A hierarchy of operads on words

Samuele Giraudo
LIGM, Université Gustave Eiffel

Joint work with Camille Combe

Séminaire ADA
LMPA, Calais

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1. The associative symmetric operad
Composition of permutations

Given two permutations $\sigma$ and $\nu$, and $i \in [|\sigma|]$, let $\sigma \square_i \nu$ be the permutation having as permutation matrix the one obtained by inserting the matrix of $\nu$ onto the $i$-th point of the matrix of $\sigma$.

**– Example –**

\[
\begin{array}{c}
35412 \\ 132
\end{array}
\square_3
\begin{array}{c}
3746512
\end{array}
\]

This composition is a map

\[\square_i : \mathcal{S}(n) \times \mathcal{S}(m) \to \mathcal{S}(n + m - 1), \quad n, m \geq 1, \quad i \in [n],\]

where $\mathcal{S}(n)$ is the set of all permutations of size $n \geq 1$. 
Algebraic point of view

The pair $\text{Per} := (\mathcal{G}, \boxplus_i)$ is an operad: any permutation $\sigma$ of size $n \geq 1$ is seen as an operator of arity $n$ and the composition map behaves like the usual composition of operators.

One can ask about a minimal generating set of this operad, that is a smallest set $\mathcal{G}$ such that each permutation $\sigma$ can be obtained by composing elements of $\mathcal{G}$ with each other.

- Proposition -

The set $\mathcal{G}$ of all simple permutations of size $n \geq 2$ is a minimal generating set of $\text{Per}$.

A permutation $\sigma$ is simple if no factor of $\sigma$ of length between 2 and $|\sigma| - 1$ is an interval.

- Examples -

The permutation $\begin{array}{c}71435628 \Rightarrow 513426 \end{array} \ 3 \ 213$ is not simple; $\begin{array}{c}4135726\end{array}$ is simple.

The elements of $\mathcal{G}$ are enumerated, arity by arity, by Sequence $A111111$ beginning by

$0, 2, 0, 2, 6, 46, 338, 2926, 28146, 298526$. 
Linearization of Per

In order to pursue the algebraic study of this operad, we consider its linearized version: from now, \( \text{Per} \) is the linear span \( \text{Span}(\mathcal{G}) \) of \( \mathcal{G} \) over a field \( \mathbb{K} \) of characteristic zero.

The set \( \{ E_\sigma : \sigma \in \mathcal{G} \} \) is hence a basis of \( \text{Per} \), called elementary basis. We set also

\[
E_\sigma \circ_i E_\nu := E_{\sigma \sqcup_i \nu}.
\]

### Examples

In \( \text{Per} \),

\[
E_{35412} \circ_3 E_{132} = E_{3746512},
\]

\[
(E_{312} - E_{2341}) \circ_2 (2E_{12} - E_{321}) = 2E_{4123} - E_{53214} - 2E_{23451} + E_{254361}.
\]

The main interest to consider such a linearized version of \( \text{Per} \) is that we can consider change of bases, quotients, and linear maps and operad morphisms to other operads.
Partial order on permutations

Let \( \preceq \) be the partial order relation on \( \mathcal{S} \) satisfying \( \sigma \preceq \nu \) if \( \text{Inv}(\sigma) \subseteq \text{Inv}(\nu) \), where
\[
\text{Inv}(\pi) := \{(i, j) : i < j \text{ and } \pi(i) > \pi(j)\}.
\]

This is the left weak order on permutations.

**Example**

Let \( \sigma := 23154 \) and \( \nu := 25143 \).

Since \( \text{Inv}(\sigma) = \{(1, 3), (2, 3), (4, 5)\} \) and \( \text{Inv}(\nu) = \{(1, 3), (2, 3), (2, 4), (2, 5), (4, 5)\} \), we have \( \sigma \preceq \nu \).

Let the elements \( F_{\sigma}, \sigma \in \mathcal{S} \), of \( \text{Per} \) defined by
\[
F_{\sigma} := \sum_{\sigma \preceq \sigma'} \mu_{\preceq}(\sigma, \sigma')E_{\sigma'},
\]

where \( \mu_{\preceq} \) is the Möbius function of the poset \( (\mathcal{S}, \preceq) \).

**Example**

\[
F_{4123} = E_{4123} - E_{4132} - E_{4213} + E_{4321}
\]
By Möbius inversion, for any $\sigma \in S$,

$$E_\sigma = \sum_{\sigma \lessdot \sigma'} F_{\sigma'}.$$ 

Therefore, by triangularity, the family $\{F_\sigma : \sigma \in S\}$ is a basis of $\text{Per}$.

A natural question now is to describe the composition $\circ_i$ of $\text{Per}$ on this new basis.

--- **Theorem** [Aguiar, Livernet, 2007] ---

For any $\sigma, \nu \in S$,

$$F_\sigma \circ_i F_\nu = \sum_{\sigma \square_i \nu \preceq \pi \preceq \sigma \boxdot_i \nu} F_\pi,$$

for a certain composition operation $\square_i$ on $S$.

In other terms, the support of any composition expressed in the $F$-basis has, as support, an interval of the left weak order.
There are other algebraic and combinatorial structures on permutations:
- symmetric groups;
- lattices for the left and right weak order;
- Hopf bialgebra of Malvenuto-Reutenauer;
- dendriform algebra.

The main motivation of this work is to construct an alternative operad structure on $\mathcal{G}$.

We obtain
- a new operad on permutations;
- operads on some generalizations of permutations;
- operads on Fuss-Catalan objects.

We introduce several bases for these operads, analogs of the compositions operations $\square_i$ and $\blacksquare_i$, and analogs of the left weak order on permutations.
2. Cliffs and related objects
A range map is a map $\delta : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ and a $\delta$-cliff is a word $u$ on $\mathbb{N}$ such that for all $i \in [|u|]$, $0 \leq u(i) \leq \delta(i)$.

The length of $u$ is denoted by $\ell(u)$. The size of $u$ is denoted by $|u|$ and is $\ell(u) + 1$.

Let $\text{Cl}_\delta$ be the set of all $\delta$-cliffs.

**Example**

If $\delta = 102222\ldots$, then

$$\text{Cl}_\delta(5) = \{0000, 0001, 0002, 0010, 0011, \ldots, 1022\}.$$  

We will consider mainly two sorts of range maps:

- for any $m \in \mathbb{N}$, $m$ as the range map satisfying $m(i) = (i - 1)m$;
- for any $c \in \mathbb{N}$, $c$ as the range map satisfying $c(i) = c$.
A $\delta$-hill is a weakly increasing $\delta$-cliff. Let $H_\delta$ be the set of all $\delta$-hills.

Some sets of cliffs or hills are in one-to-one correspondence with combinatorial families:

- **Example**
  - $\text{Cl}_1(n)$ with integers compositions of $n$.
    
    $1100010 \in \text{Cl}_1(8) \leftrightarrow (1, 1, 4, 2)$

- **Example**
  - $\text{Cl}_1(n)$ with permutations of size $n - 1$.
    
    $002323 \in \text{Cl}_1(7) \leftrightarrow 436512$

- **Example**
  - $\text{Cl}_m(n)$ with incr. $m+1$-ary trees of $n-1$ nodes.
    
    $0230228 \in \text{Cl}_2(8) \leftrightarrow$

- **Example**
  - $H_m(n)$ with $m$-Dyck paths with $n - 1$ rising steps.
    
    $02366 \in H_2(6) \leftrightarrow$

- **Example**
  - $H_c(n)$ with N/E paths from $(0, 0)$ to $(n - 1, c)$.
    
    $1123 \in H_3(5) \leftrightarrow$
Let $\preceq$ be the order relation on $\text{Cl}_\delta$ such that $u \preceq v$ if $u(i) \leq v(i)$ for all $i$.

- **Example** –

Hasse diagram of $(\text{Cl}_1(5), \preceq)$:

This is the Boolean lattice.

Hasse diagram of $(\text{Hi}_1(5), \preceq)$:

Also known as the Stanley lattice [Stanley, 1975].
3. Operad structures on cliffs
Interstice operads

For any alphabet $A$, let

$$I(A) := \bigoplus_{n \geq 1} \text{Span}(A^{n-1}).$$

The set $\{E_u : u \in A^*\}$ is a basis of $I(A)$. The arity of any $E_u$ in $I(A)$ is $\ell(u) + 1$ where $\ell(u)$ is the length of the word $u$.

Let the composition $\circ_i$ defined by $E_u \circ_i E_v := E_{u \Box_i v}$ where $u \Box_i v := u(1, i - 1) \; v \; u(i, \ell(u))$.

- Example -

For $A := \{a, b, c\}$, we have

$$E_{aabacb} \circ_4 E_{cbaa} = E_{aab\;cbaa\;acb}.$$

in $I(A)$.

The pair $(I(A), \circ_i)$ is an operad, called $A$-interstice operad.
Let the subspace $\text{Cl}_\delta := \text{Span}(\{E_u : u \in \text{Cl}_\delta\})$ of $\mathbf{I}(\mathbb{N})$.

Two observations:

- when $\delta$ is a weakly increasing, if $u, v \in \text{Cl}_\delta$, then $u \square_i v \in \text{Cl}_\delta$. For this reason, $\text{Cl}_\delta$ is a suboperad of $\mathbf{I}(\mathbb{N})$;

- when $\delta$ is not weakly increasing, $\text{Cl}_\delta$ is not a suboperad of $\mathbf{I}(\mathbb{N})$. For instance, if $\delta = 0110 \ldots$, even if $01 \in \text{Cl}_\delta$, we have $01 \square_2 01 = 0101 \notin \text{Cl}_\delta$.

**Objective**

Build substructures of $\mathbf{I}(\mathbb{N})$ on $\delta$-cliffs for the largest possible class of range maps $\delta$.

A first try consists in considering quotients $\mathbf{I}(\mathbb{N})/\mathcal{V}_\delta$, where $\mathcal{V}_\delta := \text{Span}(\{E_u : u \in \mathbb{N}^* \setminus \text{Cl}_\delta\})$.

This does not work in general. One has a counter-example for $\delta := 0110 \ldots$ since $E_{11} \in \mathcal{V}_\delta$ and $E_0 \in \mathbf{I}(\mathbb{N})$ but $E_{11} \circ_1 E_0 = E_{011} \notin \mathcal{V}_\delta$. 
Quotients of cliff operads

Given a range map $\delta$, let $\bar{\delta}$ be the range map defined by $\bar{\delta}(i) := \max\{\delta(1), \ldots, \delta(i)\}$.

By construction, $\bar{\delta}$ is weakly increasing, and thus, $\text{Cl}_{\bar{\delta}}$ is an operad.

Let the subspace $\mathcal{V}_\delta := \text{Span}(\{E_u : u \in \text{Cl}_{\bar{\delta}} \setminus \text{Cl}_\delta\})$ of $\text{Cl}_{\bar{\delta}}$.

**Theorem** [Combe, G., 2021]

For any range map $\delta$, the space $\mathcal{V}_\delta$ is an operad ideal of the operad $\text{Cl}_{\bar{\delta}}$ iff $\delta$ is unimodal. Therefore, when $\delta$ is unimodal, the space $\text{Cl}_{\bar{\delta}} / \mathcal{V}_\delta$ is an operad.

A range map $\delta$ is **unimodal** if there is no $i_1 < i_2 < i_3$ such that $\delta(i_1) > \delta(i_2) < \delta(i_3)$.

For any unimodal range map $\delta$, we set $\text{Cl}_\delta := \text{Cl}_{\bar{\delta}} / \mathcal{V}_\delta$. The composition of this operad satisfies

$$E_u \circ_i E_v = \chi_\delta(u \square_i v)E_u \square_i v,$$

where $\chi_\delta : \mathbb{N}^* \rightarrow \mathbb{K}$ is the map defined for any $u \in \mathbb{N}^*$ by $\chi_\delta(u) := 1$ if $u \in \text{Cl}_\delta$ and by $\chi_\delta(u) := 0$ otherwise.
Examples of quotients of cliff operads

– Example –

When $\delta$ is weakly increasing, we have $\bar{\delta} = \delta$, so that $V_\delta$ is the null space and $\text{Cl}_\delta$ is a suboperad of $I(\mathbb{N})$.

– Example –

When $\delta := 1232 \ldots$, we have $\bar{\delta} = 1233 \ldots$. In $\text{Cl}_\delta$,

$$E_{002} \circ_3 E_{10} = E_{00102} \quad \text{and} \quad E_{002} \circ_3 E_{1311} = 0.$$ 

– Example –

When $\delta := 00233421 \ldots$, we have $\bar{\delta} = 00233444 \ldots$. In $\text{Cl}_\delta$,

$$E_{0011} \circ_4 E_{002} = E_{0010021} \quad \text{and} \quad E_{0011} \circ_5 E_{002} = 0.$$
Due to the partial composition on the E-basis, when $\delta$ is unimodal, the E-basis is a set-operad of $\text{Cl}_\delta$ iff $\delta$ is weakly increasing.

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**Objective**

Show that when $\delta$ is unimodal, $\text{Cl}_\delta$ admits a set-operad basis so that this operad is a set-operad.

Let the elements $F_u$, $u \in \text{Cl}_\delta$, of $\text{Cl}_\delta$ defined by

$$F_u := \sum_{u \preceq u'} \mu_{\preceq}(u, u') E_{u'}.$$

Let also the elements $H_u$, $u \in \text{Cl}_\delta$, of $\text{Cl}_\delta$ defined by

$$H_u := \sum_{u' \preceq u} F_{u'}.$$

By Möbius inversion and by triangularity, the families $\{F_u : u \in \text{Cl}_\delta\}$ and $\{H_u : u \in \text{Cl}_\delta\}$ are bases of $\text{Cl}_\delta$. 
**F-basis and composition**

**– Example –**

In $\text{Cl}_{224}$... we have

$$F_{1221} = E_{1221} - E_{1222} - E_{1231} - E_{2221} + E_{1232} + E_{2222} + E_{2231} - E_{2232}.$$  

For any $u, v \in \text{Cl}_\delta$, let $u \boxtimes_i v$ be the word obtained from $u \boxtimes_i v$ by setting to the maximal value the letters of $u$ and $v$ which are locally maximal.

**– Examples –**

Let $\delta := 11321 \ldots$. We have

- $1022\square_3 101 = 1030121$;
- $1022\square_4 003 = 1020011$.

**– Proposition [Combe, G., 2021] –**

For any unimodal range map $\delta$,

$$F_u \circ_i F_v = \chi_\delta(u \boxtimes_i v) \sum_{u \boxtimes_i v \prec w \preceq u \boxtimes_i v} F_w.$$  

**– Example –**

In $\text{Cl}_{123454}$...,

$$F_{013} \circ_2 F_{103} = F_{010313} + F_{010314} + F_{010413} + F_{010414} + F_{020313} + F_{020314} + F_{020413} + F_{020414}.$$
H-basis and composition

- Examples -

In $\text{Cl}_{321}$... we have

$$H_{2101} = F_{0000} + F_{0001} + F_{0101} + F_{1001} + F_{1100} + F_{1101} + F_{2000} + F_{2001} + F_{2100} + F_{2101}.$$ 

The $\delta$-reduction of $u \in \mathbb{N}^*$ is the $\delta$-cliff $r_\delta(u)$ defined by $(r_\delta(u))(i) := \min\{u(i), \delta(i)\}$ for all $i$.

- Example -

- $r_1(212066) = 012045$
- $r_2(212066) = 012066$

- Proposition [Combe, G., 2021] -

For any unimodal range map $\delta$,

$$H_u \circ_i H_v = H_{r_\delta(u \circ_i v)}.$$ 

- Examples -

In $\text{Cl}_{2234}$... we have

- $H_{01} \circ_3 H_{221} = H_{01341};$
- $H_{2033} \circ_3 H_{12} = H_{201422}.$
From now, we have constructed three operads, fitting in the diagram

\[ \mathcal{I}(\mathbb{N}) \leftarrow \mathbf{Cl}_\delta \rightarrow \mathbf{Cl}_\delta \]

where \( \delta \) is any unimodal range map.

In this case, \( \mathbf{Cl}_\delta \) is an operad and admits three bases: the E-basis, F-basis, and the H-basis. This last one is a set-operad basis, showing that \( \mathbf{Cl}_\delta \) is a set-operad.

Question:
Describe a minimal generating set \( \mathcal{G}_\delta \) of \( \mathbf{Cl}_\delta \) and a minimal generating set of the space \( \mathcal{R}_\delta \) of nontrivial relations of \( \mathbf{Cl}_\delta \).
Minimal generating set

A nonempty $\delta$-cliff $w$ is $\delta$-prime if $w = u \square_1 v$ with $u, v \in \text{Cl}_\delta$ implies $(u, v) \in \{(w, \epsilon), (\epsilon, w)\}$.

We denote by $\mathcal{P}_\delta$ the set of all $\delta$-prime $\delta$-cliffs.

- Examples -

Let $\delta := 12233211 \ldots$. We have

- $10033 \in \mathcal{P}_\delta$;
- $121332 \in \mathcal{P}_\delta$;
- $11 \notin \mathcal{P}_\delta$ since $11 = 1 \square_1 1$;
- $11222 \notin \mathcal{P}_\delta$ since $11222 = 122 \square_2 12$.

From the behavior if the composition of $\text{Cl}_\delta$ over the $E$-basis, $\{E_u : u \in \mathcal{P}_\delta\}$ is a minimal generating set of $\text{Cl}_\delta$.

- Examples -

- $\mathfrak{S}_2 = \{E_0, E_1, E_2\}$;
- $\mathfrak{S}_{12211} = \{E_0, E_1, E_{02}, E_{12}, E_{022}, E_{122}\}$;
- $\mathfrak{S}_1 = \{E_0, E_{01}, E_{002}, E_{011}, E_{012}, E_{0003}, E_{0013}, E_{0021}, E_{0022}, E_{0023}, E_{0103}, E_{0111}, E_{0112}, E_{0113}, E_{0121}, E_{0122}, E_{0123}, \ldots\}$. 
Finite and infinite presentations

A range map $\delta$ is 1-dominated if there is a $k \geq 1$ such that for all $k' \geq k$, $\delta(1) \geq \delta(k')$.

### Examples

The range map
- $c, c \in \mathbb{N}$, is 1-dominated;
- $23567732 \ldots$ is 1-dominated;
- $21 \ldots$ is 1-dominated;
- $m, m \geq 1$, is not 1-dominated;
- $01 \ldots$ is not 1-dominated;
- $2356773 \ldots$ is not 1-dominated.

### Theorem [Combe, G., 2021]

For any unimodal range map $\delta$, $\mathcal{G}_\delta$ is finite iff $\delta$ is 1-dominated.

### Proposition [Combe, G., 2021]

For any unimodal range map $\delta$, if $\delta$ is not 1-dominated, then $\mathcal{R}_\delta$ is not finitely generated.
Operads on $m$-increasing trees

Immediately from the definition of $m$-cliffs,

$$\dim \mathbf{Cl}_m(n) = \prod_{i \in [n-1]} 1 + (i - 1)m.$$ 

Since $\dim \mathbf{Cl}_1(n) = (n - 1)!$, $\mathbf{Cl}_1$ is an operad on permutations with shifted arities.

--- Proposition [Combe, G., 2021] ---

For any $m \geq 0$, $\# \mathfrak{S}_m(1) = 0$, $\# \mathfrak{S}_m(2) = 1$, and, for any $n \geq 3$,

$$\# \mathfrak{S}_m(n) = \frac{m}{m + 1} \dim \mathbf{Cl}_m(n).$$

The space $\mathcal{R}_m$ is not finitely generated.

The sequence of the numbers of generators of $\mathcal{R}_1$ begins with $0, 0, 1, 2, 7, 33, 185, 1211$.

The space $\mathcal{R}_1$ contains nonhomogeneous and nonquadratic nontrivial relations, as e.g.,

$$E_{002} \circ_3 E_{01} - (E_0 \circ_2 E_0) \circ_3 E_{012}.$$
Quotient operads

– Objective –

Given a subset $S$ of $\text{Cl}_\delta$, describe a general construction for a substructure of the operad $\text{Cl}_\delta$ on $S$.

Let the subspace $\mathcal{V}_S := \text{Span}(\{F_u : u \in \text{Cl}_\delta \setminus S\})$ of $\text{Cl}_\delta$.

– Proposition [Combe, G., 2021] –

Let $\delta$ be a unimodal range map $\delta$ and $S$ be a nonempty graded subset of $\text{Cl}_\delta$. If $S$ is closed by subword reduction, then $\mathcal{V}_S$ is an operad ideal of $\text{Cl}_\delta$ and $\text{Cl}_S$ is a quotient operad of $\text{Cl}_\delta$.

The set $S$ is closed by subword reduction if for any $w \in S$, all subwords $w'$ of $w$ satisfy $r_\delta(w') \in S$. 
Alternative bases and composition

**Theorem [Combe, G., 2021]**

Let $\delta$ be a unimodal range map and $S$ be a nonempty graded subset of $\text{Cl}_\delta$ such that $S$ is closed by subword reduction. For any $u, v \in S$ and $i \in [|u|]$, 

$$F_u \circ_i F_v = \chi_\delta(u \square_i v) \sum_{\substack{w \in S \ni \ u \square_i v \ll w \ll u \square_i v}} F_w.$$  \hfill (1)

Moreover, when for any $n \geq 1$, $S(n)$ is a sublattice of $\text{Cl}_\delta(n)$, if (1) is different from 0, the support of this element is an interval of the poset $S$.

Let $\theta_S : \text{Cl}_\delta \to \text{Cl}_S$ be the canonical projection map satisfying

$$\theta_S(F_w) := \begin{cases} F_w & \text{if } w \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Let the bases $\{E_u : u \in S\}$ and $\{H_u : u \in S\}$ of $\text{Cl}_S$ defined by

$$E_w := \theta_S(E_w) = \sum_{w' \in S, w \ll w'} F_{w'} \quad \text{and} \quad H_w := \theta_S(H_w) = \sum_{w' \in S, w' \ll w} F_{w'}.$$
Operads on $m$-Dyck paths

For any unimodal range map $\delta$, let $\mathbf{H}_\delta := \text{Cl}_{\mathbf{H}_\delta}$. Since $\mathbf{H}_\delta$ is closed by subword reduction, $\mathbf{H}_\delta$ is a well-defined operad.

Since $\mathbf{H}_m(n)$ is in one-to-one correspondence with $m + 1$-ary trees with $n - 1$ internal nodes,

$$\dim \mathbf{H}_m(n) = \text{cat}_m(n - 1)$$

where

$$\text{cat}_m(n) := \frac{1}{mn + 1} \binom{mn + n}{n}$$

is the $n$-th $m$-Fuss-Catalan number.

\[ - \text{ Proposition [Combe, G., 2021] - } \]

If $n \geq 2$, then $\# \mathcal{G}_{\mathbf{H}_1}(n) = \text{cat}_1(n - 2)$.

The sequence of the numbers of generators of $\mathcal{G}_{\mathbf{H}_2}$ begins with 0, 1, 2, 7, 29, 133, 654, 3383, 18179.

The sequence of the cardinalities of $\mathcal{R}_{\mathbf{H}_1}$ begins by 0, 0, 1, 2, 6, 18, 60, 197.

The space $\mathcal{R}_{\mathbf{H}_1}$ contains nonhomogeneous and nonquadratic nontrivial relations, as e.g.,

$$(E_0 \circ_1 E_0) \circ_1 E_{01} - E_{01} \circ_3 E_{01}.$$
Operads on paths in $c$-rectangles

Since $H_{c}(n)$ is in one to one correspondence with paths from $(0,0)$ to $(n-1,c)$ made of north and east steps,

$$\dim H_{c}(n) = \binom{n+c-1}{c}.$$ 

--- Theorem [Combe, G., 2021] ---

The operad $H_{c}$ admits the following presentation. It is minimally generated by $G_{H_{c}} = \{E_{0}, \ldots, E_{c}\}$ and the space of nontrivial relations $R_{H_{c}}$ is generated by

$$E_{a} \circ_{1} E_{b} = E_{b} \circ_{2} E_{a'}, \quad 0 \leq b \leq c, \quad 0 \leq a, a' \leq b,$$

$$E_{b} \circ_{1} E_{a} = E_{a} \circ_{2} E_{b}, \quad 0 \leq b \leq c, \quad 0 \leq a \leq b$$

of the free operad generated by $G_{H_{c}}$.

As a side remarks,
- $H_{1}$ is the Koszul dual of the duplicial operad [Brouder, Frabetti, 2003];
- $H_{2}$ is the Koszul dual of the triplicial operad [Leroux, 2011].
Conclusion and open questions

We have introduced a hierarchy of operads on words

\[ \text{I}(\mathbb{N}) \quad \xrightarrow{} \quad \text{Cl}_\delta \quad \xrightarrow{} \quad \text{Cl}_\delta \quad \xrightarrow{} \quad \text{Hi}_\delta \]

\[ \xleftarrow{} \quad \text{Cl}_m \quad \xrightarrow{} \quad \text{Hi}_m \]

\[ \xrightarrow{} \quad \text{Cl}_\varepsilon \quad \xrightarrow{} \quad \text{Hi}_\varepsilon \]

where \( \delta \) is any unimodal range map.

Some of these, like \( \text{Cl}_m \) and \( \text{Hi}_m \) are very singular objects since they have complicated presentations.

Some questions:

1. provide a sufficient condition for the fact that \( \mathcal{R}_\delta \) is not finitely generated;
2. describe the presentations of \( \text{Cl}_m \) and \( \text{Hi}_m \);
3. find morphisms involving the operads \( \text{Cl}_\delta \) (or its quotients) and other operads arising in algebraic combinatorics (like \( \text{Per} \), the dendriform operad, the pre-Lie operad, etc.).