \gg, \quad \gg x_1 x_2 x_3 \gg R \gg x_1 x_2 x_3 \gg.
\]

The algebras of this variety are duplicial algebras [Brouder, Frabetti, 2003].

– Example –

On \(\mathbb{N}^+\) (the set of nonempty sequences of nonnegative integers), let \(\ll\) and \(\gg\) be the operations defined by

\[
u \ll v := u( v \uparrow_{\text{max}(u)} ), \quad u \gg v := u( v \uparrow_{|u|} ).
\]

Then, for instance,

\[
0211 \ll 14 = 021136, \quad 0211 \gg 14 = 021158.
\]

This structure is a duplicial algebra [Novelli, Thibon, 2013].
Orientation of duplicial relations

Let the orientation $\rightarrow$ of $\mathcal{R}$ defined by

We have for instance the following sequence of rewrites:

The rewrite relation $\rightarrow$ is terminating and confluent.
Encoding duplicial operations

– Proposition [Loday, 2008] –

The set of normal forms for $\rightarrow$ with $n \geq 0$ inputs is in one-to-one correspondence with the set of all binary trees with $n$ internal nodes where internal nodes are decorated on $X$.

A possible bijection puts the following two trees in correspondence:

Therefore, there are

$$
\frac{1}{n+1} \binom{2n}{n} k^n
$$

pairwise nonequivalent duplicial operations with $n$ inputs on variables of $X_k$. 

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Thanks to previous properties, we obtain a clone realization of the variety of duplicial algebras.

Let \( \text{Dup} \) be the clone such that

- for any \( n \geq 0 \), \( \text{Dup}(n) \) is the set of the binary trees where internal nodes are decorated on \( X_n \);

- for any such trees \( t \) and \( s_1, \ldots, s_n \), the superposition \( t \diamond [s_1, \ldots, s_n] \) is obtained by replacing in \( t \) each node \( u \) labeled by \( x_i \) by \( s_i \) and by grafting onto the leftmost (resp. rightmost) leaf of \( s_i \) the left (resp. right) child of \( u \).

\[ \begin{align*}
\begin{array}{ccc}
X_1 & X_2 & X_3 \\
\circ & & \\
& X_2 & \\
& & X_1
\end{array}
\end{align*} \]

\[ \begin{align*}
\begin{array}{ccc}
X_1 & X_2 & X_3 \\
\circ & & \\
& X_2 & \\
& & X_1
\end{array}
\end{align*} \]

\[ = \]

\[ \begin{align*}
\begin{array}{ccc}
X_1 & X_1 & X_1 \\
\circ & & \\
& X_2 & \\
& & X_1
\end{array}
\end{align*} \]
3. Clone of colored words
The variety of $\mathcal{M}$-semigroups

Let $(\mathcal{M}, \cdot, \epsilon)$ be a monoid.

Let the variety $(\mathcal{G}_\mathcal{M}, \mathcal{R}_\mathcal{M})$ defined by

- $\mathcal{G}_\mathcal{M} := \mathcal{G}_\mathcal{M}(1) \sqcup \mathcal{G}_\mathcal{M}(2)$ where $\mathcal{G}_\mathcal{M}(1) := \mathcal{M}$ and $\mathcal{G}_\mathcal{M}(2) := \{\star\}$;

- $\mathcal{R}_\mathcal{M}$ is the set of $\mathcal{G}_\mathcal{M}$-equations satisfying

\[
\begin{align*}
\star & \ x_1 \ x_2 \ x_3 \\
\alpha & \ x_1 \ x_2
\end{align*}
\]

\[
\begin{align*}
\alpha & \ x_1 \ x_2 \\
\alpha_1 \cdot \alpha_2 & \ x_1
\end{align*}
\]

\[
\begin{align*}
\epsilon & \ x_1
\end{align*}
\]

for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

This is the variety of $\mathcal{M}$-semigroups in the sense that any $\mathcal{M}$-semigroup is an algebra of $(\mathcal{G}_\mathcal{M}, \mathcal{R}_\mathcal{M})$ and any algebra of $(\mathcal{G}_\mathcal{M}, \mathcal{R}_\mathcal{M})$ is an $\mathcal{M}$-semigroup.
Let the orientation $\rightarrow$ of $\mathcal{R}_M$ satisfying
\[
\begin{align*}
\ast & \xrightarrow{} x_1 \ast \quad , \quad \alpha \xleftarrow{} \ast \quad , \quad \alpha_1 \ast \xrightarrow{} \alpha_1 \cdot \alpha_2 \quad , \quad \epsilon \xrightarrow{} x_1 \cdot .
\end{align*}
\]

– Proposition [G., 2020–] –

For any monoid $M$, the orientation $\rightarrow$ of $\mathcal{R}_M$ is terminating and confluent.

The set of normal forms for $\rightarrow$ of planar $\mathcal{S}_M$-terms is the set of the terms avoiding the left members of $\rightarrow$. These are the terms of the form

\[
\begin{align*}
\ast & \xrightarrow{} x_{k_i} \ast \quad , \quad \ast \xrightarrow{} x_{k_i} \alpha_{k_i} \ast \\
\ast & \xrightarrow{} \ast \quad , \quad \ast & \xrightarrow{} \ast \quad , \quad \ast & \xrightarrow{} \ast
\end{align*}
\]

where $s_i = x_{k_i}$ or $s_i = \alpha_{k_i}$, for $\alpha_{k_i} \in M \setminus \{\epsilon\}$. 
Colored words

Let \((\mathcal{M}, \cdot, \epsilon)\) be a monoid.

Let \(\mathcal{W}\mathcal{M}\) be the graded set of all \(\mathcal{M}\)-colored words defined for any \(n \geq 0\) by

\[
\mathcal{W}\mathcal{M}(n) := \bigsqcup_{\ell \geq 1} \left\{ (u) : (u, c) \in [n]\ell \times \mathcal{M}\ell \right\}.
\]

- Example -

\[
\begin{pmatrix}
1 & 2 & 1 & 6 \\
\epsilon & ab & bab & b
\end{pmatrix}
\]

is a \(\{a, b\}^*, \cdot, \epsilon\)-colored word of arity 6 (or greater).

Let \(\odot\) be the superposition map defined by

\[
\begin{pmatrix}
(u) \\
(c)
\end{pmatrix} \odot \left[ \begin{pmatrix}
v_1 \\
d_1
\end{pmatrix}, \ldots, \begin{pmatrix}
v_n \\
d_n
\end{pmatrix} \right] := \begin{pmatrix}
v_{u(1)} & \cdots & v_{u(\ell)} \\
(c(1) \cdot d_{u(1)}) & \cdots & (c(\ell) \cdot d_{u(\ell)})
\end{pmatrix}
\]

where for any \(\alpha \in \mathcal{M}\) and \(w \in \mathcal{M}^*\), \(\alpha \odot w := (\alpha \cdot w(1)) \ldots (\alpha \cdot w(|w|))\).

Let also \(\mathbb{1}_{i, n} := \begin{pmatrix}
i \\
\epsilon
\end{pmatrix}\).
Clone of colored words

– Example –

In $W(\{a, b\}^*, \epsilon)$,

\[
\begin{bmatrix}
2 & 2 & 3 \\
ba & aa & \epsilon
\end{bmatrix} \otimes \begin{bmatrix}
2 & 1 \\
ba & aa
\end{bmatrix}, \begin{bmatrix}
1 & 1 & 2 \\
bbb & \epsilon & b
\end{bmatrix}, \begin{bmatrix}
2 & 2 \\
ba & a
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\
ba.bbb & ba.\epsilon & ba.b & aa.bbb & aa.\epsilon & aa.b & \epsilon.aa & \epsilon.a
\end{bmatrix}.
\]

– Theorem [G., 2020–] –

For any monoid $\mathcal{M}$, $(\mathcal{W}_\mathcal{M}, \otimes, 1_{i,n})$ is a clone and is a clone realization of the variety $(\mathcal{G}_\mathcal{M}, \mathcal{R}_\mathcal{M})$.

– Example –

Here is a normal for $\rightarrow$ of the variety $(\mathcal{G}_\mathcal{M}, \mathcal{R}_\mathcal{M})$ where $\mathcal{M}$ is the monoid $(\{a, b\}^*, \epsilon)$ and the $\mathcal{M}$-colored word in correspondence:

\[
\begin{bmatrix}
2 & 3 & 2 & 4 & 4 \\
\epsilon & bbb & b & \epsilon & ab
\end{bmatrix}.
\]
Let us focus on the case where $M$ is the trivial monoid $\{\epsilon\}$.

Let $\text{Word} := W\{\epsilon\}$. We can forget about the colors of the elements of $\text{Word}$ without any loss of information.

For any $n \geq 0$, $\text{Word}(n)$ is the set of the nonempty words on the alphabet $[n]$.

--- Example ---

In $\text{Word}$,

$$311434 \otimes [221, 33, 2, 1] = 2 221 221 1 2 1 = 2221221121$$

--- Proposition [G., 2020–] ---

The clone $\text{Word}$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

Therefore, $\text{Word}$ is a clone realization of the variety of semigroups.
Let \( \equiv_{st} \) be the equivalence relation on \textit{Word} wherein \( u \equiv_{st} v \) if \( u \) and \( v \) have both the same \textit{sorted} version.

Let \( \equiv_{lo} \) (resp. \( \equiv_{ro} \)) be the equivalence relation on \textit{Word} wherein \( u \equiv_{lo} v \) (resp. \( u \equiv_{ro} v \)) if the versions of \( u \) and \( v \) obtained by keeping only the \textit{leftmost} (resp. \textit{rightmost}) among the multiple occurrences of a same letter are equal.

Let \( \equiv_{ll} \) (resp. \( \equiv_{rl} \)) be the equivalence relation on \textit{Word} wherein \( u \equiv_{ll} v \) (resp. \( u \equiv_{rl} v \)) if \( u_1 = v_1 \) (resp. \( u_{|u|} = v_{|v|} \)).

\[ \begin{array}{c}
\text{\textit{Examples}} \\
47 \equiv_{st} 74, \quad 311322 \equiv_{st} 131232, \quad 211 \not\equiv_{st} 122
\end{array} \]

\[ \begin{array}{c}
\text{\textit{Examples}} \\
223111352 \equiv_{lo} 2333315, \quad 5142144 \equiv_{ro} 552214, \\
3113 \not\equiv_{lo} 113
\end{array} \]

\[ \begin{array}{c}
\text{\textit{Examples}} \\
1 \equiv_{ll} 12, \quad 3114 \equiv_{ll} 32233, \quad 211535 \equiv_{rl} 5
\end{array} \]

\[ \text{\textit{Proposition}} [G., 2020–] \]

The equivalence relations \( \equiv_{st}, \equiv_{lo}, \equiv_{ro}, \equiv_{ll}, \) and \( \equiv_{rl} \) are clone congruences of \textit{Word}. 
Multisets

Let $\text{MSet} := \text{Word}/\equiv_{\text{st}}$.

For any $n \geq 0$, the elements of $\text{MSet}(n)$ can be seen as nonempty multisets on $[n]$. By encoding any such multiset $M = \{1^{a(1)}, \ldots, n^{a(n)}\}$ by the tuple $a = (a(1), \ldots, a(n))$, the superposition map of $\text{MSet}$ expresses as a matrix multiplication

$$a \odot [b_1, \ldots, b_n] = (a(1) \ldots a(n)) \begin{pmatrix} b_1(1) & \ldots & b_1(m) \\ \vdots & \ldots & \vdots \\ b_n(1) & \ldots & b_n(m) \end{pmatrix}.$$ 

– Proposition [G., 2020–] –

The clone $\text{MSet}$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

Therefore, $\text{MSet}$ is a clone realization of the variety of commutative semigroups.
Let $\equiv := \equiv_{st} \cap \equiv_{rl}$ and $\text{RMSet}_l := \text{Word}/\equiv$.

For any $n \geq 0$, the elements of $\text{RMSet}_l(n)$ can be seen as pairs $(M, i)$ where $M$ is a nonempty multiset on $[n]$ and $i \in M$.

\textbf{Proposition [G., 2021–]}

The clone $\text{RMSet}_l$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

\[
\begin{align*}
\star & \not\rhd x_3 & x_1 & \not\rhd x_2 x_3, & x_1 & \not\rhd x_2 x_3 & x_1 & \not\rhd x_3 x_2.
\end{align*}
\]

Therefore, $\text{RMSet}_l$ is a clone realization of the variety of \textbf{right-commutative semigroups}, that are semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_3 = x_1 \star x_3 \star x_2$.

Analog properties hold for the quotient $\text{RMSet}_r := \text{Word}/\equiv'$, where $\equiv' := \equiv_{st} \cap \equiv_{rl}$. 


Pairs of integers

Let \( \equiv := \equiv_{\|} \cap \equiv_{\|} \) and \( \text{PInt} := \text{Word}/\equiv \).

For any \( n \geq 0 \), the set \( \text{PInt}(n) \) can be identified with \([n]^2\). The superposition of \( \text{PInt} \) expresses as

\[
(i, i') \odot [(j_1, j'_1), \ldots, (j_n, j'_n)] = (j_i, j'_i).
\]

Moreover, \( \#\text{PInt}(n) = n^2 \).

\[\textbf{– Proposition [G., 2021–] –}\]

The clone \( \text{PInt} \) admits the presentation \((\mathcal{G}, \mathcal{R})\) where \( \mathcal{G} := \mathcal{G}(2) := \{\ast\} \) and \( \mathcal{R} \) satisfies

\[
\begin{align*}
\ast & \, \mathcal{R} \, \ast, \\
x_1 & \, \mathcal{R} \, x_2 \, \mathcal{R} \, x_3,
\end{align*}
\]

Therefore, \( \text{PInt} \) is a clone realization of the variety of \textbf{rectangular bands}, that are idempotent semigroups wherein the operation \( \ast \) satisfies the relation \( x_1 \ast x_2 \ast x_3 = x_1 \ast x_3 \).
Arrangements

Let $\text{Arr}_1 := \text{Word}/\equiv_{l_0}$. For any $n \geq 0$, the elements of $\text{Arr}_1(n)$ can be seen as nonempty arrangements (nonempty words without repetitions) on $[n]$. Moreover,

$$\#\text{Arr}_1(n) = \sum_{0 \leq k \leq n-1} \frac{n!}{k!}$$

and this sequence starts by 0, 1, 4, 15, 64, 325, 1956, 13699, 109600 (Sequence A007526).

\[\text{– Proposition [G., 2020–] –}\]

The clone $\text{Arr}_1$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

Therefore, $\text{Arr}_1$ is a clone realization of the variety of left-regular bands, that are idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_1 = x_1 \star x_2$.

Analog properties hold for the quotient $\text{Arr}_r := \text{Word}/\equiv_{r_0}$, leading to right-regular bands.
Sets

<table>
<thead>
<tr>
<th>− Lemma [G., 2020–] −</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \equiv_{\text{st}} \circ \equiv_{\text{lo}} = \equiv_{\text{lo}} \circ \equiv_{\text{st}} )</td>
</tr>
</tbody>
</table>

Therefore, this composition is a clone congruence of Word. Let us set it as \( \equiv_{\text{in}} \) and let \( \text{Set} := \text{Word}/\equiv_{\text{in}} \).

For any \( n \geq 0 \), the elements of \( \text{Set}(n) \) can be seen as nonempty subsets of \([n]\). On such objects, the superposition map of \( \text{Set} \) expresses as

\[
U \odot [V_1, \ldots, V_n] = \bigcup_{j \in U} V_j.
\]

Moreover, \( \#\text{Set}(n) = 2^n - 1 \).

<table>
<thead>
<tr>
<th>− Proposition [G., 2020–] −</th>
</tr>
</thead>
<tbody>
<tr>
<td>The clone ( \text{Set} ) admits the presentation ((\mathcal{G}, \mathcal{R})) where ( \mathcal{G} := \mathcal{G}(2) := {\ast} ) and ( \mathcal{R} ) satisfies</td>
</tr>
</tbody>
</table>

Therefore, \( \text{Set} \) is a clone realization of the variety of semilattices.
Maps are surjective clone morphisms.
The framed clones are combinatorial.
Some of the previous constructions can be generalized at the level of the clone $W\mathcal{M}$.

Let $\equiv_{st}$, $\equiv_{lo}$, $\equiv_{ro}$, $\equiv_{ll}$, and $\equiv_{rl}$ be the equivalence relations on $W\mathcal{M}$ defined in the same way as before where each color goes with its letter.

**Proposition** [G., 2021–]

For any monoid $\mathcal{M}$, the equivalence relations $\equiv_{st}$, $\equiv_{lo}$, $\equiv_{ro}$, $\equiv_{ll}$, and $\equiv_{rl}$ are clone congruences of $W\mathcal{M}$.

Let $\equiv_1 := \equiv_{st} \circ \equiv_{lo}$ and $\equiv_2 := \equiv_{lo} \circ \equiv_{st}$. If $\mathcal{M}$ has two different elements $a$ and $b$, one has

$$
\begin{pmatrix}
1 & 2 & 1 \\
a & a & b
\end{pmatrix}
\equiv_1
\begin{pmatrix}
1 & 2 \\
b & a
\end{pmatrix}
$$

but

$$
\begin{pmatrix}
1 & 2 & 1 \\
a & a & b
\end{pmatrix}
\not\equiv_2
\begin{pmatrix}
1 & 2 \\
b & a
\end{pmatrix}.
$$

For this reason, in this general case, $\equiv_1$ and $\equiv_2$ are not clone congruences of $W\mathcal{M}$. 
Monoids on two elements

The two monoids on two elements are $M_1 := (\mathbb{Z}/2\mathbb{Z}, +, 0)$ and $M_2 := \{0, 1\}, \text{max}, 0$.

The clone $WM_1$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(1) \sqcup \mathcal{G}(2)$, $\mathcal{G}(1) := \{1\}$, $\mathcal{G}(2) := \{\star\}$, and $\mathcal{R}$ satisfies

$$\begin{align*}
\star &\mathcal{R} x_3, \\
\star &\mathcal{R} x_1, \\
\star &\mathcal{R} \begin{array}{c}
1 \\
1 \\
1
\end{array}.
\end{align*}$$

Any algebra on this clone is a semigroup endowed with an involutive semigroup endomorphism.

The clone $WM_2$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(1) \sqcup \mathcal{G}(2)$, $\mathcal{G}(1) := \{1\}$, $\mathcal{G}(2) := \{\star\}$, and $\mathcal{R}$ satisfies

$$\begin{align*}
\star &\mathcal{R} x_3, \\
\star &\mathcal{R} x_1, \\
\star &\mathcal{R} \begin{array}{c}
1 \\
1 \\
1
\end{array}.
\end{align*}$$

Any algebra on this clone is a semigroup endowed with an idempotent semigroup endomorphism.
Conclusion and future work

In this work,

- we use clones as a framework to study varieties of algebras;
- we use rewrite systems on terms to build term realizations of varieties;
- we introduce a new functorial construction $W$ from monoids to clones;
- we build quotients of the clone of monochrome words providing clone realizations of special classes of semigroups.

Future work include

- the discovery of other congruences of $WM$;
- the exploration of the previous constructions for colored words;
- the study of subclones of $WM$ generated by finite sets of colored words.
4. Appendix
Bi-rooted multisets

Let \( \equiv := \equiv_{st} \cap \equiv_{ll} \cap \equiv_{rl} \) and \( \text{BRMSet} := \text{Word}/\equiv \).
For any \( n \geq 0 \), the elements of \( \text{BRMSet}(n) \) can be seen as triples \((M, i, i')\) where \( M \) is a nonempty multiset on \([n]\) and \( i', i \in M \).

\[\begin{align*}
\text{Proposition [G., 2021]} & \quad \text{The clone } \text{RMSet}_1 \text{ admits the presentation } (\mathcal{G}, \mathcal{R}) \text{ where } \mathcal{G} := \mathcal{G}(2) := \{\star\} \text{ and } \mathcal{R} \text{ satisfies}
\end{align*}\]

\begin{align*}
\text{Therefore, } \text{BRMSet} & \quad \text{is a clone realization of the variety of } \text{medial semigroups}, \text{ that are}
\end{align*}

\text{semigroups wherein the operation } \star \text{ satisfies the relation } x_1 \star x_2 \star x_3 \star x_1 = x_1 \star x_3 \star x_2 \star x_1.
Integers

Let $\text{Int}_l := \text{Word}/_{\equiv_l}$.

For any $n \geq 0$, the set $\text{Int}_l(n)$ can be identified with $[n]$. The superposition of $\text{Int}_l$ expresses as

$$i \odot [j_1, \ldots, j_n] = j_i.$$

Moreover, $\#\text{Int}_l(n) = n$.

--- Proposition [G., 2021–] ---

The clone $\text{Int}_l$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

$$\begin{array}{c}
\star \\
\downarrow \\
x_1 \\
\mathcal{R} \\
\downarrow \\
x_1
\end{array}, \begin{array}{c}
\star \\
\downarrow \\
x_2 \\
\mathcal{R} \\
\downarrow \\
x_1
\end{array}.$$

This is the trivial clone.

Therefore, $\text{Int}_l$ is a clone realization of the variety of left-zero bands, that are semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 = x_1$.

Analog properties hold for the quotient $\text{Int}_r := \text{Word}/_{\equiv_r}$, leading to right-zero bands.
Rooted arrangements

Let $\equiv := \equiv_{\text{lo}} \cap \equiv_{\text{rl}}$ and $\text{RArr}_1 := \text{Word}/\equiv$.

For any $n \geq 0$, the elements of $\text{RArr}_1(n)$ can be seen as pairs $(a, i)$ where $a$ is a nonempty arrangement on $[n]$ and $i$ occurs in $a$. Moreover,

$$\#\text{RArr}_1(n) = \sum_{0 \leq k \leq n-1} \frac{n!(n-k)}{k!}$$

and this sequence starts by 0, 1, 6, 33, 196, 1305, 9786, 82201, 767208 (Sequence A093964).

\begin{quote}
– Proposition [G., 2021–] –
\end{quote}

The clone $\text{RArr}_1$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

\[
\begin{array}{c}
\star \quad \star \\
\star \quad \star \\
\star \quad \star
\end{array}
\]

Therefore, $\text{RArr}_1$ is a clone realization of the variety of idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_1 \star x_3 = x_1 \star x_2 \star x_3$.

Analog properties hold for the quotient $\text{RArr}_r := \text{Word}/\equiv'$ where $\equiv' := \equiv_{\text{ro}} \cap \equiv_{\text{ll}}$
Let $\equiv := \equiv_{\text{in}} \cap \equiv_{\text{rl}}$ and $\text{RSet}_1 := \text{Word}/\equiv$.

For any $n \geq 0$, the elements of $\text{RSet}_1(n)$ can be seen as pairs $(S, i)$ where $S$ is a nonempty subset of $[n]$ and $i \in S$. Moreover,

$$\# \text{RSet}_1(n) = n \cdot 2^{n-1}$$

and this sequence starts by $0, 1, 4, 12, 32, 80, 192, 448, 1024$ (Sequence A001787).

--- Proposition [G., 2021$-$] ---

The clone $\text{RSet}_1$ admits the presentation $(\mathcal{G}, R)$ where $\mathcal{G} := \mathcal{G}(2) := \{\ast\}$ and $R$ satisfies

$$x_1 \ast x_2 \ast x_3 \ast x_3 \ast x_2 \ast x_2 \ast x_2 \ast x_2.$$ 

Therefore, $\text{RSet}_1$ is a clone realization of the variety of idempotent semigroups wherein the operation $\ast$ satisfies the relation $x_1 \ast x_2 \ast x_3 = x_1 \ast x_3 \ast x_2$.

Analog properties hold for the quotient $\text{RSet}_r := \text{Word}/\equiv'$, where $\equiv' := \equiv_{\text{in}} \cap \equiv_{\text{rl}}$. 


Bi-rooted sets

Let \( \equiv := \equiv_{\text{in}} \cap \equiv_{\|} \cap \equiv_{\text{rl}} \) and \( \text{BRSet} := \text{Word}/\equiv \).

For any \( n \geq 0 \), the elements of \( \text{BRSet}(n) \) can be seen as triples \( (S, i, i') \) where \( S \) is a nonempty subset of \( [n] \) and \( i, i' \in S \). Moreover,

\[
\# \text{BRSet}(n) = n(n + 1) \, 2^{n-2}
\]

and this sequence starts by \( 0, 1, 6, 24, 80, 240, 672, 1792, 4608 \) (Sequence \( \text{A001788} \)).

**Proposition** [G., 2021–]

The clone \( \text{BRSet} \) admits the presentation \( (\mathcal{G}, \mathcal{R}) \) where \( \mathcal{G} := \mathcal{G}(2) := \{\star\} \) and \( \mathcal{R} \) satisfies

\[
\begin{align*}
\star & \underset{\text{R}}{\rightarrow} x_1, \\
\star & \underset{\text{R}}{\rightarrow} x_2, \\
\star & \underset{\text{R}}{\rightarrow} x_3,
\end{align*}
\]

Therefore, \( \text{BRSet} \) is a clone realization of the variety of normal bands, that are idempotent semigroups wherein the operation \( \star \) satisfies the relation \( x_1 \star x_2 \star x_3 \star x_1 = x_1 \star x_3 \star x_2 \star x_1 \).
Arrangements of runs

Let \( \equiv := \equiv_{\text{st}} \cap \equiv_{\text{lo}} \) (stalactite congruence [Hivert, Novelli, Thibon, 2007]) and \( \text{ArrR}_1 := \text{Word}/\equiv \).

For any \( n \geq 0 \), the elements of \( \text{ArrR}_1(n) \) can be seen as nonempty arrangements of runs on \([n]\).

– Examples –

The word 33 111 5 5 2 6 is an element of \( \text{ArrR}_1(9) \). The word 22222 333 1 1 2 is not an element of \( \text{ArrR}_1 \).

– Proposition [G., 2020–] –

The clone \( \text{ArrR}_1 \) admits the presentation \((\mathcal{G}, \mathcal{R})\) where \( \mathcal{G} := \mathcal{G}(2) := \{\star\} \) and \( \mathcal{R} \) satisfies

\[
\begin{align*}
\star & \mathcal{R} \star, \\
x_1 & \mathcal{R} x_2, \\
x_2 & \mathcal{R} x_3, \\
x_1 & \mathcal{R} x_1, \\
x_1 & \mathcal{R} x_2.
\end{align*}
\]

Therefore \( \text{ArrR}_1 \) is a clone realization of semigroups wherein the operation \( \star \) satisfies the relation\( x_1 \star x_2 \star x_1 = x_1 \star x_1 \star x_2 \).

Analog properties hold for the quotient \( \text{ArrR}_r := \text{Word}/\equiv', \) where \( \equiv' := \equiv_{\text{st}} \cap \equiv_{\text{ro}} \).
Pairs of compatible arrangements

Let $\equiv := \equiv_{lo} \cap \equiv_{ro}$ and $\text{PArr} := \text{Word}/\equiv$.

For $n \geq 0$, the elements of $\text{PArr}(n)$ can be seen as pairs $(u, v)$ such that $u$ and $v$ are nonempty arrangements on $[n]$ with $j$ appears in $u$ iff $j$ appears in $v$.

Moreover,

$$\#\text{PArr}(n) = \sum_{k \in [n]} \frac{n!k!}{(n - k)!}$$

and this sequence starts by $0, 1, 6, 51, 748, 17685, 614226, 29354311, 1844279256$ (linked with Sequence A046662).

**– Example –**

$(3261, 1263)$ is an element of $\text{PArr}(6)$.

**– Proposition [G., 2020–] –**

The clone $\text{PArr}$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\ast\}$ and $\mathcal{R}$ satisfies

\[
\begin{align*}
& x_1 x_2 x_3 \xrightarrow{\mathcal{R}} \ast x_2 x_3, \\
& x_1 x_2 x_3 \xrightarrow{\mathcal{R}} \ast x_1, \\
& x_1 x_2 x_3 \xrightarrow{\mathcal{R}} \ast x_3 x_1, \\
& x_1 x_2 x_3 x_4 \xrightarrow{\mathcal{R}} \ast x_3 x_1 x_2, \\
& x_1 x_2 x_3 x_4 \xrightarrow{\mathcal{R}} \ast x_1 x_2 x_3.
\end{align*}
\]

Therefore, $\text{PArr}$ is a clone realization of the variety of regular bands.
Pairs of compatible arrangements of runs

Let $\equiv := \equiv_{st} \cap \equiv_{lo} \cap \equiv_{ro}$ and $\text{PArr}_R := \text{Word}/\equiv$.

For any $n \geq 0$, the elements of $\text{PArr}_R(n)$ can be seen as pairs $(u, v)$ such that $u$ and $v$ are nonempty arrangements of runs of repeated letters on $[n]$, with $u$ and $v$ having the same number of occurrences of any letter.

**Example**

$(3222611, 22211263)$ is an element of $\text{PArr}_R(6)$.

$(221, 12)$ is not.

**Proposition [G., 2020–]**

The clone $\text{PArr}_R$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

Therefore, $\text{PArr}_R$ is a clone realization of semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_1 \star x_2 \star x_3 \star x_1 = x_1 \star x_2 \star x_1 \star x_3 \star x_1 = x_1 \star x_2 \star x_3 \star x_1 \star x_1$. 
Rooted arrangements of runs

Let $\equiv := \equiv_{st} \cap \equiv_{lo} \cap \equiv_{rl}$ and $\text{RArrR}_l := \text{Word}/\equiv$.

For any $n \geq 0$, the elements of $\text{RArrR}_l(n)$ can be seen as nonempty arrangements of runs on $[n]$ wherein the rightmost run is marked or a run of length two or more is marked.

− Proposition [G., 2021–] −

The clone $\text{RArrR}_l$ admits the presentation $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} := \mathcal{G}(2) := \{\star\}$ and $\mathcal{R}$ satisfies

![Diagram of presentation]

Therefore, $\text{RArrR}_l$ is a clone realization of semigroups wherein the operation $\star$ satisfies the relation $x_1 x_2 x_1 x_3 = x_1 x_1 x_2 x_3$.

Analog properties hold for the quotient $\text{RArrR}_r := \text{Word}/\equiv'$, where $\equiv' := \equiv_{st} \cap \equiv_{ro} \cap \equiv_{ll}$. 