Operads on graphs: extending the pre-Lie operad and general construction

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Abstract

The overall aim of this paper is to define a structure of graph operads, thus generalizing the celebrated pre-Lie operad on rooted trees. More precisely, we define two operads on multigraphs, and exhibit a non trivial link between them and the pre-Lie and Kontsevich-Willwacher operads. We study one of these operads in more detail. While its structure is too involved to exhibit a description by generators and relations, we show that it has interesting finitely generated sub-operads, with links with the commutative and the magmatic commutative operads. In particular, one of them is Koszul and this allows us to compute its Koszul dual. Finally, we introduce a new framework on species and operads and a general way to define operads on multigraphs.

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Contents

Introduction

Operads are mathematical structures which were first introduced as a way to formalize the notion of type of algebra: given a set of multilinear maps and relations between them, the associated operad is composed of all the multilinear maps obtained by composing those in the initial set. These form in fact the generators of the operad. As detailed in [14], operad theory was first used in algebraic topology in the

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1960s but had a 'renaissance' in the 1990s where it began to be used in many other fields, see for example [12] for a very general algebraic approach of the theory. In particular, in combinatorics, operads provide the right framework to define the embedding of a combinatorial object in another [4, 7]. In this context, operads are defined by directly describing their objects and how to embed them. This contrast with the original way to define them by generators and relations, and passing from one definition to the other is often a difficult question and provides a lot of insight on the structure of the operad.

A particularly successful example is the pre-Lie operad which was first defined by being generated by the pre-Lie operator i.e. an antisymmetric bilinear map $x \triangleleft y$ such that $(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) = (x \triangleleft z) \triangleleft y - x \triangleleft (z \triangleleft y)$. Chapoton and Livernet later proved in [5] that the elements of the pre-Lie operad could be seen as rooted trees and described the composition of two elements in a purely combinatorial way.

The pre-Lie operad along with the nonassociative permutative operad [11] are two examples of interesting operads for which the combinatorial interpretation is given by trees. While there exist also in the literature interesting operads on graphs [10, 9, 17, 13], none of them is studied in a purely combinatorial framework. We propose to do so in this paper by extending the pre-Lie operad from trees to graphs. Within our approach, we find it is more natural to work on multigraphs and define an operad structure MG on the species of multigraphs, and an operad structure $\mathrm{MG}^{\bullet}_{or}$ on the species of pointed oriented multigraphs. The operad MG in particular restricts to the Kontsevich-Willwacher operad [13] on the species of graphs G. We show that these operads and the pre-Lie operad **PLie** relate to each other by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{PLie} \cap \mathbb{K}\mathcal{O}/\mathcal{I} \arrow[r, equal] \arrow[d, hook] & \mathbf{PLie} \cap \mathcal{O} \arrow[r, hook] \arrow[d, hook] & \mathbf{PLie} \arrow[d, hook] \\
\mathrm{G}_c \arrow[r, "\sim"] \arrow[d, hook] & \mathcal{O} \cap \mathrm{G}^{\bullet}_{orc}/\mathcal{I} \cap \mathrm{G}^{\bullet}_{orc} \arrow[d, hook] & \mathrm{G}^{\bullet}_{orc} \cap \mathcal{O} \arrow[l, two heads] \arrow[r, hook] \arrow[d, hook] & \mathrm{G}^{\bullet}_{orc} \cap \mathbf{ST} \arrow[d, hook] \\
\mathrm{MG}_c \arrow[r, "\sim"] & \mathcal{O}/\mathcal{I} & \mathcal{O} \arrow[l, two heads] \arrow[r, hook] & \mathrm{MG} \times \mathbf{PLie}
\end{tikzcd},$$

where T, G_c and MG_c are respectively the sub-operads of trees, connected graphs and connected multigraphs of MG, $\mathrm{G}^{\bullet}_{orc}$ is the sub-operad of pointed connected oriented graphs of $\mathrm{MG}^{\bullet}_{or}$, and **ST** is an operad on spanning trees with $\mathcal{O}$ and $\mathcal{I}$ sub-species of **ST** generated by ad hoc sums of spanning trees. We show that while, contrarily to **PLie**, the operad MG does not admit a simple presentation by generators and relation, its sub-operads generated by combinations of the empty graph over two vertices and the segment graph, have interesting links with the commutative operad **Com** and the magmatic operad **ComMag**. In particular, one of them is Koszul, which allows us to exhibit its Koszul dual. We end our study by providing new constructions on species and operads. These constructions enable us to define a general way of constructing operads on graphs, which we use to justify the operad structure of MG and $\mathrm{MG}^{\bullet}_{or}$. More specifically, we give a general way to construct operads on multigraphs where the partial composition of two elements $g_1 \circ_* g_2$ is given by the following steps:

1. take the disjoint union of g_1 and g_2,
2. remove the vertex $*$ from g_1, the edges previously connected to $*$ have now one or more loose end,
3. connect <u>independently</u> each loose ends of g_1 to g_2 in a certain way,

where independently means here that the way of connecting one loose end does not depend on the way we connect the other loose ends.

This paper is organized as follows. In Section 1 we give some general definitions of species theory and of operad theory, as well as some general definitions on graphs and multigraphs. In Section 2 we define the aforementioned two operads on multigraphs and pointed oriented multigraphs and we exhibit the link between these operads and the pre-Lie and Kontsevich-Willwacher operads. We then proceed to study the operad on multigraphs and some of its sub-operads. Finally, in Section 3 we exhbit our proposal of constructing species and operads and we apply them in order to obtain our general construction of operads on multigraphs.

This paper is an extended version of the extended abstract for FPSAC 2020 [1] with proofs and additional results.

1 Context

In all this paper, otherwise stated, V denotes a finite set and V_1 and V_2 two disjoint sets such that $V = V_1 \sqcup V_2$. The letter n always denotes a non negative integer and we denote by $[n]$ the set $\{1, \dots, n\}$. All vector spaces appearing in this paper are defined over a field of characteristic 0 denoted by $\mathbb{K}$. Finally, for any set E, $\mathbb{K}E$ denotes the free vector space over E.

1.1 Definitions and background on species

We recall here basic definitions on species. We refer the interested reader to [2] for a detailed presentation of combinatorial species.

Definition 1.1. A *linear species* S consists of the following data:

- For each finite set V, a vector space $\mathrm{S}[V]$ of finite dimension,
- For each bijection of finite sets $\sigma : V \to V'$, a linear map $\mathrm{S}[\sigma] : \mathrm{S}[V] \to \mathrm{S}[V']$. These maps should be such that $\mathrm{S}[\sigma_1 \circ \sigma_2] = \mathrm{S}[\sigma_1] \circ \mathrm{S}[\sigma_2]$ and $\mathrm{S}[Id] = Id$.

Furthermore if $\mathrm{S}[\emptyset] = \{0\}$, then S is say to be *positive*.

We will use the term *species* to refer to linear species.

When defining a species, the maps $\mathrm{S}[\sigma]$ are often clear under context and we do not mention them. Let S be a species. A *sub-species* of S is a species R such that $\mathrm{R}[V]$ is a sub-space of $\mathrm{S}[V]$ for every finite set V and $\mathrm{R}[\sigma] = \mathrm{S}[\sigma]$ for every bijection of finite sets σ. For E a set of elements of S, the *species generated by* E is the smallest sub-species of S containing E. We denote by S_n the sub-species of S defined by $\mathrm{S}_n[V] = \mathrm{S}[V]$ if $|V| = n$ and $\mathrm{S}_n[V] = \emptyset$ otherwise, and by S_{n+} the sub-species of S defined by $\mathrm{S}_{n+}[V] = \mathrm{S}[V]$ if $|V| \geq n$ and $\mathrm{S}_{n+}[V] = \{0\}$ otherwise. We also denote by S_+ the species S_{1+}. The *Hilbert series* of S is the formal power series defined by $\mathcal{H}_{\mathrm{S}}(x) = \sum_{n\geq 0} \frac{\dim \mathrm{S}[[n]]}{n!} x^n$.

A *morphism of species* from a species R to S is a collection of linear maps $f_V : \mathrm{R}[V] \to \mathrm{S}[V]$ such that for each bijection $\sigma : V \to V'$, we have $f_{V'} \circ \mathrm{R}[\sigma] = \mathrm{S}[\sigma] \circ f_V$. For easier reading, we will often forget the index V.

Example 1.2.
- The *Identity*, *exponential* and *singleton species* are respectively defined by $\mathrm{I\scriptsize D}[V] = \mathbb{K}V$, $\mathrm{E}[V] = \mathbb{K}\{V\}$ and $\mathrm{X} = \mathrm{E}_1$.
- For V a finite set, we denote by $\mathrm{P{\scriptsize OL}}[V]$ the *set* (not the vector space) of polynomials with coefficients in $\mathbb{N}$, variables in V and null constant coefficient. To consider the species $\mathbb{K}\mathrm{P{\scriptsize OL}}$, we must take into consideration the fact that we need to differentiate the plus of polynomials and the addition of vectors. We will thus denote by $\oplus$ the former and keep $+$ for the latter and we will denote by $0_V \in \mathrm{P{\scriptsize OL}}[V]$ the polynomial constant to 0 and keep the notation 0 for the null vector. For example, $ab \oplus c$ is an element of $\mathrm{P{\scriptsize OL}}[\{a, b, c\}]$, but $a \oplus b + c$ is a vector in $\mathbb{K}\mathrm{P{\scriptsize OL}}[\{a, b, c\}]$.
- We have a natural morphism from $\mathrm{I\scriptsize D}$ to $\mathbb{K}\mathrm{P{\scriptsize OL}}$ given by $v \mapsto v$ and three natural morphisms from E to $\mathbb{K}\mathrm{P{\scriptsize OL}}$ respectively given by $V \mapsto \prod_{v\in V} v$, $V \mapsto \bigoplus_{v\in V} v$ and $V \mapsto \sum_{v\in V} v$. In particular, the image of $\mathrm{I\scriptsize D}$ by this morphism is the species $\mathbb{K}\mathrm{P{\scriptsize OL}}^1$ of homogeneous polynomials of degree 1.

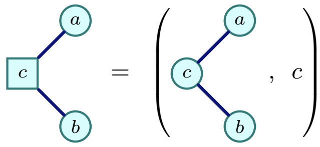
One strong point of species is the different operations on them which enable to construct new species from existing ones. Let R and S be two species. We can then construct new species which are defined as follows: *sum* $(\mathrm{R} + \mathrm{S})[V] = \mathrm{R}[V] \bigoplus \mathrm{S}[V]$, *product* $\mathrm{R} \cdot \mathrm{S}[V] = \bigoplus_{V_1 \sqcup V_2 = V} \mathrm{R}[V_1] \otimes \mathrm{S}[V_2]$, *Hadamard product* $(\mathrm{R} \times \mathrm{S})[V] = \mathrm{R}[V] \otimes \mathrm{S}[V]$, *derivative* $\mathrm{S}'[V] = \mathrm{S}[V + \{*\}]$ (where $* \notin V$), *second derivative* $\mathrm{S}''[V] = \mathrm{S}[V + \{*_1, *_2\}]$ ($*_1, *_2 \notin V$) and *pointing* $\mathrm{S}^\bullet[V] = \mathrm{S}[V] \otimes \mathbb{K}V$. Furthermore if S is positive we can also define the *composition* of R and S by

$$\mathrm{R}(\mathrm{S})[V] = \bigoplus_{P \text{ partition of } V} \mathrm{R}[P] \bigotimes_{P_i \in P} \mathrm{S}[P_i],$$

where $\bigotimes_{P_i\in P} \mathrm{S}[P_i] = (\mathrm{S}[P_1] \otimes \cdots \otimes \mathrm{S}(P_k))_{\mathfrak{S}_k}$ should be seen as an unordered tensor product. We call *assemblies* of S the elements of this unordered tensor product.

Remark 1. When considering the product of two derivatives $\mathrm{R}' \cdot \mathrm{S}'$, to avoid confusion we will use the notations $\mathrm{R}'[V] = \mathrm{R}[V + \{*_1\}]$ and $\mathrm{S}'[V] = \mathrm{S}[V + \{*_2\}]$.

Example 1.3. • Let T be the species of trees i.e. $\mathrm{T}[V]$ is the free vector space over abstract trees with vertex set V. Then the species $\mathrm{T}^\bullet$ is the species of rooted tree. We distinguish the root vertex by changing its shape. For example, the element $(\{\{a,c\},\{c,b\}\},c) \in \mathrm{T}^\bullet[\{a,b,c\}]$ will be represented as follows:

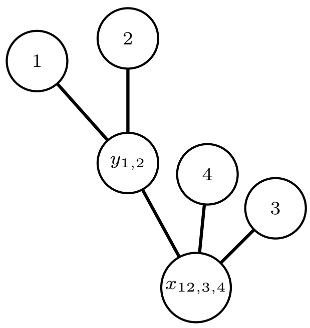


(1)

- Since $\mathrm{E}[V]$ is always reduced to a space of dimension 1, the species $\mathrm{E}(\mathrm{S})$ can be interpreted as the species of assemblies of S.

A particularly interesting species we can construct using the composition of positive species is the species of Schröder trees enriched with another species. For S a positive species such that $\mathrm{S} = \mathrm{X} + \mathrm{S}_{2+}$, the species $\mathscr{S}_\mathrm{S}$ of *Schröder trees enriched with* S is defined as the species satisfying the equation $\mathscr{S}_\mathrm{S} = \mathrm{X} + \mathrm{S}_{2+}(\mathscr{S}_\mathrm{S})$. The vector space $\mathscr{S}_\mathrm{S}[V]$ is then generated by abstract rooted trees with internal vertices labelled by elements of S and set of leaves V. In particular, remark that the internal node have a linear behaviour: if W is a subset of V, x and y are two elements of $\mathrm{S}[W]$ and k is an element of $\mathbb{K}$, a tree t with a node labelled by $x + ky$ is equal to the sum $t_1 + kt_2$ where t_1 and t_2 are identical to t except with the node which was labelled by $x + ky$ in t being respectively labelled by x and y. In this context, the elements of S are identified with the corollas of $\mathscr{S}_\mathrm{S}$.

Example 1.4. Here is an example of an element in $\mathscr{S}_\mathrm{S}[[4]]$ where $\mathrm{S}[V] = \mathbb{K}x$ if $|V| = 3$, $\mathrm{S}[V] = \mathbb{K}y$ if $|V| = 2$ and $\mathrm{S}[V] = \{0\}$ otherwise.



(2)

1.2 Definitions and background on operads

We give here basic definitions and results of the theory as well as some classical examples. We refer the reader to [15] and [12] for a more general approach to the theory of operads.

A *partial composition* of a species S is a collection of linear maps $\circ_v : \mathrm{S}[V_1] \otimes \mathrm{S}[V_2] \to \mathrm{S}[V_1 + V_2 - \{v\}]$, for V_1 and V_2 disjoint finite sets and $v \in V_1$. These maps must be natural: for σ_1 and σ_2 bijections with respective domain V_1 and V_2, we have $\circ_{\sigma_1(v)} \circ \mathrm{S}[\sigma_1] \otimes \mathrm{S}[\sigma_2] = \mathrm{S}[\sigma_1 \circ \sigma_2] \circ \circ_v$. In particular, a partial composition induces a morphism of linear species $\circ_* : \mathrm{S}' \cdot \mathrm{S} \to \mathrm{S}$, from which we can recover all the maps of the collection by naturality, and hence can be defined as such a morphism.

Definition 1.5. A *symmetric linear operad* is a positive linear species $\mathcal{O}$ equipped with an *unity* $e : \mathrm{X} \to \mathcal{O}$ and a *partial composition* $\circ_* : \mathcal{O}' \cdot \mathcal{O} \to \mathcal{O}$ such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{O}'' \cdot \mathcal{O}^2 \arrow[r, "\circ_{*_1}"] \arrow[d, "\circ_{*_2} \circ Id \cdot \tau"] & \mathcal{O}' \cdot \mathcal{O} \arrow[d, "\circ_{*_2}"] \\
\mathcal{O}' \cdot \mathcal{O} \arrow[r, "\circ_{*_1}"] & \mathcal{O}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}' \cdot \mathcal{O}' \cdot \mathcal{O} \arrow[r, "\circ_{*_1} \cdot Id"] \arrow[d, "Id \cdot \circ_{*_2}"] & \mathcal{O}' \cdot \mathcal{O} \arrow[d, "\circ_{*_2}"] \\
\mathcal{O}' \cdot \mathcal{O} \arrow[r, "\circ_{*_1}"] & \mathcal{O}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}' \cdot \mathbb{K}X \arrow[r, "\mathcal{O}' \cdot e"] \arrow[dr, "p"'] & \mathcal{O}' \cdot \mathcal{O} \arrow[d, "\circ_*"] & \mathbb{K}X' \cdot \mathcal{O} \arrow[l, "e' \cdot \mathcal{O}"'] \arrow[dl, "\cong"] \\
& \mathcal{O} &
\end{tikzcd}$$

where $\tau : x \otimes y \mapsto y \otimes x$ and $p_V : x \mapsto \mathcal{O}[\sigma](x)$ with σ the bijection that sends $*$ on v and is the identity on $V \setminus \{v\}$.

A *sub-operad* of an operad $\mathcal{O}$ is a sub-species of $\mathcal{O}$ containing the image of e and stable under partial composition. For $\mathcal{O}$ an operad, the sub-operad of $\mathcal{O}$ *generated by* a sub-species S is the smallest sub-operad of $\mathcal{O}$ containing S and the sub-operad generated by a set E of elements of $\mathcal{O}$ is the sub-operad generated by the species generated by E. A *morphism of operads* $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of species stable under the structure maps: $f \circ e = e$ and $f(x \circ_* y) = f(x) \circ_* f(y)$.

In practice the map e is often trivial and we do not mention it. Let us now give a series of examples of operads.

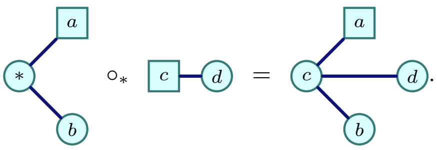
Identity. The identity species has a natural operad structure given by $v_1 \circ_* v_2 = v_1$ if $v_1 = *$ and $v_1 \circ_* v_2 = v_2$ else.

Com. The species E_+ has a natural operad structure given by $(V_1 + \{*\}) \circ_* V_2 = V_1 + V_2$. This operad is called the *commutative operad* and denoted by **Com**. In this context we denote by $\mu_V = V$ the basis element of $\mathbf{Com}[V]$.

NAP. Let be t_1 and t_2 be two rooted trees and let $t_1 \circ_* t_2$ be the rooted tree obtained by the following operation.

1. Take the disjoint union of t_1 and t_2
2. Remove the vertex $*$ from t_1.
3. Add an edge between the neighbours of $*$ in t_1 and the root of t_2.

This operation is a partial composition and turns the species $\mathrm{T}^\bullet$ of rooted trees into an operad. This operad is called *non associative permutative operad* [11] and is denoted by **NAP**. For instance we have:

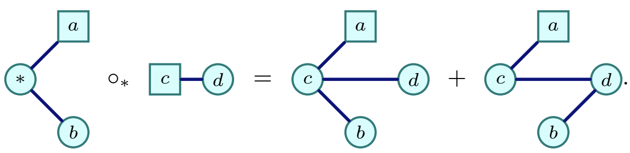


(3)

PreLie. Let be t_1 and t_2 be two rooted trees and let $t_1 \circ_* t_2$ be the sum over all rooted trees obtained by the following operation.

1. Take the disjoint union of t_1 and t_2.
2. Remove the vertex $*$ from t_1.
3. Add an edge between the parent of $*$ in t_1 and the root of t_2.
4. For each child of $*$ in t_1, add an edge between this vertex and any vertex of t_2.

This operation is a partial composition and turns the species $\mathrm{T}^\bullet$ into an operad. This operad is called *pre-Lie operad* [5] and is denoted by **PLie**. Remark that the partial composition of t_1 and t_2 as elements of **NAP** is always in the support of the partial composition of t_1 and t_2 as elements of **PLie**. For instance, we have:



(4)

Polynomials. The species $\mathbb{K}\textsc{Pol}_+$ has a natural partial composition given by the composition of polynomials: for $p_1(v_1, \dots, v_k, *)$ and $p_2(v'_1, \dots, v'_l)$ two polynomials over disjoint sets of variables,

$$(p_1 \circ_* p_2)(v_1, \dots, v_k, v'_1, \dots, v'_l) = p_1|_{* \leftarrow p_2} = p_1(v_1, \dots, v_k, p_2(v'_1, \dots, v'_l)). \tag{5}$$

One can directly check that this partial composition satisfies the commutative diagrams of Definition 1.5. This turns $\mathbb{K}\textsc{Pol}_+$ into an operad where the units are the singleton polynomials $v \in \textsc{Pol}_+[\{v\}]$. Remark that since $\circ_*$ is a linear map, the product and addition of polynomials act as bilinear maps. Restricting the four species morphisms from Example 1.2 to E_+ and $\mathbb{K}\textsc{Pol}_+$ when necessary makes them operad morphisms.

Hadamard product. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two operads, then the species $\mathcal{O}_1 \times \mathcal{O}_2$ is also an operad with partial composition: $(x_1 \otimes x_2) \circ_* (y_1 \otimes y_2) = (x_1 \circ_* y_1) \otimes (x_2 \circ_* y_2)$.

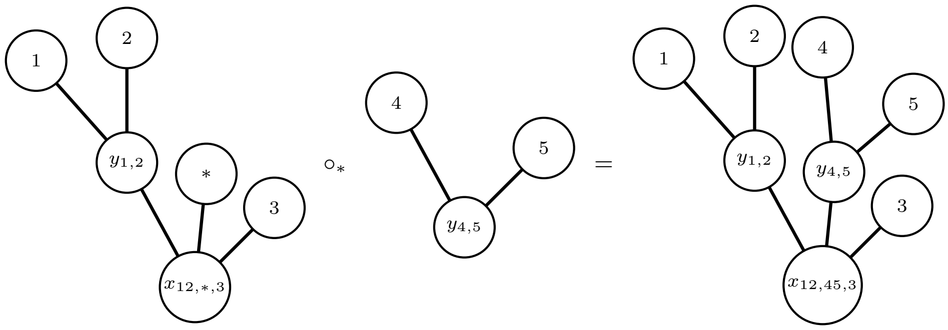
Assemblies. For $\mathcal{O}$ an operad, the species $\mathbf{Com}(\mathcal{O})$ of assemblies of $\mathcal{O}$ has a natural operad structure. Let $V_1, \dots V_n$ and $W_1, \dots, W_k$ be $n+k$ disjoint sets such that $* \in V_1$. Let be $x_i \in \mathcal{O}[V_i]$ for $1 \leq i \leq n$ and $y_j \in \mathcal{O}[W_j]$ for $1 \leq j \leq k$. Then the partial composition of the assemblies $x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_k$ is defined by

$$(x_1 \otimes \cdots \otimes x_n) \circ_* (y_1 \otimes \cdots \otimes y_k) = \sum_{i=1}^{k} y_1 \otimes \cdots \otimes y_{i-1} \otimes x_1 \circ_* y_i \otimes y_{i+1} \otimes \cdots \otimes y_k \otimes x_2 \otimes \cdots \otimes x_n.$$

In all the above examples, we defined operads by explicitly describing the vector spaces $\mathcal{O}[V]$ and the partial compositions. There is another way to present an operad which is as a quotient of a free operad.

Let S be a positive linear species such that $\mathrm{S} = \mathrm{X} + \mathrm{S}_{2+}$. Recall that $\mathscr{S}_{\mathrm{S}}[V]$ is generated by trees with internal vertices decorated with elements of S and set of leaves V. The species $\mathscr{S}_{\mathrm{S}}$ has then a natural operad structure given by the grafting of trees. For $t_1 \in \mathscr{S}'_{\mathrm{S}}[V_1]$ and $t_2 \in \mathscr{S}_{\mathrm{S}}[V_2]$, the partial composition $t_1 \circ_* t_2$ is the tree obtained by grafting t_2 on the leaf $*$ of t_1 and relabeling the nodes of t_1 accordingly. This operad is called *free operad over* S and we denote it by $\mathbf{Free}_{\mathrm{S}}$.

Example 1.6. We give here an example of partial composition in a free operad over the same species than in Example 1.4.



(6)

In the sequel, we consider free operads over species which are sub-species of an operad $\mathcal{O}$. When this happens, we denote by $\circ_*^{\xi}$ the partial composition in the free operad in order to not confuse it with the partial composition in $\mathcal{O}$.

Example 1.7 (**ComMag**)**.** The free operad over one symmetric generator $\mathbf{Free}_{\mathrm{E}_2}$ is the operad of abstract binary trees with partial composition the grafting of trees. This operad is called *commutative magmatic operad* [3] and is denoted by **ComMag**. In this context, for V a set of size 2, we denote by s_V the generating element of $\mathrm{E}_2[V]$: $\mathrm{E}_2[V] = \mathbf{Free}_{\mathrm{E}_2}[V] = \mathbb{K}s_V$.

An *ideal* of an operad $\mathcal{O}$ is a sub-species $\mathcal{I}$ such that the image of the products $\mathcal{O}' \cdot \mathcal{I}$ and $\mathcal{I}' \cdot \mathcal{O}$ by the partial composition maps are in $\mathcal{I}$. The *quotient species* $\mathcal{O}/\mathcal{I}$ defined by $(\mathcal{O}/\mathcal{I})[V] = \mathcal{O}[V]/\mathcal{I}[V]$ is then an operad with the natural partial composition and unit : $[x] \circ_* [y] = [x \circ_* y]$ where $[x]$ is the equivalence class of x. For $\mathcal{G}$ a species, if $\mathcal{R}$ is a sub-species of $\mathbf{Free}_{\mathcal{G}}$, we denote by $(\mathcal{R})$ the smallest ideal of $\mathbf{Free}_{\mathcal{G}}$ containing $\mathcal{R}$ and write that $(\mathcal{R})$ is *generated by* $\mathcal{R}$.

Denote by $\mathbf{Free}_{\mathcal{G}}^{(2)}$ the sub-species of $\mathbf{Free}_{\mathcal{G}}$ of trees with two internal node.

Definition 1.8. Let $\mathcal{G}$ be a species and $\mathcal{R}$ be a sub-species of $\mathbf{Free}_{\mathcal{G}}$. We denote by $\mathrm{Ope}(\mathcal{G}, \mathcal{R}) = \mathbf{Free}_{\mathcal{G}}/(\mathcal{R})$ the operad *generated by* $\mathcal{G}$ *and with relation* $\mathcal{R}$. The operad $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$ is *binary* if the species $\mathcal{G}$ of *generators* is concentrated in cardinality 2 (i.e. $\mathcal{G} = \mathcal{G}_2$). This operad is *quadratic* if the species $\mathcal{R}$ of *relations* is a sub-species of $\mathbf{Free}_{\mathcal{G}}^{(2)}$.

Example 1.9. Denote by $\mathcal{R}$ the sub-species of $\mathbf{ComMag}^{(2)}$ generated by the *associativity relation* $s_{\{a,*\}} \circ_*^{\xi} s_{\{b,c\}} - s_{\{c,*\}} \circ_*^{\xi} s_{\{a,b\}}$. Then $\mathbf{Com} = \mathrm{Ope}(\mathrm{E}_2, \mathcal{R})$ and **Com** is hence binary and quadratic. Remark that as a consequence we have that μ_V is the image of s_V under the projection $\mathbf{ComMag} \to \mathbf{Com}$.

Two advantages of defining an operad by its generators and relations are that it is possible to construct (under some conditions) its Koszul dual and that it is possible to check if the operad is Koszul. These notions are too involved to be presented in a simple reminder and we only refer to them in Proposition 2.12 and Proposition 2.13. We refer the reader not versed in operad theory to Appendix A for more information on these.

1.3 Graphs and multigraphs

A *graph* or *simple graph* over V is a set of non ordered pairs of distinct elements of V and a *multigraph* is a multiset of non ordered pairs of elements of V. In this context, the elements of a (multi)graph are called *edges*, the elements of V are called vertices and the vertices of an edge are called its *ends*. Graphs can be seen as multigraphs with at most one edge between two vertices and no *loop* edges, which are edges with equal ends. We denote by G the species of graphs and by MG the species of multigraphs. We have the usual definitions of *connectedness* and *restriction* to subset of vertices. We respectively denote by G_c and by MG_c the species of connected graphs and connected multigraphs and by $g|_W \in \mathrm{MG}[W]$ the restriction of $g \in \mathrm{MG}[V]$ to $W \subseteq V$

The four species presented here as well as the species of trees T are related by the following diagram:

$$\begin{tikzcd} \mathrm{T} \arrow[r, hook] & \mathrm{G}_c \arrow[r, hook] \arrow[d, hook] & \mathrm{MG}_c \arrow[d, hook] \\ & \mathrm{G} \arrow[r, hook] & \mathrm{MG} \end{tikzcd} \tag{7}$$

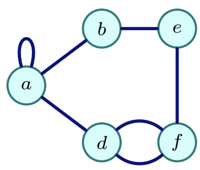
An *orientation* of a multigraph g is a pair of maps $\mathtt{o} = (\mathtt{s}, \mathtt{t})$ from g to $\mathcal{P}(V)$ such that $e = \mathtt{s}(e) \cup \mathtt{t}(e)$ for any edge e of g. The maps $\mathtt{s}$ and $\mathtt{t}$ are respectively called *source* and *target*. An *oriented* multigraph is then a pair of a multigraph and an orientation. We often represent oriented multigraphs by adding arrow heads to the target ends. We add the index *or* to the five preceding species to designate their oriented counterpart e.g. G_{orc} is the species of oriented connected graphs.

Let us finish this presentation of graphs and multigraphs by mentioning that there is an monomorphism Φ from the species MG to $\mathbb{K}\textsc{Pol}$. It is defined as follows:

- the empty graph $\emptyset_V \in \mathrm{MG}[V]$ is sent on the null polynomial $\Phi(\emptyset_V) = 0_V$;
- an edge $e = \{v_1, v_2\}$ is sent on the monomial $\Phi(e) = v_1 v_2$;
- an element $g \in \mathrm{MG}[V]$ is sent on the polynomial $\Phi(g) = \bigoplus_{e \in g} \Phi(e)$.

This enables us to see the species of multigraphs as the sub-species $\mathbb{K}\textsc{Pol}^2$ of homogeneous polynomials of degree 2. This will be useful in the following sections to do computations on multigraphs since it is easier to formally write operations, and in particular composition, on polynomials than on multigraphs.

Example 1.10. With this identification, the following multigraph g writes as the polynomial $\Phi(g) = a^2 \oplus ab \oplus ad \oplus be \oplus ef \oplus 2df$.



(8)

2 Extending the pre-Lie operad

As announced, we want to extend the pre-Lie operad structure to graphs. As we will see later, it is more natural to search an extension to multigraphs. We recall from Example 1.2 how that the partial composition of **PLie** works: for t_1 and t_2 two rooted trees, $t_1 \circ_* t_2$ the sum of all rooted tree obtained as follows:

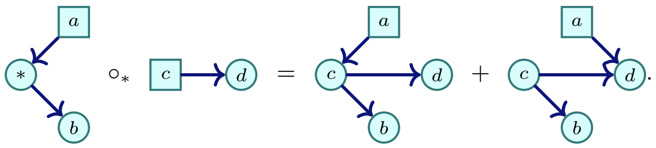
1. take the disjoint union of t_1 and t_2;
2. remove the vertex $*$ from t_1;
3. add an edge between the parent of $*$ in t_1 and the root of t_2;
4. for each child of $*$ in t_1, add an edge between this vertex and any vertex of t_2.

2.1 Two canonical operads

If we want to extends the above construction to multigraphs, we are faced with the problem that multigraphs do not have a root vertex. Our first solution is then to decide that it is more natural to try to extend it to the pointed multigraphs $\mathrm{MG}^\bullet$. But this alone is not enough: indeed the partial composition of **PLie** also use the notion of parent and child vertex/end which can not be defined on a general pointed multigraph. To replace this, we will consider oriented pointed multigraphs, where the targets will play the same role than the parents ends in **PLie** and the sources the same role than the children ends. Let then be $(g_1, v_1) \in (\mathrm{MG}^\bullet_{or})'[V_1]$ and $(g_2, v_2) \in \mathrm{MG}^\bullet_{or}[V_2]$ and define the partial composition $(g_1, v_1) \circ_* (g_2, v_2)$ as the sum over all elements obtained as follows:

1. take the disjoint union of g_1 and g_2;
2. remove the vertex $*$, we then have some edges with a loose end;
3. connect each loose source end to v_2;
4. connect each loose target end to any vertex in V_2;
5. the new root is $v_1 \circ_* v_2$, with the partial composition of the identity operad.

For instance, we have:



(9)

Theorem 2.1. The species $\mathrm{MG}^\bullet_{or}$, endowed with the preceding partial composition, is an operad.

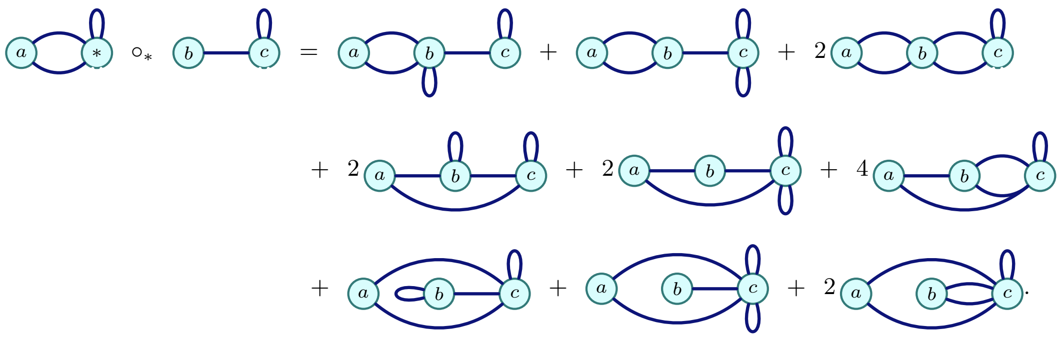
We give a proof of this theorem in Section 3 where we provide a general way to define operads on graphs and multigraphs.

It is straightforward to note that the subspecies of connected components $\mathrm{MG}^\bullet_{orc}$ and the species $\mathrm{G}^\bullet_{or}$ are sub-operads of $\mathrm{MG}^\bullet_{or}$ and that $\mathrm{G}^\bullet_{orc}$ is a sub-operad of $\mathrm{G}^\bullet_{or}$. Given a rooted tree t with root r, we have a natural orientation which consist of choosing the parent ends as targets and child ends as sources. We denote by $\mathtt{o}_r = (\mathtt{s}_r, \mathtt{t}_r)$ this orientation and $t_r = (t, \mathtt{o}_r)$ the associated oriented tree. This induces an operad monomorphism $(t, r) \mapsto (t_r, r)$ from **PLie** to $\mathrm{G}^\bullet_{orc}$ and hence makes **PLie** a sub-operad of $\mathrm{G}^\bullet_{orc}$.

Our second solution in extending **PLie** is to ignore the third step of the construction of the partial composition which involves the root. The resulting partial composition is then much more natural than the previous one over $\mathrm{MG}^\bullet_{or}$: let $g_1 \in \mathrm{MG}'[V_1]$ and $g_2 \in \mathrm{MG}[V_2]$ be two multigraphs and define the partial composition $g_1 \circ_* g_2$ as the sum over all multigraphs obtained as follows:

1. take the disjoint union of g_1 and g_2;
2. remove the vertex $*$ from g_1;
3. for each neighbour of $*$ in g_1, add an edge between this vertex and any vertex of g_2.

For instance, we have:



(10)

Theorem 2.2. The species MG endowed with the preceding partial composition is an operad.

As for Theorem 2.1, we give a proof of this theorem in Section 3.

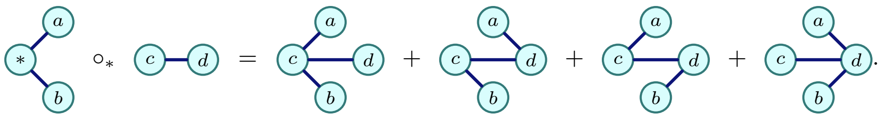
This operad structure makes all the species of the diagram 7 operads and its maps operad monomorphisms. In particular, we recover the Kontsevich-Willwacher operad [13] on G. Recall now from Section 1 that we can identify multigraphs with polynomials. The partial composition we just defined can then be formally written as $g_1 \circ_* g_2 = g_1|_{*\leftarrow \sum V_2} \oplus g_2$ (using the same notation for the composition of polynomials than in (5)), which can then be expanded as follows:

$$
\begin{aligned}
g_1 \circ_* g_2 &= g_1|_{*\leftarrow \sum V_2} \oplus g_2 \\
&= g_1|_{V_1} \oplus \bigoplus_{v\in n(*)} v\left(\sum V_2\right) \oplus \left(\left(\sum V_2\right)^2\right)^{\oplus g_1(**)} \oplus g_2 \\
&= \sum_{f:n(*)\to V_2} \sum_{l:[g_1(**)]\to V_2 V_2} g_1|_{V_1} \oplus \bigoplus_{v\in n(*)} v f(v) \oplus \bigoplus_{i=1}^{g_1(**)} l(i) \oplus g_2,
\end{aligned}
\tag{11}
$$

where $n(*)$ is the multiset of neighbours of $*$ in g_1 and $g_1(**)$ is the number of loops on $*$ in g_1. Each of the three terms of the second line, without counting g_2, have a combinatorial interpretation: $g_1|_{V_1}$ is g_1 to which we removed $*$, $\bigoplus_{n\in n(*)} v(\sum V_2)$ can be understood as "for all vertices in $n(*)$, sum over the ways of connecting it to g_2" and the term $((\sum V_2)^2)^{\oplus g_1(**)}$ as "for each loop over $*$, add an edge between any two elements of V_2". This partial composition expands in a simpler way on G because of the absence of loops. Indeed, if g_1 and g_2 are now graphs, Equ. (11) rewrites as

$$
\begin{aligned}
g_1 \circ_* g_2 &= g_1|_{*\leftarrow \sum V_2} \oplus g_2 \\
&= g_1|_{V_1} \oplus \bigoplus_{n\in n(*)} v\left(\sum V_2\right) \oplus g_2 \\
&= \sum_{f:n(*)\to V_2} g_1|_{V_1} \oplus \bigoplus_{v\in n(*)} v f(v) \oplus g_2.
\end{aligned}
\tag{12}
$$

For instance, we have:



(13)

In particular, we observe that all graphs appearing in $g_1 \circ_* g_2$ have 1 as coefficient.

2.2 Link with PLie

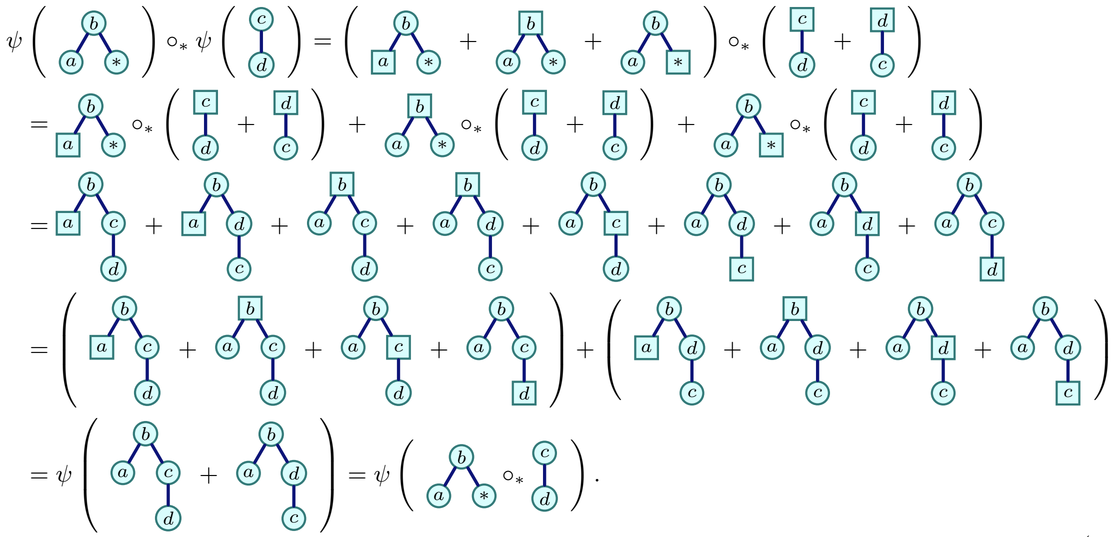
While **PLie** is a sub-operad of $\mathrm{MG}^{\bullet}_{or}$, the operad structure on MG does not seems to keep any relation with **PLie**. In fact, as we will see at the end of this subsection, there is a non trivial link between the four operads $\mathrm{MG}^{\bullet}_{or}$, MG, **PLie**, and the Kontsevich-Willwacher operad G. Let us begin with the following result which gives a link between the sub-operad T, of MG and **PLie**.

Proposition 2.3. The monomorphism of species $\psi : \mathrm{T} \to \mathrm{T}^{\bullet}$ defined by, for any tree $t \in \mathrm{T}[V]$,

$$
\psi(t) = \sum_{r\in V} (t, r),
\tag{14}
$$

is a monomorphism of operads from T to **PLie**.

Before giving the proof of this proposition, we illustrate it on an example:



(15)

We can note that the case when $*$ is the root plays a particular role.

Proof. For $t \in \mathrm{T}[V]$ a tree and $r, v \in V$, we denote by $n_t(v)$ the set of neighbours of v in t, by $c_{t,r}(v)$ the set of children of v in the rooted tree (t, r) and if $r \neq v$, we further denote by $p_{t,r}(v)$ the parent of v in (t, r).

Let be $t_1 \in \mathrm{T}'[V_1]$ and $t_2 \in \mathrm{T}[V_2]$. We now make full use of the correspondence between graphs and polynomials:

$$\begin{aligned}
\psi_{V_1}(t_1) \circ_* \psi_{V_2}(t_2) &= \sum_{r_1 \in V_1 + \{*\}} (t_1, r_1) \circ_* \sum_{r_2 \in V_2} (t_2, r_2) \\
&= \sum_{r_1 \in V_1 + \{*\}} \sum_{r_2 \in V_2} (t_1, r_1) \circ_* (t_2, r_2) \\
&= \sum_{r_1 \in V_1} \sum_{r_2 \in V_2} \left(t_1|_{V_1} \oplus p_{t_1,r_1}(*) r_2 \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,r_1}(*)} v \left(\sum V_2\right), r_1 \right) \\
&\quad + \sum_{r_2 \in V_2} \left(t_1|_{V_1} \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,*}(*)} v \left(\sum V_2\right), r_2 \right) \\
&= \sum_{r_1 \in V_1} \left(t_1|_{V_1} \oplus p_{t_1,r_1}(*) \left(\sum V_2\right) \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,r_1}(*)} v \left(\sum V_2\right), r_1 \right) \\
&\quad + \sum_{r_2 \in V_2} \left(t_1|_{V_1} \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,*}(*)} v \left(\sum V_2\right), r_2 \right) \\
&= \sum_{r \in V_1 + V_2} \left(t_1|_{V_1} \oplus \bigoplus_{v \in n_{t_1}(*)} v \left(\sum V_2\right) \oplus t_2, r \right) \\
&= \sum_{r \in V_1 + V_2} \left(t_1|_{* \leftarrow \sum V_2} \oplus t_2, r \right) \\
&= \psi_{V_1 + V_2}(t_1 \circ_* t_2).
\end{aligned} \tag{16}$$

□

The map ψ naturally extends to a morphism of species $\mathrm{MG}_c \to \mathrm{MG}_c^\bullet$. A natural question to ask is if it also possible to find an operad structure on $\mathrm{MG}_c^\bullet$ in order to make the following commutative diagram of species a commutative diagram of operads:

$$\begin{tikzcd} \mathrm{T} \arrow[r, hook, "\psi"] \arrow[d, hook] & \mathbf{PLie} \arrow[d, hook] \\ \mathrm{MG}_c \arrow[r, hook, "\psi"] & \mathrm{MG}_c^\bullet \end{tikzcd} . \tag{17}$$

But as explained before, there does not seem to be a natural way to extend **PLie** to $\mathrm{MG}_c^\bullet$ and we must rather consider $\mathrm{MG}_{orc}^\bullet$, from which **PLie** is indeed a sub-operad. By doing this we are now faced with the problem to find a natural way to embed $\mathrm{MG}_c^\bullet$ in $\mathrm{MG}_{orc}^\bullet$ which would make the species $\psi(\mathrm{MG}_c)$ a sub-operad of $\mathrm{MG}_{orc}^\bullet$ containing **PLie**. This would then require to find a canonical orientation for each pointed multigraph compatible with $\mathrm{MG}_{orc}^\bullet$ operad structure, which again does not seems possible.

Fortunately, while it does not seems possible to make the diagram (17) a diagram of operads, we can obtain a similar result albeit with a more involved diagram. To do this let us first introduce three new species. For $g \in \mathrm{MG}_c[V]$, $r \in V$, and $t \in \mathrm{T}[V]$ a spanning tree of g, we denote by $\mathsf{o}_{t,r}$ the orientation defined as follows: the targets and sources of the edges in t are the same than in t_r and the all the ends of the remaining edges are targets.

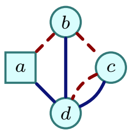
Let us define $\mathcal{I} \subset \mathcal{O} \subset \mathbf{ST}$ (**ST** standing for "spanning tree") three sub-species of $\mathrm{MG}_{orc}^\bullet$ by

$$\mathbf{ST}[V] = \mathbb{K}\left\{(g, \mathsf{o}_{(t,r)}, r) : g \in \mathrm{MG}_c[V], r \in V \text{ and } t \text{ a spanning tree of } g\right\}, \tag{18}$$

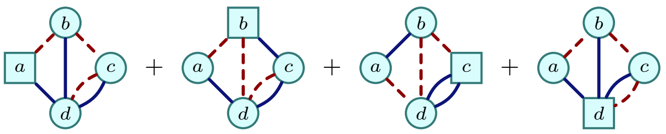
$$\mathcal{O}[V] = \mathbb{K}\left\{\sum_{r \in V}(g, \mathsf{o}_{(t(r),r)}, r) : g \in \mathrm{MG}_c[V] \text{ and for each } r,\ t(r) \text{ a spanning tree of } g\right\}, \tag{19}$$

$$\mathcal{I}[V] = \mathbb{K}\left\{(g, \mathsf{o}_{(t_1,r)}, r) - (g, \mathsf{o}_{(t_2,r)}, r) : g \in \mathrm{MG}_c[V], r \in V, \right.$$
$$\left. \text{and } t_1 \text{ and } t_2 \text{ two spanning trees of } g\right\}. \tag{20}$$

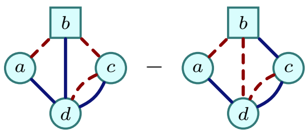
Example 2.4. Let $V = \{a, b, c, d\}$. We give example of elements in $\mathbf{ST}[V]$, $\mathcal{O}[V]$ and $\mathcal{I}[V]$. For the sake of an easier reading, instead of representing the orientations of the edges, we just colored the spanning trees in red. The blue edges should have both ends with an arrow shape and the red edges only the end directing to the root by only considering red dashed edges.



$\in \mathbf{ST}[V]$ (21)



$\in \mathcal{O}_1[V]$ (22)



$\in \mathcal{O}_2[V]$ (23)

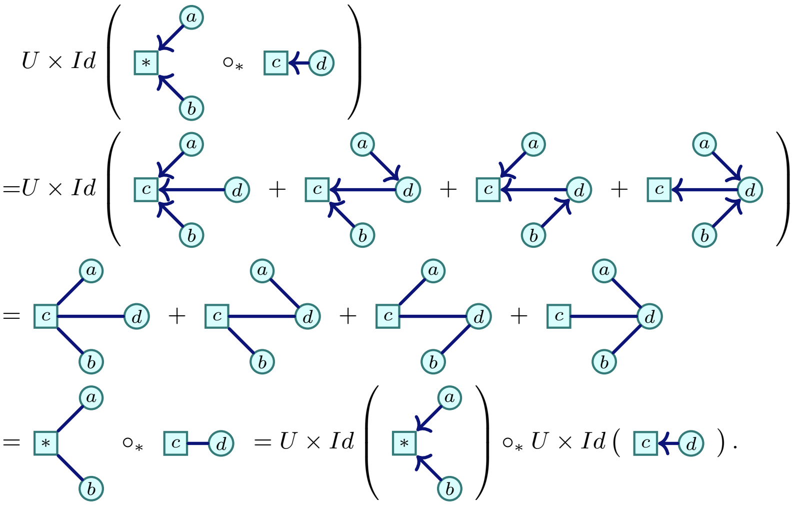
Lemma 2.5. The following properties hold:

(i) **ST** is a sub-operad of $\mathrm{MG}_{orc}^\bullet$ isomorphic to $\mathrm{MG} \times \mathbf{PLie}$,

(ii) $\mathcal{O}$ is a sub-operad of **ST**,

(iii) $\mathcal{I}$ is an ideal of $\mathcal{O}$.

Proof. Before proving these three items, we first give two equalities which will help us for the two last items. Let $U : \mathrm{MG}_{or} \to \mathrm{MG}$ be the forgetful functor which sends an oriented graph on the graph obtained by forgetting the orientation. Let $g_1 \in \mathrm{MG}'_c[V_1]$ and $g_2 \in \mathrm{MG}_c[V_2]$ be two connected multigraphs, t a spanning tree of g_1 and for each $v \in V_2$, $t(v)$ a spanning tree of g_2. When $*$ is the root of the spanning tree t, all the ends pointing to $*$ are targets. Since the target ends in an oriented multigraph behave the same than the normal ends in a multigraph, the forgetful functor preserves the partial composition. For example we have:



(24)

More formally, for $r \in V_2$, we have:

$$
\begin{aligned}
&U \times Id\left((g_1, \circ_{(t,*)}, *) \circ_* (g_2, \circ_{(t(r),r)}, r)\right) \\
&\quad = \left(g_1|_{V_1} \oplus \bigoplus_{v \in n(*)} v \left(\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r \right) \\
&\quad = (g_1 \circ_* g_2, r).
\end{aligned} \tag{25}
$$

Let now r be a vertex in V_1. Denote by p the parent of $*$ in the rooted tree (t, r), by $c(*)$ its children, by $n_{g_1 \setminus t}(*)$ the multiset of neighbours of $*$ in $g_1 \setminus t$ and by $n(*)$ the multiset of neighbours of $*$ in g_1, so that $n(*) = n_{g_1 \setminus t}(*) \cup c(*) \cup \{p\}$. We then have

$$
\begin{aligned}
&U \times Id\left((g_1, \circ_{(t,r)}, r) \circ_* \sum_{v \in V_2} (g_2, \circ_{(t(v),v)}, v)\right) \\
&\quad = \sum_{v \in V_2} U \times Id\left((g_1, \circ_{(t,r)}, r) \circ_* (g_2, \circ_{(t(v),v)}, v)\right) \\
&\quad = \sum_{v \in V_2} \left(g_1|_{V_1} \oplus pv \oplus \bigoplus_{v \in c(*)} v \left(\sum V_2\right) \oplus \bigoplus_{v \in n_{g_1 \setminus t}(*)} v \left(\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r \right) \\
&\quad = \left(g_1|_{V_1} \oplus p \left(\sum V_2\right) \oplus \bigoplus_{v \in c(*)} v \left(\sum V_2\right) \oplus \bigoplus_{v \in n_{g_1 \setminus t}(*)} v \left(\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r \right) \\
&\quad = \left(g_1|_{V_1} \oplus \bigoplus_{v \in n(*)} \left(\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r \right) \\
&\quad = (g_1 \circ_* g_2, r).
\end{aligned} \tag{26}
$$

Proof of (i) The species morphism from MG $\times$ **PLie** to $\mathrm{MG}^\bullet_{orc}$ given by $(g, t, r) \mapsto (g, \circ_{(t,r)}, r)$ is an operad morphism and hence its image **ST** is a sub-operad of $\mathbb{K}\mathrm{MG}^\bullet_{orc}$.

Proof of (ii) Let V_1 and V_2 be two disjoint sets, $g_1 \in \mathrm{MG}'_c[V_1]$ and $g_2 \in \mathrm{MG}_c[V_2]$ be two connected multigraphs and for each $v \in V_1 + \{*\}$, $t(v)$ a spanning tree of g_1 and for each $v \in V_2$, $t(v)$ a spanning tree of g_2. We have

$$
\sum_{r_1 \in V_1 + \{*\}} (g_1, \circ_{(t(r_1), r_1)}, r_1) \circ_* \sum_{r_2 \in V_2} (g_2, \circ_{(t(r_2), r_2)}, r_2) = \sum_{r_1 \in V_1 + \{*\}} \sum_{r_2 \in V_2} (g_1, \circ_{(t(r_1), r_1)}, r_1) \circ_* (g_2, \circ_{(t(r_2), r_2)}, r_2) \tag{27}
$$

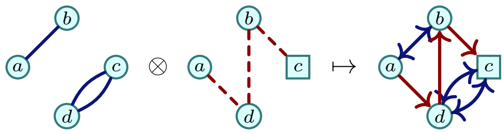


Figure 1: An example of the isomorphism of item i.

Then from (25) and (26) we know that applying $U \times Id$ to the preceding sum gives us:

$$\sum_{r \in V_1 + V_2} (g_1 \circ_* g_2, r). \tag{28}$$

To conclude note that $\mathcal{O}[V]$ is the reciprocal image of $\mathbb{K}\{\sum_{v \in V}(g, v) \mid g \in \mathrm{MG}_c[V]\}$ by $U \times Id : \mathbf{ST} \to \mathrm{MG}^\bullet$.

Proof of (iii) It is easy to see that $\mathcal{I}$ is a left ideal of $\mathbf{ST}$ and hence of $\mathcal{O}$. Let be $g_1 \in \mathrm{MG}'_c[V_1]$, $g_2 \in \mathrm{MG}_c[V_2]$, $r \in V_1$, t a spanning tree of g_1 and for every $v \in V_2$, $t(v)$ a spanning tree of g_2. Then from (25) and (26) we know that $U \times Id((g_1, \circ_{(t,r)}, r) \circ_* \sum_{v \in V_2}(g_2, \circ_{(t(v),v)}), v)$ is of the form $(g_1 \circ_* g_2, r)$ if $r \neq *$, and of the form $\sum_{v \in V_2}(g_1 \circ_* g_2, v)$ otherwise. In both cases it does not depend on t. This concludes this proof since $\mathcal{I}[V]$ is the kernel of $(U \times Id)_V : \mathbf{ST}[V] \to \mathrm{MG}_c^\bullet[V]$. $\square$

We can see $\mathbf{PLie}$ as a sub-operad of $\mathbf{ST}$ by the monomorphism $(t, r) \mapsto (t, \circ_r, r)$. The image of the operad morphism ψ of Proposition 2.3 is then $\mathcal{O} \cap \mathbf{PLie}$ and we have that $\mathcal{I} \cap \mathbf{PLie} = \{0\}$ and hence $\mathcal{O} \cap \mathbf{PLie}/\mathcal{I} \cap \mathbf{PLie} = \mathcal{O} \cap \mathbf{PLie}$.

Proposition 2.6. The operad isomorphism $\psi : \mathrm{T} \to \mathbf{PLie} \cap \mathcal{O}$ extends into an operad isomorphism $\psi : \mathrm{MG}_c \to \mathcal{O}/\mathcal{I}$ satisfying, for any $g \in \mathrm{MG}_c[V]$,

$$\psi(g) = \sum_{r \in V} (g, \circ_{(t(r),r)}, r), \tag{29}$$

where for each $r \in V$, $t(r)$ is a spanning tree of g. Furthermore, this isomorphism restricts itself to an isomorphism $\mathrm{G}_c \to \mathcal{O} \cap \mathrm{G}^\bullet_{orc}/\mathcal{I} \cap \mathrm{G}^\bullet_{orc}$.

Proof. This statement is a direct consequence of Lemma 2.5 and its proof. $\square$

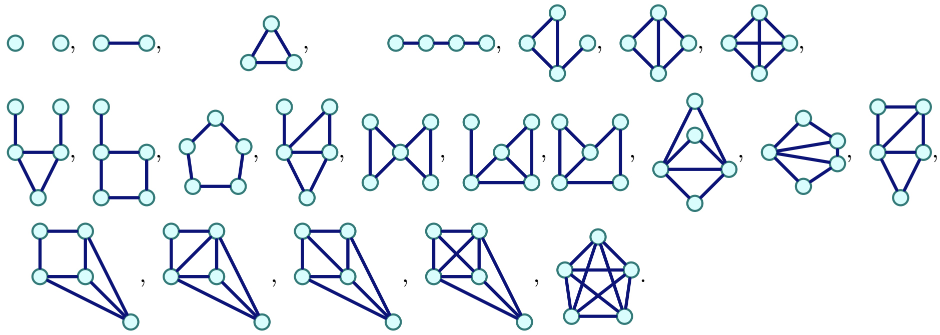
The last results are summarized in the following commutative diagram of operad morphisms.

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{PLie} \cap \mathbb{K}\mathcal{O}/\mathcal{I} \arrow[r, equal] \arrow[d, hook] & \mathbf{PLie} \cap \mathcal{O} \arrow[r, hook] \arrow[d, hook] & \mathbf{PLie} \arrow[d, hook] \\
\mathrm{G}_c \arrow[r, "\sim"] \arrow[d, hook] & \mathcal{O} \cap \mathrm{G}^\bullet_{orc}/\mathcal{I} \cap \mathrm{G}^\bullet_{orc} \arrow[d, hook] & \mathrm{G}^\bullet_{orc} \cap \mathcal{O} \arrow[l, two heads] \arrow[r, hook] \arrow[d, hook] & \mathrm{G}^\bullet_{orc} \cap \mathbf{ST} \arrow[d, hook] \\
\mathrm{MG}_c \arrow[r, "\sim"] & \mathcal{O}/\mathcal{I} & \mathcal{O} \arrow[l, two heads] \arrow[r, hook] & \mathrm{MG} \times \mathbf{PLie}
\end{tikzcd} \tag{30}$$

2.3 The MG operad

As shown in the previous subsection, MG and its sub-operads have some interesting properties and links with $\mathbf{PLie}$. While MG was defined by explicitly describing $\mathrm{MG}[V]$ for each V, the operad $\mathbf{PLie}$ was first defined by its generator and the pre-Lie relation and later proved to be an operad on rooted tree in [5]. It is then natural to search for generators and relations of MG and its sub-operads.

We first search for a smallest family of generators of G. The search for such a family is computationally hard. Using computer algebra, we obtain a family which generates simple graphs of arity less than 5:



(31)

Due to the symmetric group action on G, only the knowledge of the shapes of the graphs is significant. While (31) does not provide to us any particular insight on a possible characterisation of the generators, it does suggest that any graph with “enough” edges must be a generator. This is confirmed by the following lemma. We say that a graph $g \in \mathrm{G}$ is generated by a set E of graphs if g is in the sub-operad generated by E.

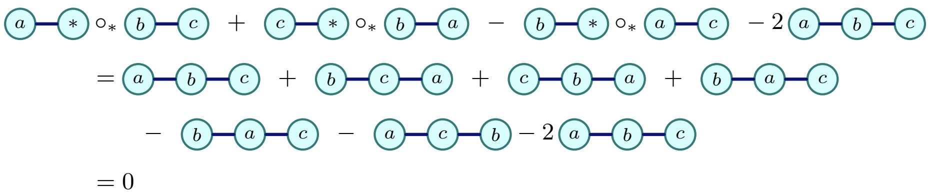
Lemma 2.7. Let S be a sub-species of G and let g be a graph in $\mathrm{G}[V]$ with at least $\binom{n-1}{2}+1$ edges, where $n = |V|$. Then g is generated by S if and only if $g \in \mathrm{S}[V]$.

Proof. Suppose that $g \notin \mathrm{S}[V]$. It is sufficient to show that g cannot appear in the support of any vector of the form $g_1 \circ_* g_2$ for any g_1 and g_2 different of g. Hence let $g_1 \in \mathrm{G}'[V_1]$ and $g_2 \in \mathrm{G}[V_2]$ be two graphs, and denote by e_1 the number of edges of g_1 and by e_2 the number of edges of g_2. Then the graphs in the sum $g_1 \circ_* g_2$ have $e_1 + e_2$ edges. This is maximal when g_1 and g_2 are both complete graphs and is then equal to $\binom{x}{2} + \binom{n+1-x}{2} = x^2 - (n+1)x + \binom{n+1}{2}$ where $1 \leq x = |V_1| \leq n$.

If $x = 1$ then necessarily $g_1 = \emptyset_*$ and $g_1 \circ_* g_2 = g_2$ and g appears in the sum $g_1 \circ_* g_2$ if and only if $g = g_2$. This is impossible, hence $x \neq 1$. Similarly we have $x \neq n$. The expression $x^2 - (n+1)x + \binom{n+1}{2}$ is then maximal for $x = 2$ or $x = n-1$ and is equal in both cases to $\binom{n-1}{2} < \binom{n-1}{2} + 1$. This implies that g can not be part of the sum of graphs $g_1 \circ_* g_2$ and concludes the proof. □

Proposition 2.8. The operad G is not free and has an infinite number of generators.

Proof. The fact that G has an infinite number of generators is a direct consequence of Lemma 2.7. Moreover, the relation



(32)

shows that G is not free. □

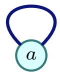
As a consequence of Proposition 2.8, it seems particularly involved to find a definition of G by generator and relation. We did not have any more success in describing T in such a way even exploiting the monomorphism $\psi : \mathrm{T} \to \mathbf{PLie}$. We at least managed to compute a family of generators of T with arity less than 6:

(33)

As with the previous family (31), this does not give to us any insight on a possible family of generators.

2.4 Finitely generated sub-operads

Since finding family of generators seems out of reach, let us now directly focus on finitely generated sub-operads of MG. In particular we will study the operads generated by:

1. $\left\{ a \quad b \right\}$ which we denote by $\mathrm{G}_\emptyset$ and which is isomorphic to **Com**,
2. $\left\{ a - b \right\}$ which we denote by **Seg** and which is isomorphic to **ComMag**,
3. $\left\{ a \quad b, a - b \right\}$ which we denote by **SP**,
4. $\Bigg\{$  $, a \quad b \Bigg\}$ which we denote by **LP**.

First, note that the sub-operad $\mathrm{G}_\emptyset$ generated by $\left\{ a \quad b \right\}$ is isomorphic to the commutative operad **Com**. Indeed, recall from Example 1.9 that **Com** is the quotient of the free operad over one symmetric element of size two E_2 by the associativity relation. By definition $\mathrm{G}_\emptyset$ is also generated by one symmetric element of size 2, and furthermore we have:

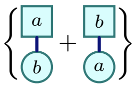
$$a \quad * \ \circ_* \ b \quad c \; = \; a \quad b \quad c \; = \; * \quad c \ \circ_* \ a \quad b, \tag{34}$$

which is the associativity relation. Hence $\mathrm{G}_\emptyset \cong$ **Com**. This could also be observed from the fact that we clearly have $\mathrm{G}_\emptyset[V] = \mathbb{K}\emptyset_V$ and hence the map $\emptyset_V \mapsto \mu_V$ implies an isomorphism from $\mathrm{G}_\emptyset$ to **Com**.

The other three cases are more involved.

Proposition 2.9. The sub-operad **Seg** of G generated by $\left\{ a - b \right\}$ is isomorphic to **ComMag**.

Proof. We know from Proposition 2.3 that **Seg** is isomorphic to the sub-operad of **PLie** generated by

 (35)

Then [3] gives us that the map which sends the above element to $s_{\{a,b\}} \in \mathrm{E}_2[\{a, b\}]$ induces an isomorphism between the sub-operad and **ComMag**. This concludes the proof. $\square$

Remark that while **ComMag** $\cong$ **Seg**, the image of an element in the canonical basis of **ComMag**$[V]$, i.e. a binary tree with V as set of leaves, is not a single graph but a sum of trees with vertex set V.

This result suggests the following more general conjecture.

Conjecture 2.10. The sub-operad of G generated by the complete graphs K_V is isomorphic to $\mathbf{Free}_{\mathrm{E}_n}$, the free operad over one symmetric element of size $n = |V|$.

Proving this would require showing that there are no relations involving only complete graphs of size n in G which is highly non trivial. In fact this was avoided in the proof of Proposition 2.9 by using the results of [3] which is somewhat equivalent in the case of $n = 2$.

The isomorphisms **Com** $\cong \mathrm{G}_\emptyset$ and **ComMag** $\cong$ **Seg** allow us to see **Com** and **ComMag** as disjoint sub-operads of G and hence gives us a natural way to define the smallest operad containing these two as disjoint sub-operads. Denote by $\mathcal{G}$ the sub-species of G generated by $\left\{ a \quad b, a - b \right\}$ and **SP** the sub-operad of $\mathcal{G}$ generated by these two elements. This operad has some interesting properties. Recall from Remark 1.2 that we use the notation $\circ_*^\xi$ for the grafting of tree in a free operad and that we denote the equivalence class of x by $[x]$.

Proposition 2.11. The three following operads are isomorphic

- **SP**
- $\mathrm{Ope}(\mathcal{G}, \mathrm{R})$ where R is the subspecies of $\mathbf{Free}_{\mathcal{G}}$ generated by

$$c \quad * \circ_*^\xi a \quad b \; - \; a \quad * \circ_*^\xi b \quad c, \tag{36a}$$

and

$$a - * \circ_*^\xi b \quad c \; - \; c \quad * \circ_*^\xi a - b \; - \; b \quad * \circ_*^\xi a - c. \tag{36b}$$

- **Com**(**ComMag**), the assemblies of **ComMag**.

In particular, **SP** is binary and quadratic.

Proof. The element $\left[(a) \quad (b) \right]$ of $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$ is symmetric of size 2 and follows the associativity relation (36a). Hence the sub-operad of $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$ generated by $\left[(a) \quad (b) \right]$ is equal to **Com** and all the trees in $\mathbf{Free}_{\mathcal{G}}$ with V as leaves and whose labels are all empty graphs over two points are sent over μ_V when passing to the quotient. In the same way, the element $\left[(a)\!-\!(b) \right]$ of $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$ is symmetric of size 2 and do not follow any relation involving only itself, hence the sub-operad of $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$ generated by $\left[(a)\!-\!(b) \right]$ is equal to **ComMag** and $\left[(a)\!-\!(b) \right] = s_{\{a,b\}} \in \mathbf{ComMag}[\{a, b\}]$.

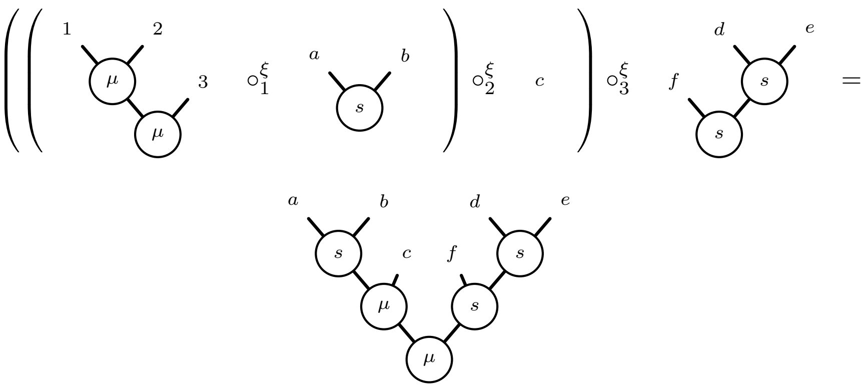
There is a natural epimorphism ϕ from $\mathbf{Free}_{\mathcal{G}}$ to **SP** which is the identity on $(a) \quad (b)$ and $(a)\!-\!(b)$ and which sends a partial composition $g_1 \circ_*^{\xi} g_2$ on the partial composition $g_1 \circ_* g_2$. We already proved by (34) that the vector (36a) is in the kernel ϕ. The case of (36b) is also straightforward:

$$\begin{aligned} (a)\!-\!(*) \circ_* (b) \quad (c) &= (a)\!-\!(b) \quad (c) + (a)\!-\!(c) \quad (b) \\ &= (c) \quad (*) \circ_* (a)\!-\!(b) + (b) \quad (*) \circ_* (a)\!-\!(c). \end{aligned} \tag{37}$$

To conclude that $\mathbf{SP} \cong \mathrm{Ope}(\mathcal{G}, \mathrm{R})$, we must now show that for any $w \in \mathrm{Ope}(\mathbb{K}\mathcal{G}, \mathcal{R})[V]$, $\phi(w) = 0$ implies $w = 0$. To do this, we first prove the bijection $\mathrm{Ope}(\mathcal{G}, \mathrm{R}) \cong \mathbf{Com}(\mathbf{ComMag})$. Because of (36b), we have that the vector $s_{\{a,*\}} \circ_*^{\xi} \mu_{\{b,c\}}$ is equal to the vector $\mu_{\{b,*\}} \circ_*^{\xi} s_{\{a,c\}} + \mu_{\{c,*\}} \circ_*^{\xi} s_{\{a,b\}}$ in $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$. Hence, by iterating this process, we get that all elements of $\mathrm{Ope}(\mathcal{G}, \mathcal{R})$ can be written as a sum of equivalence classes of trees where no s vertex has a μ vertex as descendent. This means that $\mathrm{Ope}(\mathcal{G}, \mathcal{R})[V]$ has the following generating family (cf Figure 39 for an example):

$$\left\{ \mu_\pi \circ^{\xi} (t_1, t_2 \dots t_k) \,|\, \mu_\pi \in \mathbf{Com}[\pi],\, t_i \in \mathbf{ComMag}[V_i] \right\}_{\pi = \{V_1, \dots V_k\} \text{ partition of } V}, \tag{38}$$

where $\mu_\pi \circ^{\xi} (t_1, t_2 \dots t_k)$ stands for $(\dots((\mu_\pi \circ_{V_1}^{\xi} t_1) \circ_{V_2}^{\xi} t_2) \dots) \circ_{V_k}^{\xi} t_k$. To conclude that $\mathrm{Ope}(\mathcal{G}, \mathrm{R}) \cong$



(39)

Figure 2: An element in the generating family of $\mathrm{Ope}(\mathcal{G}, \mathcal{R})[\{a, b, c, d, e, f\}]$.

Com(**ComMag**) just remark that the partial composition of $\mathrm{Ope}(\mathcal{G}, \mathrm{R})$ acts the same way than the partial composition over assemblies of **ComMag**.

Let now be w of the form $\sum_{i=1}^{l} a_i w_i$ where for each $1 \leq i \leq l$, $a_i \in \mathbb{K}$ and there is a partition of $\pi_i = \{V_{i,1}, \dots, V_{i,k_i}\}$ of V such that $w_i = (\dots(\mu_{\pi_i} \circ^{\xi} (t_{i,1} \dots t_{i,k_i})$ with $t_{i,j} \in \mathbf{ComMag}[V_{i,j}]$. For $i \in [l]$, the image of μ_{π_i} by ϕ is the empty graph over π_i, and so the image of w_i is equal to: $\bigsqcup_{j=1}^{k_i} \phi(t_{i,j})$. Hence, for $i \neq j$ two indices, if $\pi_i \neq \pi_j$ the support of $\phi(w_i)$ and $\phi(w_j)$ are disjoint. We can then restrict ourselves to the case where all the w_i are on the same partition of V *i.e.* $\pi_i = \{V_1, \dots, V_k\}$ for all indices i.

Denote by $\mathrm{G}[V_1, \dots, V_k]$ the vector space $\mathbb{K}\{g_1 \cup \dots \cup g_k \,|\, g_i \in \mathrm{G}[V_i]\}$. Then there is an isomorphism from $\mathrm{G}[V_1, \dots, V_k]$ to $\mathrm{G}[V_1] \otimes \dots \otimes \mathrm{G}[V_k]$ defined by $g_1 \cup \dots \cup g_k \mapsto g_1 \otimes \dots \otimes g_k$ which sends $\phi(w)$ on $\sum_{i=1}^{l} a_i \bigotimes_{j=1}^{k} \phi(t_{i,j})$. By definition, ϕ sends the elements $t_{i,j}$ on images of the basis elements of $\mathbf{ComMag}[V_i] \hookrightarrow \mathrm{G}[V_i]$ and hence form a free family in $\mathrm{G}[V_i]$. Hence the tensor products $\bigotimes_{j=1}^{k} \phi(t_{i,j})$ also form a free family of $\mathrm{G}[V_1] \otimes \dots \otimes \mathrm{G}[V_k]$ and so $\phi(w) = 0$ if and only if $w = 0$. This concludes the proof. $\square$

Remark 2. The elements of **Com**(**ComMag**) are assemblies of **ComMag** which can be interpreted as forest of binary trees.

From now on we identify $\textcircled{a}\ \textcircled{b}$ and $\textcircled{a}\!-\!\textcircled{b}$ with their respective image in **Com** and **ComMag**: $\mu_{\{a,b\}}$ and $s_{\{a,b\}}$. We now exhibit the Koszul dual of **SP**.

Proposition 2.12. The operad **SP** admits as Koszul dual the operad $\mathbf{SP}^!$ which is isomorphic to the operad $\mathrm{Ope}((\mathcal{G})^\vee, \mathrm{R})$ where R is the subspecies of $\mathbf{Free}_{\mathcal{G}^\vee}$ generated by

$$\textcircled{a}\!-\!\textcircled{*}^\vee \circ_*^\xi \textcircled{b}\!-\!\textcircled{c}^\vee, \tag{40a}$$

$$\textcircled{a}\ \textcircled{*}^\vee \circ_*^\xi \textcircled{b}\!-\!\textcircled{c}^\vee + \textcircled{c}\!-\!\textcircled{*}^\vee \circ_*^\xi \textcircled{a}\ \textcircled{b}^\vee + \textcircled{b}\!-\!\textcircled{*}^\vee \circ_*^\xi \textcircled{a}\ \textcircled{c}^\vee, \tag{40b}$$

$$\textcircled{a}\ \textcircled{*}^\vee \circ_*^\xi \textcircled{b}\ \textcircled{c}^\vee + \textcircled{c}\ \textcircled{*}^\vee \circ_*^\xi \textcircled{a}\ \textcircled{b}^\vee + \textcircled{b}\ \textcircled{*}^\vee \circ_*^\xi \textcircled{c}\ \textcircled{a}^\vee. \tag{40c}$$

Proof. Let us respectively denote by r_1 and r_2 and r_1', r_2', and r_3' the vectors (36a), (36b), (40a), (40b), and (40c). Denote by $\mathcal{I}$ the operad ideal generated by r_1 and r_2. As a vector space, $\mathcal{I}[[\{a,b,c\}]]$ is then the linear span of the set

$$\{r_1, (ab)\cdot r_1, r_2, (abc)\cdot r_2, (acb)\cdot r_2\}, \tag{41}$$

where $\cdot$ is the action of the symmetric group, *e.g* $r_1\cdot(ab) = \mathbf{Free}_{\mathcal{G}}[(ab)](r_1)$. This space is a sub-space of dimension 5 of $\mathbf{Free}_{\mathcal{G}}[\{a,b,c\}]$, which is of dimension 12. Hence, since as a vector space we have

$$\mathbf{Free}_{\mathcal{G}^\vee}[\{a,b,c\}] \cong \mathbf{Free}_{\mathcal{G}^*}[\{a,b,c\}] \cong \mathbf{Free}_{\mathcal{G}}[\{a,b,c\}], \tag{42}$$

we conclude that $\mathcal{I}^\perp[\{a,b,c\}]$ must be of dimension 7.

Denote by $\mathcal{J}$ the ideal generated by r_1', r_2' and r_3'. As a vector space ,$\mathcal{J}[\{a,b,c\}]$ is then the linear span of the set

$$\{r_1', (ab)\cdot r_1', (ac)\cdot r_1', r_2', (abc)\cdot r_2', (acb)\cdot r_2', r_3'\}. \tag{43}$$

This vector space is of dimension 7. To conclude, we need to show that for any elements f of $\mathcal{J}[\{a,b,c\}]$ and x of $\mathcal{I}[\{a,b,c\}]$ we have $\langle f\,|\,x\rangle = 0$. For every pair $\alpha < \beta$ in $\{a,b,c,*\}$ ordered with the alphabetical order ($c < *$) denote by $s^\vee_{\alpha\beta}$ the dual of $s_{\{\alpha,\beta\}}$ and by $\mu^\vee_{\alpha\beta}$ the dual of $\mu_{\{\alpha,\beta\}}$. Among the 21 cases to check, we have for example:

$$\begin{aligned}\langle r_1'\,|\,r_1\rangle &= \langle s^\vee_{a*} \circ_*^\xi s^\vee_{bc}\,|\,\mu_{\{*,c\}} \circ_*^\xi \mu_{\{a,b\}} - \mu_{\{a,*\}} \circ_*^\xi \mu_{\{b,c\}}\rangle \\ &= \langle s^\vee_{a*} \circ_*^\xi s^\vee_{bc}\,|\,\mu_{\{*,c\}} \circ_*^\xi \mu_{\{a,b\}}\rangle - \langle s^\vee_{a*} \circ_*^\xi s^\vee_{bc}\,|\,\mu_{\{a,*\}} \circ_*^\xi \mu_{\{b,c\}}\rangle \\ &= s^\vee_{a*}(\mu_{\{*,c\}})s^\vee_{bc}(\mu_{\{a,b\}}) - s^\vee_{a*}(\mu_{\{a,*\}})s^\vee_{bc}(\mu_{\{b,c\}}) = 0,\end{aligned} \tag{44}$$

and

$$\begin{aligned}&\langle (abc)\cdot r_2')\,|\,r_2\rangle = \\ &\quad \langle \mu^\vee_{b*} \circ_*^\xi s^\vee_{ca} + s^\vee_{a*} \circ_*^\xi \mu^\vee_{bc} + s^\vee_{c*} \circ_*^\xi \mu^\vee_{ab}\,|\,s_{\{a,*\}} \circ_* \mu_{\{b,c\}} - \mu_{\{c,*\}}) \circ_* s_{\{a,b\}} - \mu_{\{b,*\}} \circ_* s_{\{c,a\}}\rangle \\ &\quad = \mu^\vee_{b*}(s_{\{a,*\}})s^\vee_{ca}(\mu_{\{b,c\}}) - \mu^\vee_{b*}(\mu_{\{c,*\}}))s^\vee_{ca}(s_{\{a,b\}}) - \mu^\vee_{b*}(\mu_{\{b,*\}})s^\vee_{ca}(s_{\{c,a\}}) \\ &\quad + s^\vee_{a*}(s_{\{a,*\}})\mu^\vee_{bc}(\mu_{\{b,c\}}) - s^\vee_{a*}(\mu_{\{c,*\}}))\mu^\vee_{bc}(s_{\{a,b\}}) - s^\vee_{a*}(\mu_{\{b,*\}})\mu^\vee_{bc}(s_{\{c,a\}}) \\ &\quad + s^\vee_{c*}(s_{\{a,*\}})\mu^\vee_{ab}(\mu_{\{b,c\}}) - s^\vee_{c*}(\mu_{\{c,*\}}))\mu^\vee_{ab}(s_{\{a,b\}}) - s^\vee_{c*}(\mu_{\{b,*\}})\mu^\vee_{ab}(s_{\{c,a\}}) \\ &\quad = -1 + 1 = 0.\end{aligned} \tag{45}$$

We leave the verification of the 19 remaining cases as an exercise to the interested reader. $\square$

In order to compute the Hilbert series of $\mathbf{SP}^!$ we need to use identity (89) and hence to prove that the operad **SP** is Koszul.

Proposition 2.13. The operad **SP** is Koszul.

Proof. Let $\mathcal{R}$ be the species defined as in Proposition 2.11 so that $\mathbf{SP} \cong \mathrm{Ope}(\mathbb{K}\mathcal{G}, \mathcal{R})$. Denote by $\mathfrak{p}(a,b)$ and $\mathfrak{s}(a,b)$ the elements of $\mathcal{G}^{\mathscr{F}}[\{a,b\}, ab]$. Then the following vectors form a basis $\mathcal{B}$ of $\mathcal{R}^{\mathscr{F}}[\{a,b,c\}, abc]$:

$$\mathfrak{v}_1 = \mathfrak{p}(\mathfrak{p}(a,b),c) - \mathfrak{p}(a,\mathfrak{p}(b,c))\ \ ,\ \ \mathfrak{v}_2 = \mathfrak{p}(\mathfrak{p}(a,c),b) - \mathfrak{p}(a,\mathfrak{p}(b,c)) \tag{46}$$

$$\mathfrak{v}'_1 = \mathfrak{s}(\mathfrak{p}(a,b),c) - \mathfrak{p}(\mathfrak{s}(a,c),b) - \mathfrak{p}(a,\mathfrak{s}(b,c)) \tag{47}$$

$$\mathfrak{v}'_2 = \mathfrak{s}(\mathfrak{p}(a,c),b) - \mathfrak{p}(\mathfrak{s}(a,b),c) - \mathfrak{p}(a,\mathfrak{s}(b,c)) \tag{48}$$

$$\mathfrak{v}'_3 = \mathfrak{s}(a,\mathfrak{p}(b,c)) - \mathfrak{p}(\mathfrak{s}(a,b),c) - \mathfrak{p}(\mathfrak{s}(a,c),b). \tag{49}$$

We need to show that it is a Gröbner bases of $(\mathcal{R}^{\mathscr{F}})$. Let now consider the path-lexicographic ordering presented in subsubsection A with $s > p$. Then the leading terms of $\mathfrak{v}_1$, $\mathfrak{v}_2$, $\mathfrak{v}'_1$, $\mathfrak{v}'_2$ and $\mathfrak{v}'_3$ are respectively $\mathfrak{p}(\mathfrak{p}(a,b),c)$, $\mathfrak{p}(\mathfrak{p}(a,c),b)$, $\mathfrak{s}(\mathfrak{p}(a,b),c)$, $\mathfrak{s}(\mathfrak{p}(a,c),b)$ and $\mathfrak{s}(a,p(b,c))$. We conclude with Proposition A.10. Indeed, it is shown in [6] that the S-polynomials of pairs of elements in $\{\mathfrak{v}_1, \mathfrak{v}_2\}$ are congruent to zero modulo $\mathcal{B}$. We show for example that the S-polynomial of $\mathfrak{v}_1$ and $\mathfrak{v}'_1$ corresponding to $\mathfrak{c} = \mathfrak{s}(\mathfrak{p}(\mathfrak{p}(a,b),c),d) \in \mathbf{Free}^{sh}_{\mathcal{G}}[\{a,b,c,d\}, abcd]$ is congruent to zero. We have

$$\begin{aligned} m_{\mathfrak{c},\mathrm{lt}(\mathfrak{v}_1)}(\mathfrak{v}_1) &= \mathfrak{c} - \mathfrak{s}(\mathfrak{p}(a,\mathfrak{p}(b,c)),d) \\ m_{\mathfrak{c},\mathrm{lt}(\mathfrak{v}'_1)}(\mathfrak{v}'_1) &= \mathfrak{c} - \mathfrak{p}(\mathfrak{s}(\mathfrak{p}(a,b),d),c) - \mathfrak{p}(\mathfrak{p}(a,b),\mathfrak{s}(c,d)), \end{aligned} \tag{50}$$

which gives us

$$s_{\mathfrak{c}}(\mathfrak{v}_1,\mathfrak{v}'_1) = \mathfrak{p}(\mathfrak{s}(\mathfrak{p}(a,b),d),c) + \mathfrak{p}(\mathfrak{p}(a,b),\mathfrak{s}(c,d)) - \mathfrak{s}(\mathfrak{p}(a,\mathfrak{p}(b,c)),d). \tag{51}$$

Let us look how each of the terms of $s_{\mathfrak{c}}(\mathfrak{v}_1,\mathfrak{v}'_1)$ reduces modulo $\mathcal{B}$:

$$\begin{aligned} \mathfrak{p}(\mathfrak{s}(\mathfrak{p}(a,b),d),c) &\equiv_{\mathfrak{v}'_1} \mathfrak{p}(\mathfrak{p}(\mathfrak{s}(a,d),b),c) + \mathfrak{p}(\mathfrak{p}(a,\mathfrak{s}(b,d)),c) \\ &\equiv_{\mathfrak{v}_1} \mathfrak{p}(\mathfrak{s}(a,d),\mathfrak{p}(b,c)) + \mathfrak{p}(a,\mathfrak{p}(\mathfrak{s}(b,d),c), \end{aligned}$$

$$\mathfrak{p}(\mathfrak{p}(a,b),\mathfrak{s}(c,d)) \equiv_{\mathfrak{v}_1} \mathfrak{p}(a,\mathfrak{p}(b,\mathfrak{s}(c,d)), \tag{52}$$

$$\begin{aligned} \mathfrak{s}(\mathfrak{p}(a,\mathfrak{p}(b,c)),d) &\equiv_{\mathfrak{v}'_1} \mathfrak{p}(\mathfrak{s}(a,d),\mathfrak{p}(b,c)) + \mathfrak{p}(a,\mathfrak{s}(\mathfrak{p}(b,c),d)) \\ &\equiv_{\mathfrak{v}'_1} \mathfrak{p}(\mathfrak{s}(a,d),\mathfrak{p}(b,c)) + \mathfrak{p}(a,\mathfrak{p}(\mathfrak{s}(b,d),c)) + \mathfrak{p}(a,\mathfrak{p}(b,\mathfrak{s}(c,d))). \end{aligned}$$

Putting this together in (51) gives us that $s_{\mathfrak{c}}(\mathfrak{v}_1,\mathfrak{v}'_1)$ reduces to 0 modulo $\mathcal{B}$. We leave the verification of the other cases to the interested reader. □

Proposition 2.14. The Hilbert series of $\mathbf{SP}^!$ is given

$$\mathcal{H}_{\mathbf{SP}^!}(x) = \frac{(1-\log(1-x))^2 - 1}{2}. \tag{53}$$

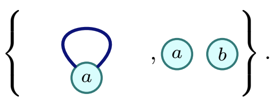
Proof. The Hilbert series of **ComMag** is $\mathcal{H}_{\mathbf{ComMag}}(x) = 1 - \sqrt{1-2x}$ hence the Hilbert series of $\mathbf{SP} \cong \mathbf{Com}(\mathbf{ComMag})$ is $\mathcal{H}_{\mathbf{SP}}(x) = e^{1-\sqrt{1-2x}} - 1$, where the -1 comes from the fact that we consider positive species. We deduce the Hilbert series of $\mathbf{SP}^!$ from $\mathcal{H}_{\mathbf{SP}}$ and the identity (89). □

The first dimensions $\dim \mathbf{SP}^![[n]]$ for $n \geq 1$ are

$$1, 2, 5, 17, 74, 394, 2484, 18108, 149904. \tag{54}$$

This is sequence **A000774** of [16]. This sequence is in particular linked to some pattern avoiding signed permutations and mesh patterns.

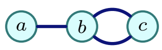
Before ending this section let us mention the sub-operad **LP** of MG generated by

 (55)

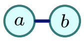
This operad seems particularly interesting to us since its two generators can be considered as minimal elements in the sense that a partial composition with the two isolated vertices adds exactly one vertex and no edge, while a partial composition with the loop adds exactly one edge and no vertex. A natural question to ask at this point concerns the description of the multigraphs generated by these two minimal elements.

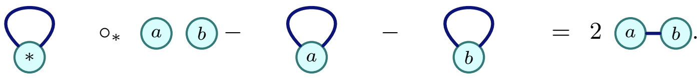
Proposition 2.15. The following properties hold:

- the operad **SP** is a sub-operad of **LP**;
- the operad **LP** is a strict sub-operad of MG. In particular, the multigraph

 (56)

is in MG but is not in **LP**.

Proof. • The following identity shows that  is in $\mathbf{LP}[\{a,b\}]$ and hence that $\mathbf{SP}$ is a sub-operad of $\mathbf{LP}$:


(57)

- Using computer algebra, one generates all vectors in $\mathbf{LP}[\{a,b,c\}]$ with three edges and shows that the announced multigraph is not a linear combination of these.

$\square$

3 Graph insertion operads

The goal of this section is to give a general construction of operads on multigraphs where the partial composition of two elements $g_1 \circ_* g_2$ is given by

1. taking the disjoint union of g_1 and g_2;
2. removing the vertex $*$ from g_1;
3. connecting independently each loose ends of g_1 to g_2 in a certain way.

What we mean by independently is that the way of connecting one end does not depend on how we connect the other ends. Note that the "certain way" in which an end can be connected may include duplication of edges.

3.1 Constructions on species and operads

We begin by defining new constructions on species and operads. We define three constructions: the augmentation, the semi-direct product and the maps from a set to an operad.

Definition 3.1. Let A be a set and S be a species. An *A-augmentation* of S is a species A-S such that $A\text{-}\mathrm{S}[V] \cong \mathrm{S}[A \times V]$ for every finite set V.

Example 3.2. Let A be a set.

- Instead of considering an A-augmented multigraph on V as a multigraph on $V \times A$, we consider them as multigraphs on V where the ends of the edges are labelled with elements of A. In particular, the species of oriented multigraphs MG_{or} is in bijection with the species of $\{\mathtt{s},\mathtt{t}\}$-augmented multigraphs $\{\mathtt{s},\mathtt{t}\}$-MG. For $g \in \mathrm{MG}$ and $\mathtt{o}$ an orientation of g, the pair $(g,\mathtt{o})$ is sent on the multigraph obtained by respectively labeling by t and s the targets and sources of the edges.
- Instead of seeing the elements in $A\text{-}\mathrm{POL}_+[V]$ as polynomials with set of variables the couples $(v,a) \in V \times A$, we consider them as polynomials with set of variables $\{v_a \mid v \in V, a \in A\}$ of elements of V indexed by elements of A.

Remark 3. For S and R any two species and $f : \mathrm{R} \to \mathrm{S}$ a morphism, f extends to a morphism between any two A-augmentation of R and S by $A\text{-}\mathrm{R}[V] \cong \mathrm{R}[A \times V] \xrightarrow{f} \mathrm{S}[A \times V] \cong A\text{-}\mathrm{S}$. In particular the morphism $\mathrm{MG} \to \mathrm{POL}_+$ given in subsubsection 1.3 extends to a morphism which sends an edge (u_a, v_b) on the monomial $u_a v_b$.

In the following proposition, we give an operad structure to a Hadamard product $\mathrm{S} \times \mathcal{O}$ where S is a species and $\mathcal{O}$ is an operad.

Proposition 3.3. Let S be a linear species and $\mathcal{O}$ an operad. Let φ be a morphism from $\mathrm{S}' \cdot (\mathrm{S} \times \mathcal{O})$ to S and denote by $x \circ_*^f y = \varphi(x \otimes y \otimes f)$. Suppose φ satisfies the following hypotheses.

Commutativity. For x an element S'' and $y \otimes f$ and $z \otimes g$ two elements of $\mathrm{S} \times \mathcal{O}$,

$$(x \circ_{*_1}^f y) \circ_{*_2}^g z = (x \circ_{*_2}^g z) \circ_{*_1}^f y. \tag{58}$$

Associativity. For x an element of S', $y \otimes f$ an element of $(\mathrm{S} \times \mathcal{O})'$ and $z \otimes h$ an element of $\mathrm{S} \times \mathcal{O}$,

$$(x \circ_{*_1}^f y) \circ_{*_2}^g z = x \circ_{*_1}^{f \circ_{*_2} g} (y \circ_{*_2}^g z). \tag{59}$$

Unity. There exists a map $e : \mathrm{X} \to \mathrm{S}$ such that

$$x \circ_*^{e_{\mathcal{O}}(v)} e(v) = \mathrm{S}[\sigma](x) \quad \text{and} \quad e(*) \circ_*^f x = x, \tag{60}$$

where $e_{\mathcal{O}}$ is the unit of $\mathcal{O}$ and σ is the bijection which sends $*$ on v and is the identity on the rest of the set on which x is defined.

Then the partial composition $\circ_*^\varphi$ defined by

$$\begin{aligned} \circ_*^\varphi : (\mathrm{S} \times \mathcal{O})' \cdot \mathrm{S} \times \mathcal{O} &\to \mathrm{S} \times \mathcal{O} \\ (x \otimes f) \otimes (y \otimes g) &\mapsto x \circ_*^g y \otimes f \circ_* g \end{aligned} \tag{61}$$

makes $\mathrm{S} \times \mathcal{O}$ an operad with unit e. We call this operad the *semi-direct product of* S *and* $\mathcal{O}$ *over* φ and we denote it by $\mathrm{S} \ltimes_\varphi \mathcal{O}$.

Proof. We must verify that the three diagrams (1.5) commutes.

- Let V_1, V_2, V_3 be three disjoint sets and $x \otimes f \in (\mathrm{S} \times \mathcal{O})''[V_1]$, $y \otimes g \in \mathrm{S} \times \mathcal{O}[V_2]$ and $z \otimes h \in \mathrm{S} \times \mathcal{O}[V_3]$. We then have

$$\begin{aligned} \big((x \otimes f) \circ_{*_1}^\varphi (y \otimes g)\big) \circ_{*_2}^\varphi (z \otimes h) &= \big((x \circ_{*_1}^g y) \circ_{*_2}^h z\big) \otimes \big((f \circ_{*_1} g) \circ_{*_2} h\big) \\ &= \big((x \circ_{*_2}^h z) \circ_{*_1}^g y\big) \otimes \big((f \circ_{*_2} h) \circ_{*_1} g\big) \\ &= \big((x \otimes f) \circ_{*_1}^\varphi (z \otimes h)\big) \circ_{*_2}^\varphi (y \otimes g), \end{aligned} \tag{62}$$

where the second equality follows from (58) and the fact that $\mathcal{O}$ is an operad.

- Let V_1, V_2, V_3 be three disjoint sets and $x \otimes f \in (\mathrm{S} \times \mathcal{O})'[V_1]$, $y \otimes g \in (\mathrm{S} \times \mathcal{O})'[V_2]$ and $z \otimes h \in \mathrm{S} \times \mathcal{O}[V_3]$. We then have

$$\begin{aligned} \big((x \otimes f) \circ_{*_1}^\varphi (y \otimes g)\big) \circ_{*_2}^\varphi (z \otimes h) &= \big((x \circ_{*_1}^g y) \circ_{*_2}^h z\big) \otimes \big((f \circ_{*_1} g) \circ_{*_2} h\big) \\ &= \big(x \circ_{*_1}^{g \circ_{*_1} h} (y \circ_{*_2}^h z)\big) \otimes f \circ_{*_1} (g \circ_{*_2} h)\big) \\ &= (x \otimes f) \circ_{*_1}^\varphi \big((y \otimes g) \circ_{*_2}^\varphi (z \otimes h)\big), \end{aligned} \tag{63}$$

where the second equality follows from (59) and the fact that $\mathcal{O}$ is an operad.

- Let be $x \otimes f \in (\mathrm{S} \times \mathcal{O})'[V]$ and $v \notin V$. We then have

$$(x \otimes f) \circ_*^\varphi (e(v) \otimes e_{\mathcal{O}}(v)) = x \circ_*^{e_{\mathcal{O}}(v)} e(v) \otimes f \circ_* e(v) = \mathrm{S}[\sigma](x \otimes f) \tag{64}$$

where σ is the bijection which sends $*$ to v and is the identity on V. The last equality follows from the first equality from (60) and the fact that $\mathcal{O}$ is an operad. Let now be $x \otimes f \in \mathrm{S} \times \mathcal{O}[V]$. Then

$$(e(*) \otimes e_{\mathcal{O}}(*)) \circ_* (x \otimes f) = e(*) \circ_*^f x \otimes e_{\mathcal{O}}(*) \circ_* f = x \otimes f \tag{65}$$

where the last equality comes from second equality from (60) and the fact that $\mathcal{O}$ is an operad.

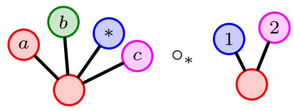
□

When it is clear from the context, we do not mention φ and just write semi-direct product of S and $\mathcal{O}$ and denote it by $\mathrm{S} \ltimes \mathcal{O}$. In practice, the operad structure $\mathcal{O}$ of $\mathrm{S} \ltimes \mathcal{O}$ is transparent and we are just interested in what happens on S, that it is to say the "pseudo partial composition" $x \circ_*^g y$.

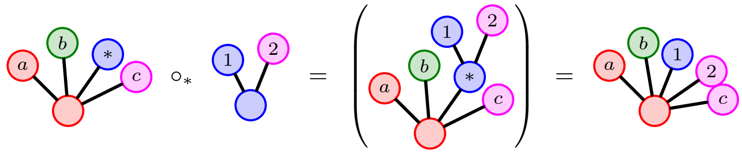
Example 3.4. For C a finite set, let C be the trivial species given by $\mathrm{C}[V] = \mathbb{K}C$ for all set V and $\mathcal{C} = \mathrm{X} + \mathrm{C}_{2+}$. This second species has an operad structure given defined by, for $c_1 \in \mathrm{C}'[V_1]$ and $c_2 \in \mathrm{C}[V_2]$: $c_1 \circ_* c_2 = c_1$ if $V_1 \neq \emptyset$ and $* \circ_* c_2 = c_2$ when $V_1 = \emptyset$ and $* \in \mathrm{X}[\{*\}]$.

Let $\mathcal{F}^C = \mathrm{X} + \mathcal{F}_{2+}^C$ be the species of maps with co-domain C: $\mathcal{F}^C[V] = \mathbb{K}\{f : V \to C\}$ for $|V| > 1$. We define a semi-direct product structure $\mathcal{F}^C \ltimes_\varphi \mathcal{C}$. Suppose that $|V_1 + \{*\}|, |V_2| > 1$ and let be $f \in \mathcal{F}^C[V_1 + \{*\}]$ and $g \otimes x \in \mathcal{F}^C \times \mathcal{C}[V_2]$. We then have $f \circ_*^c g = 0$ if $f(*) \neq c$ and $f \circ_*^c g(v) = \begin{cases} f(v) & \text{if } v \in V_1 \\ g(v) & \text{if } v \in V_2 \end{cases}$ else. When $V_1 = \emptyset$ or V_2 is a singleton, the action of φ is implied by the unit hypothesis.

We call this operad the C*-coloration* operad. When this operad is considered alone, one can see an element of $(f, c) \in \mathcal{F}^C \ltimes C[V]$ as a corolla on V with its root colored by c and its leaves $v \in V$ colored by $f(v)$. The partial composition consists then in grafting two corollas if the root and the leaf on which it must be grafted share the same colors. For instance we have, by representing the elements of C with colors:



$= 0$ and (66)



(67)

A way to define *colored operads* (see [18] for more details on the theory of colored operads) is then to define them as any Hadamard product of a C-coloration operad with another operad.

Let us now define our last construction.

Definition 3.5. Let A be a set and S be a species. The set species of *functions from A to* S is defined by $\mathcal{F}_A^{\mathrm{S}}[V] = \mathbb{K}\{f : A \to \mathrm{S}[V]\}$.

The following proposition then tells us that if $\mathcal{O}$ has an operad structure, it naturally reflects on $\mathcal{F}_A^{\mathcal{O}}$.

Proposition 3.6. If $\mathcal{O}$ is an operad with unit e, $\mathcal{F}_A^{\mathcal{O}}$ has an operad structure with the elements $e_v : A \to \{e(v)\} \in \mathcal{F}_A^{\mathcal{O}}[\{v\}]$ as units and partial composition defined by $f_1 \circ_* f_2(a) = f_1(a) \circ_* f_2(a)$.

Proof. We must verify that the diagrams (1.5) are indeed commutative.

- Let be $f_1 \in (\mathcal{F}_A^{\mathcal{O}})''[V_1]$, $f_2 \in \mathcal{F}_A^{\mathcal{O}}[V_2]$ and $f_3 \in \mathcal{F}_A^{\mathcal{O}}[V_3]$. Then for all $a \in A$, we have

$$\begin{aligned}\big((f_1 \circ_{*_1} f_2) \circ_{*_2} f_3\big)(a) &= (f_1 \circ_{*_1} f_2(a)) \circ_{*_2} f_3(a)\\ &= \big(f_1(a) \circ_{*_1} f_2(a)\big) \circ_* f_3(a)\\ &= f_1(a) \circ_{*_2} f_3(a) \circ_* f_2(a)\\ &= (f_1 \circ_{*_2} f_3(a)) \circ_{*_1} f_2(a)\\ &= \big((f_1 \circ_{*_2} f_3) \circ_{*_1} f_2\big)(a),\end{aligned} \tag{68}$$

  where the third equality follows from the fact that $\mathcal{O}$ is an operad. Hence, we have $(f_1 \circ_{*_1} f_2) \circ_{*_2} f_3 = (f_1 \circ_{*_2} f_3) \circ_{*_1} f_2$.

- Let be $f_1 \in (\mathcal{F}_A^{\mathcal{O}})'[V_1]$, $f_2 \in (\mathcal{F}_A^{\mathcal{O}})'[V_2]$ and $f_3 \in \mathcal{F}_A^{\mathcal{O}}[V_3]$. Then for all $a \in A$:

$$\begin{aligned}\big((f_1 \circ_{*_1} f_2) \circ_{*_2} f_3\big)(a) &= (f_1 \circ_{*_1} f_2(a)) \circ_{*_2} f_3(a)\\ &= f_1(a) \circ_{*_1} f_2(a) \circ_* f_3(a)\\ &= f_1(a) \circ_{*_2} f_3(a) \circ_* f_2(a)\\ &= (f_1 \circ_{*_2} f_3(a)) \circ_{*_1} f_2(a)\\ &= \big((f_1 \circ_{*_2} f_3) \circ_{*_1} f_2\big)(a),\end{aligned} \tag{69}$$

  where the third equality comes from the fact that $\mathcal{O}$ is an operad. Hence, we have $(f_1 \circ_{*_1} f_2) \circ_{*_2} f_3 = (f_1 \circ_{*_2} f_3) \circ_{*_1} f_2$.

- Let be $f \in (\mathcal{F}_A^{\mathcal{O}})'[V]$ and $v \notin V$. Then for all $a \in A$, we have the equalities $f \circ_* e_v(a) = f(a) \circ_* e(v) = \mathcal{O}[\sigma](f(a))$ and so $f \circ_* e_v = \mathcal{O}[\sigma](f)$, where σ is the bijection which sends $*$ to v and is the identity over V. If now $f \in \mathcal{F}_A^{\mathcal{O}}$, we have for all $a \in A$: $e_* \circ_* f(a) = e(*) \circ_* f(a) = f(a)$ and so $e_* \circ_* f = f$. □

Note that if A is a singleton then $\mathcal{F}_A^{\mathcal{O}} \cong \mathcal{O}$. Let A, B, C, D four sets such that A and B are disjoint and $f : A \to C$ and $g : B \to D$ two maps. We denote by $f \uplus g$ the map from $A \sqcup B$ to $C \cup D$ defined by $f \uplus g(a) = f(a)$ for $a \in A$ and $f \uplus g(b) = g(b)$ for $b \in B$.

Proposition 3.7. Let A and B be two disjoint sets and $\mathcal{O}_1$ and $\mathcal{O}_2$ be two operads. Then the species $\mathcal{F}_{A,B}^{\mathcal{O}_1,\mathcal{O}_2}$ defined by $\mathcal{F}_{A,B}^{\mathcal{O}_1,\mathcal{O}_2}[V] = \left\{f \uplus g \mid f \in \mathcal{F}_A^{\mathcal{O}_1}, g \in \mathcal{F}_B^{\mathcal{O}_2}\right\}$ is an operad with same partial composition than in Proposition 3.6.

Proof. We remark that since A and B are disjoint, $f_1 \uplus f_2 \circ_* g_1 \uplus g_2 = (f_1 \circ_* g_1) \uplus (f_2 \circ_* g_2)$. To conclude we apply what was already shown in the proof of Proposition 3.6. □

3.2 Application to multigraphs

We now use the construction of the previous subsection to define operad structures on multigraphs.

Recall from Remark 3 that there is a monomorphism from A-MG to A-$\textsc{Pol}_+$. We now make use of this monomorphism and consider the elements of A-MG as both multigraphs and polynomials. Let $p \in \textsc{Pol}_+[V]$ be a sum of polynomials. Then for A a set and $a \in A$, we denote by $p_a \in A$-$\textsc{Pol}_+$ the sum of polynomials obtained by indexing all the variables in p by a. Let now p be any polynomial and $x_1, \dots, x_n$ a subset of its variables. Then for $q_1, \dots, q_n$ n polynomials, we expand the notation introduced in equation (5) and denote by $p|_{\{x_i \leftarrow q_i\}}$ the polynomial $\big(\dots((p_1 \circ_{x_1} q_1) \circ_{x_2} q_2) \dots\big) \circ_{x_n} q_n$. This notation generalize to sum of polynomials by recalling that the multiplication and addition of polynomials act as bilinear maps.

Example 3.8. For A the singleton $\{a\}$ and p the polynomial $xy \oplus x^2 + zy \in \textsc{Pol}_+[\{x, y, z\}]$ we have $p_a = x_a y_a \oplus x_a^2 + z_a y_a$. Let now be $p = x \oplus yz$, $q_1 = u + v$ and $q_2 = x_1 \oplus x_2$. Then

$$\begin{aligned} p|_{x \leftarrow q_1, y \leftarrow q_2} &= q_1 \oplus q_2 z = (u + v) \oplus (x_1 \oplus x_2) z \\ &= u \oplus x_1 z \oplus x_2 z + v \oplus x_1 z \oplus x_2 z. \end{aligned} \tag{70}$$

Theorem 3.9. Let A be a set and φ be the morphism from A-MG $\cdot$ (A-MG $\times \mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$) to A-MG given by

$$\varphi(g_1 \otimes g_2 \otimes f) = g_1 \circ_*^f g_2 = g_1|_{\{*_a \leftarrow f(a)_a\}} \oplus g_2. \tag{71}$$

Then φ satisfies the hypotheses of Proposition 3.3 and we can consider the semidirect product of A-MG and $\mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$ over φ.

Proof. We need to check that φ satisfies the three hypotheses of Proposition 3.3. The first two are simply computations over polynomials.

Commutativity. Let g_1 be an element of A-MG$''$ and $g_2 \otimes f$ and $g_3 \otimes h$ two elements of A-MG$\times \mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$. Then

$$\begin{aligned} (g_1 \circ_{*_1}^f g_2) \circ_{*_2}^h g_3 &= (g_1|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus g_2)|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_3 \\ &= g_1|_{\{*_{1a} \leftarrow f(a)_a\}}|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_2 \oplus g_3 \\ &= g_1|_{\{*_{2a} \leftarrow h(a)_a\}}|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus g_3 \oplus g_2 \\ &= (g_1|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_3)|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus g_2 \\ &= (g_1 \circ_{*_2}^h g_3) \circ_{*_1}^f g_2. \end{aligned} \tag{72}$$

Associativity. Let be g_1 an element of A-MG$'$, $g_2 \otimes f$ an element of (A-MG $\times \mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$)$'$ and $g_3 \otimes h$ an element of A-MG $\times \mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$. Then

$$\begin{aligned} (g_1 \circ_{*_1}^f g_2) \circ_{*_2}^h g_3 &= (g_1|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus g_2)|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_3 \\ &= g_1|_{\{*_{1a} \leftarrow f(a)_a\}}|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_2|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_3 \\ &= g_1|_{\left\{*_{1a} \leftarrow f(a)_a|_{\{*_{2a} \leftarrow h(a)_a\}}\right\}} \oplus g_2|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_3 \\ &= g_1|_{\left\{*_{1a} \leftarrow f \circ_{*_2} h(a)_a\right\}} \oplus g_2|_{\{*_{2a} \leftarrow h(a)_a\}} \oplus g_3 \\ &= g_1 \circ_{*_1}^{f \circ_{*_1} g} (g_2 \circ_{*_2}^h g_3). \end{aligned} \tag{73}$$

Unity. Let $e : \mathrm{X} \to A$-MG be defined by $e(v) = \emptyset_{\{v\}}$ and let $e_{\mathcal{F}}$ the unit of $\mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$. Let be $g \in A$-MG$'[V]$ and $v \notin V$. Then

$$\begin{aligned} g \circ_*^{e_{\mathcal{F}}(v)} e(v) &= g|_{\{*_a \leftarrow e_{\mathcal{F}}(v)(a)_a\}} \oplus \emptyset_{\{v\}} \\ &= g|_{\{*_a \leftarrow v_a\}} = A\text{-MG}[\sigma](g), \end{aligned} \tag{74}$$

where σ is the bijection which sends $*$ to v and is the identity over V. If now $g \in A$-MG$[V]$, then for any f,

$$e(*) \circ_*^f g = \emptyset_{\{*\}}|_{\{*_a \leftarrow f(a)_a\}} \oplus g = g. \tag{75}$$

□

In all the following we only consider this semi-direct, albeit not exactly on A-MG and $\mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$, and we will hence omit the φ index. We call *graph insertion operad* any operad which can be written with this semi-direct product.

Remark 4. This notion of graph insertion operad is different than the one mentioned in [10], in the context of Feynman graph insertions in quantum field theory.

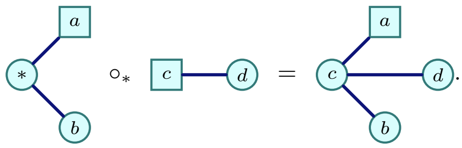
As mentioned above, this construction is supposed to encode partial compositions of the form: take the disjoint union, forget the vertex on which we compose, and independently reconnect loose edges. Let $g_1 \otimes f_1$ and $g_2 \otimes f_2$ be two elements of A-MG $\ltimes \mathcal{F}_A^{\mathbb{K}\textsc{Pol}_+^1}$. The ends of g_1 are labelled by elements of A. When considering $g_1 \circ_*^{f_2} g_2$, the map f_2 is there to tell us how we should connect the loose ends obtained by forgetting $*$: each end labelled with a will be connected to the elements of $f_2(a)$. More pratically, when defining a partial composition of the above form, one will have certain types of ends 1,2,3... and for each type of edge a way to reconnect them. For example 'edges of type i connect to vertices a and b or to vertex c'. This translates by $f(i) = a \oplus b + c$. For a partial composition defined this way, we then just need to verify that the sums of polynomials $f(i)$ are a stable sub-species of $\mathbb{K}\textsc{Pol}_+^1$ by composition of polynomials i.e. a sub-operads of $\mathbb{K}\textsc{Pol}_+^1$.

Let us give two simple examples of operads which can be defined using this semi-direct product. Recall from Section 1 that we have a natural embedding of $\textsc{Id}$ in $\mathbb{K}\textsc{Pol}_+$, and three natural embeddings of E_+ in $\mathbb{K}\textsc{Pol}_+$.

Example 3.10. $\mathrm{G}^\bullet$ has a natural operad structure given by $\mathrm{G}^\bullet \cong \mathrm{G} \times \textsc{Id} \cong \{0\}\text{-G} \ltimes \mathcal{F}_{\{0\}}^{\textsc{Id}}$. For (g_1, v_1) and (g_2, v_2) two pointed graphs, the partial composition $(g_1, v_1) \circ_* (g_2, v_2)$ is then equal to $(g_3, v_1|_{*\leftarrow v_2})$ where g_3 is the graph obtained by connecting all the ends on $*$ to v_2. More formally,

$$\begin{aligned}(g_1, v_1) \circ_* (g_2, v_2) &= (g_1|_{*\leftarrow v_2} \oplus g_2, v_1|_{*\leftarrow v_2}) \\ &= (\mathrm{G}[\sigma](g_1) \oplus g_2, v_1|_{*\leftarrow v_2}),\end{aligned} \tag{76}$$

where σ is the bijection which sends $*$ on v_2 and which is the identity on the rest of its domain. For instance, we have:



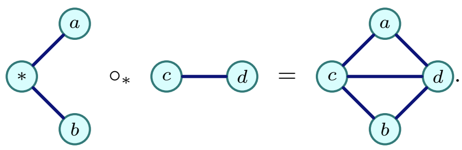
(77)

Remark that the operad **NAP** [11] is a sub-operad of the operad above and hence is a graph insertion operad.

Example 3.11. G has a natural operad structure given by $\mathrm{G} \cong \mathrm{G} \times \mathrm{E}_+ \cong \{0\}\text{-G} \ltimes \mathcal{F}_{\{0\}}^{\mathrm{E}}$, where we consider here the embedding $V \mapsto \bigoplus_{v\in V} v$. For g_1 and g_2 two graphs, the partial composition $g_1 \circ g_2$ is then the graph obtained by adding an edge between each neighbour of $*$ and each vertex of g_2. More formally, for $g_1 \in G'[V_1]$ and $g_2 \in G[V_2]$:

$$\begin{aligned}g_1 \circ_* g_2 &= g_1|_{*\leftarrow \bigoplus V_2} \oplus g_2 \\ &= g_1|_{V_1} \oplus \bigoplus_{v\in n(*)} v \bigoplus_{v\in V_2} \oplus g_2 \\ &= g_1|_{V_1} \oplus \bigoplus_{v_1\in n(*), v_2\in V_2} v_1 v_2 \oplus g_2,\end{aligned} \tag{78}$$

where $n(*)$ is the set of neighbours of $*$. Each of terms of the last line have a combinatorial interpretation: $g_1|_{V_1}$ is g_1 to which we removed $*$, the term $\bigoplus_{v_1\in n(*), v_2\in V_2} v_1 v_2$ means that we add an edge between any element in $n(*)$ and any element in V_2, and finally the term g_2 means that we keep all the edges of g_2. For instance, we have:



(79)

Let us now prove that the two partial composition introduced in subsection 2.1 do indeed make $\mathrm{MG}_{or}^\bullet$ and MG operads.

proof of Theorem 2.1. Recall from Example 3.2 that we have a bijection $\mathrm{MG}_{or} \cong \{\mathtt{s},\mathtt{t}\}$-MG. The isomorphism $\mathrm{MG}_{or}^{\bullet} \cong \{\mathtt{s},\mathtt{t}\}\text{-MG} \times \mathrm{I}\textsc{d} \times \mathrm{E}_{+} \cong \{\mathtt{s},\mathtt{t}\}\text{-MG} \ltimes \mathcal{F}^{\mathrm{I}\textsc{d},\mathrm{E}_{+}}_{\{s\},\{t\}}$, give the desired operad structure on $\mathrm{MG}_{or}^{\bullet}$ when considering the embedding $V \mapsto \sum_{v\in V} v$ of E_{+} in $\mathbb{K}\mathrm{P}\textsc{ol}_{+}^{1}$. $\square$

proof of Theorem 2.2. This is the operad structure on MG given by $\mathrm{MG} \cong \mathrm{MG} \times \mathrm{E}_{+} \cong \{0\}\text{-MG} \ltimes \mathcal{F}^{\mathrm{E}_{+}}_{\{0\}}$ when considering the embedding $V \mapsto \sum_{v\in V} v$ of E_{+} in $\mathbb{K}\mathrm{P}\textsc{ol}_{+}^{1}$. $\square$

We end this section by mentioning that while we restricted ourselves to multigraphs in this paper, all the work done in this section naturally generalize to the very general framework of multi-hypergraphs, whose edges can contain any non null number of vertices and can be appear more than once. This is done by considering replacing MG by MHG, the species of multi-hypergraphs, and $\mathbb{K}\mathrm{P}\textsc{ol}_{+}^{1}$ by $\mathbb{K}\mathrm{P}\textsc{ol}_{+}$ in Theorem 3.9. In this more general context, the connection of ends can also change the number of vertices of the edges.

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A Koszul duality and Koszul operads

As said at the end of Section 1, two advantages of defining an operad by its generators and relations are that it is possible to construct (under some conditions) its Koszul dual and that it is possible to check if the operad is Koszul.

Koszul duality

Let us begin by defining the Koszul dual of an operad. To do this, we consider from now on that we have an arbitrary order for every finite set V in order to consider the signature of a bijection between two different sets. Without loss of generality, we consider these order to follow the alphabetical order when applicable.

For S a linear species, we denote by S^* the *dual* species of S which is defined by $\mathrm{S}^*[V] = \mathrm{S}[V]^*$ and $\mathrm{S}^*[\sigma](f) = f \circ \mathrm{S}[\sigma^{-1}]$. We denote by $\mathrm{S}^\vee$ the species defined by $\mathrm{S}^\vee[V] = \mathrm{S}^*[V]$ and $\mathrm{S}^\vee[\sigma](f) = \mathrm{sign}(\sigma) f \circ \mathrm{S}[\sigma^{-1}]$, where $\mathrm{sign}(\sigma)$ is the signature of σ.

Definition A.1. Let $\mathcal{O} = \mathrm{Ope}(\mathcal{G}, \mathcal{R})$ be a binary quadratic operad. Define the linear form $\langle - \,|\, - \rangle$ on $\mathbf{Free}^{(2)}_{\mathcal{G}^\vee} \times \mathbf{Free}^{(2)}_{\mathcal{G}}$ by

$$\langle f_1 \circ_* f_2 \,|\, x_1 \circ_* x_2 \rangle = f_1(x_1) f_2(x_2), \tag{80}$$

The *Koszul dual* of $\mathcal{O}$ is then the operad $\mathcal{O}^! = \mathrm{Ope}(\mathcal{G}^\vee, \mathcal{R}^\perp)$ where $\mathcal{R}^\perp$ is the orthogonal of $\mathcal{R}$ for $\langle - \,|\, - \rangle$.

Example A.2. The *Lie operad* **Lie** is the quotient of the free operad over one antisymmetric generator by the Jacobi relation. More formally, denote by $[a,b]$ the generating element of $\mathrm{E}_2^\vee[\{a,b\}]$ and by $[b,a] = (ab) \cdot [a,b]$ so that $[b,a] = -[a,b]$. Then $\mathbf{Lie} = \mathrm{Ope}(\mathrm{E}_2^\vee, \mathcal{R})$ with $\mathcal{R}$ the sub-species of $\mathbf{Free}^{(2)}_{\mathrm{E}_2^\vee}$ generated by the *Jacobi relation* $[a,[b,c]] + [c,[a,b]] + [b,[c,a]$, where $[a,[b,c]]$ stands for $[a,*]\circ_{b,c}$ (one can check that this is indeed stable under the action of $\mathfrak{S}_{\{a,b,c\}}$ and hence $\mathcal{R}$ is indeed a species). This operad is the Koszul dual of **Com**. Indeed, for every pair $\alpha < \beta$ in $\{a,b,c,*\}$ ordered with the alphabetical order ($c < *$) denote by $s^\vee_{\alpha\beta} = [\alpha,\beta]$ the dual of $s_{\{\alpha,\beta\}} \in \mathbf{ComMag}[\{\alpha,\beta\}]$. Then we have, for example:

$$\begin{aligned}
\langle s^\vee_{a*} \circ_* s^\vee_{bc} + s^\vee_{c*} \circ_* s^\vee_{ab} + s^\vee_{b*} \circ_* s^\vee_{ca} \,|\, s_{\{a,*\}} \circ_* s_{\{b,c\}} - s_{\{c,*\}} \circ_* s_{\{a,b\}} \rangle& \\
= \langle s^\vee_{a*} \circ_* s^\vee_{bc} \,|\, s_{\{a,*\}} \circ_* s_{\{b,c\}} \rangle + \langle s^\vee_{c*} \circ_* s^\vee_{ab} \,|\, s_{\{a,*\}} \circ_* s_{\{b,c\}} \rangle + \langle s^\vee_{b*} \circ_* s^\vee_{ca} \,|\, s_{\{a,*\}} \circ_* s_{\{b,c\}} \rangle& \\
- \langle s^\vee_{a*} \circ_* s^\vee_{bc} \,|\, s_{\{c,*\}} \circ_* s_{\{a,b\}} \rangle - \langle s^\vee_{c*} \circ_* s^\vee_{ab} \,|\, s_{\{c,*\}} \circ_* s_{\{a,b\}} \rangle - \langle s^\vee_{b*} \circ_* s^\vee_{ca} \,|\, s_{\{c,*\}} \circ_* s_{\{a,b\}} \rangle& \\
= s^\vee_{a*}(s_{\{a,*\}}) s^\vee_{bc}(s_{\{b,c\}}) + s^\vee_{c*}(s_{\{a,*\}}) s^\vee_{ab}(s_{\{b,c\}}) + s^\vee_{b*}(s_{\{a,*\}}) s^\vee_{ca}(s_{\{b,c\}})& \\
- s^\vee_{a*}(s_{\{c,*\}}) s^\vee_{bc}(s_{\{a,b\}}) - s^\vee_{c*}(s_{\{c,*\}}) s^\vee_{ab}(s_{\{a,b\}}) - s^\vee_{b*}(s_{\{c,*\}}) s^\vee_{ca}(s_{\{a,b\}})& \\
= 1 + 0 + 0 - 1 - 0 - 0 = 0.&
\end{aligned} \tag{81}$$

Koszul operads

Koszulity is an important aspect of operad theory. We only give here a very quick overview of Koszulity and Gröbner bases for operads which hides a lot of the theory. We do not give the general results but only restricted versions which suffice for our use. We refer the reader to the literature; for a broader approach of the topic, see for example [12], [15], [8] and [6]. In particular, all the examples presented here come from [6].

In order to give the characterisation which interests us, we need to introduce the concepts of $\mathscr{L}$-species, shuffle operads and Gröbner bases. Informally, we can see these objects as the same as species and operads, except with a total order on every set of vertices.

$\mathscr{L}$-species

A *linear positive $\mathscr{L}$-species* consists of the following data:

- for each finite set V and total order l on V, a vector space $\mathrm{S}[V,l]$, such that $\mathrm{S}[\emptyset,\emptyset] = \{0\}$.
- For each increasing bijection $\sigma : (V,l) \to (V',l')$, a linear map $\mathrm{S}[\sigma] : \mathrm{S}[V,l] \to \mathrm{S}[V',l']$. These maps should be such that $\mathrm{S}[\sigma_1 \circ \sigma_2] = \mathrm{S}[\sigma_1] \circ \mathrm{S}[\sigma_2]$ and $\mathrm{S}[Id] = Id$.

In the sequel, we write order to designate a total order and $\mathscr{L}$-species to designate linear positive $\mathscr{L}$-species. For S an $\mathscr{L}$-species and l an order on V, we also denote by $\mathrm{S}[l] = \mathrm{S}[V,l]$. We can do this since the data of V is included in l. As for species, we denote by X the $\mathscr{L}$-species defined $\mathrm{X}[V,l] = \{0\}$ if V is not a singleton and $\mathrm{X}[\{v\}, v] = \mathbb{K}v$ else.

As in the case of classical species, $\mathscr{L}$-species also have constructions on them, but before giving them, let us give some notations.

- For l an order on V and $W \subseteq V$ a subset of V, we denote by l_W the order on W induced by l.
- For $l = l_1 \dots l_n$ an order on a set V of size n, $i \in [n]$ and $* \notin V$, we denote by $l \xleftarrow{i} *$ the order $l_1 \dots l_{i-1} * l_i \dots l_n$ on $V + *$.
- For $l^1, \dots l^k$ k orders on pairwise disjoint sets $V_1, \dots, V_k$, we denote by $sh(l^1, \dots, l^k) = \{w \mid w_{V_i} = l^i\}$ the set of *shuffles* of $l^1, \dots, l^k$. Note that this is an "associative operation" in the sense that for l^1, l^2, l^3 three orders, the union of the shuffles of l^1 with the elements of $sh(l^2, l^3)$ is exactly $sh(l^1, l^2, l^3)$.
- The *shuffle compositions* $\textsc{Comp}_{sh}[V, l]$ of an ordered set (V, l) are the compositions $P = P_1, \dots, P_k$ of V such that, for every $1 \leq i < j \leq k$, $\min_l P_i < \min_l P_j$.

Let R and S be two linear $\mathscr{L}$-species and l a total order on V. Denote by $n = |V|$ and let be $i \in [n]$. We define the following operations.

$$\textit{Product} \quad \mathrm{R} \cdot \mathrm{S}[V, l] = \bigoplus_{l \in sh(l', l'')} \mathrm{R}[l'] \otimes \mathrm{S}[l''],$$

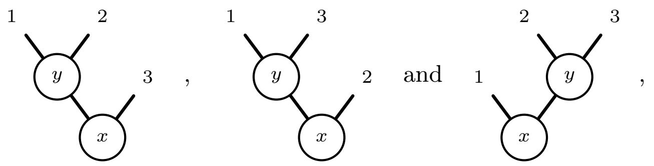
$$\textit{i-thDerivative} \quad \mathrm{S}^i[V, l] = \mathrm{S}[V + *, l \xleftarrow{i} *],$$

$$\textit{Composition} \quad \mathrm{R}(\mathrm{S})[V, l] = \bigoplus_{P \in \textsc{Comp}_{sh}[V, l]} \mathrm{R}[\{P_1, \dots, P_k\}, P] \otimes \mathrm{S}[P_1, l_{P_1}] \otimes \dots \otimes \mathrm{S}[P_k, l_{P_k}].$$

Since we have the $\mathscr{L}$-species X and the notion of composition, we can define Schröder tree on $\mathscr{L}$-species in the same way as for species: if $\mathrm{S} = \mathrm{X} + \mathrm{S}_{2+}$, then $\mathscr{S}_\mathrm{S} = \mathrm{X} + \mathrm{S}_{2+}(\mathscr{S}_\mathrm{S})$. In this case, for S a $\mathscr{L}$-species, $\mathscr{S}_\mathrm{S}[V]$ is the vector space of rooted *planar* trees with internal vertices decorated with elements of S and set of leaves V. This is because of the orders: instead of indexing with partitions in the composition we use shuffle compositions.

For S a $\mathscr{L}$-species and l an order on V, the fact that we have an order enables us an easier notation of the elements of $\mathscr{S}_\mathrm{S}[V, l]$ as operations *e.g.* $\alpha(l_1, \dots, l_n)$.

Example A.3. Let S be the $\mathscr{L}$-species defined by $\mathrm{S}[V, l] = \{0\}$ when $|V| \neq 2$ and $\mathrm{S}[V, l] = \mathbb{K}\{1, \dots n\}$ else, for n an integer greater than 1. Then the generators of $\mathscr{S}_\mathrm{S}^{sh}[[3], 123]$ are of the form



(82)

where $1 \leq x, y \leq n$. This is because the only shuffle compositions of size 2 of [3] are $(12, 3)$, $(13, 2)$ and $(1, 23)$. These trees can be denoted in a more compact way by $\mu_x(\mu_y(1, 2), 3)$, $\mu_x(\mu_y(1, 3), 2)$ and $\mu_x(1, \mu_y(2, 3))$.

Let us end this brief presentation of $\mathscr{L}$-species by giving their link to classical species. Let $\mathscr{F}$ be the forgetful functor which send a species S on the $\mathscr{L}$-species $\mathrm{S}^\mathscr{F}$ defined by:

- for l an order on V, $\mathrm{S}^\mathscr{F}[V, l] = \mathrm{S}[V]$.
- For $\sigma : (V, l) \to (V', l')$ an increasing bijection, $\mathrm{S}^\mathscr{F}[\sigma]$ is given by

$$\mathrm{S}^\mathscr{F}[V, l] = \mathrm{S}[V] \xrightarrow{\sigma} \mathrm{S}[W] = \mathrm{S}^\mathscr{F}[W, l']. \tag{83}$$

- For $f : \mathrm{S} \to \mathrm{R}$ a species morphism, $f^\mathscr{F}$ is the $\mathscr{L}$-species morphism given by

$$f^\mathscr{F}_{V,l} : \mathrm{S}^\mathscr{F}[V, l] = \mathrm{S}[V] \xrightarrow{f} \mathrm{R}[V] = \mathrm{R}^\mathscr{F}[V, l]. \tag{84}$$

This is a forgetful functor in the sense that we forget the action σ_V on $\mathrm{S}[V]$. We have the following fundamental proposition from [6].

Proposition A.4 (Proposition 3 in [6])**.** Let S and R be two species. Then

$$(\mathrm{R}(\mathrm{S}))^\mathscr{F} = \mathrm{R}^\mathscr{F}(\mathrm{S}^\mathscr{F}). \tag{85}$$

Shuffle operads

We would like to define shuffle operads as $\mathscr{L}$-species satisfying the same axioms that a linear species must satisfy to be an operad. For now we can not do this because we do not have the notion of derivative of a shuffle operad $\mathcal{O}'$. Fortunately, there is a way to make sense of the different species appearing in the diagrams (1.5).

For l an order on V and $v \in V$, denote by $\arg_l v$ the index of v in l: $l_{\arg_l v} = v$. For R and S two $\mathscr{L}$-species we define the $\mathscr{L}$-species $\mathrm{R}' \cdot \mathrm{S}$ and $\mathrm{R}'' \cdot \mathrm{S}^2$ as follow:

$$\begin{aligned} \mathrm{R}' \cdot \mathrm{S}[V,l] &= \bigotimes_{l \in sh(l^1,l^2)} \mathrm{R}^{\arg_l l^2_1}[l^1] \otimes \mathrm{S}[l^2], \\ \mathrm{R}'' \cdot \mathrm{S}^2[V,l] &= \bigotimes_{l \in sh(l^1,l^2,l^3)} \mathrm{R}^{\arg_l l^2_1, \arg_l l^3_1}[l^1] \otimes \mathrm{S}[l^2] \otimes \mathrm{S}[l^3]. \end{aligned} \tag{86}$$

A *shuffle operad* is then a $\mathscr{L}$-species $\mathcal{O}$ with a unity e and a partial composition $\circ^{sh}_* : \mathcal{O}' \cdot \mathcal{O} \to \mathcal{O}$ such that the diagrams (1.5) commutes. For S a $\mathscr{L}$-species, the *free shuffle operad over* S is denoted by $\mathbf{Free}^{sh}_{\mathrm{S}}$ and defined in the same way as the free operad over a species. The same goes with the *ideal of a shuffle operad* and the notation $\mathrm{Ope}_{sh}(\mathcal{G}, \mathcal{R})$.

Remark 5. With the notations of elements of the free operad as operations, the partial composition of the free operad is then the composition of operation: $\mu_x(\dots, *, \dots) \circ^{\xi}_* \mu_y(\dots) = \mu_x(\dots, \mu_y(\dots), \dots)$.

We have the following corollary from Proposition A.4.

Theorem A.5 (Corollary 1 in [6]). Let $\mathcal{G}$ be a species. The image by $\mathscr{F}$ of the free operad generated by $\mathcal{G}$ is isomorphic to the free shuffle operad generated by $\mathscr{F}(\mathcal{G})$. For $\mathcal{R}$ a sub-species of $\mathbf{Free}_{\mathcal{G}}$, the image by $\mathscr{F}$ of the operad ideal generated by $\mathcal{R}$ is isomorphic to the shuffle operad ideal generated by $\mathscr{F}(\mathcal{R})$. This writes as $\mathbf{Free}^{\mathscr{F}}_{\mathcal{G}} \cong \mathbf{Free}^{sh}_{\mathcal{G}^{\mathscr{F}}}$ and $(\mathcal{R})^{\mathscr{F}} = (\mathcal{R}^{\mathscr{F}})$.

Hence $\mathrm{Ope}(\mathcal{G}, \mathcal{R})^{\mathscr{F}} \cong \mathrm{Ope}_{sh}(\mathcal{G}^{\mathscr{F}}, \mathcal{R}^{\mathscr{F}})$.

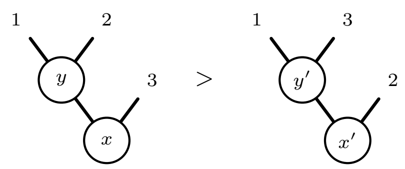
Admissible order

Let S be a $\mathscr{L}$-species. In order to define the notion of Gröbner bases, we need to introduce an order on the trees generating $\mathbf{Free}^{sh}_{\mathrm{S}}[V,l]$. Instead of giving the broader notion of admissible order defined in [6], we only give a small variation of the *path-lexicographic ordering*.

First for every order l on V, fix a basis of $\mathrm{S}[l]$ and an order on this basis such that for every x in the chosen basis of $\mathrm{S}[l]$ and $\sigma : l \to l'$, $\sigma \cdot x$ is also in the chosen basis of $\mathrm{S}[l']$ and for y an other element of the basis greater than x we have $\sigma \cdot x < \sigma \cdot y$. That is to say, the order does not depend on the labels (but it can depend on their relative order). Given an ordered basis, we also have an order on the words on elements of the basis given by the lexicographic order. Let now be $t \in \mathbf{Free}^{sh}_{\mathrm{S}}[l_1 \dots l_n]$ such that every internal node is labelled by an element of the chosen bases. For all $i \in [n]$, there is a unique path from l_i to the root of t. Denote by a_i the word composed, from left to right, of the labels of the nodes of this path, from the root to the leaf. We associate to t the sequence $(a_1, \dots, a_i, w)$, where w is the word obtained by reading the leaves of t from left to right.

For two trees $t, t' \in \mathbf{Free}^{sh}_{\mathrm{S}}$, with associated sequences $(a_1, \dots, a_n, w)$ and $(b_1 \dots, b_n, w')$ we then compare t and t' by lexicographicaly comparing a_1 with b_1 then a_2 with b_2 etc and reverse lexicographicaly comparing w with w' if $a_i = b_i$ for all i.

Example A.6. The $\mathscr{L}$-species S of Example A.3 have natural ordered bases equal to the set $\{1, \dots, n\}$ with the natural order. The sequences attached to the given trees are then respectively $(xy, xy, 123)$, $(xy, x, xy, 132)$ and $(x, xy, xy, 123)$ and we have



(87)

if $x > x'$ or $x = x'$ and $y > y'$ or $x = x'$ and $y = y'$.

Now remark that trees in $\mathbf{Free}_{\mathrm{S}}^{sh}[V,l]$ with basis elements as internal node labels make a basis $\mathbf{Free}_{\mathrm{S}}^{sh}[V,l]$. Hence any $x \in \mathbf{Free}_{\mathrm{S}}^{sh}$ can be written as a sum of such elements and we define $\mathrm{lt}(x)$ the *leading term* of x as the maximal element in this sum.

Divisibility and S-polynomials

Let S be a $\mathscr{L}$-species. A tree t of $\mathbf{Free}_{\mathrm{S}}^{sh}$ is *divisible* by another tree t' of $\mathbf{Free}_{\mathrm{S}}^{sh}$ if t' is a sub-tree of t. Here a sub-tree must also conserve the order of the leaves. A tree u of $\mathbf{Free}_{\mathrm{S}}^{sh}$ is a *small common multiple* of two tree t and t' if it is divisible by both t and t' and its number of vertices is less than the total number of vertices of t and t'.

Example A.7. Let S be a $\mathscr{L}$-species, $l = l_1, \dots l_n$ an order and $\alpha(\beta(l_1,l_3),\gamma(\beta(l_2,l_6),l_4,l_5))$ an element of $\mathbf{Free}_{\mathrm{S}}^{sh}$. This tree has among its divisors $\alpha(\beta(l_1,l_3),l_2)$ and $\gamma(\beta(l_1,l_4),l_2,l_3)$ but not $\gamma(\beta(l_1,l_3),l_2,l_4)$.

If t is divisible by t', then there exists trees α and $\beta_1,\dots,\beta_k$ such that $t = \alpha(\dots,t'(\beta_1,\dots,\beta_k),\dots)$. We denote by $m_{t,t'}$ the operation on any tree with same number of leaves than t' which associate to a tree u the tree $\alpha(\dots,u(\beta_1,\dots,\beta_k),\dots)$. Let now V be a finite set, l an order on V and $x,y \in \mathbf{Free}_{\mathrm{S}}^{sh}[V,l]$. Assume $\mathrm{lt}(x)$ and $\mathrm{lt}(y)$ have a small common multiple u. Then we have $m_{u,\mathrm{lt}(x)}(\mathrm{lt}(x)) = u = m_{u,\mathrm{lt}(y)}(\mathrm{lt}(y))$. We call *S-polynomial of x and y (corresponding to u)* the element

$$s_u(x,y) = m_{u,\mathrm{lt}(x)}(x) - \frac{c_x}{c_y} m_{u,\mathrm{lt}(y)}(y), \tag{88}$$

where c_x and c_y are the respective coefficient of the leading terms of x and y.

Gröbner bases and Koszulity

We can finally give the definition of a Gröbner bases and a Koszul operad.

Definition A.8 (Definition 13 in [6])**.** Let $\mathcal{G}$ be a $\mathscr{L}$-species and $\mathcal{R}$ be a $\mathscr{L}$-sub-species of $\mathbf{Free}_{\mathcal{G}}^{sh}$. Let $\mathcal{B}$ be basis of $\mathcal{R}$. We say that $\mathcal{B}$ is a *Gröbner bases* of $\mathcal{R}$ if for every $x \in (\mathcal{R})$, the leading term of x is divisible by the leading term of one element in $\mathcal{B}$.

Definition A.9 (Corollary 3 in [6])**.** Let $\mathcal{G}$ be a $\mathscr{L}$-species and $\mathcal{R}$ be a quadratic $\mathscr{L}$-sub-species of $\mathbf{Free}_{\mathcal{G}}^{sh}$. We say that $\mathrm{Ope}_{sh}(\mathcal{G},\mathcal{R})$ is *Koszul* if $\mathcal{R}$ admits a Gröbner bases.

Let $\mathcal{G}$ be a set species and $\mathcal{R}$ be a quadratic sub-species of $\mathbf{Free}_{\mathcal{G}}^{sh}$. We say that $\mathrm{Ope}(\mathcal{G},\mathcal{R})$ is *Koszul* if $\mathrm{Ope}_{sh}(\mathcal{G}^{\mathscr{F}},\mathcal{R}^{\mathscr{F}})$ is Koszul.

When $\mathcal{O}$ is a Koszul symmetric operad, it admits a Koszul dual $\mathcal{O}^!$. In this case the Hilbert series of $\mathcal{O}$ and $\mathcal{O}^!$ are related by the identity:

$$\mathcal{H}_{\mathcal{O}}(-\mathcal{H}_{\mathcal{O}^!}(-t)) = t. \tag{89}$$

Let us finish by a characterisation of Gröbner bases.

Proposition A.10 (Theorem 1 in [6])**.** Let $\mathcal{G}$ be a $\mathscr{L}$-species and $\mathcal{R}$ be a $\mathscr{L}$-sub-species of $\mathbf{Free}_{\mathcal{G}}^{sh}$. Let $\mathcal{B}$ be basis of $\mathcal{R}$. Then $\mathcal{B}$ is a Gröbner bases if and only if for all pair of elements in $\mathcal{B}$, their S-polynomials are congruent to zero modulo $\mathcal{B}$ (*i.e.* they are in $(\mathcal{R})$).