Three interacting families of Fuss-Catalan posets

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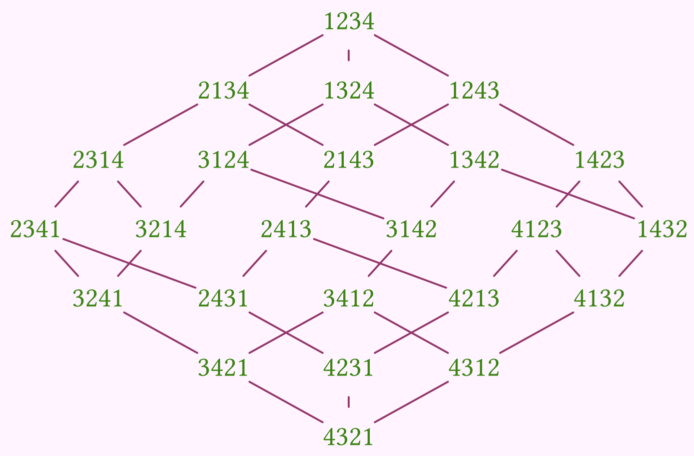
Motivations: associative algebras and posets

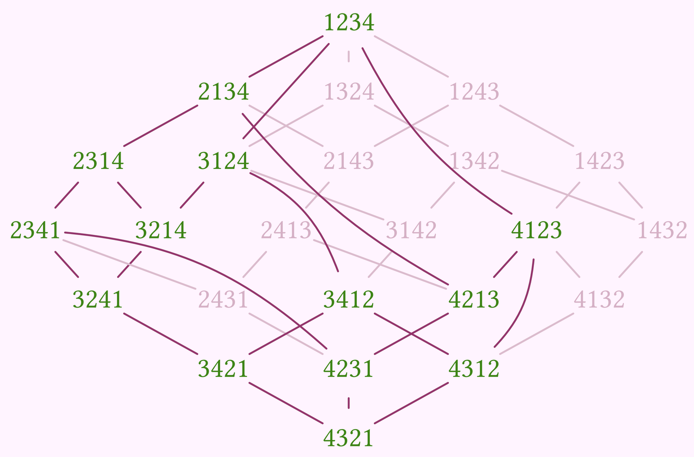
Weak order on permutations

Poset $(\mathfrak{S}(n), \preccurlyeq_{\mathrm{w}})$ where $\preccurlyeq_{\mathrm{w}}$ is the order relation with the covering relation $\lessdot$ defined by $u\, ab\, v \lessdot u\, ba\, v$ where $a < b$, and u and v are any words.



Tamari order on 132-avoiding permutations

Poset $(\mathcal{B}(n), \preccurlyeq_{\mathrm{t}})$ where $\mathcal{B}(n)$ is the subset of $\mathfrak{S}(n)$ of permutations avoiding 132, and $\preccurlyeq_{\mathrm{t}}$ is the restriction of $\preccurlyeq_{\mathrm{w}}$ on $\mathcal{B}(n)$.



Malvenuto-Reutenauer algebra

Algebra FQSym on the linear span of $\{\mathsf{F}_\sigma : \sigma \in \mathfrak{S}\}$, endowed with the **shifted shuffle product**, defined by

$$\mathsf{F}_\sigma \cdot \mathsf{F}_\nu := \sum_{\pi \in \sigma \,\bar{\sqcup\!\sqcup}\, \nu} \mathsf{F}_\pi.$$

For instance,

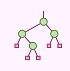
$$\mathsf{F}_{312} \cdot \mathsf{F}_{21} = \mathsf{F}_{31254} + \mathsf{F}_{31524} + \mathsf{F}_{31542} + \mathsf{F}_{35124} + \mathsf{F}_{35142} + \mathsf{F}_{35412} + \mathsf{F}_{53124} + \mathsf{F}_{53142} + \mathsf{F}_{53412} + \mathsf{F}_{54312}.$$

Loday-Ronco algebra

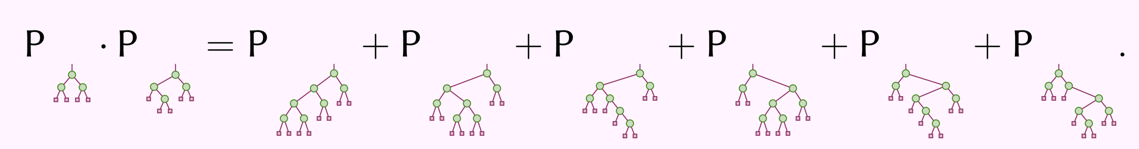
Algebra PBT, defined as the subalgebra of FQSym spanned by the elements

$$\mathsf{P}_{\mathfrak{t}} := \sum_{\substack{\sigma \in \mathfrak{S} \\ \mathrm{bst}(\sigma) = \mathfrak{t}}} \mathsf{F}_\sigma$$

For instance,

P  $= \mathsf{F}_{2143} + \mathsf{F}_{2413} + \mathsf{F}_{4213}$

and



Link between FQSym and the weak order

The product of FQSym rephrases as

$$\mathsf{F}_\sigma \cdot \mathsf{F}_\nu = \sum_{\pi \in [\sigma \,/\, \nu,\ \sigma \,\backslash\, \nu]_{\preccurlyeq_{\mathrm{w}}}} \mathsf{F}_\pi,$$

where $\sigma \,/\, \nu := \sigma\bar{\nu}$, $\sigma \,\backslash\, \nu = \bar{\nu}\sigma$, and $\bar{\nu}$ is obtained by incrementing by $|\sigma|$ each letter of ν.

Link between PBT and the Tamari order

A similar property holds for PBT: its product expresses as

$$\mathsf{P}_{\mathfrak{t}} \cdot \mathsf{P}_{\mathfrak{s}} = \sum_{\mathfrak{r} \in [\mathfrak{t} \,/\, \mathfrak{s},\ \mathfrak{t} \,\backslash\, \mathfrak{s}]_{\preccurlyeq_{\mathrm{t}}}} \mathsf{P}_{\mathfrak{r}},$$

where $/$ and $\backslash$ are some grafting operations on binary trees.

Motivations and main results

- In what extent **different orders** on permutations lead to similar constructions and properties?
- We construct a **generalization of FQSym** based on posets involving generalizations of Lehmer codes of permutations.
- We construct **two analogues and generalizations of PBT** in this context based on two Fuss-Catalan posets.

Cliff posets and three Fuss-Catalan posets

δ-cliffs

- A *range map* is a map $\delta : \mathbb{N} \setminus \{0\} \to \mathbb{N}$.
- A *δ-cliff* of size n is a word u of length n s.t. $u_i \in [0, \delta(i)]$, for all $i \in [n]$.
- For any $m \in \mathbb{N}$, let $\mathbf{m}$ be the range map satisfying $\mathbf{m}(i) := m(i-1)$.

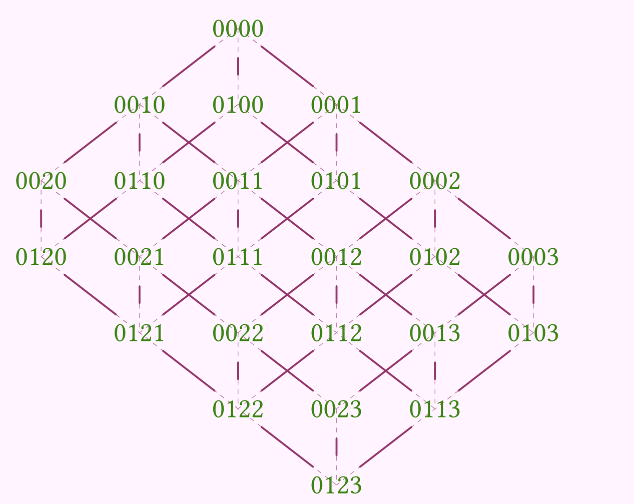
The set of $\mathbf{1}$-cliffs of size n is in one-to-one correspondence with $\mathfrak{S}(n)$: a possible bijection sends any permutation to its **Lehmer code**, which is a $\mathbf{1}$-cliff.

δ-cliff posets

Let Cl_δ be the set of δ-cliffs.

Let $\preccurlyeq$ be the partial order relation on each $\mathrm{Cl}_\delta(n)$ s.t. $u \preccurlyeq v$ if $u_i \leqslant v_i$ for all $i \in [n]$.

Since there is a bijective morphism from $(\mathfrak{S}(n), \preccurlyeq_{\mathrm{w}})$ to $(\mathrm{Cl}_{\mathbf{1}}(n), \preccurlyeq)$, this last is an **order extension** of the first.



The poset $(\mathrm{Cl}_{\mathbf{1}}(4), \preccurlyeq)$.

Subposets and properties

Let $\mathcal{S}$ be a subset of Cl_δ. This subset is endowed with the restriction of $\preccurlyeq$ on $\mathcal{S}$.

The poset $\mathcal{S}$ is

- *straight* if the covering relation of $\mathcal{S}$ changes exactly one letter;
- *coated* if for any $u, v \in \mathcal{S}$ such that $u \preccurlyeq v$, for all $i \in [|u|]$, $u_1 \dots u_i v_{i+1} \dots v_{|v|} \in \mathcal{S}$;
- *closed by prefix* if for any $u \in \mathcal{S}$, all prefixes of u belong to $\mathcal{S}$;
- *minimally extendable* if $\epsilon \in \mathcal{S}$ and for any $u \in \mathcal{S}$, $u0 \in \mathcal{S}$;
- *maximally extendable* if $\epsilon \in \mathcal{S}$ and for any $u \in \mathcal{S}$, $u\,\delta(|u|+1) \in \mathcal{S}$.

Order theoretic properties

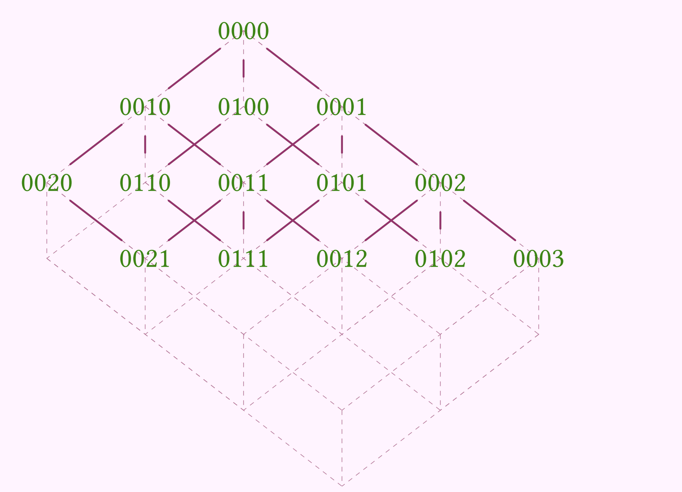
- If $\mathcal{S}$ is straight and coated, then $\mathcal{S}(n)$ admits an **EL-labeling**.
- If $\mathcal{S}$ is minimally (resp. maximally) extendable, then $\mathcal{S}(n)$ is a **meet** (resp. **join**) **semi-lattice**.
- If $\mathcal{S}$ is nested and closed by prefix, then $\mathcal{S}$ is a lattice **constructible by interval doubling**.

Avalanche posets

Let Av_δ be the subset of Cl_δ containing all δ-cliffs u s.t. for all nonempty prefixes $u_1 \dots u_k$ of u, $u_1 + \dots + u_k \leqslant \delta(k)$, called *$\delta$-avalanches*.

In general, these posets are

- graded;
- meet semi-sublattices of Cl_δ;
- admit EL-labelings.



The poset $(\mathrm{Av}_{\mathbf{1}}(4), \preccurlyeq)$.

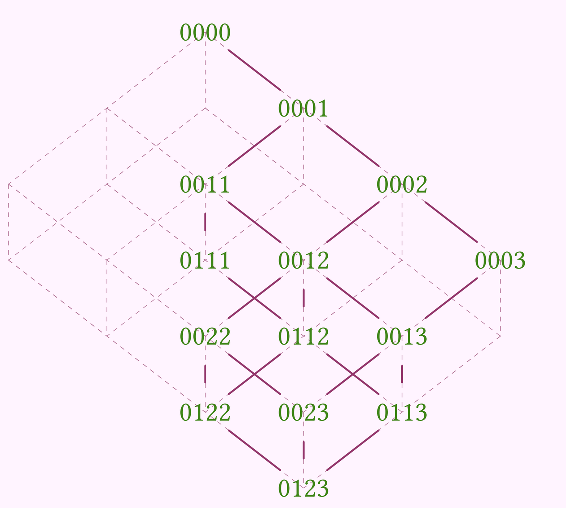
Hill posets

Let Hi_δ be the subset of Cl_δ containing all weakly increasing δ-cliffs, called *δ-hills*.

In general, when δ is weakly increasing, these posets are

- sublattices of Cl_δ;
- EL-shellable;
- constructible by interval doubling.

When $\delta = \mathbf{1}$, these are **Stanley lattices**.



The poset $(\mathrm{Hi}_{\mathbf{1}}(4), \preccurlyeq)$.

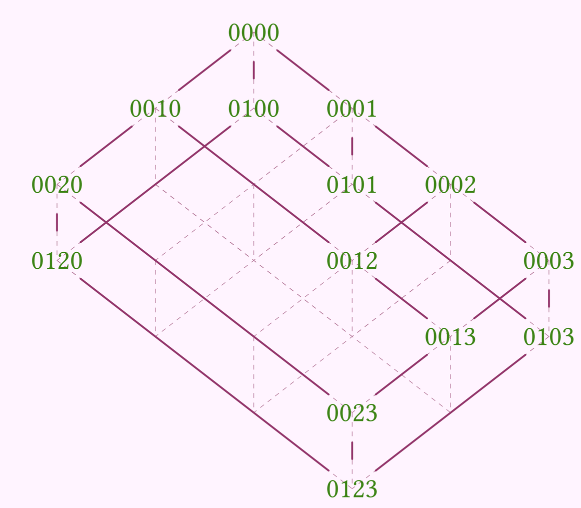
Canyon lattices

Let Ca_δ be the subset of Cl_δ containing all δ-cliffs u s.t. $u_{i-j} \leqslant u_i - j$, for all $i \in [|u|]$ and $j \in [u_i]$ satisfying $i - j \geqslant 1$, called *δ-canyons*.

In general, when δ is increasing, these posets are

- lattices (but not sublattices of Cl_δ);
- EL-shellable;
- constructible by interval doubling.

When $\delta = \mathbf{1}$, these are **Tamari lattices**.



The poset $(\mathrm{Ca}_{\mathbf{1}}(4), \preccurlyeq)$.

Interactions and common properties

For any $m \geqslant 0$ and $n \geqslant 0$,

$$\#\mathrm{Av}_{\mathbf{m}}(n) = \#\mathrm{Hi}_{\mathbf{m}}(n) = \#\mathrm{Ca}_{\mathbf{m}}(n) = \frac{1}{mn+1}\binom{mn+n}{n}.$$

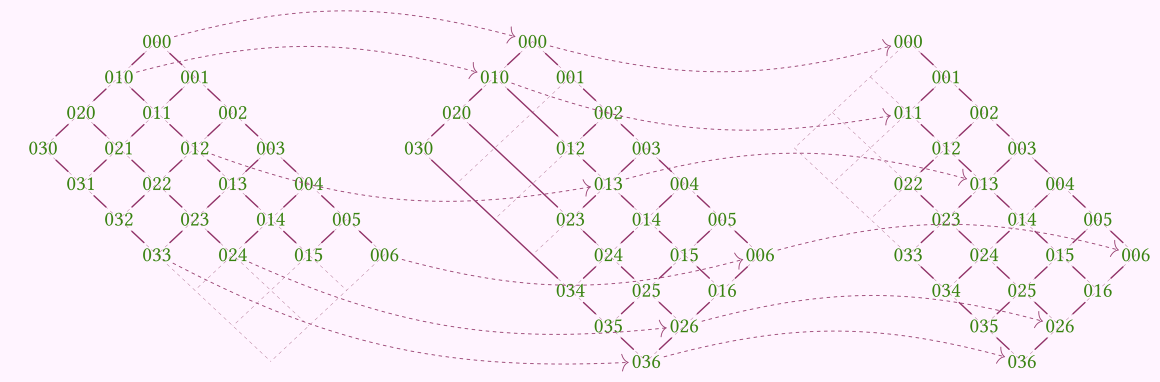
This is the n-th m-**Fuss-Catalan number**.

When δ is an increasing range map, the three posets fit into the diagram

$$\mathrm{Av}_\delta(n) \longrightarrow \mathrm{Ca}_\delta(n) \longrightarrow \mathrm{Hi}_\delta(n)$$

of bijective poset morphisms, so that $\mathrm{Ca}_\delta(n)$ is an **order extension** of $\mathrm{Av}_\delta(n)$, and $\mathrm{Hi}_\delta(n)$ is an **order extension** of $\mathrm{Ca}_\delta(n)$.

For instance, for $\delta = \mathbf{3}$ and $n = 3$,



Besides, for any $m \geqslant 1$ and $n \geqslant 0$, there is a poset embedding

$$\mathrm{Hi}_{\mathbf{m-1}}(n) \longrightarrow \mathrm{Ca}_{\mathbf{m}}(n).$$

Algebras of cliffs

Associative algebras of δ-cliffs

The *δ-reduction map*

$$\mathrm{r}_\delta : \mathbb{N}^n \to \mathrm{Cl}_\delta(n)$$

is defined for any word $u \in \mathbb{N}^n$ and any $i \in [n]$ by

$$(\mathrm{r}_\delta(u))_i := \begin{cases} u_i & \text{if } u_i \leqslant \delta(i), \\ \delta(i) & \text{otherwise.} \end{cases}$$

For instance, $\mathrm{r}_{\mathbf{1}}(212066) = 012045$ and $\mathrm{r}_{\mathbf{2}}(212066) = 012066$.

Let $\mathbf{Cl}_\delta$ be the linear span of $\{\mathsf{F}_u : u \in \mathrm{Cl}_\delta\}$, endowed with the product

$$\mathsf{F}_u \cdot \mathsf{F}_v = \sum_{\substack{v' \in \mathrm{r}_\delta^{-1}(v) \\ uv' \in \mathrm{Cl}_\delta}} \mathsf{F}_{uv'},$$

For instance, in $\mathbf{Cl}_{\mathbf{1}}$,

$$\mathsf{F}_{00} \cdot \mathsf{F}_{011} = \mathsf{F}_{00011} + \mathsf{F}_{00021} + \mathsf{F}_{00031} + \mathsf{F}_{00111} + \mathsf{F}_{00121} + \mathsf{F}_{00131} + \mathsf{F}_{00211} + \mathsf{F}_{00221} + \mathsf{F}_{00231},$$

and in $\mathbf{Cl}_{\mathbf{2}}$,

$$\mathsf{F}_{00} \cdot \mathsf{F}_{011} = \mathsf{F}_{00011} + \mathsf{F}_{00111} + \mathsf{F}_{00211} + \mathsf{F}_{00311} + \mathsf{F}_{00411}.$$

Some properties

- The product $\cdot$ is **associative** iff δ is unimodal.
- The δ-cliff poset plays for $\mathbf{Cl}_\delta$ the same role as the one played by the weak order for FQSym, that is
$$\mathsf{F}_u \cdot \mathsf{F}_v = \chi_\delta(u \,/\, v) \sum_{w \in [u \,/\, v,\, u \,\backslash\, v]_{\preccurlyeq}} \mathsf{F}_w,$$
where $\chi_\delta(w)$ is 1 if $w \in \mathrm{Cl}_\delta$ and is 0 otherwise, and $/$ and $\backslash$ are some concatenation operations on δ-cliffs.
- The algebra $\mathbf{Cl}_\delta$ is **free as an associative algebra** iff δ is weakly increasing.

Quotient algebras

Given a subset $\mathcal{S}$ of Cl_δ, let the quotient space $\mathbf{Cl}_{\mathcal{S}} := \mathbf{Cl}_\delta / \mathcal{V}_{\mathcal{S}}$ where $\mathcal{V}_{\mathcal{S}}$ is the linear span of the set $\{\mathsf{F}_u : u \in \mathrm{Cl}_\delta \setminus \mathcal{S}\}$.

A subset $\mathcal{S}$ of Cl_δ is *closed by suffix reduction* if for any $u \in \mathcal{S}$, for all suffixes u' of u, $\mathrm{r}_\delta(u') \in \mathcal{S}$.

When δ is unimodal and $\mathcal{S}$ is closed by prefix and closed by suffix reduction, $\mathbf{Cl}_{\mathcal{S}}$ is a **quotient associative algebra** of $\mathbf{Cl}_\delta$.

Two Fuss-Catalan quotient algebras

Let $\mathbf{Hi}_m$ and $\mathbf{Ca}_m$ be respectively the quotients $\mathbf{Cl}_{\mathrm{Hi}_{\mathbf{m}}}$ and $\mathbf{Cl}_{\mathrm{Ca}_{\mathbf{m}}}$.

For instance, in $\mathbf{Hi}_1$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{01} = \mathsf{F}_{0111} + \mathsf{F}_{0112} + \mathsf{F}_{0113} + \mathsf{F}_{0122} + \mathsf{F}_{0123}, \qquad \mathsf{F}_{01} \cdot \mathsf{F}_{00} = 0,$$

in $\mathbf{Hi}_2$,

$$\mathsf{F}_{02} \cdot \mathsf{F}_{023} = \mathsf{F}_{02223} + \mathsf{F}_{02233} + \mathsf{F}_{02333}, \qquad \mathsf{F}_{011} \cdot \mathsf{F}_{01} = \mathsf{F}_{01111},$$

in $\mathbf{Ca}_1$,

$$\mathsf{F}_{0} \cdot \mathsf{F}_{01} = \mathsf{F}_{001} + \mathsf{F}_{002} + \mathsf{F}_{012}, \qquad \mathsf{F}_{0} \cdot \mathsf{F}_{002} = \mathsf{F}_{0002} + \mathsf{F}_{0003} + \mathsf{F}_{0103}.$$

and in $\mathbf{Ca}_2$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{0014} = 0, \qquad \mathsf{F}_{01} \cdot \mathsf{F}_{0013} = \mathsf{F}_{010013}.$$

- The dimensions of these associative algebras are provided by Fuss-Catalan numbers.
- The support of any **product** in $\mathbf{Ca}_m$ or in $\mathbf{Hi}_m$ is an **interval** of a canyon poset or of a hill poset.
- $\mathbf{Ca}_1$ is isomorphic to PBT.
- For all $m \geqslant 2$, $\mathbf{Ca}_m$ is not free.
- For all $m \geqslant 1$, $\mathbf{Hi}_m$ is not free and is not isomorphic to $\mathbf{Ca}_m$.

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