1. Motivation and goals

- Many combinatorial objects like permutations, binary trees, integer compositions and partitions are endowed with a combinatorial Hopf algebra structure.
- An approach to construct most of these structures in a unified way relies on the definition of a plastic-like congruence on words satisfying some structure conditions.
- Every congruence on words leads to the definition of a monoid of combinatorial objects, and, in addition to the construction of combinatorial Hopf algebras, this construction often comes with partial orders, combinatorial algorithms and Robinson-Schensted-like algorithms.

2. Baxter permutations, pairs of twin binary trees

- A Baxter permutation is a permutation avoiding the generalized permutation patterns 2 – 41 – 3 and 3 – 14 – 2. For example, 43675218 is a Baxter permutation but 4217365 is not.
- A pair \((T_a, T_b)\) of binary trees with the same number of nodes is a pair of twin binary trees if the canopies (i.e. the orientation of internal leaves in binary trees) of \(T_a\) and \(T_b\) are complementary.
- There is a bijection between Baxter permutations and pairs of twin binary trees.
- First numbers of Baxter permutations by size are 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240.

3. The Baxter monoid satisfies some structure properties:

**Proposition**

The Baxter monoid is compatible with the desynchronization process, i.e., for all \(u, v \in A^*\), \(u \equiv_{\text{B}} v\) if \(\text{std}(u) \equiv_{\text{B}} \text{std}(v)\) and \(\text{eval}(u) \equiv \text{eval}(v)\).

**Proposition**

The Baxter monoid is compatible with the restriction of alphabet indices, i.e., for all interval \(I\) of \(A\) and for all \(u, v \in A^*\), \(u \equiv_{\text{B}} v\) implies \(u|_I \equiv_{\text{B}} v|_I\).

**Proposition**

The Baxter monoid is compatible with the Schützenberger involution, i.e., for all \(u, v \in A^*\), \(u \equiv_{\text{B}} v\) implies \(u^\mathcal{S} \equiv_{\text{B}} v^\mathcal{S}\).

4. A Robinson-Schensted-like algorithm

- Given an \(A\)-labeled pair of twin binary trees \((T_a, T_b)\), one can insert a letter \(a \in A\) into \((T_a, T_b)\) with the following algorithm:

**Algorithm:** Insertion\((T_a, T_b, a)\).

1. Make a leaf insertion of \(a\) into the binary tree \(T_a\).
2. Make a root insertion of \(a\) into the binary tree \(T_b\).

- The \(\mathcal{P}\) symbol of a word \(u \in A^*\) is the \(A\)-labeled pair of twin binary trees obtained by iteratively inserting, from left to right, the letters of \(u\) into the empty pair of twin binary trees \((\lambda, \lambda)\).

- The \(\mathcal{P}\) symbol allows to decide if two words are \(\equiv_{\text{B}}\) equivalent.

**Proposition**

Let \(u, v \in A^*\). Then, \(u \equiv_{\text{B}} v\) if \(\text{P}(u) = \text{P}(v)\).

**Example:**

![Diagram of the \(\mathcal{P}\) symbol of a word]

5. The Baxter lattice

- The quotient of the permutahedron by the Baxter equivalence relation defines a lattice over the set of pairs of twin binary trees of a given size.

**Proposition**

The Baxter equivalence relation is a lattice congruence of the permutahedron, i.e., every \(\equiv_{\text{B}}\) equivalence class of permutations is an interval of the permutahedron, and for all permutations \(u, v\) such that \(u \equiv_{\text{B}} v\), the minimal (resp. maximal) elements \(u\)' and \(v\)' of the \(\equiv_{\text{B}}\) equivalence classes of \(u\) and \(v\) satisfy \(u\)' \(\leq_{\text{B}} v\)'.

- Covering relations are similar to those of the Tamari lattice and can be described using left and right binary tree rotations.

**Examples:**

![Diagram of the Baxter lattice]

6. The Hopf algebra Baxter

- The family \((F_j)_{j \in \mathbb{Z}}\) forms the fundamental basis of \(\mathcal{FQSym}\), the combinatorial Hopf algebra of permutations. Its product and its coproduct are defined as follows:

**Proposition**

Let us define the following elements of \(\mathcal{FQSym}\), indexed by pairs of twin trees:

**Proposition**

Since \(\equiv_{\text{B}}\) is a congruence compatible with the desynchronization process and also with the restriction of alphabet intervals, we have the following theorem:

**Theorem**

The family \((P_j)_{j \in \mathbb{Z}}\) spans a Hopf subalgebra of \(\mathcal{FQSym}\), namely the Hopf algebra Baxter.

**The product of Baxter** is deduced from the product of \(\mathcal{FQSym}\) and is expressed as follows:

**In the same way, the coproduct of Baxter is expressed as follows:**

**Examples:**

![Diagram of the Hopf algebra Baxter]

7. References

- Samuele Giraud, Algebraic and combinatorial structures on Baxter permutations, FPSAC, 2011.