The music box operad:
Random generation of musical phrases from patterns

Samuele Giraudo
LIGM, Université Gustave Eiffel, CNRS, ESIEE Paris, F-77454 Marne-la-Vallée, France.
samuele.giraudo@u-pem.fr

ABSTRACT. We introduce the notion of multi-pattern, a combinatorial abstraction of polyphonic musical phrases. The interest of this approach to encode musical phrases lies in the fact that it becomes possible to compose multi-patterns in order to produce new ones. This dives the set of musical phrases into an algebraic framework since the set of multi-patterns has the structure of an operad. Operads are algebraic structures offering a formalization of the notion of operators and their compositions. Seeing musical phrases as operators allows us to perform computations on phrases and admits applications in generative music. Indeed, given a set of short patterns, we propose various algorithms to randomly generate a new and longer phrase inspired by the inputted patterns.

Contents

Introduction 3
1. The music box model 4
  1.1. Notes and scales 4
  1.2. Patterns 5
2. Operad structures 8
  2.1. A primer on operads 8
  2.2. Music box operads 10
  2.3. Operations on musical phrases 17
3. Generation and random generation 20
  3.1. Colored operads and bud operads 20
  3.2. Bud generating systems 22
  3.3. Random generation 24
4. Applications: exploring variations of patterns 27
  4.1. Patterns to patterns 27
  4.2. Patterns to multi-patterns 29
Conclusion and perspectives 31
References 32

Date: 2021-04-27.
2020 Mathematics Subject Classification. 00A65, 18M60, 68Q42.
Key words and phrases. Generative music; Musical phrases; Formal grammars; Random generation; Operads; Bud generating systems.
Generative music is a subfield of computational musicology in which the focus lies on the automatic creation of musical material [LJ96]. This creation is based on algorithms accepting inputs to influence the result obtained, and having a randomized behavior in the sense that two executions of the algorithm with the same inputs produce different results. Several very different approach exist. For instance, some of them use Markov chains, others genetic algorithms [Mat10], still others neural networks [BHP20], or even formal grammars [Hol81, McC96, HMU06, HQ18]. In the case of the Markov chain approach, the input is a corpus of musical pieces. The algorithm builds a Markov chain for the probabilities of a note to be played according to some previous ones, and then, uses it to build randomly a new musical piece which inherits in some sense from the inputted ones. In the case of the formal grammar approach, the input data is a formal grammar specifying the possible results. The algorithm builds a musical piece by performing a random generation of a word of the language specified by the grammar by selecting, using various strategies, a rule of the grammar at random at each step.

The way in which such algorithms represent and manipulate musical data is crucial. Indeed, the data structures used to represent musical phrases orient the nature of the operations we can define of them. Considering operations producing new phrases from old ones is important to specify algorithms to randomly generate music. A possible way for this purpose consists to give at input some musical phrases and the algorithm creates a new one by blending them through operations. Therefore, the willingness to endow the infinite set of all musical phrases with operations in order to obtain suitable algebraic structures is a promising approach. Such interactions between music and algebra is a fruitful field of investigation [Hud04, And18, Jed19].

In this work, we propose to use tools coming from combinatorics and algebraic combinatorics to represent musical phrases and operations on them, in order to introduce generative music algorithms close to the family of those based on formal grammars. More precisely, we introduce the music box model, a very simple model to represent polyphonic phrases, by combinatorial objects called multi-patterns. The infinite set of all these objects admits the structure of an operad. Such structures originate from algebraic topology and are used nowadays also in algebraic combinatorics and in computer science [Mén15, Gir18]. Roughly speaking, in these algebraic structures, the elements are operations with several inputs and the composition law mimics the usual composition of operators. Since the set of multi-patterns forms an operad, one can regard each pattern as an operation. The fallout of this is that each pattern is, at the same time, a musical phrase and an operation acting on musical phrases. In this way, our music box model and its associated operad provide an algebraic and combinatorial framework to perform computations on musical phrases.

All this admits direct applications to design random generation algorithms since, as introduced by the author in [Gir19], given an operad there exist algorithms to generate some of its elements. These algorithms are based upon bud generating systems, which are general formal grammars based on colored operads [Yau16]. In the present work, we propose three different variations of these algorithms to produce new musical phrases from old ones. More precisely, our algorithm works as follows. It takes as input a finite set of multi-patterns and
an integer value to influence the size of the output. It works iteratively by choosing patterns from the initial collection in order to alter the current one by performing a composition using the operad structure. As we shall explain, the initial patterns are endowed with colors in order to forbid some compositions and to avoid in this way some musical intervals or particular rhythmic motives for instance. These generation algorithms are not intended to write complete musical pieces; they are for, from short primitive patterns, obtain a new longest similar one, bringing to the human composer possibly new ideas.

This text is organized as follows. Section 1 is devoted to set our context and notations about music theory and to introduce the music box model. In Section 2, we begin by presenting a brief overview of operad theory and we build step by step the music box operad. For this, we introduce first an operad on sequences of scale degrees (depending on a monoid structure in order to encode how to compute on degrees), an operad on rhythm patterns, and then an operad on monophonic patterns to end with the operad of multi-patterns. Three random generation algorithms for multi-patterns are introduced in Section 3. Finally, Section 4 provides some concrete applications of the previous algorithms. We focus here on random variations of a monophonic musical phrase as input leading to monophonic results obtained, among others, by changing its rhythmic motif, or to polyphonic results obtained, among others, by harmonizing or arpeggiating it.

A computer implementation of all the presented algorithms is, as well as its source code and examples, available at [Gir21].

This paper is an extended version of [Gir20] containing some new results (like the complete study of the introduced operads) and their proofs. Moreover, we describe in this present version a more general model than the one presented in the previous work: now, the degrees of the phrases are elements of a monoid, allowing us to consider different ways to compose them.

General notations and conventions. For any integers $i$ and $j$, $[i,j]$ denotes the set $\{i, i + 1, \ldots, j\}$. For any integer $i$, $[i]$ denotes the set $\{1, i\}$. A word is a finite sequence of elements. Given a word $u$, $\ell(u)$ is the length of $u$, and for any $i \in [\ell(u)]$, $u_i$ is the $i$-th letter of $u$. If $a$ is a letter and $n$ is a nonnegative integer, $a^n$ is the word consisting in $n$ occurrences of $a$. In particular, $a^0$ is the empty word $\epsilon$.

1. THE MUSIC BOX MODEL

The purpose of this section is to set some definitions and conventions about music theory, and introduce multi-patterns that are abstractions of musical phrases. This encoding of musical phrases forms the music box model.

1.1. Notes and scales. We begin by providing definitions about notes, scales, rooted scales, and sets of notes associated with a rooted scale.
1.1.1. Notes. We fit into the context of an $\eta$ tone equal temperament, also written as $\eta$-TET, where $\eta$ is any positive integer. An $\eta$-note is a pair $(k, n)$ where $0 \leq k \leq \eta - 1$ and $n \in \mathbb{Z}$. We shall write $k_n$ instead of $(k, n)$. The integer $n$ is the octave index and $k$ is the step index of $k_n$. The set of all $\eta$-notes is denoted by $N^{(\eta)}$. Despite this level of generality, and even if all the concepts developed in the sequel work for any $\eta$, in most applications and examples we shall consider that $\eta = 12$. Therefore, under this convention, we simply call note any 12-note and write $N^\ast$ for $N^{(12)}$. We set in this context of 12-TET the “middle C”, as the note $0_w$, which is the first step of the octave of index 4.

1.1.2. Scales. An $\eta$-scale is an integer composition $\lambda$ of $\eta$, that is a sequence $(\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ of nonnegative integers satisfying $\lambda_1 + \cdots + \lambda_{\ell(\lambda)} = \eta$. The length $\ell(\lambda)$ of $\lambda$ is the length of $\lambda$ as a word. We simply call scale any $\eta$-note $\lambda$.

For instance, $(2, 2, 1, 2, 2, 1)$ is the major natural scale, $(2, 1, 2, 2, 1, 2, 2)$ is the minor natural scale, $(2, 1, 2, 1, 3, 1)$ is the harmonic minor scale, $(3, 2, 2, 3, 2)$ is the minor pentatonic scale, and $(2, 1, 4, 1, 4)$ is the Hirajoshi scale. This encoding of a scale by an integer composition $\lambda$ wherein each $\lambda_i$ encode the distance in steps between two consecutive notes is also known under the terminology of interval pattern.

1.1.3. Rooted scales. A rooted scale is a pair $(\lambda, r)$ where $\lambda$ is a scale and $r$ is a note called root. This rooted scale describes a subset $N_{(\lambda, r)}$ of $N^\ast$ consisting in the notes reachable from $r$ by following the steps prescribed by the values $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}$ of $\lambda$. More formally, for any note $k_n$,

$$N_{(\lambda, k_n)} := \bigcup_{m \in \mathbb{Z}} \left\{ k_m^{(0)} m, k_m^{(1)} m, \ldots, k_m^{(\ell(\lambda) - 1)} m \right\}$$

(1.1.1)

where for any $i \in [0, \ell(\lambda) - 1]$, $k_m^{(i)} := (k + \lambda_1 + \cdots + \lambda_i) \mod \eta$. Observe of course that $N_{(\lambda, k_n)} = N_{(\lambda, k_m)}$ for any notes $k_n$ and $k_m$ having different octave indices but the same step index $k$.

For instance, if $\lambda$ is the minor pentatonic scale, then

$$N_{(\lambda, 0)} = \{ \ldots, 0_5, 3_5, 5_5, 7_5, 10_5, 0_4, 3_4, 5_4, 7_4, 10_4, 0_3, 3_3, \ldots \}. \quad (1.1.2)$$

If $\lambda$ is the major natural scale and $\lambda'$ is the minor natural scale, then

$$N_{(\lambda', 9_5)} = \{ \ldots, 9_2, 11_2, 0_5, 3_5, 5_5, 7_5, 9_5, 11_5, 0_4, 2_4, 4_4, 5_4, 7_4, 9_4, 11_4, 0_3, 2_3, \ldots \} \quad (1.1.3)$$

and

$$N_{(\lambda', 9_4)} = \{ \ldots, 9_2, 11_2, 0_5, 3_5, 5_5, 7_5, 9_5, 11_5, 0_4, 2_4, 4_4, 5_4, 7_4, 9_4, 11_4, \ldots \}, \quad (1.1.4)$$

so that $N_{(\lambda, 0_4)} = N_{(\lambda', 9_5)}$. This is a consequence of the fact that the minor natural scale rooted at the “middle A”, that is the note $9_5$, specifies the same notes as the major natural scale rooted at the “middle C”. This is due to the fact that these two rooted scales are in a relative relationship. Let us emphasize however that $(\lambda, 0_4)$ and $(\lambda', 9_5)$ are different rooted scales.

1.2. Patterns. We now introduce degree patterns, rhythm patterns, patterns, and finally multi-patterns which are the components of music box model.
1.2.1. **Degree patterns.** A *degree* $d$ is any element of $\mathbb{Z}$. Negative degrees are denoted by putting a bar above their absolute value. For instance, $-3$ is denoted by $\bar{3}$. A *degree pattern* is a nonempty word $d$ of degrees. The *arity* of $d$, also denoted by $|d|$, is the length of $d$ as a word. For instance, $d := 01210$ is a degree pattern of arity 5.

Given a rooted scale $(\lambda, r)$, a degree pattern $d$ specifies a sequence of notes in the following way. Let $\preceq$ be the lexicographic order on $\mathcal{N}_{(\lambda, r)}$ satisfying $k_n \preceq k_{n'}$ if $n = n'$ and $k \leq k'$, or if $n < n'$. This is equivalent to say that the pitch of $k_n$ is lower than or equal as the one of $k_{n'}$. Now, we set that each degree $d$ encodes the root note $r$ if $d = 0$, the $d$-th greater note w.r.t. $\preceq$ obtained from $r$ if $d \geq 1$, and the $d$-th smaller note w.r.t. $\preceq$ obtained from $r$ if $d \leq 1$.

For instance, let the degree pattern $d := 1023507$. By choosing the rooted scale $(\lambda, 0_4)$ where $\lambda$ is the minor pentatonic scale, one obtains the sequence of notes

$$3_4, 0_4, 7_5, 5_5, 0_5, 0_4, 5_5.$$  

By choosing instead the rooted scale $(\lambda, 0_4)$ where $\lambda$ is the major natural scale, the same degree pattern specifies the sequence of notes

$$2_4, 0_4, 9_5, 7_5, 9_4, 0_4, 0_5.$$  

1.2.2. **Rhythm patterns.** A *rhythm pattern* $r$ is a word on the alphabet $\{\text{□, ■}\}$ having at least one occurrence of ■. The symbol □ is a *rest* and the symbol ■ is a *beat*. The *length* $\ell(r)$ of $r$ is the length of $r$ as a word and the *arity* $|r|$ of $r$ is its number of occurrences of beats. For instance, $r := \text{□□□□□□□}$ is a rhythm pattern of length 6 and arity 2.

The *duration sequence* of a rhythm pattern $r$ is the unique sequence $\sigma := (\sigma_1, \ldots, \sigma_{|r|+1})$ of nonnegative integers such that

$$r = \square^{\sigma_1} \square^{\sigma_2} \ldots \square^{\sigma_{|r|+1}}. \tag{1.2.1}$$

The rhythm pattern $r$ specifies a rhythm wherein each beat has a relative duration: the rhythm begins with a silence of $\sigma_1$ units of time, followed by a first beat sustained $1 + \sigma_2$ units of time, and so on, and finishing by a last beat sustained $1 + \sigma_{|r|+1}$ units of time. We adopt here the convention that each rest and beat last each the same amount of time of one eighth of the duration of a whole note. Therefore, given a tempo specifying how many there are rests and beats by minute, any rhythm pattern encodes a rhythm.

For instance, let us consider the rhythm pattern $r := \text{□□□□□□□□□□□□□}. The duration sequence of $r$ is $(1, 0, 1, 3, 0, 1, 0)$ so that $r$ specifies the rhythm consisting in an eighth rest, an eighth note, a quarter note, a half note, an eighth note, a quarter note, and finally an eighth note.

1.2.3. **Patterns.** A *pattern* is a pair $p := (d, r)$ such that $|d| = |r|$. The *arity* $|p|$ of $p$ is the arity of both $d$ and $r$, and the *length* $\ell(p)$ of $d$ is the length $\ell(r)$ of $r$. We call $d$ the *underlying degree pattern* of $p$ and $r$ the *underlying rhythm pattern* of $p$. By extension, the *duration sequence* of $p$ is the duration sequence of $r$. For instance, $p := (1f2, \text{□□□□□□□□□})$ is a pattern of arity 3 and of length 7.
In order to handle concise notations, we shall write any pattern \((d, r)\) as a word \(p\) on the alphabet \(\{\square\} \cup \mathbb{Z}\) where the subword of \(p\) obtained by removing all occurrences of \(\square\) is the degree pattern \(d\), and the word obtained by replacing in \(p\) each integer by \(\square\) is the rhythm pattern \(r\). For instance,

\[1 \square \bar{2} \square 12\]  \hspace{1cm} (1.2.2)

is the concise notation for the pattern

\[(1 \bar{2} 12, \square \square \square \square \square \square \square).\]  \hspace{1cm} (1.2.3)

For this reason, in the sequel, we shall see and treat any pattern \(p\) as a word on the alphabet \(\{\square\} \cup \mathbb{Z}\). Therefore, for any \(i \in [\ell(p)]\), \(p_i\) is the \(i\)-th letter of \(p\) which is either \(\square\) or a degree. Remark that the length of \(p\) is the length of \(p\) as a word and that the arity of \(p\) is the number of letters of \(\mathbb{Z}\) it has.

Given a rooted scale \((\lambda, r)\) and a tempo, a pattern \(p := (d, r)\) specifies a musical phrase, that is a sequence of notes arranged into a rhythm. The notes of the musical phrase are the ones specified by the degree pattern \(d\) as explained in Section 1.2.1 and their relative durations are specified by the rhythm pattern \(r\) as explained in Section 1.2.2.

For instance, consider the pattern \(p := 0 \square 1 2 \bar{1} 0 \square 1 \bar{2} \bar{1} \bar{0} 0\) By choosing the rooted scale \((\lambda, 93)\) where \(\lambda\) is the harmonic minor scale, and by setting 128 as tempo, one obtains the musical phrase

\[\begin{array}{cccccccccccc}
\text{\(\lambda\)} = 192 \\
\end{array}\]

1.2.4. Multi-patterns. A multi-pattern is a tuple \(m := (m_1, \ldots, m_m)\) of length \(m \geq 1\) of patterns such that all \(m_i, i \in [m]\), have the same arity and the same length. The arity \(|m|\) of \(m\) is the common arity of all the \(m_i\), the length \(\ell(m)\) of \(m\) is the common length of all the \(m_i\) and the multiplicity \(m(m)\) of \(m\) is \(m\).

A multi-pattern \(m\) is denoted through a matrix of dimension \(m(m) \times \ell(m)\), where the \(i\)-th row contains the pattern \(m_i\) for any \(i \in [m(m)]\). For any \(i \in [m(m)]\) and any \(j \in [\ell(m)]\) we denote simply by \(m_{i,j}\) the element \(p_j\) where \(p\) is the pattern \(m_i\). For instance,

\[m := \begin{bmatrix}
0 & \square & 1 & \square & 1 \\
\square & 2 & 3 & \square & 0
\end{bmatrix}\]  \hspace{1cm} (1.2.4)

is a multi-pattern of multiplicity 2, having 3 as arity, and 5 as length. It satisfies \(m_{1,1} = 0\), \(m_{1,4} = \square\) and \(m_{2,3} = 3\). The fact that all patterns of a multi-pattern must have the same length ensures that they last the same amount of units of time. This is important since multi-patterns of multiplicity \(m\) are intended to handle musical sequences consisting in \(m\) stacked voices. The condition about the arities of the patterns, and hence, about the number of degrees appearing in these, is a particularity of our model and comes from algebraic reasons. This will be clarified later in the text.

Given a rooted scale \((\lambda, r)\) and a tempo, a multi-pattern \(m\) specifies a musical phrase obtained by considering the musical phrases specified by each pattern \(m_i, i \in [m(m)]\), each forming a voice.
For instance, consider the multi-pattern

\[ m := \begin{bmatrix} 0 & 4 & 4 & 0 & 0 \\ 7 & 7 & 0 & 3 & 3 \end{bmatrix}. \] (1.2.5)

By choosing the rooted scale \((\lambda, 9)\) where \(\lambda\) is the minor natural scale and by setting 128 as tempo, one obtains the musical phrase

Due to the fact that multi-patterns evoke paper tapes of a programmable music box, we call music box model the model just described to represent musical phrases by multi-patterns within the context of a rooted scale and a tempo.

2. Operad structures

The purpose of this section is to introduce an operad structure on multi-patterns, called music box operad. The main interest of endowing the set of multi-patterns with the structure of an operad is that this leads to an algebraic framework to perform computations on musical phrases.

2.1. A primer on operads. We begin by setting here the elementary notions of operad theory used in the sequel. Most of them come from [Gir18].

2.1.1. Graded sets. A graded set is a set \(O\) decomposing as a disjoint union

\[ O := \bigsqcup_{n \in \mathbb{N}} O(n), \] (2.1.1)

where the \(O(n), n \in \mathbb{N}\), are sets. For any \(x \in O\), there is by definition a unique \(n \in \mathbb{N}\) such that \(x \in O(n)\). This integer \(n\) is the arity of \(x\) and is denoted by \(|x|\). Let \(O'\) be a second graded set. A map \(\phi : O \to O'\) is a graded set morphism if for any \(x \in O\), \(\phi(x) \in O'(|x|)\). The identity graded set morphism is denoted by \(I\). Besides, \(O'\) is a graded subset of \(O\) if \(O'(n) \subseteq O(n)\) for any \(n \in \mathbb{N}\).

The Hadamard product of two graded sets \(O\) and \(O'\) is the graded set \(O \boxtimes O'\) defined, for any \(n \in \mathbb{N}\), by \((O \boxtimes O')(n) := O(n) \times O'(n)\). Observe that if \(O''\) is another graded set, the graded sets \((O \boxtimes O') \boxtimes O''\) and \(O \boxtimes (O' \boxtimes O'')\) are isomorphic, so that \(\boxtimes\) is an associative operation. By a slight abuse of notation, for any other graded sets \(O_1\) and \(O'_1\), and any graded set morphisms \(\phi : O \to O_1\) and \(\phi' : O' \to O'_1\), we denote by \(\phi \boxtimes \phi'\) the map from \(O \boxtimes O'\) to \(O_1 \boxtimes O'_1\) defined, for any \((x, x') \in \phi \boxtimes \phi'\), by \((O \boxtimes O')(x, x') := (\phi(x), \phi(x'))\). This map is a graded set morphism.
2.1.2. Nonsymmetric operads. A nonsymmetric operad, or an operad for short, is a triple \((\mathcal{O}, \circ, 1)\) such that \(\mathcal{O}\) is a graded set, \(\circ\) is a map
\[
\circ : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \quad 1 \leq i \leq n,
\]
called partial composition map, and \(1\) is a distinguished element of \(\mathcal{O}(1)\), called unit. This data has to satisfy, for any \(x, y, z \in \mathcal{O}\), the three relations
\[
(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \quad 1 \leq i \leq |x|, \quad 1 \leq j \leq |y|,
\]
\[
(x \circ_i y) \circ_{i+|y|-1} z = (x \circ_j z) \circ_i y, \quad 1 \leq i < j \leq |x|,
\]
\[
1 \circ_i x = x = x \circ_i 1, \quad 1 \leq i \leq |x|.
\]

2.1.3. Abstract operators and intuition. From an intuitive point of view, an operad is an algebraic structure wherein each element can be seen as an operator having \(|x|\) inputs and one output. Such an operator is depicted as
\[
\begin{array}{c}
  x \\
  \circ_i \\
  1 \quad \ldots \quad |x| \\
\end{array}
\]
where the inputs are at the bottom and the output at the top. Given two operations \(x\) and \(y\) of \(\mathcal{O}\), the partial composition \(x \circ_i y\) is a new operator obtained by composing \(y\) onto the \(i\)-th input of \(x\). Pictorially, this partial composition expresses as
\[
\begin{array}{c}
  x \\
  \circ_i \\
  1 \quad \ldots \quad |x| \\
\end{array}
+ \begin{array}{c}
  y \\
  \circ_i \\
  1 \quad \ldots \quad |y| \\
\end{array} = \begin{array}{c}
  x \\
  \circ_i \\
  1 \quad \ldots \quad |x|+|y|-1 \\
\end{array}
\]

Relations (2.1.3a), (2.1.3b), and (2.1.3c) become clear when they are interpreted into this context of abstract operators and rooted trees.

2.1.4. Elementary definitions. Let \((\mathcal{O}, \circ, 1)\) be an operad. The full composition map of \(\mathcal{O}\) is the map
\[
\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n),
\]
defined, for any \(x \in \mathcal{O}(n)\) and \(y_1, \ldots, y_n \in \mathcal{O}\) by
\[
x \circ [y_1, \ldots, y_n] := (\cdots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \cdots) \circ_1 y_1.
\]
Intuitively, \(x \circ [y_1, \ldots, y_n]\) is obtained by grafting simultaneously the outputs of all the \(y_i\) onto the \(i\)-th inputs of \(x\). Similarly, the homogeneous composition map of \(\mathcal{O}\) is the map
\[
\circ : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(nm),
\]
defined, for any \(x \in \mathcal{O}(n)\) and \(y \in \mathcal{O}(m)\) by
\[
x \circ y := x \circ [y, \ldots, y].
\]
Let \( (Ω, o_1, 1') \) be a second operad. A graded set morphism \( φ : Ω \to Ω' \) is an operad morphism if \( φ(1) = 1' \) and for any \( x, y \in Ω \) and \( i \in |x| \),

\[
φ(x \circ_i y) = φ(x) \circ_i φ(y).
\tag{2.1.10}
\]

If instead (2.1.10) holds by replacing the second occurrence of \( i \) by \( |x| - i + 1 \), then \( φ \) is an operad antimorphism. We say that \( Ω' \) is a suboperad of \( Ω \) if \( Ω' \) is a graded subset of \( Ω \), \( 1 = 1' \), and for any \( x, y \in Ω \) and \( i \in |x| \) \( x \circ_i y = x \circ'_i y \). For any subset \( G \) of \( Ω \), the operad generated by \( G \) is the smallest suboperad \( Ω^G \) of \( Ω \) containing \( G \). When \( Ω^0 = Ω \) and \( G \) is minimal with respect to the inclusion among the subsets of \( G \) satisfying this property, \( G \) is a minimal generating set of \( Ω \) and its elements are generators of \( Ω \).

The Hadamard product of \( Ω \) and \( Ω' \) is the graded set \( Ω \odot Ω' \) endowed with the partial composition map \( o'' \) defined, for any \( (x, x'), (y, y') \in Ω \odot Ω' \) and \( i \in |(x, x')| \), by

\[
(x, x') \odot'' (y, y') := (x \circ_i y, x' \circ_i y'),
\tag{2.1.11}
\]

and having \( (1, 1') \) as unit. This graded set \( Ω \odot Ω' \) is an operad. Moreover, for any other operads \( Ω_1 \) and \( Ω'_1 \), and any operads morphisms (resp. antimorphisms) \( φ : Ω \to Ω_1 \) and \( φ' : Ω' \to Ω'_1 \), the graded set morphism \( φ \odot φ' \) is also an operad morphism (resp. antimorphism) from \( Ω \odot Ω' \) to \( Ω_1 \odot Ω'_1 \).

2.2. Music box operads. We build operads on multi-patterns step by step by introducing operads on degree patterns and an operad on rhythm patterns. The operads of patterns are constructed as the Hadamard product of the two previous ones. Finally, the operads of multi-patterns are suboperads of iterated Hadamard products of operads of patterns with themselves. These operads (except the operad of rhythm patterns) depend on a monoid structure in order on \( \mathbb{Z} \) to encode how to compute on degrees.

2.2.1. Operads of degree patterns. The construction of an operad on degree patterns in based upon the construction \( T \), a construction from monoids to operads introduced in [Gir15] which we recall now. Given a monoid \( (M, \ast, e) \), let \( T.M \) be the graded set of all nonempty words on \( M \), where the arity of a word is its length. This graded set is endowed with a partial composition map \( o_1 \) defined for any \( u, u' \in T.M \) and \( i \in |u| \), by

\[
u \circ_i u' := \left( u_1, \ldots, u_{i-1}, u_i \ast u'_1, \ldots, u_i \ast u'_{\ell(u')} , u_{i+1}, \ldots, u_{\ell(u)} \right).
\tag{2.2.1}
\]

It is shown in [Gir15] that \( (T.M, o_1, 1) \), where \( 1 \) is the element \( (e) \) of \( T.M(1) \), is an operad. Moreover, \( T.M \) admits as a minimal generating set the set

\[
\{(g) : g \in G \} \cup \{(e, e)\}
\tag{2.2.2}
\]

where \( G.M \) is a minimal generating set of \( M \) as a monoid. Besides, if \( M' \) is another monoid and \( φ : M \to M' \) is a monoid morphism, let \( T.φ : T.M \to T.M' \) be the map defined, for any \( u \in T.M \), by

\[
T.φ(u) := (φ(u_1), \ldots, φ(u_{\ell(u)})).
\tag{2.2.3}
\]

It is also shown in [Gir15] that \( T.φ \) is an operad morphism preserving injections and surjections.
Let \( \text{mir}: T_M \to T_M \) be the map defined, for any \( u \in T_M \), by
\[
\text{mir}(u) := (u_{\ell(u)}, \ldots, u_1).
\] (2.2.4)
The word \( \text{mir}(u) \) is the *mirror* of \( u \).

**Proposition 2.2.1.** For any monoid \( M \), the map \( \text{mir} \) is an operad anti-automorphism of \( T_M \).

**Proof.** This is a straightforward verification based upon the fact that for any \( u \in T_M \), the \( i \)-th letter of \( \text{mir}(u) \) is \( u_{\ell(u) - i + 1} \). \qed

A **degree monoid** is any monoid \((D, \ast, e)\) such that \( D \subseteq \mathbb{Z} \). By construction, for any \( n \geq 1 \), each \( d \in TD(n) \) is a sequence of integers of length \( n \), and thus \( d \) is a degree pattern of arity \( n \). For this reason, \( TD \) is an operad having as underlying graded set the graded set of all degree patterns having elements of \( D \) as degrees. We denote by
\[
\text{DP}^D := TD
\] (2.2.5)
this operad, called *\( D \)-degree pattern operad*.

Let us consider three important examples depending on different natural degree monoids.

1. By denoting by \( \mathbb{Z} \) the additive monoid \( (\mathbb{Z}, +, 0) \), \( \text{DP}^\mathbb{Z} \) contains all degree patterns. In \( \text{DP}^\mathbb{Z} \), we have
\[
10132_{\mathbb{Z}} 211 = 1011222.
\] (2.2.6)
   Moreover, since \( \mathbb{Z} \) admits \( \{-1, 1\} \) as a minimal generating set, the operad \( \text{DP}^\mathbb{Z} \) admits \( \{1, 1, 00\} \) as a minimal generating set.

2. By denoting, for any \( k \geq 1 \), by \( \mathbb{C}_k \) the cyclic monoid \( (\mathbb{Z}/k\mathbb{Z}, +, 0) \), \( \text{DP}^\mathbb{C}_k \) contains all degree patterns having degrees between 0 and \( k - 1 \). In \( \text{DP}^\mathbb{C}_3 \) we have
\[
20010_{\mathbb{C}_3} 2120 = 20002010.
\] (2.2.7)
   Moreover, since \( \mathbb{C}_k \) admits \( \{1\} \) as a minimal generating set, the operad \( \text{DP}^\mathbb{C}_k \) admits \( \{1, 00\} \) as a minimal generating set.

3. By denoting, for any subset \( Z \) of \( \mathbb{Z} \) having a lower bound \( z \), by \( M_Z \) the monoid \( (Z, \text{max}, z) \), \( \text{DP}^{M_Z} \) contains all degree patterns having degrees in \( Z \). In \( \text{DP}^{M_{\{0,1\}}} \) we have
\[
20010_{\mathbb{C}_3} 2120 = 20021210.
\] (2.2.8)
   Moreover, since \( M_Z \) admits \( Z \setminus \{z\} \) as a minimal generating set, the operad \( \text{DP}^{M_Z} \) admits \( (Z \setminus \{z\}) \cup \{00\} \) as a minimal generating set.

By Proposition 2.2.1, for any degree monoid \( D \), the map \( \text{mir}: \text{DP}^D \to \text{DP}^D \) is an operad anti-automorphism. Moreover, one can consider on the operads \( \text{DP}^\mathbb{Z} \), \( \text{DP}^\mathbb{C}_k \), and \( \text{DP}^{M_Z} \) the following morphisms.

1. For any \( \alpha \in \mathbb{Z} \), let \( \text{mul}_\alpha : \text{DP}^\mathbb{Z} \to \text{DP}^\mathbb{Z} \) be the map defined by \( \text{mul}_\alpha := T_\phi \) where \( \phi \) is the monoid morphism satisfying \( \phi(d) = ad \) for any \( d \in \mathbb{Z} \). For instance,
\[
\text{mul}_{-2}(12003) = 24006.
\] (2.2.9)
   Since \( \phi \) is a monoid morphism, \( \text{mul}_\alpha \) is an operad endomorphism. Moreover, when \( \alpha \neq 0 \), \( \text{mul}_\alpha \) is injective, and when \( \alpha \in \{-1, 1\} \), \( \text{mul}_\alpha \) is bijective.
(2) For any \( k \geq 1 \), let \( \text{red}_k : \mathcal{D}P^Z \rightarrow \mathcal{D}P^{C_k} \) be the map defined by \( \text{red}_k := T\phi \) where \( \phi \) is the monoid morphism satisfying \( \phi(d) = d \mod k \) for any \( d \in \mathbb{Z} \). For instance, \( \text{red}_5(12003) = 11000 \). \( (2.2.10) \)

Since \( \phi \) is a surjective monoid morphism, \( \text{red}_k \) is a surjective operad morphism.

(3) For any subsets \( Z \) and \( Z' \) of \( \mathbb{Z} \) having respective lower bounds \( z \) and \( z' \), a map \( \theta : Z \rightarrow Z' \) is a rooted weakly increasing map if \( \theta(z) = z' \) and, for any \( d, d' \in Z \), \( d \leq d' \) implies \( \theta(d) \leq \theta(d') \). If \( \theta \) is such a map, let \( \text{incr}_\theta : \mathcal{D}P^{M_k} \rightarrow \mathcal{D}P^{M_k} \) be the map defined by \( \text{incr}_\theta := T\theta \). For instance, for \( Z := [0, 3] \) and \( Z' := [2, 5] \), and \( \theta \) satisfying \( \theta(2) = 2 + 2 \) for any \( d \in Z \),

\[ \text{incr}_\theta(12003) = 34225. \] \( (2.2.11) \)

Since \( \theta \) is a monoid morphism, \( \text{incr}_\theta \) is an operad morphism. This morphism is not necessarily injective nor surjective.

2.2.2. Operad of rhythm patterns. Let us introduce a new construction from monoids to operads, similar to the construction \( T \) recalled in Section 2.2.1. Given a monoid \( (\mathcal{M}, \ast, e) \), let \( \mathcal{U}\mathcal{M} \) be the graded set of all words on \( \mathcal{M} \) of length 2 or greater, where the arity of a word is its length minus 1. This graded set is endowed with a partial composition map \( \circ_i \), defined for any \( u, u' \in \mathcal{U}\mathcal{M} \) and \( i \in [\|u\|] \), by

\[ u \circ_i u' := \left( u_1, \ldots, u_{i-1}, u_i \ast u_1, u_2', \ldots, u_{(u)-1}', u_i', u_{i+1}, u_{i+2}, \ldots, u_{\ell(u)} \right) \] \( (2.2.12) \)

Let us also denote by \( \mathbb{1} \) the element \((e, e)\) of \( \mathcal{U}\mathcal{M}(1) \). Moreover, if \( \mathcal{M}' \) is another monoid and \( \phi : \mathcal{M} \rightarrow \mathcal{M}' \) is a monoid morphism, let \( \mathcal{U}\phi : \mathcal{U}\mathcal{M} \rightarrow \mathcal{U}\mathcal{M}' \) be the map defined, for any \( u \in \mathcal{U}\mathcal{M} \), by

\[ \mathcal{U}\phi(u) := \left( \phi(u_1), \ldots, \phi(u_{\ell(u)}) \right). \] \( (2.2.13) \)

**Proposition 2.2.2.** For any monoid \( \mathcal{M} \), \( \mathcal{U}\mathcal{M} \) is an operad. Moreover, for any monoids \( \mathcal{M} \) and \( \mathcal{M}' \), and any monoid morphism \( \phi : \mathcal{M} \rightarrow \mathcal{M}' \), \( \mathcal{U}\phi \) is an operad morphism.

**Proof.** By using the fact that the product of \( \mathcal{M} \) is associative and that \( \mathcal{M} \) has a unit, it follows by a straightforward but technical verification that Relations (2.1.3a), (2.1.3b), and (2.1.3c) are satisfied. Finally, the fact that \( \phi \) is a monoid morphism says that \( \phi \) commutes with the product of \( \mathcal{M} \). As a straightforward computation shows, this implies that \( \mathcal{U}\phi \) is an operad morphism. \( \square \)

Let \( \text{mir} : \mathcal{U}\mathcal{M} \rightarrow \mathcal{U}\mathcal{M} \) be the map defined, for any \( u \in \mathcal{U}\mathcal{M} \), by

\[ \text{mir}(u) := (u_{\ell(u)}, \ldots, u_1). \] \( (2.2.14) \)

The word \( \text{mir}(u) \) is the **mirror** of \( u \).

**Proposition 2.2.3.** For any commutative monoid \( \mathcal{M} \), the map \( \text{mir} \) is an operad anti-automorphism of \( \mathcal{U}\mathcal{M} \).

**Proof.** This is a straightforward verification based upon the fact that for any \( u \in \mathcal{U}\mathcal{M} \), the \( i \)-th letter of \( \text{mir}(u) \) is \( u_{\ell(u) - i+1} \). The commutativity of \( \mathcal{M} \) is important here. \( \square \)
By Proposition 2.2.2,
\[ \text{RP} := U \mathbb{N} \]  
(2.2.15)
where \( \mathbb{N} \) is the additive monoid \((\mathbb{N}, +, 0)\) is an operad. By construction, for any \( n \geq 1 \), each \( \sigma \in \text{RP}(n) \) is a sequence of nonnegative integers of length \( n + 1 \), and thus, \( \sigma \) is a duration sequence of a rhythm pattern of arity \( n \) (see Section 1.2.2). For this reason, \( \text{RP} \) can be seen as an operad having as underlying graded set the graded set of all rhythm patterns. This operad is the \textit{rhythm pattern operad}. We have for instance,
\[ (0, 0, 1, 2, 1) \circ_3 (1, 1, 0) = (0, 0, 2, 1, 2, 1), \]  
(2.2.16a)
\[ (1, 1, 0) \circ_1 (1, 2) = (2, 3, 0). \]  
(2.2.16b)

Directly on rhythm patterns, the partial composition rephrases as follows. For any rhythm patterns \( r \) and \( r' \), and any integer \( i \in \| r \| \), \( r \circ_i r' \) is obtained by replacing the \( i \)-th occurrence of \( \square \) in \( r \) by \( r' \). For instance, on rhythm patterns, (2.2.16a) and (2.2.16b) translate as
\[ \square\square\square\square\square\square\square\square\square\square\circ_3 \square\square\square\square\square\square\square\square\square\square \quad \Rightarrow \quad \square\square\square\square\square\square\square\square\square\square\circ_3 \square\square\square\square\square\square\square\square\square\square, \]  
(2.2.17a)
\[ \square\square\square\square\square\square\square\square\square\square \circ_1 \square\square\square\square\square\square\square\square\square\square = \square\square\square\square\square\square\square\square\square\square. \]  
(2.2.17b)

\textbf{Proposition 2.2.4.} \textit{The operad} \( \text{RP} \) \textit{admits} \{\square, \square, \square\} \textit{as a minimal generating set.}

\textit{Proof.} Let us denote by \( \mathcal{G} \) the candidate minimal generating set for \( \text{RP} \) described in the statement of the proposition. First, notice that since, for any \( \sigma_1, \sigma_2 \in \mathbb{N} \),
\[ \square^{\sigma_1} \square^{\sigma_2} = (\underbrace{\square \square \cdots \square}_{\sigma_1 \text{ terms}}) \circ_{\sigma_2} (\underbrace{\square \square \cdots \square}_{\sigma_2 \text{ terms}}), \]  
(2.2.18)
all rhythm patterns of arity one belong to \( \text{RP}^{\square} \). Now, let \( r \in \text{RP}(n), n \geq 1 \), and let \( \sigma \) be the duration sequence of \( r \). Since
\[ r = (\underbrace{\square \sigma_1 \cdots \sigma_{n-1}}_{n-1 \text{ terms}}) \circ_1 (\underbrace{\square \sigma_1 \cdots \sigma_{n-1}}_{n-1 \text{ terms}}), \]  
(2.2.19)
This shows that \( r \) belongs to \( \text{RP}^{\square} \) and hence, that \( \mathcal{G} \) is a generating set of \( \text{RP} \). Finally, this generating set is minimal since no element of \( \mathcal{G} \) can be expressed by partial compositions of some different other ones. \( \square \)

For the next lemma, recall that when \( r \) is a rhythm pattern, \( \ell(r) \) is the length of \( r \) that is the number of occurrences of \( \square \) plus the number of \( \square \) it has.

\textbf{Lemma 2.2.5.} \textit{For any two rhythm patterns} \( r \) \textit{and} \( r' \), \textit{and} \( i \in \| r \| \), \textit{in the operad} \( \text{RP} \), \( \ell(r \circ_i r') = \ell(r) + \ell(r') - 1. \)

\textit{Proof.} This is a direct consequence of the interpretation of the partial composition of \( \text{RP} \) in terms of rhythm patterns: the rhythm pattern \( r \circ_i r' \) is obtained by replacing a \( \square \) of \( r \) by \( r' \). \( \square \)

By Proposition 2.2.3, since \( \mathbb{N} \) is a commutative monoid, the map \( \text{mir} : \text{RP} \to \text{RP} \) is an operad anti-automorphism. On rhythm patterns, given \( r \in \text{RP} \), \( \text{mir}(r) \) is the rhythm pattern obtained by reading \( r \) from right to left. For instance,
\[ \text{mir}(\square\square\square\square\square\square\square\square\square\square) = \square\square\square\square\square\square\square\square\square\square. \]  
(2.2.20)
For any $\beta \in \mathbb{N}$, let $\text{dil}_\beta : \text{RP} \to \text{RP}$ be the map defined by $\text{dil}_\beta := U\phi$ where $\phi$ is the monoid morphism satisfying $\phi(s) = \beta s$ for any $s \in \mathbb{N}$. Since $\phi$ is a monoid morphism of $\mathbb{N}$, by Proposition 2.2.3, $\text{dil}_\beta$ is an operad endomorphism. When $\beta \neq 0$, $\text{dil}_\beta$ is injective and $\text{dil}_\beta$ is surjective if and only if $\beta = 1$. Interpreted on rhythm patterns, $\text{dil}_\beta(r)$ is obtained by replacing each occurrence of $\square$ in $r$ by $\square^\beta$. For instance,
\begin{equation}
\text{dil}_2(\square \square \square \square \square) = \square \square \square \square \square, \tag{2.2.21a}
\end{equation}
\begin{equation}
\text{dil}_0(\square \square \square \square \square) = \square \square. \tag{2.2.21b}
\end{equation}

2.2.3. Operads of patterns. For any degree monoid $(D, \ast, e)$, let $P^D$ be the operad defined as
\begin{equation}
P^D := D\text{-pattern operad}. \tag{2.2.22}
\end{equation}

By construction, for any $n \geq 1$, each $p \in P^D(n)$ is a pair $(d, r)$ such that $d$ is a degree pattern of arity $n$ having elements of $D$ as degrees and $r$ is a rhythm pattern of arity $n$. For this reason, $P^D$ is an operad having as underlying graded set the graded set of all patterns having elements of $D$ as degrees. We call $P^D$ then $D$-pattern operad. In $P^e$, we have for instance
\begin{equation}
\begin{pmatrix} 231, \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \square \spherical harmonics. Since $\mathcal{G}$ is a generating set of $\mathcal{D}$, $d$ decomposes as $d = g_1 \ast \cdots \ast g_k$ for a $k \in \mathbb{N}$ and elements $g_1, \ldots, g_k$ of $\mathcal{G}$. Therefore, $\square^e d \square^e$ belongs to $P^D$. Now, let $p \in P^D(n)$, $n \geq 1$, and let $\sigma$ be the duration sequence of $p$. Since
\begin{equation}
p = (ee \ast \cdots \ast ee) \circ [\square^e d_1 \square^e, \square^e d_2 \square^e, \ldots, \square^e d_n \square^e], \tag{2.2.25}
\end{equation}
this shows that $d$ belongs to $P^D$ and hence, that $\mathcal{G}$ is a generating set of $P^D$. Finally, this generating set is minimal since no element of $\mathcal{G}$ can be expressed by partial compositions of some different other ones.

By Proposition 2.2.6,
§ Operads of multi-patterns

Music generation by operads

S. Giraudo

1. the operad \( P^Z \) admits \( \{ 1, \mathbb{0}, \mathbb{0}, \mathbb{0}, \mathbb{0}, \mathbb{0} \} \) as a minimal generating set;
2. for any \( k \geq 1 \), the operad \( P^{C_k} \), admits \( \{ 1, \mathbb{0}, \mathbb{0}, \mathbb{0}, \mathbb{0} \} \) as a minimal generating set;
3. for any subset \( Z \) of \( \mathbb{Z} \) having a lower bound \( z \), \( P^{M_z} \) admits \( (Z \setminus \{ z \}) \cup \{ \mathbb{0}, z, \mathbb{0}, \mathbb{0} \} \) as a minimal generating set.

Due to the construction of \( P^D \) as the Hadamard product of \( D \) and \( P^D \), by Proposition 2.2.3, the map \( \text{mir} \otimes \text{mir} \) is an operad anti-automorphism of \( P^D \). Moreover, again due to the construction of \( P^D \) and the existence of the operad morphisms involving \( D \) and \( P^D \), presented in Sections 2.2.1 and 2.2.2, one can consider on the operads \( P^Z \), \( P^{C_k} \), and \( P^{M_z} \) the following morphisms.

1. For any \( \alpha \in \mathbb{Z} \), the map \( \text{mul}_\alpha \otimes I \) is an operad endomorphism of \( P^Z \).
2. For any \( k \geq 1 \), the map \( \text{red}_k \otimes I \) is an operad morphism from \( P^Z \) to \( P^{C_k} \).
3. For any subsets \( Z \) and \( Z' \) of \( Z \) having lower bounds and any rooted weakly increasing map \( \theta : Z \rightarrow Z' \), the map \( \text{inc}_\alpha \otimes I \) is an operad morphism from \( P^{M_z} \) to \( P^{M_{z'}} \).
4. For any degree monoid \( D \) and \( \beta \in \mathbb{N} \), the map \( I \otimes \text{dil}_\beta \) is an operad endomorphism of \( P^D \).

Let us describe some morphisms involving the operad \( P^D \) and the previous operads \( D \) and \( P^D \). Let the map \( \text{inc}_{DP^D, P^D} : D \otimes P^D \rightarrow P^D \) defined, for any \( d \in D \) by

\[
\text{inc}_{DP^D, P^D}(d) := \left( d, [d] \right).
\]

This map is an injective operad morphism. Let also the map \( \text{inc}_{RP^D, P^D} : RP \rightarrow P^D \) defined, for any \( r \in RP \) by

\[
\text{inc}_{RP^D, P^D}(r) := \left( e^{|r|}, r \right),
\]

where \( e \) is the unit of \( D \). This map is an injective operad morphism.

2.2.4. Operads of multi-patterns. For any degree monoid \( D \) and any positive integer \( m \), let \( D^m \) be the operad defined as

\[
P^D := \underbrace{P^D \otimes \cdots \otimes P^D}_{m \text{ terms}}.
\]

Let also \( D^m \) be the graded subset of the underlying graded set of \( D^m \) restrained to the \( m \)-tuples \( (m_1, \ldots, m_m) \) such that \( \ell(m_1) = \cdots = \ell(m_m) \).

\[\text{Theorem 2.2.7.} \quad \text{For any degree monoid } D \text{ and any positive integer } m, P^D_m \text{ is an operad.} \]

\[\text{Proof.} \quad \text{We have to prove that the set } P^D_m \text{ forms a suboperad of } P^D. \text{ Let us denote by } e \text{ the unit of } D. \text{ Since } e \text{ is also the unit of the operad } P^D, \text{ the } m \text{-tuple } (e, \ldots, e) \text{ belongs to } P^D_m(1) \text{ and is the unit of } P^D_m. \text{ Moreover, by Lemma 2.2.5, given two patterns } p \text{ and } p', \text{ for any } i \in \llbracket |p| \rrbracket, \text{ the length of } p \circ_i p' \text{ is } \ell(p) + \ell(p') - 1. \text{ This shows that all the patterns of the multi-pattern resulting as a partial composition of two multi-patterns have the same length. This implies that the partial composition of two elements of } P^D_m \text{ is also in } P^D_m \text{ and hence, that } P^D_m \text{ is an operad.} \]

By construction, for any \( n \geq 1 \), each \( m \in P^D_m(n) \) is an \( m \)-tuple of patterns having all \( n \) as arity, having all elements of \( D \) as degrees, and having all the same length. For this reason, \( P^D_m \) is an operad having as underlying graded set the graded set of all multi-patterns of multiplicity \( m \) and having elements of \( D \) as degrees. By Theorem 2.2.7, \( P^D_m \) is an operad,
called \textit{D-music box operad} of order $m$. This construction of $P^D_m$ explains why all the patterns of a multi-pattern must have the same arity. This is a consequence of the general definition of the Hadamard product of operads.

By using the matrix notation for multi-patterns described in Section 1.2.4, we have for instance respectively in $P_{Z^2}$ and in $P_{Z^3}$,

\[
\begin{bmatrix}
\Box & 2 & 1 & \Box & 0 \\
0 & 1 & \Box & \Box & 1
\end{bmatrix} \circ_2 \begin{bmatrix}
\Box \\
3
\end{bmatrix} = \begin{bmatrix}
\Box & 2 & 0 & \Box & 0 \\
0 & 2 & \Box & \Box & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
0 & \Box & 0 \\
2 & \Box & 0 \\
4 & 4 & \Box
\end{bmatrix} \circ_2 \begin{bmatrix}
7 & 7 \\
0 & 0 \\
4 & 3 & 3
\end{bmatrix} = \begin{bmatrix}
0 & \Box & 0 \\
2 & \Box & 0 \\
4 & 3 & 3
\end{bmatrix}.
\]

Due to the construction of $P^D_m$ as a suboperad of an operad obtained by an iterated Hadamard product of $P^m$ and to the fact that, as explained in Section 2.2.3, $\text{mir} \boxtimes \text{mir}$ is an anti-automorphism of $P$, the map

\[
\text{mir} := (\text{mir} \boxtimes \text{mir}) \boxtimes \cdots \boxtimes (\text{mir} \boxtimes \text{mir})
\]

is an operad anti-automorphism of $P^D_m$. Moreover, again due to the construction of $P^D_m$ and the operad endomorphisms of $P^D$ presented in Section 2.2.3, one can consider on the operads $P^Z_m$, $P^C_m$, and $P^{M_2}_m$ the following morphisms.

(1) For any $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$, the map

\[
\text{mul}_{\alpha_1, \ldots, \alpha_m} := (\text{mul}_{\alpha_1} \boxtimes \text{I}) \boxtimes \cdots \boxtimes (\text{mul}_{\alpha_m} \boxtimes \text{I})
\]

is an operad endomorphism of $P^Z_m$.

(2) For any $k \geq 1$, the map

\[
\text{red}_k := (\text{red}_k \boxtimes \text{I}) \boxtimes \cdots \boxtimes (\text{red}_k \boxtimes \text{I})
\]

is an operad morphism from $P^Z_m$ to $P^C_m$.

(3) For any subsets $Z$ and $Z'$ of $\mathbb{Z}$ having lower bounds and any rooted weakly increasing maps $\theta_i : Z \to Z'$, $i \in [m]$, the map

\[
\text{incr}_{\theta_1, \ldots, \theta_m} := (\text{incr}_{\theta_1} \boxtimes \text{I}) \boxtimes \cdots \boxtimes (\text{incr}_{\theta_m} \boxtimes \text{I})
\]

is an operad morphism from $P^{M_2}_m$ to $P^{M_{2r}}_m$.

(4) For any degree monoid $D$ and $\beta \in \mathbb{N}$,

\[
\text{dil}_\beta := (1 \boxtimes \text{dil}_\beta) \boxtimes \cdots \boxtimes (1 \boxtimes \text{dil}_\beta)
\]

is an operad endomorphism of $P^D_m$.

Let also, for any $m \geq 1$, the map $\text{copy}_m : P^D \to P^D_m$ defined, for any $p \in P^D$, by

\[
\text{copy}_m(p) := \langle p, \ldots, p \rangle.
\]

This map is an injective operad morphism.
The full diagram involving the operads $\mathbb{D}^D$, $\mathbb{R}^D$, $\mathbb{P}^D$, and $\mathbb{P}_m^D$ is

\[
\begin{array}{c}
\mathbb{D}^D \\
\mathbb{P}_m^D \\
\text{inc}_{\mathbb{D}^D,\mathbb{P}_m^D} \\
\mathbb{R}^D \\
\text{inc}_{\mathbb{R}^D,\mathbb{P}_m^D}
\end{array}
\]

where $D$ is a degree monoid, $m \geq 1$, and arrows $\rightarrow$ are injective operad morphisms.

2.3. Operations on musical phrases. Thanks to the $D$-music box operads and more precisely, to the operad structures on multi-patterns, we can see any multi-pattern as an operator acting on musical phrases.

2.3.1. Decomposing multi-patterns. Instead of using the operads $\mathbb{P}_m^D$ to build multi-patterns, we can use this structure in the opposite way to decompose them. More precisely, given a multi-pattern $m$, we can search for decompositions of $m$ in an operad $\mathbb{P}_m^D$ where $m := m(m)$. A decomposition can be encoded as a planar rooted tree where internal nodes are decorated by elements of $\mathbb{P}_m^D$ (see Section 2.1.3). Such a decomposition is maximal w.r.t. a minimal generating set $\mathcal{G}$ of $\mathbb{P}_m^D$ if the multi-patterns intervening in the decomposition are elements of $\mathcal{G}$.

For instance, the pattern $p := \square \bar{1} \bar{1} \bar{2} \square \bar{1} \square$ decomposes in $\mathbb{P}_1^Z = \mathbb{P}^Z$ as the tree

\[
\begin{array}{c}
\square \\
\bar{0} \\
\bar{1} \\
\bar{0} \\
\bar{1} \\
\square
\end{array}
\]

on the minimal generating set of $\mathbb{P}^Z$ described in Section 2.2.3. Therefore, this decomposition is maximal w.r.t. this minimal generating set. Besides, the multi-pattern

\[
m := \begin{bmatrix}
0 & 0 & \square & 1 & \square & \bar{2} \\
\square & 0 & 0 & 0 & 3 & \square
\end{bmatrix}
\]

admits in $\mathbb{P}_2^Z$ the decomposition

\[
\begin{array}{c}
\begin{bmatrix}
0 & \square \\
\square & 0
\end{bmatrix}
\end{array}
\]
Therefore, we do not have an effective way to propose maximal decompositions of multi-patterns of multiplicity greater than 1 for the time being.

2.3.2. **Particular multi-patterns.** A multi-pattern \( m \) is a **chord** if \( m(m) \geq 2, |m| = 1, \) and \( \ell(m) = 1 \). For instance,

\[
\begin{bmatrix}
? \\
0 \\
2 \\
4
\end{bmatrix}
\]

is a chord. A multi-pattern \( m \) is **flat** if all degrees of its patterns \( m_i, i \in [m(m)] \), are 0. For instance,

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

is flat. A multi-pattern \( m \) is an **arpeggio shape** if \( m \) is flat, \( m(m) \geq 2, |m| = 1, \) and, for any \( j \in [\ell(m)] \) and \( i, i' \in [m(m)] \), \( m_{i,j} \neq \Box \neq m_{i',j} \) implies \( i = i' \). For instance,

\[
\begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix}
\]

is an arpeggio shape. A multi-pattern \( m \) is an **arpeggio** if \( m \) writes as \( m' \odot m'' \) where \( m' \) is a chord and \( m'' \) is an arpeggio shape. For instance, since

\[
\begin{bmatrix}
\Box & \Box & \Box \\
0 & 0 & 0 \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} = \begin{bmatrix}
\Box & \Box & \Box \\
0 & 0 & 0 \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} \odot \begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix},
\]

the multi-pattern on left-hand side is an arpeggio.

2.3.3. **Operations.** Let us describe here some transformations on musical phrases by using the operad \( P^Z_m \) and the related operad morphisms.

(1) **Mimesis.** For any multi-patterns \( m \) and \( m' \) of the same multiplicity, \( m \odot m' \) is the **mimesis** of \( m \) according to \( m' \). For instance

\[
\begin{bmatrix}
\Box & \Box & \Box \\
0 & 0 & 0 \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} \odot \begin{bmatrix}
\Box & ? & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} = \begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix}.
\]

Observe that since

\[
\begin{bmatrix}
\Box & \Box & \Box \\
0 & 0 & 0 \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} \odot \begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} = \begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix},
\]

this operation is not commutative. Moreover, when \( m \) is a chord and \( m' \) is an arpeggio shape, the mimesis of \( m \) according to \( m' \) is by definition an arpeggio. For instance,

\[
\begin{bmatrix}
? \\
0 \\
2 \\
4
\end{bmatrix} \odot \begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix} = \begin{bmatrix}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box 
\end{bmatrix}.
\]
(2) **Concatenation.** For any two multi-patterns \( m \) and \( m' \) of the same multiplicity \( m \),
\[
\text{conc}(m, m') := \text{copy}_{m}(00) \circ [m, m']
\]
(2.3.11)
is the concatenation of \( m \) and \( m' \). For instance,
\[
\text{conc}
\left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}
\right)
, \left(\begin{array}{c}
\begin{array}{ccc}
0 & 0 & 1
\end{array} \\
\begin{array}{ccc}
1 & 1 & 3
\end{array} \\
\begin{array}{ccc}
\bar{1} & \bar{1} & 3
\end{array}
\end{array}
\right)
= \left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}
\right).
\]
(2.3.12)

(3) **Repetition.** For any multi-pattern \( m \) and any \( k \geq 1 \),
\[
\text{rep}_k(m) := \text{copy}_{m/m}(0^k) \circ m
\]
(2.3.13)
is the \( k \)-fold repetition of \( m \). For instance,
\[
\text{rep}_3\left(\left(\begin{array}{c}
\begin{array}{ccc}
0 & 0 & 1
\end{array} \\
\begin{array}{ccc}
1 & 1 & 3
\end{array} \\
\begin{array}{ccc}
\bar{1} & \bar{1} & 3
\end{array}
\end{array}\right)\right) = \left(\begin{array}{c}
\begin{array}{ccc}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array} \\
\begin{array}{ccc}
1 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 1 & 3
\end{array} \\
\begin{array}{ccc}
\bar{1} & \bar{1} & 3 & \bar{1} & \bar{1} & 3 & \bar{1} & \bar{1} & 3 & \bar{1} & \bar{1} & 3 & \bar{1} & \bar{1} & 3
\end{array}
\end{array}\right).
\]
(2.3.14)

(4) **Transposition.** For any multi-pattern \( m \) and any \( d \in \mathbb{Z} \),
\[
\text{tran}_d(m) := \text{copy}_{m/m}(d) \circ m
\]
(2.3.15)
is the transposition of \( m \) by \( d \) degrees. For instance,
\[
\text{tran}_{-2}\left(\left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right)\right) = \left(\begin{array}{c}
\begin{array}{ccc}
\bar{4} & \bar{1} & \bar{1}
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array} \\
\begin{array}{ccc}
\bar{1} & \bar{1} & \bar{1}
\end{array}
\end{array}\right).
\]
(2.3.16)

(5) **Temporization.** For any multi-pattern \( m \) and any \( k \geq 0 \),
\[
\text{temp}_k(m) := m \circ \text{copy}_{m/m}(0^k)
\]
(2.3.17)
is the temporization of \( m \) by \( k \) units of time. For instance,
\[
\text{temp}_2\left(\left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right)\right) = \left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right).
\]
(2.3.18)

(6) **Inverse.** For any multi-pattern \( m \),
\[
\text{inv}(m) := \text{mul}_{-1, \ldots, -1}(m)
\]
(2.3.19)
is the inverse of \( m \). For instance,
\[
\text{inv}\left(\left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right)\right) = \left(\begin{array}{c}
\begin{array}{ccc}
2 & \bar{1} & \bar{1}
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right).
\]
(2.3.20)

(7) **Retrograde.** For any multi-pattern \( m \), \( \text{mir}(m) \) is the retrograde of \( m \). For instance,
\[
\text{mir}\left(\left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right)\right) = \left(\begin{array}{c}
\begin{array}{ccc}
\bar{1} & \bar{1} & \bar{2}
\end{array} \\
\begin{array}{ccc}
1 & 1 & 2
\end{array} \\
\begin{array}{ccc}
3 & 3 & 2
\end{array}
\end{array}\right).
\]
(2.3.21)

(8) **Retrograde inverse.** For any multi-pattern \( m \),
\[
\text{minv}(m) := \text{mir}(\text{inv}(m)).
\]
(2.3.22)
is the retrograde inverse of \( m \). Notice that \( \text{minv}(m) \) is also equal to \( \text{inv}(\text{mir}(m)) \) since the two maps \( \text{mir} \) and \( \text{inv} \) commute. For instance,
\[
\text{minv}\left(\left(\begin{array}{c}
\begin{array}{ccc}
\bar{2} & \bar{1} & 2
\end{array} \\
\begin{array}{ccc}
2 & 3 & 3
\end{array} \\
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\end{array}\right)\right) = \left(\begin{array}{c}
\begin{array}{ccc}
\bar{1} & \bar{1} & \bar{2}
\end{array} \\
\begin{array}{ccc}
1 & 1 & 2
\end{array} \\
\begin{array}{ccc}
3 & 3 & 2
\end{array}
\end{array}\right).
\]
(2.3.23)
3. Generation and random generation

We exploit now the music box operads introduced in the previous section to design three random generation algorithms devoted to generate new musical phrases from a finite set of multi-patterns. This relies on colored operads and bud generating systems, a sort of formal random generation algorithms devoted to generate new musical phrases from a finite set of multi-patterns. For any pattern \( p \) and any chord multi-pattern \( m \),

\[
\text{har}(p, m) := \text{copy}_{m|m}(p) \odot m
\]

(23.24)

is the harmonization of \( p \) according to \( m \). For instance,

\[
\text{har}\left(\begin{bmatrix} 1 \ 1 \ 0 \ 0 \ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \ 1 \ 0 \ 2 \\ 1 \ 3 \ 2 \ 4 \\ 3 \ 5 \ 4 \ 6 \end{bmatrix}.
\]

(23.25)

(9) **Harmonization.** For any pattern \( p \) and any chord multi-pattern \( m \),

\[
\text{har}(p, m) := \text{copy}_{m|m}(p) \odot m
\]

(23.24)

is the harmonization of \( p \) according to \( m \). For instance,

\[
\text{har}\left(\begin{bmatrix} 1 \ 1 \ 0 \ 2 \end{bmatrix}, \begin{bmatrix} 1 \ 4 \\ 2 \ 4 \end{bmatrix}\right) = \begin{bmatrix} 0 \ 2 \ 3 \ 4 \ 5 \ 6 \\ 0 \ 3 \ 4 \ 5 \ 6 \\ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}.
\]

(23.27)

(10) **Arpeggiation.** For any pattern \( p \) and any arpeggio multi-pattern \( m \),

\[
\text{arp}(p, m) := \text{copy}_{m|m}(p) \odot m
\]

(23.26)

is the arpeggiation of \( p \) according to \( m \). For instance,

\[
\text{arp}\left(\begin{bmatrix} 1 \ 1 \ 0 \ 2 \end{bmatrix}, \begin{bmatrix} 1 \ 4 \\ 2 \ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \ 2 \ 4 \\ 5 \ 6 \end{bmatrix}.
\]

(3.1.2)

Any operation preserving the lengths and the arities is *homogeneous*. The operations \( \text{tran}_d \), \( d \in \mathbb{Z} \), \( \text{inv} \), \( \text{mir} \), and \( \text{minv} \) are homogeneous.

3.1. Colored operads and bud operads. We provide here the elementary notions about colored operads \([Yau16]\). We also explain how to build colored operads from an operad.

3.1.1. Colored operads. A *set of colors* is any nonempty finite set \( \mathcal{C} := \{b_1, \ldots, b_k\} \) wherein elements are called *colors*. A \( \mathcal{C} \)-colored set is a set \( G \) decomposing as a disjoint union

\[
G := \bigsqcup_{a \in \mathcal{C}} G(a, u), \quad (3.1.1)
\]

where \( \mathcal{C}^* \) is the set of all finite sequences of elements of \( \mathcal{C} \), and the \( G(a, u) \) are sets. For any \( x \in G \), there is by definition a unique pair \( (a, u) \in \mathcal{C} \times \mathcal{C}^* \) such that \( x \in G(a, u) \). The *arity* \( |x| \) of \( x \) is the length \( \ell(u) \) of \( u \) as a word, the *output color* \( \text{out}(x) \) of \( x \) is \( a \), and for any \( i \in [|x|] \), the *i-th input color* \( \text{in}_i(x) \) of \( x \) is the \( i \)-th letter of \( u \). We also denote, for any \( n \in \mathbb{N} \), by \( G(n) \) the set of all elements of \( G \) of arity \( n \). Therefore, a colored graded set is in particular a graded set.

A \( \mathcal{C} \)-colored operad is a triple \( (G, \circ_i, 1) \) such that \( G \) is a \( \mathcal{C} \)-colored set, \( \circ_i \) is a map

\[
\circ_i : G(a, u) \times G(u_i, v) \to G(a, u \circ_i v), \quad 1 \leq i \leq |u|,
\]

(3.1.2)

called partial composition map, where \( u \circ_i v \) is the word on \( \mathcal{C} \) obtained by replacing the \( i \)-th letter of \( u \) by \( v \), and \( 1 \) is a map

\[
1 : \mathcal{C} \to G(a, a),
\]

(3.1.3)
called *colored unit* map. This data has to satisfy Relations (2.1.3a) and (2.1.3b) when their left and right members are both well-defined, and, for any \( x \in \mathcal{G} \), the relation

\[
1(\text{out}(x)) \circ_i x = x = x \circ_i 1(\text{in}_i(x)), \quad 1 \leq i \leq |x|.
\] (3.1.4)

Intuitively, an element \( x \) of a colored operad having \( a \) as output color and \( u_i \) as \( i \)-th input color for any \( i \in [|x|] \) can be seen as an abstract operator wherein colors are assigned to its output and to each of its inputs. Such an operator is depicted as

\[
\begin{array}{c}
  \begin{array}{c}
    a \\
    u_i \\
    \vdots \\
    u_{|x|}
  \end{array} \\
  x
\end{array}
\]

(3.1.5)

where the colors of the output and inputs are put on the corresponding edges. The partial composition of two elements \( x \) and \( y \) in a colored operad expresses pictorially as

\[
\begin{array}{c}
  \begin{array}{c}
    x \\
    u_i \\
    \vdots \\
    u_{|x|}
  \end{array} \\
  \circ_i \\
  \begin{array}{c}
    y \\
    v_i \\
    \vdots \\
    v_{|y|}
  \end{array}
\end{array}
\]

\[
= 1
\]

(3.1.6)

Besides, most of the definitions about operads recalled in Section 2.1.4 generalize straightforwardly to colored operads. In particular, one can consider the full composition map of a colored operad defined by (2.1.7) when its right member is well-defined.

The situation is specific for the homogeneous composition for colored operads. Let \((\mathcal{G}, \circ_i, 1)\) be a colored operad. The homogeneous composition map of \( \mathcal{G} \) is the map

\[
\circ : \mathcal{G}(a, u) \times \mathcal{G}(b, v) \to \mathcal{G}, \quad a, b \in \mathcal{C}, \; u, v \in \mathcal{C}^*.
\] (3.1.7)

defined, for any \( x \in \mathcal{G}(a, u) \) and \( y \in \mathcal{G}(b, v) \), by using the full composition map, by

\[
x \circ y := x \circ \left[ y^{(1)}, \ldots, y^{(|x|)} \right],
\] (3.1.8)

where for any \( i \in [|x|] \),

\[
y^{(i)} := \begin{cases} y & \text{if} \; \text{in}_i(x) = \text{out}(y), \\ 1(\text{in}_i(x)) & \text{otherwise}. \end{cases}
\] (3.1.9)

Intuitively, \( x \circ y \) is obtained by grafting simultaneously the outputs of copies of \( y \) into all the inputs of \( x \) having the same color as the output color of \( y \). If the set of colors of \( \mathcal{G} \) is a singleton, the homogeneous composition map of \( \mathcal{G} \) is the homogeneous composition map of operads described in Section 2.1.4.
3.1.2. Bud operads. Let us describe a general construction building a colored operad from a noncolored one introduced in [Gir19]. Given a noncolored operad \((\emptyset, \circ_1, \circ_0)\) and a set of colors \(\mathcal{C}\), the \(\mathcal{C}\)-bud operad of \(\emptyset\) is the \(\mathcal{C}\)-colored operad \(\mathcal{B}_\emptyset\) defined in the following way. First, \(\mathcal{B}_\emptyset\) is the \(\mathcal{C}\)-colored set defined, for any \(a \in \mathcal{C}\) and \(u \in \mathcal{C}^*\), by
\[
(\mathcal{B}_\emptyset)(a, u) := \{(a, x, u) : x \in \emptyset([u])\}.
\] (3.1.10)
Second, the partial composition maps \(\circ_i\) of \(\mathcal{B}_\emptyset\) are defined, for any \((a, x, u), (u_i, y, v) \in \mathcal{B}_\emptyset\) and \(i \in [[u]]\), by
\[
(a, x, u) \circ_i (u_i, y, v) := (a, x \circ_i y, u \circ_i v)
\] (3.1.11)
where the first occurrence of \(\circ_i\) in the right member of (3.1.11) is the partial composition map of \(\emptyset\) and the second one is a substitution of words: \(u \circ_i v\) is the word obtained by replacing in \(u\) the \(i\)-th letter of \(u\) by \(v\). Finally, the colored unit map \(\mathbb{1}\) of \(\mathcal{B}_\emptyset\) is defined by \(\mathbb{1}(a) := (a, \mathbb{1}, a)\) for any \(a \in \mathcal{C}\), where \(\mathbb{1}\) is the unit of \(\emptyset\). The pruning \(\text{pr}(\langle a, x, u \rangle)\) of an element \(\langle a, x, u \rangle\) of \(\mathcal{B}_\emptyset\) is the element \(x\) of \(\emptyset\).

Intuitively, this construction consists in forming a colored operad \(\mathcal{B}_\emptyset\) out of \(\emptyset\) by surrounding its elements with an output color and input colors coming from \(\mathcal{C}\) in all possible ways.

For a fixed degree monoid \(D\), we apply this construction to the \(D\)-music box operad by setting, for any set \(\mathcal{C}\) of colors,
\[
\mathcal{B}_m^D := \mathcal{B}_\emptyset^D.
\] (3.1.12)
We call \(\mathcal{B}_m^D\) the \(\mathcal{C}\)-bud \(\mathcal{D}\)-music box operad. The elements of \(\mathcal{B}_m^D\) are called \(\mathcal{C}\)-colored multi-patterns. For instance, for \(\mathcal{C} = \{b_1, b_2, b_3\}\),
\[
\begin{pmatrix}
  b_1,
  \begin{bmatrix}
    \square & 0 & 0 & 1 \\
    \checkmark & 0 & 0 & \square 
  \end{bmatrix},
  b_2b_2b_1
\end{pmatrix}
\] (3.1.13)
is a \(\mathcal{C}\)-colored multi-pattern. Moreover, in the colored operad \(\mathcal{B}_2^Z\), one has
\[
\begin{pmatrix}
  b_3,
  \begin{bmatrix}
    0 & 1 & \square \\
    \checkmark & 0 & \square 
  \end{bmatrix},
  b_2b_1 \end{pmatrix} \circ_2 \begin{pmatrix}
  b_1,
  \begin{bmatrix}
    1 & 1 & 2 \\
    2 & 1 & 2 
  \end{bmatrix},
  b_3b_2b_2
\end{pmatrix} = \begin{pmatrix}
  b_3,
  \begin{bmatrix}
    0 & 2 & 3 & \square \\
    \checkmark & 2 & 1 & 2 
  \end{bmatrix},
  b_2b_3b_3b_2
\end{pmatrix}.
\] (3.1.14)

The intuition that justifies the introduction of these colored versions of multi-patterns and of the \(\mathcal{D}\)-music box operad is that colors restrict the right to perform the composition of two given multi-patterns. In this way, one can for instance forbid some intervals in the musical phrases specified by the multi-patterns of a suboperad of \(\mathcal{B}_m^D\) generated by a given set of \(\mathcal{C}\)-colored multi-patterns. Moreover, given a set \(\mathcal{G}\) of \(\mathcal{C}\)-colored multi-patterns, the elements of the suboperad \(\mathcal{B}_m^D \mathcal{G}\) of \(\mathcal{B}_m^D\) generated by \(\mathcal{G}\) are obtained by composing elements of \(\mathcal{G}\). Therefore, in some sense, these elements inherit from properties of the patterns of \(\mathcal{G}\).

The next sections use these ideas to propose random generation algorithms outputting new multi-patterns from existing ones in a controlled way.

3.2. Bud generating systems. We describe here a sort of generating systems using operads and colored operads introduced in [Gir19]. Slight variations are considered in this present work.
3.2.1. Bud generating systems. A *bud generating system* [Gir19] is a tuple \((\emptyset, \mathcal{C}, \mathcal{R}, \mathbf{b})\) where

(i) \((\emptyset, \epsilon_{\mathcal{C}}, 1)\) is an operad, called *ground operad*;
(ii) \(\mathcal{C}\) is a finite set of colors;
(iii) \(\mathcal{R}\) is a finite subset of \(\mathbf{B}_{\mathcal{C}}(\emptyset)\), called *set of rules*;
(iv) \(\mathbf{b}\) is a color of \(\mathcal{C}\), called *initial color*.

For any color \(\alpha \in \mathcal{C}\), we shall denote by \(\mathcal{R}_\alpha\) the set of all rules of \(\mathcal{R}\) having \(\alpha\) as output color.

Bud generating systems are devices similar to context-free formal grammars [HMU06] wherein colors play the role of nonterminal symbols. These last devices are designed to generate sets of words. Bud generating systems are designed to generate more general combinatorial objects (here, multi-patterns). More precisely, a bud generating system \((\emptyset, \mathcal{C}, \mathcal{R}, \mathbf{b})\) allows us to build elements of \(\emptyset\) by following three different operating modes. We describe in the next sections the three corresponding random generation algorithms. These algorithms are in particular intended to work with \(P^0_m\) as ground operad in order to generate multi-patterns.

In the next sections, we shall provide some examples based upon the bud generating system \(\mathcal{B} := \{\mathbf{P}_{2}^2, \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5\}, \mathbf{b}_1\}\) where

\[
\mathbf{c}_1 := \left(\mathbf{b}_1, \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 5 & 4 & 0 & 0 & 0 \end{bmatrix}, \mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_5\right), \quad \mathbf{c}_2 := \left(\mathbf{b}_1, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right),
\]
\[
\mathbf{c}_3 := \left(\mathbf{b}_2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_1\right), \quad \mathbf{c}_4 := \left(\mathbf{b}_2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right), \quad \mathbf{c}_5 := \left(\mathbf{b}_3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_5\right). \tag{3.2.1}
\]

Moreover, to interpret the generated multi-patterns, we choose to consider a tempo of \(128\) and the rooted scale \((\lambda, 9_5)\) where \(\lambda\) is the Hirajoshi scale.

3.2.2. Partial generation. Let \(\mathcal{B} := (\emptyset, \mathcal{C}, \mathcal{R}, \mathbf{b})\) be a bud generating system. Let \(\rightarrow\) be the binary relation on \(\mathbf{B}_{\mathcal{C}}\emptyset\) such that \((a, x, u) \rightarrow (a, y, v)\) if there is a rule \(r \in \mathcal{R}\) and \(i \in [||u||]\) such that

\[(a, y, v) = (a, x, u) \circ_i r. \tag{3.2.2}\]

An element \(x\) of \(\emptyset\) is *partially generated* by \(\mathcal{B}\) if there is an element \((b, x, u)\) such that \((b, 1, b)\) is in relation with \((b, x, u)\) w.r.t. the reflexive and transitive closure of \(\rightarrow\).

For instance, by considering the bud generating system \(\mathcal{B}\) defined in Section 3.2.1, since

\[
\left(\mathbf{b}_1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_1\right) \rightarrow \left(\mathbf{b}_1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_1\right) \rightarrow \left(\mathbf{b}_1, \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_5\mathbf{b}_1\mathbf{b}_3\mathbf{b}_2\mathbf{b}_1\mathbf{b}_5\right) \rightarrow \left(\mathbf{b}_1, \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix}, \mathbf{b}_1\mathbf{b}_3\mathbf{b}_1\mathbf{b}_1\mathbf{b}_1\mathbf{b}_5\mathbf{b}_1\mathbf{b}_3\mathbf{b}_1\mathbf{b}_1\mathbf{b}_5\right), \tag{3.2.3}
\]

the multi-pattern

\[
\begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix}
\]

is partially generated by \(\mathcal{B}\).
§ Partial random generation

3.2.3. Full generation. Let $\mathcal{B} := (\mathcal{B}, \mathcal{C}, \mathcal{R}, b)$ be a bud generating system. Let $\xrightarrow{\circ}$ be the binary relation on $B_C(\mathcal{B})$ such that $(a, x, u) \xrightarrow{\circ} (a, y, v)$ if there are rules $r_1, \ldots, r_{|\mathcal{R}|} \in \mathcal{R}$ such that

$$(a, y, v) = (a, x, u) \circ [r_1, \ldots, r_{|\mathcal{R}|}].$$

(3.2.5)

An element $x$ of $\mathcal{B}$ is fully generated by $\mathcal{B}$ if there is an element $(b, x, u)$ such that $(b, 1, b)$ is in relation with $(b, x, u)$ w.r.t. the reflexive and transitive closure of $\xrightarrow{\circ}$.

For instance, by considering the bud generating system $\mathcal{B}$ defined in Section 3.2.1, since

$$
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\xrightarrow{\circ}
\begin{pmatrix}
0 & 2 & 1 & 0 & 4 \\
5 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{\circ}
\begin{pmatrix}
0 & 1 & 2 & 0 & 4 \\
5 & 0 & 0 & 1 & 0
\end{pmatrix}
\xrightarrow{\circ}
\begin{pmatrix}
0 & 1 & 2 & 0 & 4 \\
5 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

the multi-pattern

$$
\begin{bmatrix}
0 & 1 & 2 & 0 & 4 \\
5 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

is fully generated by $\mathcal{B}$.

Bud generating systems together with this scheme for generation are very similar to Lindenmayer systems, which are sorts of formal grammars [Lin68]. Such systems lead to frameworks to generate musical phrases [MS94, McC96, HQ18].

3.2.4. Homogeneous generation. Let $\mathcal{B} := (\mathcal{B}, \mathcal{C}, \mathcal{R}, b)$ be a bud generating system. Let $\xrightarrow{\bigcirc}$ be the binary relation on $B_C(\mathcal{B})$ such that $(a, x, u) \xrightarrow{\bigcirc} (a, y, v)$ if there is a rule $r \in \mathcal{R}$ such that

$$(a, y, v) = (a, x, u) \bigcirc r.$$  

(3.2.8)

An element $x$ of $\mathcal{B}$ is homogeneously generated by $\mathcal{B}$ if there is an element $(b, x, u)$ such that $(b, 1, b)$ is in relation with $(b, x, u)$ w.r.t. the reflexive and transitive closure of $\xrightarrow{\bigcirc}$.

For instance, by considering the bud generating system $\mathcal{B}$ defined in Section 3.2.1, since

$$
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\xrightarrow{\bigcirc}
\begin{pmatrix}
0 & 2 & 1 & 0 & 4 \\
5 & 0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{\bigcirc}
\begin{pmatrix}
0 & 1 & 2 & 0 & 4 \\
5 & 0 & 0 & 1 & 0
\end{pmatrix}
\xrightarrow{\bigcirc}
\begin{pmatrix}
0 & 1 & 2 & 0 & 4 \\
5 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

the multi-pattern

$$
\begin{bmatrix}
0 & 1 & 2 & 0 & 4 \\
5 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

is homogeneously generated by $\mathcal{B}$.

3.3. Random generation. We now design three random generation algorithms to produce musical phrases, each based upon the three generation schemes of bud generating systems described in Section 3.2. For any finite and nonempty set $S$, RANDOM($S$) is an element of $S$ picked uniformly at random among all the elements of $S$. 
3.3.1. Partial random generation. Algorithm 1 returns an element partially generated by $\mathcal{B}$ obtained by applying at most $k$ rules to the initial element $(b,1,b)$. The execution of the algorithm builds a composition tree of elements of $\mathcal{R}$ with at most $k$ internal nodes.

**Algorithm 1:** Partial random generation.

*Data:* A bud generating system $\mathcal{B} := (O, C, R, b)$ and an integer $k \in \mathbb{N}$.

*Result:* A randomly generated element of $O$.

1. $x \leftarrow (b, 1, b)$
2. for $j \in [k]$ do
3. $i \leftarrow \text{RANDOM}([|x|])$
4. if $\mathcal{R}_{in}(x) \neq \emptyset$ then
5. $r \leftarrow \text{RANDOM}(\mathcal{R}_{in}(x))$
6. $x \leftarrow x \circ_i r$
7. return $\text{pr}(x)$

For instance, by considering the previous bud generating system $\mathcal{B}$, this algorithm called with $k := 5$ builds the tree of colored multi-patterns

![Tree Diagram]

which produces the multi-pattern

\[
\begin{bmatrix}
0 & 1 & \square & 2 & \square & 1 & 3 & \square & 2 & \square & 1 & 5 & \square & 0 & 4 \\
6 & \square & \square & 1 & 0 & \square & 4 & \square & \square & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]

Together with the interpretation described in Section 3.2.1, the generated musical phrase is

\[
J = 128
\]

3.3.2. Full random generation. Algorithm 2 returns an element synchronously generated by $\mathcal{B}$ obtained by applying at most $k$ rules to the initial element $(b, 1, b)$. The execution of the algorithm builds a composition tree of elements of $\mathcal{R}$ of height at most $k + 1$ wherein the leaves are all at the same distance from the root. Observe that when for all colors $b \in C$, the sets $\mathcal{R}_b$ have no more than one element, this algorithm is deterministic.
Algorithm 2: Full random generation.

Data: A bud generating system \( B := (O, C, \mathcal{R}, b) \) and an integer \( k \in \mathbb{N} \).

Result: A randomly generated element of \( O \).

1 begin
2 \( x \leftarrow (b, 1, b) \)
3 for \( j \in [k] \) do
4 \( R \leftarrow \mathcal{R}_{\text{in}}(x) \times \cdots \times \mathcal{R}_{\text{in}}(x) \)
5 if \( R \neq \emptyset \) then
6 \( \{r_1, \ldots, r_{|x|}\} \leftarrow \text{RANDOM}(R) \)
7 \( x \leftarrow x \circ [r_1, \ldots, r_{|x|}] \)
8 return \( \text{pr}(x) \)

For instance, by considering the previous bud generating system \( B \), this algorithm called with \( k := 2 \) builds the tree of colored multi-patterns

\[
\begin{array}{c}
\left| c_2 \right| \\
\left| c_2 \right| \\
\left| c_1 \right| \\
\left| e_3 \right| \\
\left| e_2 \right| \\
\left| e_5 \right| \\
\left| e_1 \right| \\
\left| e_2 \right| \\
\left| e_5 \right| \\
\end{array}
\]

which produces the multi-pattern

\[
\begin{bmatrix}
2 & 4 & \square & 3 & 2 & 6 & \square & 2 & 1 & 0 & 1 & \square & 2 & 1 & \square & 0 & 4 \\
\square & \square & 0 & 0 & 0 & 0 & 1 & \square & 2 & 4 & \square & 0 & 1 & \square & 2 & 1 & 2 & 1
\end{bmatrix}.
\]

Together with the interpretation described in Section 3.2.1, the generated musical phrase is

\[
\begin{array}{c}
\text{J}_{128} \\
\text{J}_8 \\
\text{J}_4 \\
\text{J}_2 \\
\text{J}_1 \\
\text{J}_0 \\
\end{array}
\]

3.3.3. Homogeneous random generation. Algorithm 3 returns an element homogeneously generated by \( B \) obtained by applying at most \( k \) rules to the initial element \((b, 1, b)\). The execution of the algorithm builds a composition tree of elements of height at most \( k + 1 \). Observe that when the set of rules \( \mathcal{R} \) has no more than one element, this algorithm is deterministic.

For instance, by considering the previous bud generating system \( B \), this algorithm called with \( k := 3 \) builds the tree of colored multi-patterns

\[
\begin{array}{c}
\left| c_1 \right| \\
\left| c_3 \right| \\
\left| c_1 \right| \\
\left| c_1 \right| \\
\left| c_1 \right| \\
\left| e_5 \right| \\
\left| e_3 \right| \\
\left| e_5 \right| \\
\left| e_3 \right| \\
\left| e_5 \right| \\
\end{array}
\]
§ Random temporizations

Music generation by operads

S. Giraudo

Algorithm 3: Homogeneous random generation.

Data: A bud generating system \( B := (O, C, R, b) \) such that \( R \neq \emptyset \) and an integer \( k \in \mathbb{N} \).

Result: A randomly generated element of \( O \).

1 begin
2 \( x \leftarrow (b, 1, b) \)
3 for \( j \in [k] \) do
4 \( r \leftarrow \text{RANDOM}(R) \)
5 \( r \leftarrow x \odot r \)
6 return \( \text{pr}(r) \)

which produces the multi-pattern

\[
\begin{bmatrix}
0 & 1 & 1 & 2 & 2 & 1 & 5 & 0 & 1 & 1 & 1 & 0 & 4 & 4 \\
5 & 0 & 1 & 5 & 0 & 0 & 0 & 5 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

Together with the interpretation described in Section 3.2.1, the generated musical phrase is

![Musical Phrase](image)

4. Applications: exploring variations of patterns

We construct here some particular bud generated systems devoted to work with the algorithms introduced in Section 3.3. They generate variations of a single pattern \( p \) placed at input, with possibly some auxiliary data. Each is based on a precise musical transformation of \( p \).

4.1. Patterns to patterns. We describe here three methods to produce patterns from a pattern. The first one accepts additionally a positive integer, the second one, a flat pattern, and the last one, an homogeneous operation.

4.1.1. Random temporizations. Given a pattern \( p \) and an integer \( t \geq 1 \), we define the temporizer bud generating system \( B_{p,t}^{\text{temp}} \) of \( p \) and \( t \) by

\[
B_{p,t}^{\text{temp}} := \left( p^t, C, \{ c_1, c_2, c'_1, \ldots, c'_t \}, b_1 \right)
\]

where \( C \) is the set of colors \( \{ b_1, b_2, b_3 \} \) and \( c_1, c_2, c'_1, \ldots, c'_t \) are the \( C \)-colored patterns

\[
c_1 := \left( b_1, p, b_2 \right), \quad c_2 := \left( b_2, p, b_2 \right), \quad c'_j := \left( b_2, 0 \square, b_3 \right)
\]

for any \( j \in [t] \). The temporizer bud generating system of \( p \) and \( t \) generates a version of the pattern \( p \) composed with itself where the durations of some beats have been increased by at most \( t \). The colors, and in particular the color \( b_3 \), prevent multiple compositions of the colored patterns \( c'_j, j \in [t] \), in order to not overly increase the duration of some beats.
For instance, by considering the pattern
\[ p := 02 \quad 1 \quad 04 \] \hspace{1cm} (4.1.3)
and the parameter \( t := 2 \), the partial random generation algorithm run with the bud generating system \( B_{p,t}^{\text{tem}} \) and \( k := 16 \) as inputs produces the pattern
\[ 02 \quad 0 \quad 1 \quad 3 \quad 0 \quad 2 \quad 15 \quad 0 \quad 0 \quad 4. \] \hspace{1cm} (4.1.4)
Together with the interpretation consisting in a tempo of 128 and the rooted scale \( \langle \lambda, 9_3 \rangle \) where \( \lambda \) is the Hirajoshi scale, the generated musical phrase is
\[ J = 128 \]

4.1.2. Random rhythmic variations. Given a pattern \( p \) and a flat pattern \( p' \), we define the \textit{rhythmic bud generating system} \( B_{p,p'}^{\text{rhy}} \) of \( p \) and \( p' \) by
\[ B_{p,p'}^{\text{rhy}} := \{ P_1, C, \{ c_1, c_2, c_3 \}, b_1 \} \] \hspace{1cm} (4.1.5)
where \( C \) is the set of colors \( \{ b_1, b_2, b_3 \} \) and \( c_1, c_2, c_3 \) are the three \( C \)-colored patterns
\[ c_1 := \left( b_1, p, b_1^{[p]} \right), \quad c_2 := \left( b_2, p, b_2^{[p]} \right), \quad c_3 := \left( b_2, p', b_3^{[p']} \right). \] \hspace{1cm} (4.1.6)
The rhythmic bud generating system of \( p \) and \( p' \) generates a version of the pattern \( p \) composed with itself where some beats are repeated accordingly to the flat pattern \( p' \) by mimesis (see Section 2.3.3). The colors, and in particular the color \( b_3 \), prevent multiple compositions of the colored pattern \( c_3 \).

For instance, by considering the pattern
\[ p := 1 \quad 011 \quad 2 \] \hspace{1cm} (4.1.7)
and the flat pattern
\[ p' := 00 \quad 0 \] \hspace{1cm} (4.1.8)
the partial random generation algorithm run with the bud generating system \( B_{p,p'}^{\text{rhy}} \) and \( k := 8 \) as inputs produces the pattern
\[ 22 \quad 2 \quad 2 \quad 122 \quad 3 \quad 1 \quad 011 \quad 22 \quad 122 \quad 2 \quad 2 \quad 31 \quad 22 \quad 2 \quad 2. \] \hspace{1cm} (4.1.9)
Together with the interpretation consisting in a tempo of 128 and the rooted scale \( \langle \lambda, 9_3 \rangle \) where \( \lambda \) is the minor natural scale, the generated musical phrase is
\[ J = 128 \]
4.1.3. Concatenating transformations. Given a pattern \( p \) and an homogeneous operation \( \Lambda \), we define the \( \Lambda \)-concatenating bud generating system \( B_{p,\Lambda}^{\text{con}} \) of \( p \) by
\[
B_{p,\Lambda}^{\text{con}} := (P, \mathcal{C}, \{ c_1 \}, b_1)
\]
where \( \mathcal{C} \) is the set of colors \( \{ b_1 \} \), and \( c_1 \) is the \( \mathcal{C} \)-colored pattern
\[
c_1 := (b_1, \text{conc}(p, \Lambda(p)), b_2^{\|p\|}).
\]
The \( \Lambda \)-concatenating bud generating system of \( p \) generates a pattern obtained by composing the concatenation of \( p \) and \( \Lambda(p) \) with itself.

For instance, by considering the pattern
\[
p := 20 \square 11 \\square
\]
and \( \Lambda \) as the operation \( \text{mir} \), the partial random generation algorithm run with the bud generating system \( B_{p,\Lambda}^{\text{con}} \) and \( k := 3 \) as inputs produces the pattern
\[
20 \square 11 \square 31 \sqcap 20 \square 02 \sqcap 31 \sqcap 02 \sqcap 20 \square 111 \square 02.
\]
Together with the interpretation consisting in a tempo of 192 and the rooted scale \( (\lambda, 9) \) where \( \lambda \) is the Hirajoshi scale, the generated musical phrase is

\[\text{\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{musical_phrase}
\caption{Generated musical phrase.}
\end{figure}}\]

4.2. Patterns to multi-patterns. We describe here three methods to produce multi-patterns from a pattern. The first one accepts additionally a chord, the second one, an arpeggio, and the last one, an homogeneous operation.

4.2.1. Random harmonizations. Given a pattern \( p \) and a chord \( m \) of multiplicity \( m \), we define the harmonizator bud generating system \( B_{p,m}^{\text{har}} \) of \( p \) and \( m \) by
\[
B_{p,m}^{\text{har}} := (P, \mathcal{C}, \{ c_1, c_2, c_3 \}, b_1)
\]
where \( \mathcal{C} \) is the set of colors \( \{ b_1, b_2, b_3 \} \) and \( c_1, c_2, \) and \( c_3 \) are the three \( \mathcal{C} \)-colored multi-patterns
\[
c_1 := (b_1, \text{copy}_m(p), b_2^{\|p\|}), \quad c_2 := (b_2, \text{copy}_m(p), b_2^{\|p\|}), \quad c_3 := (b_2, m, b_3).
\]
The harmonizator bud generating system of \( p \) and \( m \) generates an harmonized version of the pattern \( p \) composed with itself, with chords controlled by \( m \). The colors, and in particular the color \( b_3 \), prevent multiple compositions of the colored pattern \( c_3 \).

For instance, by considering the pattern
\[
p := 2102 \square 1 \square 0 \square
\]
and the chord
\[
m := \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix},
\]

the partial random generation algorithm run with the bud generating system $\mathcal{G}_{p,m}^{\text{hor}}$ and $k := 3$ as inputs produces the multi-pattern

\[
\begin{bmatrix}
2 & 1 & 0 & 2 & \Box & 1 & 0 & \Box \\
2 & 6 & 5 & 2 & \Box & 1 & 0 & \Box \\
2 & 6 & 7 & 2 & \Box & 1 & 0 & \Box \\
\end{bmatrix}.
\] (4.2.5)

Together with the interpretation consisting in a tempo of 128 and the rooted scale $(\lambda, 9 \lambda)$ where $\lambda$ is the minor natural scale, the generated musical phrase is

\[
J = 128
\]

4.2.2. Random arpeggiations. Given a pattern $p$ and an arpeggio $m$ of multiplicity $m$, we define the arpeggiator bud generating system $\mathcal{G}_{p,m}^{\text{arp}}$ of $p$ and $m$ by

\[
\mathcal{G}_{p,m}^{\text{arp}} := (p_{m}^{\mathbb{Z}}, \mathcal{C}, \{c_1, c_2, c_3\}, b_1)
\] (4.2.6)

where $\mathcal{C}$ is the set of colors $\{b_1, b_2, b_3\}$ and $c_1$, $c_2$, and $c_3$ are the three $\mathcal{C}$-colored multi-patterns

\[
c_1 := (b_1, \text{copy}_m(p), b_2^{m}), \quad c_2 := (b_2, \text{copy}_m(p), b_2^{m}), \quad c_3 := (b_2, m, b_3).
\] (4.2.7)

The arpeggiator bud generating system of $p$ and $m$ generates an arpeggiated version of the pattern $p$ composed with itself, where the arpeggio is controlled by $m$. The colors, and in particular the color $b_3$, prevent multiple compositions of the colored pattern $c_3$.

For instance, by considering the pattern

\[
p := 0 \Box 213 \Box 1
\] (4.2.8)

and the degree pattern

\[
m := \begin{bmatrix}
0 & \Box & \Box \\
\Box & 2 & \Box \\
\Box & \Box & 4
\end{bmatrix},
\] (4.2.9)

the partial random generation algorithm run with the bud generating system $\mathcal{G}_{p,m}^{\text{arp}}$ and $k := 8$ as inputs produces the multi-pattern

\[
\begin{bmatrix}
0 & \Box & 2 & 1 & \Box & 5 & \Box & 3 \\
0 & \Box & 2 & 1 & \Box & 3 & \Box & 3 \\
0 & \Box & 2 & \Box & 1 & \Box & 2 & \Box \\
0 & \Box & 2 & \Box & 1 & \Box & 4 & \Box \\
0 & \Box & 2 & \Box & 1 & \Box & 4 & \Box \\
0 & \Box & 2 & \Box & 1 & \Box & 4 & \Box \\
0 & \Box & 2 & \Box & 1 & \Box & 4 & \Box \\
0 & \Box & 2 & \Box & 1 & \Box & 4 & \Box
\end{bmatrix}.
\] (4.2.10)

Together with the interpretation consisting in a tempo of 128 and the rooted scale $(\lambda, 0 \lambda)$ where $\lambda$ is the major natural scale, the generated musical phrase is
4.2.3. Stacking transformations. Given a pattern $p$ and an homogeneous operation $\Lambda$, we define the $\Lambda$-stacking bud generating system $\mathcal{B}_{p,\Lambda}^{\text{sta}}$ of $p$ by

$$\mathcal{B}_{p,\Lambda}^{\text{sta}} := \left(B_{m, \mathcal{C}}, \{c_1\}, b_1\right)$$

(4.2.11)

where $\mathcal{C}$ is the set of colors $\{b_1\}$, and $c_1$ is the $\mathcal{C}$-colored multi-pattern

$$c_1 := \left(b_1, \left[\begin{array}{c} p_{\Lambda(p)} \\ b_1 \end{array}\right]\right).$$

(4.2.12)

The $\Lambda$-stacking bud generating system of $p$ generates a multi-pattern of multiplicity 2 obtained by composing the superposition of $p$ and $\Lambda(p)$ with itself.

For instance, by considering the pattern

$$p := 20 \square 1 \square 1 \square$$

(4.2.13)

and $\Lambda$ as the operation $\text{mir}$, the partial generation algorithm run with the bud generating system $\mathcal{B}_{p,\Lambda}^{\text{sta}}$ and $k := 6$ as inputs produces the multi-pattern

$$\begin{bmatrix}
4 & 2 & 3 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 4 & 0 & 3 & 5 & 0 & 2
\end{bmatrix}$$

(4.2.14)

Together with the interpretation consisting in a tempo of 192 and the rooted scale $\langle \lambda, 9_3 \rangle$ where $\lambda$ is the harmonic minor scale, the generated musical phrase is

$$J = 192$$

CONCLUSION AND PERSPECTIVES

In this work, we have introduced the music box model, a framework to represent musical phrases as multi-patterns. One of the strengths of this model is that it allows us to perform computations over musical phrases. Indeed, the operad structures on multi-patterns lead to consider any multi-pattern as an operation acting on multi-patterns. Moreover, as explained, one can express well-known musical transformations like the concatenation, transposition, inverse, retrograde, and retrograde inverse transformations in terms of operad compositions or operad (anti)morphisms. This encoding of musical phrases and the introduced algebraic machinery is coupled with bud generating systems in order to propose a new kind of formal grammar based generative music system. This allows us to build relatively complex random musical data from short inputs.
This approach has several advantages: it is simple, extensible in the sense that it works for any scale and any rooted scale of an \( \eta \)-TET, and, since the composition of multi-patterns depends on degree monoids, it can be parametrized by several ways depending on the wished behavior for the multi-pattern composition. Nevertheless, this framework admits some disadvantages: due to the requirements of the algebraic constructions of the operads, all patterns of a multi-pattern must have the same arity and the same length. It is possible to modify the algebraic construction of the operad of multi-patterns in order to admit multi-patterns having patterns with different arities but the same length though the construction would become more complicated and less usable by bud generating systems.

Here are some perspectives raised by this work. The first one consists in the description of a minimal generating set for the operad \( P^D_{m} \), \( m \geq 2 \), even just for \( D = \mathbb{Z} \). The knowledge of such a minimal generating set would provide a way to decompose a multi-pattern as a tree decorated by generators. This would lead to applications such as the discovery of repeated parts of musical phrases (up to some elementary transformations like transpositions or more complex ones involving rhythmic motives), or the compression of musical data. Another perspective is to consider variations of the operad \( RP \) controlling the rhythmic part of patterns. Such a variation can potentially produce very different results than the present ones when used with bud generating systems.

References


