Fundamental and homogeneous bases of Hopf algebras built from nonsymmetric operads

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Abstract. We introduce new partial order structures on the underlying sets of free nonsymmetric operads. These posets involve decorated ordered rooted trees, and their terminal intervals are lattices. These lattices are not graded, not self-dual, and not semi-distributive, but they are EL-shellable, and their Möbius functions take values in $\{-1, 0, 1\}$. They admit sublattices on the families of m-Fuss-Catalan objects and of forests of trees. This latter order structure is used to construct two new bases for the natural Hopf algebras of free nonsymmetric operads: a fundamental basis and a homogeneous basis. Along with the already known elementary basis of these Hopf algebras, this yields a triple of bases. The situation is similar to what is observed in the Hopf algebras of Malvenuto-Reutenauer, Loday-Ronco, and noncommutative symmetric functions, each of which presents such triples of bases and basis changes involving, respectively, the right weak partial order, the Tamari partial order, and the Boolean lattice partial order.

Contents

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1 Introduction

Since the advent of modern Hopf algebra theory around the 1990s, most Hopf algebras defined on the linear spans of sets X of combinatorial objects are equipped with multiple bases. This is the case, as main examples, for the Malvenuto-Reutenauer Hopf algebra **FQSym** [MR95; DHT02] of permutations, the Loday-Ronco Hopf algebra **PBT** [LR98; LR02; HNT05] of binary trees, and the Hopf algebra of noncommutative symmetric functions **Sym** [Gel+95] of integer compositions. Each of these structures is equipped with bases $\{\mathsf{F}_x\}_{x \in X}$, $\{\mathsf{E}_x\}_{x \in X}$, and $\{\mathsf{H}_x\}_{x \in X}$. A notable fact is that, in each case, there is a partial order relation $\preccurlyeq$ on X such that

$$\mathsf{E}_x = \sum_{\substack{x' \in X \\ x \preccurlyeq x'}} \mathsf{F}_{x'} \quad \text{and} \quad \mathsf{H}_x = \sum_{\substack{x' \in X \\ x' \preccurlyeq x}} \mathsf{F}_{x'}, \tag{1.0.0.A}$$

and two binary operations $\diagup$ and $\diagdown$ on X such that

$$\mathsf{E}_{x_1} \star \mathsf{E}_{x_2} = \mathsf{E}_{x_1 \diagup x_2} \quad \text{and} \quad \mathsf{H}_{x_1} \star \mathsf{H}_{x_2} = \mathsf{H}_{x_1 \diagdown x_2}, \tag{1.0.0.B}$$

and

$$\mathsf{F}_{x_1} \star \mathsf{F}_{x_2} = \sum_{\substack{x \in X \\ x_1 \diagup x_2 \preccurlyeq x \preccurlyeq x_1 \diagdown x_2}} \mathsf{F}_x, \tag{1.0.0.C}$$

where $\star$ is the product of the Hopf algebra. For **FQSym**, $\preccurlyeq$ is the right weak partial order, for **PBT**, $\preccurlyeq$ is the Tamari partial order [Tam62], and for **Sym**, $\preccurlyeq$ is the Boolean lattice partial order. The case of **Sym** is prototypical and, by analogy with the theory of symmetric functions, the F-basis is termed the *fundamental* basis, while the E-basis (resp. H-basis) is termed the *elementary* (resp. *homogeneous*) basis. There are a lot of other known examples of Hopf algebras or associative algebras sharing these properties [NT06; Gir12; CGM15; G C17; CG22].

Besides all this, the *natural Hopf algebra* of an operad $\mathcal{O}$ is a Hopf algebra $\mathbf{N}{\cdot}\mathcal{O}$ whose bases are indexed by some words on $\mathcal{O}$, where the coproduct is inherited from the composition map of $\mathcal{O}$. This construction is considered for instance in [Laa04; Gir11; BG16; Gir24], and a noncommutative variation for nonsymmetric operads is introduced in [ML14] and already employed in [Gir11]. However, surprisingly, no alternative bases are known in general for $\mathbf{N}{\cdot}\mathcal{O}$. In this work, we focus on natural Hopf algebras $\mathbf{N}{\cdot}\lfloor\mathfrak{T}{\cdot}\mathcal{S}\rfloor$ of free nonsymmetric operads $\mathfrak{T}{\cdot}\mathcal{S}$ generated by a signature $\mathcal{S}$. These Hopf algebras are defined on the linear span of certain forests and come, through the construction $\mathbf{N}$, with an E-basis.

Our main contribution consists in the introduction of a general partial order, the *$\mathcal{S}$-easterly wind partial order*, defined on some treelike structures decorated on a signature $\mathcal{S}$ (called *$\mathcal{S}$-terms*). This poset exhibits notable properties and specializes as a poset on forests as well as on various other families of combinatorial objects. This structure leads to the definition of fundamental and homogeneous bases satisfying (1.0.0.A) for $\mathbf{N}{\cdot}\lfloor\mathfrak{T}{\cdot}\mathcal{S}\rfloor$. We also introduce two binary operations $\diagup$ and $\diagdown$ on forests, such that the triple formed by the elementary, fundamental, and homogeneous bases of $\mathbf{N}{\cdot}\lfloor\mathfrak{T}{\cdot}\mathcal{S}\rfloor$ satisfies (1.0.0.B) and (1.0.0.C).

The contents and the results of this work are presented as follows.

Section 2, after presenting preliminary notions about signatures and $\mathcal{S}$-terms, introduces the easterly wind partial order relation $\preccurlyeq$ on the set of $\mathcal{S}$-terms. This partial order relation is defined via *connection words*, which are some sequences of rational numbers associated with $\mathcal{S}$-terms. In parallel, we define a rewrite rule $\rightharpoonup$ on $\mathcal{S}$-terms, and establish, through Theorem 2.2.3.F, that $\rightharpoonup$ is the covering relation of $\preccurlyeq$. We then introduce a map tlt$\cdot\mathcal{X}$, called the *$\mathcal{X}$-tilting map*, on the easterly wind poset which turns out to be a closure operator [DP02] of this poset, as stated by Theorem 2.3.2.B. This will allow us to consider subposets of easterly wind posets on closed terms w.r.t. tlt$\cdot\mathcal{X}$, called *tilted terms*.

In Section 3, we prove via Theorem 3.1.1.E that the easterly wind posets are EL-shellable [Bjö80; BW96] and that their Möbius functions take values in the set $\{-1, 0, 1\}$. Using connections words, we also propose geometric realizations of these posets. We end this section with Theorem 3.2.2.A, which shows that any terminal interval $\downarrow\cdot\mathfrak{t}$ from a term $\mathfrak{t}$ in the easterly wind poset is a lattice. This property also holds for its subposets on tilted terms. As a side remark, these lattices $\downarrow\cdot\mathfrak{t}$ are not always join semi-distributive.

Section 4 focuses on special cases of easterly wind posets. We begin by defining a notion of *forest* as a particular kind of term. To each word w on a signature $\mathcal{S}$, we associate an interval $[\mathbf{f}^{\Uparrow}\cdot w, \mathbf{f}^{\Downarrow}\cdot w]$ of the $\mathcal{S}$-easterly wind poset. Theorem 4.1.2.A shows that such intervals are maximal. We then study these maximal intervals to construct posets on forests that are in bijection with objects from the Fuss-Catalan family. This is established in Theorem 4.2.1.D. These resulting posets, which are also lattices, are distinct from the already known posets on this combinatorial family [BP12; CG22], and to our knowledge, have not appeared before in the literature. Through a separate construction involving tilted terms, we realize the Tamari poset [Tam62] as a maximal interval of a particular easterly wind poset of tilted terms. Proposition 4.2.2.B provides an explicit poset isomorphism with the Tamari poset using the Knuth realization [Knu04] involving ordered rooted trees and scope sequences. We conclude this section by introducing *leaning forests*, a specific subclass of forests (and thus, of terms), which form the bases of the natural Hopf algebras of free nonsymmetric operads. We endow this set with two concatenation operations $\diagup$ and $\diagdown$, and define a shuffle operation $\sqcup\!\sqcup$ on leaning forests. The properties and concepts established in this section are crucial for the final one.

In the final part, Section 5, we use the easterly wind partial order on leaning forests to build a fundamental and a homogeneous basis of the natural Hopf algebra of a free nonsymmetric operad. Theorem 5.3.1.A shows that the product of two elements of the fundamental basis can be expressed as an interval of the easterly wind poset, or equivalently as a shuffle of leaning forests. Theorem 5.3.2.A shows in a similar way that the product of two elements of the homogenous basis expresses through the $\diagdown$ operation on leaning forests.

We conclude in Section 6 with some open questions raised by this work.

General notations and conventions. All functions are written in curried form: given a function $f : A_1 \times \cdots \times A_n \to A$, we denote its application by $f\cdot a_1\cdot \ldots \cdot a_n$ rather than $f(a_1, \ldots, a_n)$. Accordingly, the function type of f is $A_1 \to \cdots \to A_n \to A$; the arrow $\to$ is taken to be right-associative. Rather than enclosing sub-expressions in parentheses, we use underlining to distinguish these parts within expressions[a]. For a statement P, the Iverson bracket $[P]$ takes the value 1 if P is true and 0 otherwise. For two integers i and j, $[i, j]$ denotes the interval $\{i, \ldots, j\}$, $[i]$ denotes the set $[1, i]$ and $[\![n]\!]$ denotes the set $[0, i]$. For a set A, A^* is the set of words on A. For $w \in A^*$, the length of w is $\ell\cdot w$. The only word of length 0 is the empty word ϵ. For any $i \in [\ell\cdot w]$, $w\cdot i$ is the

[a]These two notational conventions are particularly useful when working with treelike structures and operads, as they simplify the handling of compositions and nested applications.

i-th letter of w. For $a \in A$ and $n \in \mathbb{N}$, a^n is the word of length n such that $a^n \cdot i = a$ for all $i \in [n]$. Given two words w and w', the concatenation of w and w' is denoted by $w \centerdot w'$. In graphical representations of Hasse diagrams of posets, the order relation progresses from top to bottom.

2 Easterly wind partial orders

This first section is devoted to introducing a partial order on the set of $\mathcal{S}$-terms, which are treelike structures that realize the free nonsymmetric operad on the generating set $\mathcal{S}$. We establish several properties of this partial order.

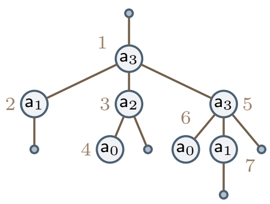
2.1 Signatures and terms

We begin with some preliminary definitions concerning particular treelike structures, called terms, which are defined from a set of allowed decorations, called signatures.

2.1.1 Signatures. A *signature* is a set $\mathcal{S}$ endowed with a map $\mathrm{ar} : \mathcal{S} \to \mathbb{N}$. For any $\mathsf{s} \in \mathcal{S}$, $\mathrm{ar} \cdot \mathsf{s}$ is the *arity* of s. For any $n \in \mathbb{N}$, let $\mathcal{S} \cdot n := \{\mathsf{s} \in \mathcal{S} : \mathrm{ar} \cdot \mathsf{s} = n\}$. For the examples that will follow, we shall consider the signature $\mathcal{S}_{\mathrm{e}} := \{\mathsf{a}_{i,j} : i, j \in \mathbb{N}\}$ where for any $\mathsf{a}_{i,j} \in \mathcal{S}_{\mathrm{e}}$, $\mathrm{ar} \cdot \mathsf{a}_{i,j} = i$. To lighten the notations, we shall write simply a_i for $\mathsf{a}_{i,0}$.

In what follows, we define several concepts "C" parameterized by a signature $\mathcal{S}$, denoted by "$\mathcal{S}$-C". To streamline the phrasing, whenever there is no ambiguity, we simply write "C".

2.1.2 Terms. Given a signature $\mathcal{S}$, an *$\mathcal{S}$-term* is either the *leaf* $\circ$ or a pair $(\mathsf{s}, (\mathsf{t}_1, \dots, \mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}))$ where $\mathsf{s} \in \mathcal{S}$, and $\mathsf{t}_1, \dots, \mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}$ are $\mathcal{S}$-terms. For brevity, we write $\mathsf{s}\, \mathsf{t}_1 \dots \mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}$ for $(\mathsf{s}, (\mathsf{t}_1, \dots, \mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}))$. By definition, an $\mathcal{S}$-term is therefore a decorated ordered rooted tree where each internal node having n children is decorated on $\mathcal{S} \cdot n$. The set of $\mathcal{S}$-terms is denoted by $\mathfrak{T} \cdot \mathcal{S}$. For instance, $\mathsf{a}_3 \underline{\mathsf{a}_1 \circ}\, \underline{\mathsf{a}_2 \mathsf{a}_0 \circ}\, \underline{\mathsf{a}_3 \mathsf{a}_0 \underline{\mathsf{a}_1 \circ} \circ}$ is an $\mathcal{S}_{\mathrm{e}}$-term and it writes as the decorated ordered rooted tree



. (2.1.2.A)

A *subterm* of an $\mathcal{S}$-term $\mathsf{t} := \mathsf{s}\, \mathsf{t}_1 \dots \mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}$ is either t itself, or recursively a subterm of t_i where $i \in [\mathrm{ar} \cdot \mathsf{s}]$. For any $\mathsf{s} \in \mathcal{S}$, the *$\mathsf{s}$-corolla* is the $\mathcal{S}$-term $\iota \cdot \mathsf{s} := \mathsf{s} \circ \dots \circ$. In other words, $\iota \cdot \mathsf{s}$ is the $\mathcal{S}$-term consisting in one single internal node decorated by s and in $\mathrm{ar} \cdot \mathsf{s}$ leaves. Let us now introduce some additional definitions about $\mathcal{S}$-terms. Below, t is an $\mathcal{S}$-term.

The *preorder traversal* of t is defined recursively as follows. If $\mathsf{t} = \circ$, then the leaf forming t is visited. Otherwise, we have $\mathsf{t} = \mathsf{s}\, \mathsf{t}_1 \mathsf{t}_2 \dots \mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}$ where $\mathsf{s} \in \mathcal{S}$ and $\mathsf{t}_1, \mathsf{t}_2, \dots$, and $\mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}$ are $\mathcal{S}$-terms. In this case, the root of t is visited first, and then, $\mathsf{t}_1, \mathsf{t}_2, \dots$, and $\mathsf{t}_{\mathrm{ar} \cdot \mathsf{s}}$ are visited from left to right according to their respective preorder traversals. This procedure induces a total order on the leaves and internal nodes of t where the first visited element is the smallest one. Note that the symbolic notation of an $\mathcal{S}$-term already lists its leaves and internal nodes in this order (see (2.1.2.A) and its symbolic notation given just before).

The *arity* $\mathrm{ar}\cdot\mathfrak{t}$ of $\mathfrak{t}$ is the number of occurrences of leaves of $\mathfrak{t}$. The leaves of $\mathfrak{t}$ are numbered consecutively, starting with 1, according to their positions in the preorder traversal of $\mathfrak{t}$. Given $i \in [\mathrm{ar}\cdot\mathfrak{t}]$, the i-th leaf of $\mathfrak{t}$ is *extreme* if all internal nodes of $\mathfrak{t}$ are visited before the i-th leaf in the preorder traversal of $\mathfrak{t}$.

The *degree* $\mathrm{dg}\cdot\mathfrak{t}$ of $\mathfrak{t}$ is the number of internal nodes of $\mathfrak{t}$. The internal nodes of $\mathfrak{t}$ are numbered consecutively, starting with 1, according to their positions in the preorder traversal of $\mathfrak{t}$. Henceforth, we identify each internal node of $\mathfrak{t}$ with the index assigned to it. When $\mathrm{dg}\cdot\mathfrak{t} \geqslant 1$, the *contraction* of $\mathfrak{t}$ is the $\mathcal{S}$-term $\partial\cdot\mathfrak{t}$ obtained by replacing the last internal node $\mathrm{dg}\cdot\mathfrak{t}$ of $\mathfrak{t}$ by a leaf. We denote by $\mathrm{N}\cdot\mathfrak{t}$ the set $[\mathrm{dg}\cdot\mathfrak{t}]$ of internal nodes of $\mathfrak{t}$. The *decoration word* of $\mathfrak{t}$ is the word $\mathrm{dc}\cdot\mathfrak{t}$ on $\mathcal{S}$ such that for any $i \in \mathrm{N}\cdot\mathfrak{t}$, $\mathrm{dc}\cdot\mathfrak{t}\cdot i$ is the decoration of i in $\mathfrak{t}$. By a slight abuse of notation, let us denote by $\mathrm{ar}\cdot\mathfrak{t}\cdot i$ the arity $\mathrm{ar}\cdot\lfloor \mathrm{dc}\cdot\mathfrak{t}\cdot i \rfloor$ of the decoration of $i \in \mathrm{N}\cdot\mathfrak{t}$.

An *edge* of $\mathfrak{t}$ is a triple (i_1, j, i_2) such that $i_1, i_2 \in \mathrm{N}\cdot\mathfrak{t}$ and i_2 is the j-th child of i_1, where the children of i_1 are numbered from left to right, starting by 1. For convenience, when $\mathrm{dg}\cdot\mathfrak{t} \geqslant 1$, we consider that $(1, 0, 1)$ is an edge of $\mathfrak{t}$. This edge can be seen as a loop on the root of $\mathfrak{t}$. We denote by $\mathrm{E}\cdot\mathfrak{t}$ the set of edges of $\mathfrak{t}$. For any internal node i_2 of $\mathfrak{t}$, there is a unique edge of $\mathfrak{t}$ of the form (i_1, j, i_2), called the *parent edge* of i_2. Under these definitions, the *parent* $\mathrm{pa}\cdot\mathfrak{t}\cdot i_2$ of i_2 is the node i_1, and the *local position* $\mathrm{lp}\cdot\mathfrak{t}\cdot i_2$ of i_2 is the integer j. With the previous convention, the internal node 1 is the parent of itself and its local position is 0.

Let us give some examples of the previous definitions. The integers near each internal node of the $\mathcal{S}_{\mathrm{e}}$-term $\mathfrak{t}$ in (2.1.2.A) are the integers with which they are identified. Moreover, we have $\mathrm{dg}\cdot\mathfrak{t} = 7$, $\mathrm{ar}\cdot\mathfrak{t} = 4$, $\mathrm{dc}\cdot\mathfrak{t} = \mathsf{a}_3\mathsf{a}_1\mathsf{a}_2\mathsf{a}_0\mathsf{a}_3\mathsf{a}_0\mathsf{a}_1$,

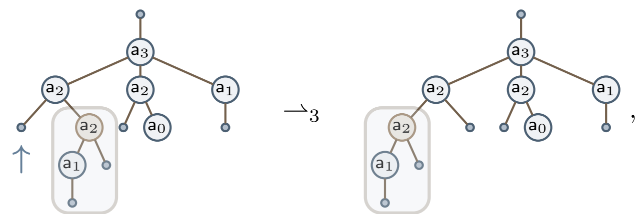
$$\mathrm{E}\cdot\mathfrak{t} = \{(1,0,1), (1,1,2), (1,2,3), (3,1,4), (1,3,5), (5,1,6), (5,2,7)\}, \tag{2.1.2.B}$$

and the extreme leaves of $\mathfrak{t}$ are the 3-rd and the 4-th ones. Besides, the contraction of $\mathfrak{t}$ is the $\mathcal{S}_{\mathrm{e}}$-term $\mathsf{a}_3\lfloor \mathsf{a}_1 \circ \rfloor\lfloor \mathsf{a}_2\mathsf{a}_0 \circ \rfloor\lfloor \mathsf{a}_3\mathsf{a}_0 \circ\circ \rfloor$.

2.2 Posets on terms

The purpose of this section is to define a partial order relation $\preccurlyeq$ on the set of $\mathcal{S}$-terms. This partial order is defined by comparing, letter by letter, certain sequences of rational numbers obtained from $\mathcal{S}$-terms, called connection words. We show that $\preccurlyeq$ admits, as its covering relation, a rewrite rule $\rightharpoonup$ on the set of $\mathcal{S}$-terms, which consists of pruning and grafting subterms in an appropriate way. We begin this section by introducing this rewrite rule.

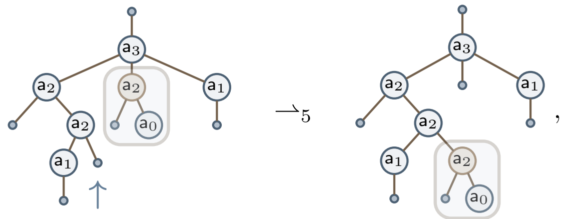
2.2.1 A rewrite rule on terms. For any $i \in \mathbb{N} \setminus \{0\}$, let $\rightharpoonup_i$ be the binary relation on $\mathfrak{T}\cdot\mathcal{S}$ defined as follows. Let $\mathfrak{t}_1$ be an $\mathcal{S}$-term of degree at least i and such that its internal node i is visited immediately after a leaf in the preorder traversal of $\mathfrak{t}_1$. Then, $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$ holds if $\mathfrak{t}_2$ is the $\mathcal{S}$-term obtained from $\mathfrak{t}_1$ by moving to this leaf the subterm rooted at i. For instance, we have



(2.2.1.A)

$$\mathsf{a}_3\,\mathsf{a}_2\circ\boldsymbol{\mathsf{a}_2\,\mathsf{a}_1\circ\circ}\;\mathsf{a}_2\circ\mathsf{a}_0\;\mathsf{a}_1\circ \quad \rightharpoonup_3 \quad \mathsf{a}_3\,\mathsf{a}_2\,\boldsymbol{\mathsf{a}_2\,\mathsf{a}_1\circ\circ}\circ\;\mathsf{a}_2\circ\mathsf{a}_0\;\mathsf{a}_1\circ$$

and



$$\mathsf{a}_3\,\mathsf{a}_2\circ\mathsf{a}_2\,\mathsf{a}_1\circ\circ\;\boldsymbol{\mathsf{a}_2\circ\mathsf{a}_0}\;\mathsf{a}_1\circ \quad \rightharpoonup_3 \quad \mathsf{a}_3\,\mathsf{a}_2\circ\mathsf{a}_2\,\mathsf{a}_1\circ\;\boldsymbol{\mathsf{a}_2\circ\mathsf{a}_0}\circ\;\mathsf{a}_1\circ. \tag{2.2.1.B}$$

In these examples, in accordance with the previous definition of $\rightharpoonup_i$, each surrounded area shows the moved subterm rooted at an internal node i, and each arrow shows the leaf which is visited immediately before i in the preorder traversal, target of the moved subterm. Observe that these transformations become particularly transparent when the terms are displayed in their symbolic notation: the subterm rooted at i, written in bold, is relocated on the leaf appearing immediately to its left. Remark that by setting $\mathfrak{t}_1$ as the $\mathcal{S}_{\mathrm{e}}$-term appearing in the left-hand sides in (2.2.1.A) and (2.2.1.B), there is no $\mathcal{S}_{\mathrm{e}}$-term $\mathfrak{t}_2$ such that $\mathfrak{t}_1 \rightharpoonup_1 \mathfrak{t}_2$ nor $\mathfrak{t}_1 \rightharpoonup_2 \mathfrak{t}_2$ because there are no leaves which are visited before the internal nodes 1 and 2 in the preorder traversal of $\mathfrak{t}_1$. Moreover, there is no $\mathcal{S}_{\mathrm{e}}$-term $\mathfrak{t}_2$ such that $\mathfrak{t}_1 \rightharpoonup_7 \mathfrak{t}_2$ because an internal node, in this case the internal node 6, is visited just before visiting the internal node 7 in the preorder traversal of $\mathfrak{t}_1$.

From now on, except at some places in Section 4.2.1, we will exclusively use the symbolic notation for $\mathcal{S}$-terms. However, readers are invited to convert these into their graphical notation if they feel more comfortable doing so.

It follows immediately from the definition of $\rightharpoonup_i$ that $\rightharpoonup_1$ is the empty relation. Another immediate property is that for any $i \in \mathbb{N} \setminus \{0\}$ and any $\mathcal{S}$-term $\mathfrak{t}_1$, there is at most one $\mathcal{S}$-term $\mathfrak{t}_2$ such that $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$. Let us also denote by $\overset{*}{\rightharpoonup}_i$ the reflexive and transitive closure of $\rightharpoonup_i$.

Given two triples $x_1 := (i_1, j_1, i)$ and $x_2 := (i_2, j_2, i)$ of integers, x_1 is *dominated* by x_2 if the pair $(i_1, -j_1)$ is lexicographically smaller than or equal to the pair $(i_2, -j_2)$ For instance, $(2, 5, 4)$ is dominated by $(3, 7, 4)$ and by $(2, 3, 4)$, but not by $(1, 1, 4)$ neither by $(2, 6, 4)$.

The following lemma is a crucial tool that is used in several subsequent proofs.

▶ **Lemma** 2.2.1.A — *Let $\mathcal{S}$ be a signature and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$ for an $i \geqslant 2$. The following properties hold:*

(i) the decoration words of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are the same;

(ii) the $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ share all edges except for the parent edge of i;

(iii) the parent edge of i in $\mathfrak{t}_1$ is dominated by the parent edge of i in $\mathfrak{t}_2$;

(iv) for any internal nodes i' and i'' of $\mathfrak{t}_1$ and $\mathfrak{t}_2$, if i'' is a descendant of i' in $\mathfrak{t}_1$, then i'' is also a descendant of i' in $\mathfrak{t}_2$.

◀ **Proof** — Let $i_1 := \mathrm{pa}\cdot\mathfrak{t}_1\cdot i$, k be the leaf which is visited immediately before i in the preorder traversal of $\mathfrak{t}_1$, and i_2 be the internal node of $\mathfrak{t}_1$ to which k is attached.

First, by definition of $\rightharpoonup_i$, since $\mathfrak{t}_2$ is obtained from $\mathfrak{t}_1$ by moving on k the subterm rooted at i, all internal nodes of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are visited in the same order. Therefore, $\mathrm{dc}\cdot\mathfrak{t}_1 = \mathrm{dc}\cdot\mathfrak{t}_2$, establishing (i).

Point (ii) is immediate since by definition of $\rightharpoonup_i$, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ differ only by the parent of their internal node i, which is i_1 and $\mathfrak{t}_1$ and i_2 in $\mathfrak{t}_2$. All other edges are identical in both $\mathfrak{t}_1$ and $\mathfrak{t}_2$.

To prove (iii), we need to distinguish two cases. First, if both k and i admits the same parent $i_1 = i_2$, then, by definition of $\rightharpoonup_i$, $(i_1, j_1 - 1, i)$ is the parent edge of i in $\mathfrak{t}_2$. Therefore, the parent edge of i in $\mathfrak{t}_1$ is dominated by the parent edge of i in $\mathfrak{t}_2$. Otherwise, due to the fact that in $\mathfrak{t}_1$, k is attached to i_2, i_1 is the parent of i, and k is visited immediately before i in the preorder traversal, it follows that i_2 is visited after i_1 in the preorder traversal of $\mathfrak{t}_1$. For this reason, $i_1 < i_2$. Since i_2 is the parent of i in $\mathfrak{t}_2$, the parent edge of i in $\mathfrak{t}_1$ is dominated by the parent edge of i in $\mathfrak{t}_2$.

Finally, by hypothesis, i is a descendant of i_1 in $\mathfrak{t}_1$, and by (ii), i is a descendant of i_2 in $\mathfrak{t}_2$. Moreover, by (iii), $i_1 \leqslant i_2$. Hence, and since $i_2 < i$, we have either that $i_1 = i_2$ or that i_2 is a descendant of i_1. Therefore, in $\mathfrak{t}_2$, i remains a descendant of i_1. This property implies that, independently from the location of two internal nodes i' and i'' of $\mathfrak{t}_1$ and $\mathfrak{t}_2$, if i'' is a descendant of i' in $\mathfrak{t}_1$, these two internal nodes enjoy the same property in $\mathfrak{t}_2$. □

To illustrate Lemma 2.2.1.A, observe that in (2.2.1.A), the edge $(2, 2, 3)$ is replaced by the dominating edge $(2, 1, 3)$, and in (2.2.1.B), the edge $(1, 2, 5)$ is replaced by the dominating edge $(3, 2, 5)$. Moreover, all $\mathcal{S}_{\mathrm{e}}$-terms of these two examples have $\mathsf{a}_3\mathsf{a}_2\mathsf{a}_2\mathsf{a}_1\mathsf{a}_2\mathsf{a}_0\mathsf{a}_1$ as decoration word.

Let us denote by $\rightharpoonup$ the binary relation on $\mathfrak{T}{\cdot}\mathcal{S}$ defined as the union of all relations $\rightharpoonup_i$ for $i \in \mathbb{N} \setminus \{0\}$. We call $\rightharpoonup$ the *$\mathcal{S}$-easterly wind rewrite rule*. Let us also denote by $\overset{*}{\rightharpoonup}$ the reflexive and transitive closure of $\rightharpoonup$.

2.2.2 Connection words. The *connection word* of an $\mathcal{S}$-term $\mathfrak{t}$ is the word $\mathrm{cnc}{\cdot}\mathfrak{t}$ on $\mathbb{Q}$ of length $\mathrm{dg}{\cdot}\mathfrak{t}$ such that for any $i_2 \in \mathrm{N}{\cdot}\mathfrak{t}$,

$$\mathrm{cnc}{\cdot}\mathfrak{t}{\cdot}i = \mathrm{pa}{\cdot}\mathfrak{t}{\cdot}i + 1 - 2^{\mathrm{lp}{\cdot}\mathfrak{t}{\cdot}i \,-\, \mathrm{ar}{\cdot}\mathfrak{t}{\cdot}\lfloor \mathrm{pa}{\cdot}\mathfrak{t}{\cdot}i \rfloor}. \tag{2.2.2.A}$$

In other words, $\mathrm{cnc}{\cdot}\mathfrak{t}{\cdot}i$ expresses in binary fixed-point notation as $\mathtt{x}.\mathtt{1}^{\ell}$, where $\mathtt{x}$ is the binary expansion of $\mathrm{pa}{\cdot}\mathfrak{t}{\cdot}i$ and ℓ is the number of internal nodes of $\mathfrak{t}$ among the siblings of i that lie to its right.

For instance, the connection words of the $\mathcal{S}_{\mathrm{e}}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of (2.2.1.B) satisfy

$$\mathrm{cnc}{\cdot}\mathfrak{t}_1 = \left(\frac{15}{8}, \frac{7}{4}, 2, \frac{7}{2}, \frac{3}{2}, 5, 1\right) \quad \text{and} \quad \mathrm{cnc}{\cdot}\mathfrak{t}_2 = \left(\frac{15}{8}, \frac{7}{4}, 2, \frac{7}{2}, \mathbf{3}, 5, 1\right). \tag{2.2.2.B}$$

Under the alternative interpretation of $\mathrm{cnc}{\cdot}\mathfrak{t}$, the binary fixed-point representation of $\mathrm{cnc}{\cdot}\mathfrak{t}_1{\cdot}2$ is $\mathtt{1}.\mathtt{1}^2 = \mathtt{1.11}$, which denotes the value $\frac{7}{4}$, and the one of $\mathrm{cnc}{\cdot}\mathfrak{t}_1{\cdot}6$ is $\mathtt{101}.\mathtt{1}^0 = \mathtt{101.0}$, which denotes the value 5 as expected.

▶ **Lemma** 2.2.2.A — *Let $\mathcal{S}$ be a signature, $\mathfrak{t}$ be an $\mathcal{S}$-term, and i be an internal node of $\mathfrak{t}$. The parent of i in $\mathfrak{t}$ is the unique integer i' such that $i' \leqslant \mathrm{cnc}{\cdot}i < i' + 1$.*

◀ **Proof** — Let (i', j, i) be the parent edge of i in $\mathfrak{t}$. From the definition (2.2.2.A) of cnc, the rational number $\mathrm{cnc}{\cdot}i$ is minimal when $j = \mathrm{ar}{\cdot}\mathfrak{t}{\cdot}i'$ and is, in this case, equal to i'. Moreover, $\mathrm{cnc}{\cdot}i$ is maximal when $j = 0$ and is, in this case, equal to $i' + 1 - 2^{-\mathrm{ar}{\cdot}\mathfrak{t}{\cdot}i}$. Since $2^{-\mathrm{ar}{\cdot}\mathfrak{t}{\cdot}i}$ is a positive number, the previous quantity is smaller than $i' + 1$. The stated property follows. □

▶ **Proposition** 2.2.2.B — *For any signature $\mathcal{S}$ and any word w on $\mathcal{S}$, the map cnc on the domain of the $\mathcal{S}$-terms having w as decoration word, is injective.*

◀ **Proof** — Let c be a word of rational numbers belonging to the image of the map cnc on the domain of $\mathcal{S}$-terms having w as decoration word. Let us show that there exists a unique antecedent $\mathfrak{t}$ of c by cnc. By denoting by n the length of w, by Lemma 2.2.2.A, for any $i \in [n]$, there is a unique $i' \in [n]$ such that $i' \leqslant c \cdot i < i' + 1$. Therefore, the parent of the internal node i of $\mathfrak{t}$ is i'. Moreover, from (2.2.2.A), we have $\mathrm{lp} \cdot \mathfrak{t} \cdot i = \mathrm{ar} \cdot \llcorner w \cdot i' \lrcorner + \log_2 \cdot \llcorner i' + 1 - c \cdot i \lrcorner$. This shows that the parent edge of i in $\mathfrak{t}$ is entirely specified by c. Since an $\mathcal{S}$-term is entirely specified by its decoration word and its set of edges, this shows the unicity of $\mathfrak{t}$ and entails the statement of the proposition. □

2.2.3 Easterly wind posets. Let $\preccurlyeq$ be the binary relation on $\mathfrak{T} \cdot \mathcal{S}$ such that, for any $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$, we have $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ if $\mathrm{dc} \cdot \mathfrak{t}_1 = \mathrm{dc} \cdot \mathfrak{t}_2$ and, for all $i \in \mathrm{N} \cdot \mathfrak{t}_1$, $\mathrm{cnc} \cdot \mathfrak{t}_1 \cdot i \leqslant \mathrm{cnc} \cdot \mathfrak{t}_2 \cdot i$. Immediately from its definition, $\preccurlyeq$ is a partial order relation on $\mathfrak{T} \cdot \mathcal{S}$. Let us call $(\mathfrak{T} \cdot \mathcal{S}, \preccurlyeq)$ the *$\mathcal{S}$-easterly wind poset*.

Let us now state a series of lemmas that will be used to establish that $\rightharpoonup$ is the covering relation of the $\mathcal{S}$-easterly wind poset.

▶ **Lemma** 2.2.3.A — *Let $\mathcal{S}$ be a signature and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms of the same degree n and the same decoration word. We have $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ if and only if for any $i \in [n]$, the parent edge of i in $\mathfrak{t}_1$ is dominated by the parent edge of i in $\mathfrak{t}_2$.*

◀ **Proof** — Let (i_1, j_1, i) (resp. (i_2, j_2, i)) be the parent edge of i in $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$). By definition of $\preccurlyeq$, the property $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ is equivalent to the fact that for any $i \in [n]$, $\mathrm{cnc} \cdot \mathfrak{t}_1 \cdot i \leqslant \mathrm{cnc} \cdot \mathfrak{t}_2 \cdot i$. By (2.2.2.A) and Lemma 2.2.2.A, this is equivalent to the fact that $i_2 > i_1$ or both $i_1 = i_2$ and $j_2 \leqslant j_1$. This says exactly that (i_1, j_1, i) is dominated by (i_2, j_2, i). The statement of the lemma follows. □

▶ **Lemma** 2.2.3.B — *For any signature $\mathcal{S}$ and any $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$, $\mathfrak{t}_1 \overset{*}{\rightharpoonup} \mathfrak{t}_2$ implies that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$.*

◀ **Proof** — Assume that $\mathfrak{t}_1 \rightharpoonup \mathfrak{t}_2$. By Lemma 2.2.1.A, $\mathrm{E} \cdot \mathfrak{t}_2$ is obtained from $\mathrm{E} \cdot \mathfrak{t}_1$ by replacing an edge (i_1, j_1, i) by a dominating edge (i_2, j_2, i). Therefore, by Lemma 2.2.3.A, $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. Finally, the statement of the lemma follows from the fact that $\preccurlyeq$ is transitive. □

▶ **Lemma** 2.2.3.C — *For any signature $\mathcal{S}$ and any $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of degree 1 or more, if $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ then $\partial \cdot \mathfrak{t}_1 \preccurlyeq \partial \cdot \mathfrak{t}_2$.*

◀ **Proof** — Assume that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two $\mathcal{S}$-terms of the same degree $n \geqslant 1$ and that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. By definition of $\preccurlyeq$, for all $i \in [n]$, $\mathrm{cnc} \cdot \mathfrak{t}_1 \cdot i \leqslant \mathrm{cnc} \cdot \mathfrak{t}_2 \cdot i$. Observe that for any $\mathcal{S}$-term $\mathfrak{t}$ of degree n, $\mathrm{E} \cdot \mathfrak{t} = \mathrm{E} \cdot \llcorner \partial \cdot \mathfrak{t} \lrcorner \cup \{(\mathrm{pa} \cdot \mathfrak{t} \cdot n, \mathrm{lp} \cdot \mathfrak{t} \cdot n, n)\}$. Therefore, this implies that for all $i \in [n-1]$, $\mathrm{cnc} \cdot \llcorner \partial \cdot \mathfrak{t}_1 \lrcorner \cdot i = \mathrm{cnc} \cdot \mathfrak{t}_1 \cdot i \leqslant \mathrm{cnc} \cdot \mathfrak{t}_2 \cdot i = \mathrm{cnc} \cdot \llcorner \partial \cdot \mathfrak{t}_2 \lrcorner \cdot i$. Hence, we have $\partial \cdot \mathfrak{t}_1 \preccurlyeq \partial \cdot \mathfrak{t}_2$ as expected. □

▶ **Lemma** 2.2.3.D — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms of the same degree $n \geqslant 1$. If there exists $i \in [n-1]$ such that $\partial \cdot \mathfrak{t}_1 \rightharpoonup_i \partial \cdot \mathfrak{t}_2$, $\mathrm{dc} \cdot \mathfrak{t}_1 \cdot n = \mathrm{dc} \cdot \mathfrak{t}_2 \cdot n$, $\mathrm{pa} \cdot \mathfrak{t}_1 \cdot n = \mathrm{pa} \cdot \mathfrak{t}_2 \cdot n$, and $\mathrm{lp} \cdot \mathfrak{t}_1 \cdot n = \mathrm{lp} \cdot \mathfrak{t}_2 \cdot n$, then $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$.*

◀ **Proof** — The $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are both obtained by adding respectively to $\partial \cdot \mathfrak{t}_1$ and $\partial \cdot \mathfrak{t}_2$ an internal node n decorated by the same element of $\mathcal{S}$ and through the same edge. Since $\partial \cdot \mathfrak{t}_2$ can be obtained from $\partial \cdot \mathfrak{t}_1$ by changing a single edge involving internal nodes smaller than n, as prescribed by the definition of $\rightharpoonup_i$, it is possible to obtain $\mathfrak{t}_2$ from $\mathfrak{t}_1$ by the same changing of edge. Therefore, $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$. □

▶ **Lemma** 2.2.3.E — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. There exists a sequence $\left(\mathfrak{t}^{(0)}, \mathfrak{t}^{(1)} \dots, \mathfrak{t}^{(n)}\right)$ of $\mathcal{S}$-terms such that*

$$\mathfrak{t}_1 = \mathfrak{t}^{(0)} \xrightharpoonup{*}_1 \mathfrak{t}^{(1)} \xrightharpoonup{*}_2 \cdots \xrightharpoonup{*}_n \mathfrak{t}^{(n)} = \mathfrak{t}_2. \tag{2.2.3.A}$$

◀ **Proof** — Let us proceed by induction on the common degree n of $\mathfrak{t}_1$ and $\mathfrak{t}_2$. If $n = 0$ or $n = 1$, then $\mathfrak{t}_1 = \mathfrak{t}_2$ and the property holds. Otherwise, we have $n \geqslant 2$ and, since $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$, by Lemma 2.2.3.C, we have $\partial \cdot \mathfrak{t}_1 \preccurlyeq \partial \cdot \mathfrak{t}_2$. By induction hypothesis, there exists a sequence $\left(\mathfrak{s}^{(0)}, \mathfrak{s}^{(1)} \dots, \mathfrak{s}^{(n-1)}\right)$ of $\mathcal{S}$-terms such that

$$\partial \cdot \mathfrak{t}_1 = \mathfrak{s}^{(0)} \xrightharpoonup{*}_1 \mathfrak{s}^{(1)} \xrightharpoonup{*}_2 \cdots \xrightharpoonup{*}_n \mathfrak{s}^{(n-1)} = \partial \cdot \mathfrak{t}_2. \tag{2.2.3.B}$$

Let, for any $i \in [\![n-1]\!]$, $\mathfrak{r}^{(k)}$ be the $\mathcal{S}$-term obtained by adding to $\mathfrak{s}^{(k)}$ an internal node n decorated by $\mathrm{dc} \cdot \mathfrak{t}_1 \cdot n$ through the edge $(\mathrm{pa} \cdot \mathfrak{t}_1 \cdot n, \mathrm{lp} \cdot \mathfrak{t}_1 \cdot n, n)$. By Lemma 2.2.3.D,

$$\mathfrak{t}_1 = \mathfrak{r}^{(0)} \xrightharpoonup{*}_1 \mathfrak{r}^{(1)} \xrightharpoonup{*}_2 \cdots \xrightharpoonup{*}_n \mathfrak{r}^{(n-1)}. \tag{2.2.3.C}$$

Now, since $\mathrm{cnc} \cdot \mathfrak{t}_1 \cdot n = \mathrm{cnc} \cdot \mathfrak{r}^{(n-1)} \cdot n \leqslant \mathrm{cnc} \cdot \mathfrak{t}_2 \cdot n$, by Lemma 2.2.3.A, $\mathfrak{t}_2$ is obtained from $\mathfrak{r}^{(n-1)}$ by replacing the parent edge of j in $\mathfrak{r}^{(n-1)}$ by an edge dominating it. By definition of $\rightharpoonup$ and Lemma 2.2.1.A, the parent edge of n in $\mathfrak{t}_2$ can be formed from the parent edge of n in $\mathfrak{r}^{(n-1)}$ by performing a sequence of applications of the $\mathcal{S}$-easterly wind rewrite rule $\rightharpoonup$ from $\mathfrak{r}^{(n-1)}$. Indeed, this consists in iteratively moving the node n of $\mathfrak{r}^{(n-1)}$ as specified by the binary relation $\rightharpoonup_n$. Therefore, we have $\mathfrak{r}^{(n-1)} \xrightharpoonup{*}_n \mathfrak{t}_2$, establishing the expected property. □

▶ **Theorem** 2.2.3.F — *For any signature $\mathcal{S}$, the binary relation $\rightharpoonup$ is the covering relation of the $\mathcal{S}$-easterly wind poset.*

◀ **Proof** — By Lemmas 2.2.3.B and 2.2.3.E, the binary relations $\preccurlyeq$ and $\xrightharpoonup{*}$ are the same. Besides, by Lemma 2.2.1.A, if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two $\mathcal{S}$-terms satisfying $\mathfrak{t}_1 \rightharpoonup \mathfrak{t}_2$, then $\mathfrak{t}_1$ and $\mathfrak{t}_2$ differ by the parent edge of a certain internal node i. Therefore, we have $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$. Now, by contradiction, assume that there is an $\mathcal{S}$-term $\mathfrak{t}_3$ such that $\mathfrak{t}_3 \neq \mathfrak{t}_2$ and $\mathfrak{t}_1 \rightharpoonup \mathfrak{t}_3 \xrightharpoonup{*} \mathfrak{t}_2$. Recall that as noticed in Section 2.2.1, $\mathfrak{t}_2$ is the unique $\mathcal{S}$-term such that $\mathfrak{t}_1 \rightharpoonup_i \mathfrak{t}_2$. Therefore, we have $\mathfrak{t}_1 \rightharpoonup_{i'} \mathfrak{t}_3$ with $i' \neq i$. The fact that $\mathfrak{t}_1 \rightharpoonup_{i'} \mathfrak{t}_3 \xrightharpoonup{*} \mathfrak{t}_2$ implies that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ differ by the parent edge of i'. This yields a contradiction with our hypotheses. This shows that $\rightharpoonup$ is the covering relation of the poset $(\mathfrak{T} \cdot \mathcal{S}, \preccurlyeq)$. □

Theorem 2.2.3.F justifies the given name for $(\mathfrak{T} \cdot \mathcal{S}, \preccurlyeq)$: this name of $\mathcal{S}$-"*easterly wind*" poset is derived from the observation that the covering relation of this poset involves detaching a subterm from the east and then attaching it to the west, as if an easterly breeze is blowing on the tree.

For any $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S}$, let $\downarrow \cdot \mathfrak{t} := \{\mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} : \mathfrak{t} \preccurlyeq \mathfrak{t}'\}$. We call $(\downarrow \cdot \mathfrak{t}, \preccurlyeq)$ the *$\mathcal{S}$-easterly wind poset of* $\mathfrak{t}$. Figure 1 shows the Hasse diagram of the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of an $\mathcal{S}_{\mathrm{e}}$-term.

2.3 Posets on tilted terms

In this section, we consider an idempotent map $\mathrm{tlt} \cdot \mathcal{X}$ on the set of $\mathcal{S}$-terms. It turns out that this map is a closure operator on the $\mathcal{S}$-easterly wind poset, so that the set of elements closed w.r.t. $\mathrm{tlt} \cdot \mathcal{X}$ forms a subposet of the $\mathcal{S}$-easterly wind poset. This construction will be useful in the final section of this paper, as such posets are used to construct bases of natural Hopf algebras of free operads.

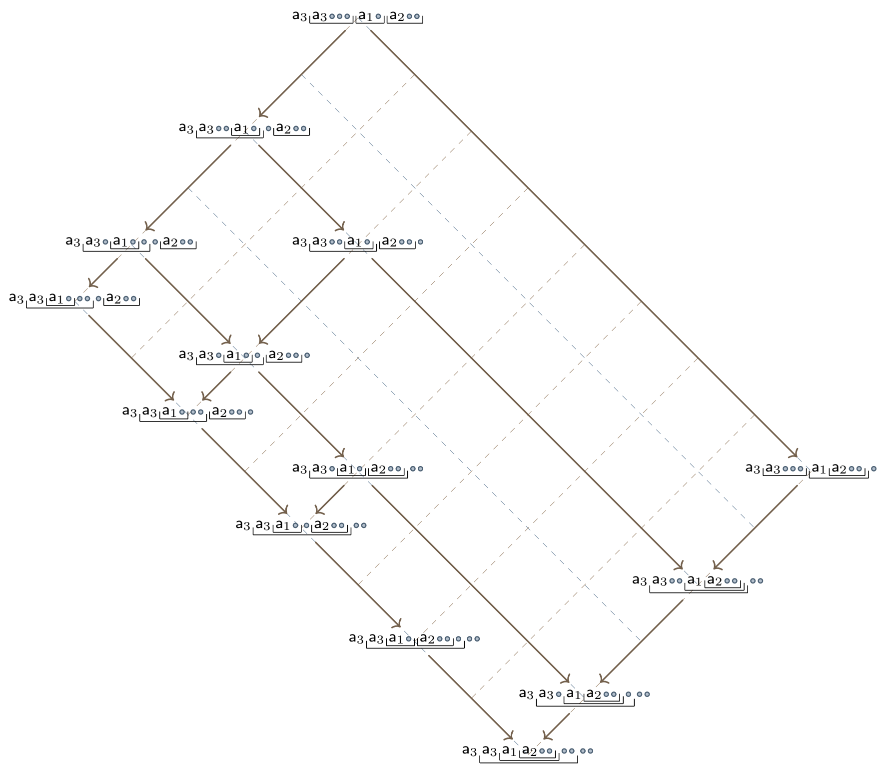


Figure 1: The Hasse diagram of the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathsf{a}_3 \mathsf{a}_3 \circ\circ\circ \, \mathsf{a}_1 \circ \, \mathsf{a}_2 \circ\circ$.

2.3.1 Tilting map. The *$\mathcal{X}$-tilting map* is the map $\mathrm{tlt}\cdot\mathcal{X} : \mathfrak{T}\cdot\mathcal{S} \to \mathfrak{T}\cdot\mathcal{S}$ defined as follows. For any $\mathfrak{t} \in \mathfrak{T}\cdot\mathcal{S}$, the $\mathcal{S}$-term $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$ is obtained from $\mathfrak{t}$ by rearranging the children of every internal node i with $i \in \mathcal{X}$ so that all non-leaf children preserve their original order and precede the children that are leaves. Let us also define the *$\mathcal{X}$-reversed tilting map* as the map $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}$ in the exact same manner as the $\mathcal{X}$-tilting map but with the difference that the children different from the leaf appear after the children which are leaves. For instance, we have

$$\mathrm{tlt}\cdot\{1,2\}\cdot \mathsf{a}_3 \, \mathsf{a}_2 \circ \mathsf{a}_2 \circ \mathsf{a}_1 \circ \circ \mathsf{a}_0 \quad = \quad \mathsf{a}_3 \, \mathsf{a}_2 \, \mathsf{a}_2 \circ \mathsf{a}_1 \circ \circ \mathsf{a}_0 \circ \tag{2.3.1.A}$$

and

$$\mathrm{tlt}^{\mathrm{r}}\cdot\{1,2\}\cdot \mathsf{a}_3 \, \mathsf{a}_2 \circ \mathsf{a}_2 \circ \mathsf{a}_1 \circ \circ \mathsf{a}_0 \quad = \quad \mathsf{a}_3 \circ \mathsf{a}_2 \circ \mathsf{a}_2 \circ \mathsf{a}_1 \circ \, \mathsf{a}_0. \tag{2.3.1.B}$$

Given an $\mathcal{S}$-term $\mathfrak{t}$ and an internal node i of $\mathfrak{t}$, let $\mathrm{lb}\cdot\mathfrak{t}\cdot i$ be the number of internal nodes of $\mathfrak{t}$ among the siblings of i that lie to its left, including i itself. For instance, by setting $\mathfrak{t}$ as the $\mathcal{S}_{\mathrm{e}}$-term appearing in the left-hand side of (2.3.1.A), we have $\mathrm{lb}\cdot\mathfrak{t}\cdot 1 = 0$, $\mathrm{lb}\cdot\mathfrak{t}\cdot 2 = 1$, and $\mathrm{lb}\cdot\mathfrak{t}\cdot 5 = 2$. Observe, of course, that if (i_1, j, i) is an edge of $\mathfrak{t}$, then $\mathrm{lb}\cdot\mathfrak{t}\cdot i \leqslant j$.

The following lemma provides a formalization of the effect of the $\mathcal{X}$-tilting map on an $\mathcal{S}$-term.

▶ **Lemma** 2.3.1.A — *Let $\mathcal{S}$ be a signature, $\mathcal{X}$ be a set of positive integers, $\mathfrak{t}$ be an $\mathcal{S}$-term and (i', j, i) be an edge of $\mathfrak{t}$. The following properties hold:*

(i) if $i' \notin \mathcal{X}$, then (i', j, i) is an edge of $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$;

(ii) if $i' \in \mathcal{X}$, then $(i', \mathrm{lb}\cdot\mathfrak{t}\cdot i, i)$ is an edge of $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$.

◀ **Proof** — By definition of the map $\mathrm{tlt}\cdot\mathcal{X}$, if $i' \notin \mathcal{X}$, then the children of i' in $\mathfrak{t}$ and in $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$ are arranged in the same way. This implies (i). Besides, when $i' \in \mathcal{X}$, in order to obtain $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$, the children of i' in $\mathfrak{t}$ are pushed to the left taking the place of potential leaves. Since the children of i' which are not leaves remain in the same relative order, each internal node i of $\mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}$ which is a children of i' appears at the $\mathrm{lb} \cdot \mathfrak{t} \cdot i$-th position. This implies (ii). □

2.3.2 A CLOSURE OPERATOR. A map $\phi : \mathcal{P} \to \mathcal{P}$ is a *closure operator* (see [DP02]) of a poset $(\mathcal{P}, \preccurlyeq_{\mathcal{P}})$ if the following three properties hold:

(C1) for any $x \in \mathcal{P}$, $x \preccurlyeq_{\mathcal{P}} \phi \cdot x$;

(C2) for any $x, x' \in \mathcal{P}$, $x \preccurlyeq_{\mathcal{P}} x'$ implies $\phi \cdot x \preccurlyeq_{\mathcal{P}} \phi \cdot x'$;

(C3) for any $x \in \mathcal{P}$, $\phi \cdot \llcorner \phi \cdot x \lrcorner = \phi \cdot x$.

Condition (C1) says that ϕ is extensive, Condition (C2) says that ϕ is order-preserving, and Condition (C3) says that ϕ is idempotent.

▶ **Lemma** 2.3.2.A — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. If i is an internal node of both $\mathfrak{t}_1$ and $\mathfrak{t}_2$ admitting the same parent in $\mathfrak{t}_1$ and $\mathfrak{t}_2$, then $\mathrm{lb} \cdot \mathfrak{t}_2 \cdot i \leqslant \mathrm{lb} \cdot \mathfrak{t}_1 \cdot i$.*

◀ **Proof** — Assume that $\mathfrak{t}_1 \rightharpoonup_{i'} \mathfrak{t}_2$ where $i' \in \mathrm{N} \cdot \mathfrak{t}_1$. It is immediate, by definition of $\rightharpoonup_{i'}$ that if i' is not a left brother of i in $\mathfrak{t}_1$, then $\mathrm{lb} \cdot \mathfrak{t}_1 \cdot i = \mathrm{lb} \cdot \mathfrak{t}_2 \cdot i$. Assume now that i' is a left brother of i in $\mathfrak{t}_1$ (including the case $i' = i$). Again by definition of $\rightharpoonup_{i'}$, if the leaf which is visited immediately before i' in the preorder traversal of $\mathfrak{t}_1$ is a child of $\mathrm{pa} \cdot \mathfrak{t}_1 \cdot i$, then $\mathrm{lb} \cdot \mathfrak{t}_2 \cdot i = \mathrm{lb} \cdot \mathfrak{t}_1 \cdot i$. Otherwise, $\mathrm{lb} \cdot \mathfrak{t}_2 \cdot i = \mathrm{lb} \cdot \mathfrak{t}_1 \cdot i - 1$. The facts that, by Theorem 2.2.3.F, $\rightharpoonup$ is the covering relation of the partial order relation $\preccurlyeq$, and that $\rightharpoonup$ is the union of all $\rightharpoonup_{i'}$ with $i' \geqslant 1$, entail the statement of the lemma. □

▶ **Theorem** 2.3.2.B — *For any signature $\mathcal{S}$ and any set $\mathcal{X}$ of positive integers, the map $\mathrm{tlt} \cdot \mathcal{X}$ is a closure operator of the $\mathcal{S}$-easterly wind poset.*

◀ **Proof** — Let $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S}$. By Lemma 2.3.1.A, for any edge (i', j_1, i) of $\mathfrak{t}$, there is an edge (i', j_2, i) of $\mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}$ such that $j_2 \leqslant j_1$. Hence, the former edge is dominated by the latter. Therefore, by Lemma 2.2.3.A, $\mathfrak{t} \preccurlyeq \mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}$, showing that $\mathrm{tlt} \cdot \mathcal{X}$ satisfies (C1).

Let $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{T} \cdot \mathcal{S}$ such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. Assume that (i_1, j_1, i) is an edge of $\mathfrak{t}_1$. By Lemma 2.2.3.A, there is an edge (i_2, j_2, i) of $\mathfrak{t}_2$ such that the former edge is dominated by the latter. Moreover, by Lemma 2.3.1.A, $\mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}_1$ has an edge (i_1, j'_1, i) with $j'_1 \in \{j_1, \mathrm{lb} \cdot \mathfrak{t} \cdot i\}$. Again by Lemma 2.3.1.A, we have also that (i_2, j'_2, i) is an edge of $\mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}_2$ with $j'_2 \in \{j_2, \mathrm{lb} \cdot \mathfrak{t}_2 \cdot i\}$. Now, we have two cases depending on how (i_1, j_1, i) is dominated by (i_2, j_2, i).

1. If $i_1 < i_2$, then (i_1, j'_1, i) is dominated by (i_2, j'_2, i).
2. Otherwise, we have necessarily that $i_1 = i_2$ and $j'_2 \leqslant j'_1$. Now, by Lemma 2.3.1.A, if $i_1 \notin \mathcal{X}$, then both $j'_1 = j_1$ and $j'_2 = j_2$ hold. Otherwise, when $i_1 \in \mathcal{X}$, we have $j'_1 = \mathrm{lb} \cdot \mathfrak{t}_1 \cdot i$ and $j'_2 = \mathrm{lb} \cdot \mathfrak{t}_2 \cdot i$. By Lemma 2.3.2.A, we have in particular that $\mathrm{lb} \cdot \mathfrak{t}_2 \cdot i \leqslant \mathrm{lb} \cdot \mathfrak{t}_1 \cdot i$. It follows that in both sub-cases, (i_1, j'_1, i) is dominated by (i_1, j'_2, i).

From all this, it follows that (i_1, j'_1, i) is dominated by (i_2, j'_2, i). Therefore, by Lemma 2.2.3.A, this implies that $\mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}_1 \preccurlyeq \mathrm{tlt} \cdot \mathcal{X} \cdot \mathfrak{t}_2$ and shows that $\mathrm{tlt} \cdot \mathcal{X}$ satisfies (C2).

Finally, the map $\mathrm{tlt} \cdot \mathcal{X}$ is, immediately from its definition, idempotent. Therefore, (C3) holds. □

Observe, contrary to the property highlighted by Theorem 2.3.2.B for the map $\mathrm{tlt} \cdot \mathcal{X}$, the map $\mathrm{tlt}^{\mathrm{r}} \cdot \mathcal{X}$ is not a closure operator of the $\mathcal{S}$-easterly wind poset. Indeed, this map is not extensive. A counterexample involves the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset, the set $\mathcal{X} := \{1\}$, and the $\mathcal{S}_{\mathrm{e}}$-term $\mathfrak{t} := \mathsf{a}_2\lfloor\mathsf{a}_1\circ\rfloor\circ$ because we have $\mathfrak{t} \not\preccurlyeq \mathsf{a}_2\circ\lfloor\mathsf{a}_1\circ\rfloor = \mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}$. Moreover, the $\mathcal{X}$-reversed tilting map is not either order-preserving. A counterexample involves the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset, the set $\mathcal{X} := [4]$, and the $\mathcal{S}_{\mathrm{e}}$-terms $\mathfrak{t}_1 := \mathsf{a}_3\lfloor\mathsf{a}_2\lfloor\mathsf{a}_1\circ\rfloor\mathsf{a}_0\rfloor\circ\circ$ and $\mathfrak{t}_2 := \mathsf{a}_3\lfloor\mathsf{a}_2\lfloor\mathsf{a}_1\mathsf{a}_0\rfloor\circ\rfloor\circ\circ$ because we have $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ but $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}_1 = \mathsf{a}_3\circ\circ\lfloor\mathsf{a}_2\lfloor\mathsf{a}_1\circ\rfloor\mathsf{a}_0\rfloor \not\preccurlyeq \mathsf{a}_3\circ\circ\lfloor\mathsf{a}_2\circ\lfloor\mathsf{a}_1\mathsf{a}_0\rfloor\rfloor = \mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}_2$.

2.3.3 Kernel of the tilting map and intervals. Let us denote by $\equiv_{\mathrm{tlt}\cdot\mathcal{X}}$ the kernel of $\mathrm{tlt}\cdot\mathcal{X}$, that is, the equivalence relation on $\mathfrak{T}\cdot\mathcal{S}$ such that for any $\mathcal{S}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \equiv_{\mathrm{tlt}\cdot\mathcal{X}} \mathfrak{t}'$ holds whenever $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t} = \mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}'$. For instance,

$$\mathsf{a}_4\lfloor\mathsf{a}_2\circ\lfloor\mathsf{a}_1\circ\rfloor\rfloor\circ\circ\lfloor\mathsf{a}_2\circ\lfloor\mathsf{a}_2\circ\circ\rfloor\rfloor \equiv_{\mathrm{tlt}\cdot\{1,4\}} \mathsf{a}_4\circ\lfloor\mathsf{a}_2\circ\lfloor\mathsf{a}_1\circ\rfloor\rfloor\circ\lfloor\mathsf{a}_2\lfloor\mathsf{a}_2\circ\circ\rfloor\circ\rfloor. \tag{2.3.3.A}$$

Observe that $\mathfrak{t} \equiv_{\mathrm{tlt}\cdot\mathcal{X}} \mathfrak{t}'$ if and only if $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t} = \mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}'$.

▶ **Proposition** 2.3.3.A — *Let $\mathcal{S}$ be a signature and $\mathcal{X}$ be a set of positive integers. The $\equiv_{\mathrm{tlt}\cdot\mathcal{X}}$-equivalence class of an $\mathcal{S}$-term $\mathfrak{t}$ is an interval of the $\mathcal{S}$-easterly wind poset. More specifically,*

$$[\mathfrak{t}]_{\equiv_{\mathrm{tlt}\cdot\mathcal{X}}} = [\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}, \mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}]. \tag{2.3.3.B}$$

◀ **Proof** — By Theorem 2.3.2.B, $\mathfrak{t} \preccurlyeq \mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$. Similarly, by very analogous arguments as the one used in the proof of this property, we have $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t} \preccurlyeq \mathfrak{t}$. Now, let $\mathfrak{t}' \in [\mathfrak{t}]_{\equiv_{\mathrm{tlt}\cdot\mathcal{X}}}$. Since $\mathfrak{t}' \equiv_{\mathrm{tlt}\cdot\mathcal{X}} \mathfrak{t}$, we have $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}' = \mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$ and $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}' = \mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}$. Therefore, from the above property, we have $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t} \preccurlyeq \mathfrak{t}' \preccurlyeq \mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$.

Assume now that $\mathfrak{t}' \in [\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}, \mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}]$ and let (i, j, i') be an edge of $\mathfrak{t}'$. If $i \in \mathcal{X}$, then by Lemma 2.2.3.A, (i, j, i') is an edge of both $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}$ and $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$. Otherwise, when $i \notin \mathcal{X}$, again by Lemma 2.2.3.A, (i, j', i') is an edge of $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}\cdot\mathfrak{t}$ and (i, j'', i') is an edge of $\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}$ with $j'' \leqslant j \leqslant j'$. Therefore, by definition of $\equiv_{\mathrm{tlt}\cdot\mathcal{X}}$, we have $\mathfrak{t}' \equiv_{\mathrm{tlt}\cdot\mathcal{X}} \mathfrak{t}$ so that $\mathfrak{t}' \in [\mathfrak{t}]_{\equiv_{\mathrm{tlt}\cdot\mathcal{X}}}$. □

2.3.4 Closed elements. An element x of a poset $\mathcal{P}$ is *closed* w.r.t. a closure operator ϕ of $\mathcal{P}$ if x is a fixed point of ϕ. In this way, since by Theorem 2.3.2.B, for any set $\mathcal{X}$ of positive integers, $\mathrm{tlt}\cdot\mathcal{X}$ is a closure operator of $(\mathfrak{T}\cdot\mathcal{S}, \preccurlyeq)$, closed elements w.r.t. $\mathrm{tlt}\cdot\mathcal{X}$ are well-defined and are called *$\mathcal{X}$-tilted*. For instance, the $\mathcal{S}_{\mathrm{e}}$-term $\mathsf{a}_3\lfloor\mathsf{a}_2\circ\lfloor\mathsf{a}_2\lfloor\mathsf{a}_1\circ\rfloor\circ\rfloor\rfloor\mathsf{a}_0\lfloor\mathsf{a}_2\circ\circ\rfloor$ is $\{1, 3, 6\}$-tilted but is not $\{2\}$-tilted.

For any set $\mathcal{X}$ of positive integers, $\preccurlyeq$ is a partial order on $\mathrm{tlt}\cdot\mathcal{X}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$. Let us call $(\mathrm{tlt}\cdot\mathcal{X}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor, \preccurlyeq)$ the *$\mathcal{X}$-tilted $\mathcal{S}$-easterly wind poset*. Observe that for any sets $\mathcal{X}_1$ and $\mathcal{X}_2$ of positive integers, if $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then $(\mathrm{tlt}\cdot\mathcal{X}_2\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor, \preccurlyeq)$ is a subposet of $(\mathrm{tlt}\cdot\mathcal{X}_1\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor, \preccurlyeq)$. Of course, $(\mathrm{tlt}\cdot\emptyset\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor, \preccurlyeq)$ is the $\mathcal{S}$-easterly wind poset introduced in Section 2.2.3. Moreover, for any $\mathfrak{t} \in \mathfrak{T}\cdot\mathcal{S}$, let $\downarrow\cdot\mathcal{X}\cdot\mathfrak{t} := \lfloor\downarrow\cdot\mathfrak{t}\rfloor \cap \lfloor\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}\rfloor$. We call $(\downarrow\cdot\mathcal{X}\cdot\mathfrak{t}, \preccurlyeq)$ the *$\mathcal{X}$-tilted $\mathcal{S}$-easterly wind poset of* $\mathfrak{t}$. Figure 2 shows the Hasse diagram of the $\mathcal{X}$-tilted $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of a term.

2.3.5 Scope sequences and fully tilted terms. The *scope sequence* of an $\mathcal{S}$-term $\mathfrak{t}$ is the word $\mathrm{sc}\cdot\mathfrak{t}$ on $\mathbb{N}$ of length $\mathrm{dg}\cdot\mathfrak{t}$ such that for any $i \in \mathrm{N}\cdot\mathfrak{t}$, $\mathrm{sc}\cdot\mathfrak{t}\cdot i$ is the number descendants of i in $\mathfrak{t}$. For instance,

$$\mathsf{a}_4\mathsf{a}_0\lfloor\mathsf{a}_2\lfloor\mathsf{a}_1\circ\rfloor\lfloor\mathsf{a}_2\circ\circ\rfloor\rfloor\mathsf{a}_1\lfloor\mathsf{a}_2\circ\circ\rfloor = 6020010. \tag{2.3.5.A}$$

▶ **Lemma** 2.3.5.A — *Let $\mathcal{S}$ be a signature and $\mathfrak{t}$ be an $\mathcal{S}$-term. For any internal node i of $\mathfrak{t}$ different from the root, the parent of i in $\mathfrak{t}$ is the greatest internal node i' of $\mathfrak{t}$ such that $i' < i$ and*

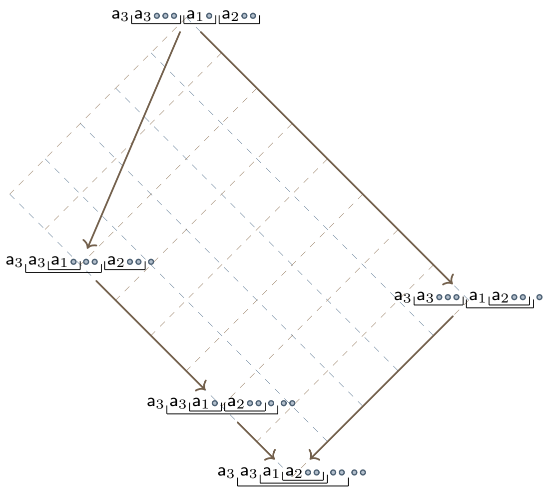


Figure 2: The Hasse diagram of the $\{1,2\}$-tilted $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathsf{a}_3\mathsf{a}_3\circ\circ\circ\mathsf{a}_1\circ\mathsf{a}_2\circ\circ$.

$i' + \mathrm{sc}{\cdot}\mathfrak{t}{\cdot}i' \geqslant i$.

◀ **Proof** — By definition of the word $\mathrm{sc}{\cdot}\mathfrak{t}$, in $\mathfrak{t}$, an internal node i is a descendant of an internal node i' if and only if $i' + 1 \leqslant i \leqslant i' + \mathrm{sc}{\cdot}\mathfrak{t}{\cdot}i'$. Therefore, each ancestor i' of i in $\mathfrak{t}$ satisfies $i' < i$ and $i' + \mathrm{sc}{\cdot}\mathfrak{t}{\cdot}i' \geqslant i$. The parent of i in $\mathfrak{t}$ is the greatest internal node among the ancestors of i. The statement of the lemma follows. □

An $\mathcal{S}$-term $\mathfrak{t}$ is *fully tilted* if $\mathfrak{t}$ is $\mathbb{N} \setminus \{0\}$-tilted.

▶ **Proposition** 2.3.5.B — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two fully tilted $\mathcal{S}$-terms of the same degree n. We have $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ if and only if, for any $i \in [n]$, $\mathrm{sc}{\cdot}\mathfrak{t}_1{\cdot}i \leqslant \mathrm{sc}{\cdot}\mathfrak{t}_2{\cdot}i$.*

◀ **Proof** — Assume that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two fully tilted $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. By Lemmas 2.2.1.A and 2.2.3.E, for any internal node i of both $\mathfrak{t}_1$ and $\mathfrak{t}_2$, the number of descendants of i in $\mathfrak{t}_1$ is smaller than or equal to the number of descendants of i in $\mathfrak{t}_2$. This implies that $\mathrm{sc}{\cdot}\mathfrak{t}_1{\cdot}i \leqslant \mathrm{sc}{\cdot}\mathfrak{t}_2{\cdot}i$.

Conversely, assume that for any $i \in [n]$, $\mathrm{sc}{\cdot}\mathfrak{t}_1{\cdot}i \leqslant \mathrm{sc}{\cdot}\mathfrak{t}_2{\cdot}i$. Let $i \in [n]$ and i_1 (resp. i_2) be the parent of i in $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$). Lemma 2.3.5.A together with the fact that each letter of $\mathrm{sc}{\cdot}\mathfrak{t}_2$ is greater than or equal to the letter at the same position of $\mathrm{sc}{\cdot}\mathfrak{t}_1$ imply that $i_2 \geqslant i_1$. Hence, the parent edge of i in $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) is (i_1, j_1, i) (resp. (i_2, j_2, i)) for some integer j_1 (resp. j_2). Since $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) is fully tilted, $j_1 = \mathrm{lb}{\cdot}\mathfrak{t}_1{\cdot}i$ (resp. $j_2 = \mathrm{lb}{\cdot}\mathfrak{t}_2{\cdot}i$). This implies that $i_1 < i_2$, or both $i_1 = i_2$ and $j_1 \geqslant j_2$, so that the edge (i_1, j_1, i) is dominated by the edge (i_2, j_2, i). By Lemma 2.2.3.A, we have $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$, as expected. □

For any fully tilted $\mathcal{S}$-term $\mathfrak{t}$, the *fully tilted $\mathcal{S}$-easterly wind poset of* $\mathfrak{t}$ is the $\mathbb{N} \setminus \{0\}$-tilted $\mathcal{S}$-easterly wind poset of $\mathfrak{t}$.

3 Geometric and lattice properties

In this section, we continue to establish properties of the easterly wind posets by focusing in particular on geometric properties and on showing that terminal intervals of these posets are lattices.

3.1 Geometric properties

We begin this section by showing that the $\mathcal{S}$-easterly wind posets are EL-shellable. To this end, we introduce an encoding of the saturated chains of these posets. Next, we propose a realization of the $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind posets as geometric objects using connection words introduced in Section 2.2.2.

3.1.1 EL-shellability.

Given two $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of common degree $n \geqslant 0$ such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$, a *$(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence* is a word u on $[n]$ such that

$$\mathfrak{t}_1 = \mathfrak{t}^{(0)} \rightharpoonup_{u \cdot 1} \mathfrak{t}^{(1)} \rightharpoonup_{u \cdot 2} \dots \rightharpoonup_{u \cdot \lfloor \ell \cdot u \rfloor} \mathfrak{t}^{(\ell \cdot u)} = \mathfrak{t}_2 \tag{3.1.1.A}$$

for some $\mathcal{S}$-terms $\mathfrak{t}^{(0)}$, $\mathfrak{t}^{(1)}$, $\dots$, $\mathfrak{t}^{(\ell \cdot u)}$. Note that, by Theorem 2.2.3.F, the sequence of these terms forms a saturated chain from $\mathfrak{t}_1$ to $\mathfrak{t}_2$ in the $\mathcal{S}$-easterly wind poset. We say that this saturated chain is *specified* by u.

▶ **Lemma** 3.1.1.A — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. The set of saturated chains between $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is in one-to-one correspondence with the set of $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequences.*

◀ **Proof** — Let ϕ be the map having the set of $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequences as domain and the set of saturated chains between $\mathfrak{t}_1$ and $\mathfrak{t}_2$ as codomain, sending any $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence u to the saturated chain $\left(\mathfrak{t}^{(0)}, \mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(\ell)}\right)$ defined accordingly with (3.1.1.A). First, recall that as noticed in Section 2.2.1, given an $\mathcal{S}$-term $\mathfrak{s}_1$ and an internal node i of $\mathfrak{s}_1$, there is at most one $\mathcal{S}$-term $\mathfrak{s}_2$ such that $\mathfrak{s}_1 \rightharpoonup_i \mathfrak{s}_2$. This implies that ϕ is a well-defined map. Moreover, as a consequence of Lemma 2.2.1.A, for any $\mathcal{S}$-terms $\mathfrak{s}_1$ and $\mathfrak{s}_2$, if $\mathfrak{s}_1 \rightharpoonup_i \mathfrak{s}_2$ and $\mathfrak{s}_1 \rightharpoonup_{i'} \mathfrak{s}_2$ for two internal nodes i and i' of $\mathfrak{s}_1$, then $i = i'$. This implies that any saturated chain $\left(\mathfrak{t}^{(0)}, \mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(\ell)}\right)$ between $\mathfrak{t}_1$ and $\mathfrak{t}_2$ admits exactly one $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence which is an antecedent by ϕ. Therefore ϕ is a bijection and the statement of the lemma follows. □

Lemma 3.1.1.A entails in particular that the notion of saturated chain specified by a $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence is well-defined.

▶ **Lemma** 3.1.1.B — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. Among all $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequences,*

(i) there is at most one which is a weakly increasing word;

(ii) there is at most one which is a weakly decreasing word;

(iii) the one specifying the saturated chain from $\mathfrak{t}_1$ to $\mathfrak{t}_2$ which is induced by Lemma 2.2.3.E is a weakly increasing word;

(iv) this weakly increasing $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence is lexicographically smaller than any other $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence.

◀ **Proof** — To prove the uniqueness of a weakly increasing $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence, assume that u and u' are two such weakly increasing $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequences. Let i be a positive integer such that u and u' have both the same number of occurrences of each letter i' for any $i' < i$. Since $i = 1$ always satisfies this condition, such an i exists. Let v be the prefix of u made of the letters smaller than i. By the previously stated property on i and the fact that u and u' are weakly increasing, v is equivalently the prefix of u' made of the letters smaller than i. By Lemma 3.1.1.A, there is a unique $\mathcal{S}$-term $\mathfrak{s}_1$ such that v is a $(\mathfrak{t}_1, \mathfrak{s}_1)$-sequence. Now, let $\mathfrak{s}_2$ be the $\mathcal{S}$-term obtained from $\mathfrak{s}_1$ by applying k times the rewrite rule $\rightharpoonup_i$ where k is the number of occurrences of i in u. By Lemma 2.2.1.A, $\mathfrak{s}_2$ is obtained from $\mathfrak{s}_1$ by moving iteratively the parent edge of i. Moreover, again by Lemma 2.2.1.A, for all $i' \neq i$, each application of $\rightharpoonup_{i'}$ on $\mathfrak{s}_2$ does not change the edge connecting i to its parent. Therefore, in order to obtaining $\mathfrak{t}_2$ from $\mathfrak{s}_2$ through both $(\mathfrak{t}_1, \mathfrak{s}_1)$-sequences u and u', all these properties imply that the number of occurrences of i is the same in u and u'. This entails $u = u'$ and (i). In a completely similar way, this also shows that there is at most one $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence which is a weakly decreasing word. Thus, (ii) holds.

Point (iii) is immediate by Lemma 2.2.3.E. Indeed, the saturated chain induced by the chain of the statement of the lemma which has just been cited is specified by the $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence u consisting in a possibly empty block of letter 1, then a possibly empty block of the letter 2, and so on. Therefore, (iii) checks out. The fact that u is lexicographically smaller than any other $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence holds by construction. Indeed, each letter of u is the smallest possible in order to specify the right saturated chain from $\mathfrak{t}_1$ to $\mathfrak{t}_2$. Hence, (iv) holds. □

We use the standard definitions about labelings of Hasse diagrams of posets and EL-labelings as given in [Bjö80; BW96], which we recall here. A *labeling* of a poset $(\mathcal{P}, \preccurlyeq_{\mathcal{P}})$ is a map $\lambda : \lessdot_{\mathcal{P}} \to \Lambda$ where $\lessdot_{\mathcal{P}}$ is the covering relation of $\mathcal{P}$ and $(\Lambda, \preccurlyeq_{\Lambda})$ is a poset. Let $\bar{\lambda}$ be the map sending any saturated chain c of length $k \geqslant 1$ of $\mathcal{P}$ to the word on Λ of length $k-1$ defined by

$$\bar{\lambda}{\cdot}c{\cdot}i := \lambda{\cdot}\lfloor c{\cdot}i \rfloor{\cdot}\lfloor c{\cdot}\lfloor i+1 \rfloor \rfloor \tag{3.1.1.B}$$

for any $i \in [k-1]$. A saturated chain of $\mathcal{P}$ is *λ-increasing* (resp. *λ-weakly decreasing*) if its image by $\bar{\lambda}$ is an increasing (resp. weakly decreasing) word w.r.t. the partial order relation $\preccurlyeq_{\Lambda}$. A saturated chain c of $\mathcal{P}$ is *λ-smaller* than a saturated chain c' of $\mathcal{P}$ if $\bar{\lambda}{\cdot}c$ is smaller than $\bar{\lambda}{\cdot}c'$ for the lexicographic order induced by $\preccurlyeq_{\Lambda}$. The labeling λ is an *EL-labeling* of $\mathcal{P}$ if for any $x, x' \in \mathcal{P}$ satisfying $x \preccurlyeq_{\mathcal{P}} x'$, there is exactly one λ-increasing saturated chain c from x to x', and c is λ-smaller than any other saturated chains from x to x'.

Let us denote by $\mathbb{Z}^3$ the set of triples of integers endowed with the lexicographic order. Let $\lambda : \rightharpoonup \to \mathbb{Z}^3$ be the map defined, for any $(\mathfrak{t}_1, \mathfrak{t}_2) \in \rightharpoonup$, by

$$\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2 := (i, i_1, -j_1) \tag{3.1.1.C}$$

where (i_1, j_1, i) is the edge of $\mathfrak{t}_1$ which is replaced by an edge (i_1', j_1', i) in order to produce $\mathfrak{t}_2$. This map is well-defined thanks to Lemma 2.2.1.A. For instance, by considering the $\mathcal{S}_{\mathrm{e}}$-term $\mathfrak{t}_1$ as the left-hand side of (2.2.1.A) and the $\mathcal{S}_{\mathrm{e}}$-term $\mathfrak{t}_2$ as its right-hand side, we have $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2 = (3, 2, -2)$. With the same conventions, in (2.2.1.B), we have $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2 = (5, 1, -2)$.

▶ **Lemma** 3.1.1.C — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$, $\mathfrak{t}_2$, and $\mathfrak{t}_2'$ be three $\mathcal{S}$-terms such that $\mathfrak{t}_1 \rightharpoonup_{i_1} \mathfrak{t}_2$ and $\mathfrak{t}_1 \rightharpoonup_{i_1'} \mathfrak{t}_2'$, where i_1 and i_1' are internal nodes of $\mathfrak{t}_1$. We have $i_1 \leqslant i_1'$ if and only if $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ is smaller than or equal to $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2'$ for the lexicographic order.*

◀ **Proof** — By Lemma 2.2.1.A, $\mathfrak{t}_2$ is obtained from $\mathfrak{t}_1$ by replacing the edge (i, j, i_1) by the edge (i_2, j_2, i_1) where i and i_2 are internal nodes of both $\mathfrak{t}_1$ and $\mathfrak{t}_2$, and j and j_2 are integers. In the same way, $\mathfrak{t}_2'$ is obtained from $\mathfrak{t}_1$ by replacing the edge (i', j', i_1') by an edge (i_2', j_2', i_1') where i' and i_2' are internal nodes of both $\mathfrak{t}_1$ and $\mathfrak{t}_2'$, and j' and j_2' are integers. By definition of λ, $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2 = (i_1, i, -j)$ and $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2' = (i_1', i', -j')$. The statement of the lemma follows immediately.

□

Observe that Lemma 3.1.1.C implies that if u and u' are two $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequences where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two $\mathcal{S}$-terms satisfying $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ and u is lexicographically smaller than u', then $\bar{\lambda}{\cdot}c$ is smaller than $\bar{\lambda}{\cdot}c'$, where c (resp. c') is the saturated chain specified by u (resp. u').

▶ **Lemma** 3.1.1.D — *Let $\mathcal{S}$ be a signature, and $\mathfrak{t}_1$, $\mathfrak{t}_2$, and $\mathfrak{t}_3$ be three $\mathcal{S}$-terms such that $\mathfrak{t}_1 \rightharpoonup_{i_1} \mathfrak{t}_2 \rightharpoonup_{i_2} \mathfrak{t}_3$, where i_1 is an internal node of $\mathfrak{t}_1$ and i_2 is an internal node of $\mathfrak{t}_2$. The following properties hold:*

(i) the triples $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ and $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$ are different;

(ii) we have $i_1 \leqslant i_2$ if and only if $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ is smaller than $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$ for the lexicographic order.

◀ **Proof** — Let us first gather some properties about the fact that $\mathfrak{t}_1 \rightharpoonup_{i_1} \mathfrak{t}_2 \rightharpoonup_{i_2} \mathfrak{t}_3$. By Lemma 2.2.1.A, $\mathfrak{t}_2$ is obtained from $\mathfrak{t}_1$ by replacing the edge (i, j, i_1) by the edge (i', j', i_1) where i and i' are internal nodes of both $\mathfrak{t}_1$ and $\mathfrak{t}_2$, and j and j' are integers. In the same way, $\mathfrak{t}_3$ is obtained from $\mathfrak{t}_2$ by replacing the edge (i'', j'', i_2) by the edge (i''', j''', i_2) where i'' and i''' are internal nodes of both $\mathfrak{t}_2$ and $\mathfrak{t}_3$, and j'' and j''' are integers. Again by Lemma 2.2.1.A, the edge (i, j, i_1) is dominated by the edge (i', j', i_1), and the edge (i'', j'', i_2) is dominated by the edge (i''', j''', i_2). Moreover, by definition of λ, we have $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2 = (i_1, i, -j)$ and $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3 = (i_2, i'', -j'')$.

First, let us assume by contradiction that $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2 = \lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$. This implies that $(i_1, i, -j) = (i_2, i'', -j'')$ so that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have both the same edge $(i, j, i_1) = (i'', j'', i_2)$. Moreover, since $\mathfrak{t}_1 \rightharpoonup_{i_1} \mathfrak{t}_2$, by Lemma 2.2.1.A, (i, j, i_1) is not an edge of $\mathfrak{t}_2$. This yields a contradiction and (i) checks out.

Besides, if $i_1 < i_2$, then we have immediately that $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ is smaller than $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$ for the lexicographic order. When $i_1 = i_2$, the edges (i', j', i_1) and (i'', j'', i_2) are equal. Hence, (i, j, i_1) differs from and is dominated by (i'', j'', i_2). For this reason, $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ is smaller than $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$. for the lexicographic order. Conversely, when $i_1 > i_2$, it follows directly that $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ is greater than $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$. Therefore, the fact that $\lambda{\cdot}\mathfrak{t}_1{\cdot}\mathfrak{t}_2$ is smaller than $\lambda{\cdot}\mathfrak{t}_2{\cdot}\mathfrak{t}_3$ implies that $i_1 \leqslant i_2$. The equivalence stated by (ii) is established. □

▶ **Theorem** 3.1.1.E — *For any signature $\mathcal{S}$,*

(i) the labeling λ is an EL-labeling of the $\mathcal{S}$-easterly wind poset;

(ii) there is at most one λ-weakly decreasing saturated chain between any pair of elements of the $\mathcal{S}$-easterly wind poset.

◀ **Proof** — First of all, by Theorem 2.2.3.F, since $\rightharpoonup$ is the covering relation of the $\mathcal{S}$-easterly wind poset, the map λ is a well-defined labeling of this poset.

Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. Assume that there exist two λ-weakly increasing (resp. λ-weakly decreasing) saturated chains c and c' between $\mathfrak{t}_1$ and $\mathfrak{t}_2$. By Lemma 3.1.1.A, there exist two $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequences u and u' such that c is specified by u and c' is specified by u'. By Lemma 3.1.1.C, u and u' are weakly increasing (resp. weakly decreasing) words. By Point (i) (resp. Point (ii)) of Lemma 3.1.1.B, we have $u = u'$. This shows that c and c' are in fact the same saturated chain. Therefore, there is at most one λ-weakly increasing (resp. λ-weakly decreasing) saturated chain from $\mathfrak{t}_1$ to $\mathfrak{t}_2$. In particular, this proves (ii).

Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$, and let u be a weakly increasing $(\mathfrak{t}_1, \mathfrak{t}_2)$-sequence. The existence of such u is ensured by Point (iii) of Lemma 3.1.1.B. By Lemma 3.1.1.D, the saturated chain specified by u is λ-increasing. The uniqueness of this λ-increasing saturated chain is shown in the previous paragraph of this proof. Finally, the fact that the saturated chain specified by u is λ-smaller than any other saturated chains from $\mathfrak{t}_1$ to $\mathfrak{t}_2$ follows from Point (iv) of Lemma 3.1.1.B and from Lemma 3.1.1.C. Therefore, (i) is established. □

An important consequence [BW96] of Point (ii) of Theorem 3.1.1.E is that the codomain of the Möbius function μ of the $\mathcal{S}$-easterly wind poset is the set $\{-1, 0, 1\}$.

Besides, since by Theorem 2.3.2.B, for any set $\mathcal{X}$ of positive integers, $\mathrm{tlt}\cdot\mathcal{X}$ is a closure operator of the $\mathcal{S}$-easterly wind poset, by the Crapo's Closure Theorem [Cra66], the Möbius function $\mu\cdot\mathcal{X}$ of the $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind poset satisfies

$$\mu\cdot\mathcal{X}\cdot\mathfrak{t}_1\cdot\mathfrak{t}_2 = \sum_{\mathfrak{t}'_2 \in \mathfrak{T}\cdot\mathcal{S}} [\mathrm{tlt}\cdot\mathcal{X}\cdot\mathfrak{t}'_2 = \mathfrak{t}_2]\ \mu\cdot\mathfrak{t}_1\cdot\mathfrak{t}'_2 \qquad (3.1.1.D)$$

for all $\mathcal{X}$-tilted $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$.

3.1.2 Geometric realization. Let $\mathcal{P}$ be an interval of the $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind poset. Since two $\mathcal{S}$-terms are comparable only if they have the same degree, let us denote by n the common degree of the $\mathcal{S}$-terms of $\mathcal{P}$. The *geometric realization* $\mathfrak{G}\cdot\mathcal{P}$ of $\mathcal{P}$ is the embedding of the Hasse diagram of $\mathcal{P}$ in the space $\mathbb{R}^n$ such that each $\mathfrak{t} \in \mathcal{P}$ gives rise to a vertex of coordinates $\mathrm{cnc}\cdot\mathfrak{t}$ and each pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of $\mathcal{S}$-terms of $\mathcal{P}$ gives rise to an edge, provided that $\mathfrak{t}_1$ is covered by $\mathfrak{t}_2$ in $\mathcal{P}$.

Moreover, when $\mathcal{X} = \emptyset$, since by Lemma 2.2.1.A and Theorem 2.2.3.F, the connection sequences of two $\mathcal{S}$-terms which are in relation for $\rightharpoonup$ differ in exactly one component, every edge of $\mathfrak{G}\cdot\mathcal{P}$ is parallel to a line passing through the origin and a point of $\mathbb{R}^n$ the form $(0, \dots, 0, 1, 0, \dots, 0)$. For this reason, the geometric realizations of $\mathcal{S}$-easterly wind posets are cubic [CG22; Com23]. In general, $\mathfrak{G}\cdot\mathcal{P}$ is not cubic when $\mathcal{P}$ is an interval of an $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind poset with $\mathcal{X} \neq \emptyset$. Figure 3 shows examples of geometric realizations of such intervals.

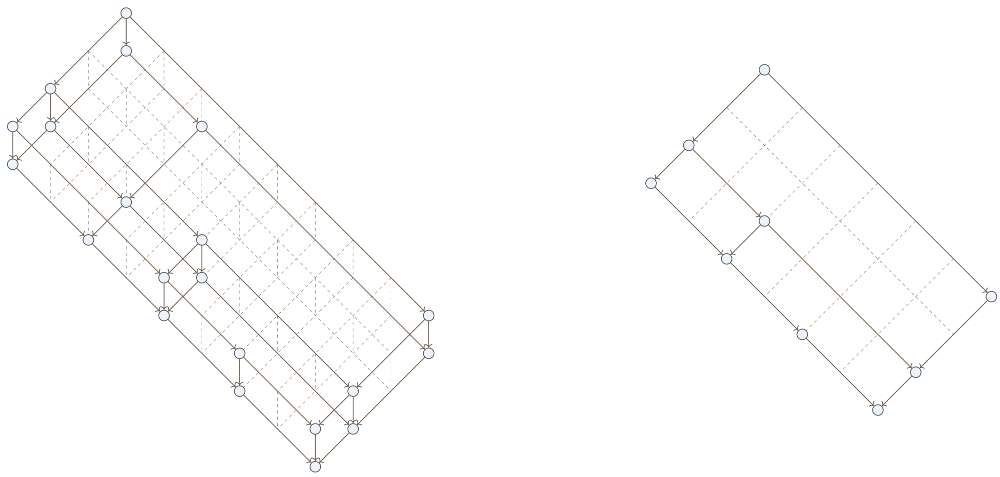


Figure 3: The geometric realization of the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathfrak{t}$ on the left, and the geometric realization of the $\{1\}$-tilted $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathrm{tlt}\cdot\{1\}\cdot\mathfrak{t}$ on the right, for $\mathfrak{t} := \mathsf{a}_3 \circ \mathsf{a}_3 \circ \circ \mathsf{a}_1 \circ \mathsf{a}_1 \circ$. The top (resp. bottom) point has $\left(\frac{15}{18}, \frac{3}{2}, 2, 1\right)$ (resp. $\left(\frac{15}{18}, \frac{7}{4}, \frac{11}{4}, 3\right)$) as coordinates on the left, and the top (resp. bottom) point has $\left(\frac{15}{18}, \frac{7}{4}, 2, \frac{3}{2}\right)$ (resp. $\left(\frac{15}{18}, \frac{7}{4}, \frac{11}{4}, 3\right)$) on the right.

3.2 Lattice properties

The purpose of this section is to show that all terminal intervals of $\mathcal{S}$-easterly wind posets are lattices. We also show that this property holds for $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind posets.

3.2.1 Join-semilattice structure.

For any words w_1 and w_2 on $\mathbb{Q}$ of the same length n, let $w_3 := w_1 \vee w_2$ be the sequence of length n such that for any $i \in [n]$, $w_3 \cdot i = \max\{w_1 \cdot i, w_2 \cdot i\}$.

Let $\mathfrak{t}$ be an $\mathcal{S}$-term and let on the set $\downarrow \cdot \mathfrak{t}$ the binary operation $\vee$ defined as follows. For any $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of $\downarrow \cdot \mathfrak{t}$, let $\mathfrak{t}_1 \vee \mathfrak{t}_2$ be the $\mathcal{S}$-term $\mathfrak{t}_3$ having $\mathrm{dc} \cdot \mathfrak{t}$ as decoration word and such that its connection word is $\mathrm{cnc} \cdot \mathfrak{t}_1 \vee \mathrm{cnc} \cdot \mathfrak{t}_2$. For instance, for $\mathfrak{t} := \mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \circ \mathsf{a}_0 \rfloor \lfloor \mathsf{a}_3 \circ \circ \circ \rfloor$, we have

$$\mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \circ \rfloor \lfloor \mathsf{a}_3 \circ \circ \circ \rfloor \vee \mathsf{a}_3 \lfloor \mathsf{a}_3 \circ \circ \mathsf{a}_0 \rfloor \lfloor \mathsf{a}_3 \circ \circ \circ \rfloor \circ = \mathsf{a}_3 \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \circ \rfloor \lfloor \mathsf{a}_3 \circ \circ \circ \rfloor \circ. \tag{3.2.1.A}$$

Note that by Lemma 2.2.1.A, since $\downarrow \cdot \mathfrak{t} \preccurlyeq \mathfrak{t}_1$ and $\downarrow \cdot \mathfrak{t} \preccurlyeq \mathfrak{t}_2$, we have $\mathrm{dc} \cdot \mathfrak{t}_1 = \mathrm{dc} \cdot \mathfrak{t} = \mathrm{dc} \cdot \mathfrak{t}_2$.

▶ **Lemma** 3.2.1.A — *Let $\mathcal{S}$ be a signature and $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two $\mathcal{S}$-terms such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$. If $\mathfrak{t}_1$ has an internal node i such that its j-th child is an extreme leaf, then in $\mathfrak{t}_2$, the j-th child of i is an extreme leaf.*

◀ **Proof** — Assume that $\mathfrak{t}_1 \rightharpoonup \mathfrak{t}_2$ and that $\mathfrak{t}_1$ has an internal node i such that its j-th child is an extreme leaf. Let us call k this leaf. Since k is extreme in $\mathfrak{t}_1$, there is no internal node of $\mathfrak{t}_1$ which is visited after k in the preorder traversal of $\mathfrak{t}_1$. For this reason, $\mathfrak{t}_2$ cannot be obtained by replacing k by any other subterm through the $\mathcal{S}$-easterly wind rewrite rule. This shows that k remains an extreme leaf in $\mathfrak{t}_2$, so that the j-th child of i is an extreme leaf. Now, the fact that, by Theorem 2.2.3.F, $\rightharpoonup$ is the covering relation of the partial order relation $\preccurlyeq$ entails the statement of the lemma. □

▶ **Proposition** 3.2.1.B — *For any signature $\mathcal{S}$ and any $\mathcal{S}$-term $\mathfrak{t}$, the operation $\vee$ is well-defined on the $\mathcal{S}$-easterly wind poset of $\mathfrak{t}$.*

◀ **Proof** — We have to show that for any $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ such that $\mathfrak{t} \preccurlyeq \mathfrak{t}_1$ and $\mathfrak{t} \preccurlyeq \mathfrak{t}_2$, the word $c_1 \vee c_2$ is the connection word of an $\mathcal{S}$-term of $\downarrow \cdot \mathfrak{t}$, having $\mathrm{dc} \cdot \mathfrak{t}$ as decoration word, where $c_1 := \mathrm{cnc} \cdot \mathfrak{t}_1$ and $c_2 := \mathrm{cnc} \cdot \mathfrak{t}_2$. Let us prove this property by induction on n, the degree of $\mathfrak{t}$. This is immediately true when $n = 0$. Assume that $n \geqslant 1$. By Lemma 2.2.3.C, we have $\partial \cdot \mathfrak{t} \preccurlyeq \partial \cdot \mathfrak{t}_1$ and $\partial \cdot \mathfrak{t} \preccurlyeq \partial \cdot \mathfrak{t}_2$. Let us set $c'_1 := \mathrm{cnc} \cdot \lfloor \partial \cdot \mathfrak{t}_1 \rfloor$, $c'_2 := \mathrm{cnc} \cdot \lfloor \partial \cdot \mathfrak{t}_2 \rfloor$, and $c' := c'_1 \vee c'_2$. By induction hypothesis, c' is the connection word of an $\mathcal{S}$-term $\mathfrak{t}'$ of $\downarrow \cdot \lfloor \partial \cdot \mathfrak{t} \rfloor$ whose decoration word is $\mathrm{dc} \cdot \lfloor \partial \cdot \mathfrak{t} \rfloor$. Without loss of generality, assume that $c_1 \cdot n \geqslant c_2 \cdot n$ and let (i_1, j_1, n) be the parent edge of n in $\mathfrak{t}_1$. Let $\mathfrak{t}''$ be the $\mathcal{S}$-term obtained by adding to $\mathfrak{t}'$ the edge (i_1, j_1, n) so that the added internal node n is decorated by $\mathrm{dc} \cdot \mathfrak{t} \cdot n$. Note that since n is the last visited internal node of $\mathfrak{t}_1$ in the preorder traversal of $\mathfrak{t}_1$, in $\partial \cdot \mathfrak{t}_1$, the j_1-th child of the internal node i_1 is an extreme leaf. Therefore, by Lemma 3.2.1.A, in $\mathfrak{t}'$, the j_1-child of $\mathfrak{t}'$ is a leaf. This ensures that it is possible to build $\mathfrak{t}''$ as stated. By construction, $\mathrm{cnc} \cdot \mathfrak{t}'' = \mathrm{cnc} \cdot \mathfrak{t}' \centerdot c_1 \cdot n$. Now, since $\partial \cdot \mathfrak{t} \preccurlyeq \mathfrak{t}'$ and $\mathrm{cnc} \cdot \mathfrak{t} \cdot n \leqslant \mathrm{cnc} \cdot \mathfrak{t}_1 \cdot n$, we have $\mathfrak{t} \preccurlyeq \mathfrak{t}''$. Moreover, by construction, $\mathrm{dc} \cdot \mathfrak{t}'' = \mathrm{dc} \cdot \mathfrak{t}' \centerdot \mathrm{dc} \cdot \mathfrak{t} \cdot n = \mathrm{dc} \cdot \mathfrak{t}$. This shows the stated property. □

3.2.2 Lattice structure.

Let us now state one of the most important results of this section.

▶ **Theorem** 3.2.2.A — *For any signature $\mathcal{S}$ and any $\mathcal{S}$-term $\mathfrak{t}$, the subposet $\downarrow \cdot \mathfrak{t}$ of the $\mathcal{S}$-easterly wind poset is a lattice. Moreover, this lattice admits $\vee$ as join operation.*

◀ **Proof** — Let us first prove that $\vee$ is the join operation of the poset $(\downarrow\cdot\mathfrak{t}, \preccurlyeq)$. First of all, by Proposition 3.2.1.B, the operation $\vee$ is well-defined on $\downarrow\cdot\mathfrak{t}$. Let us show that for any $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ such that $\mathfrak{t} \preccurlyeq \mathfrak{t}_1$ and $\mathfrak{t} \preccurlyeq \mathfrak{t}_2$, $\mathfrak{t}' := \mathfrak{t}_1 \vee \mathfrak{t}_2$ is the unique minimal element of the set $\downarrow\cdot\mathfrak{t}_1 \cap \downarrow\cdot\mathfrak{t}_2$. For this, let $\mathfrak{t}''$ be an $\mathcal{S}$-term such that $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}''$ and $\mathfrak{t}_2 \preccurlyeq \mathfrak{t}''$. By definition of $\preccurlyeq$, for any $i \in [\mathrm{dg}\cdot\mathfrak{t}]$, $\mathrm{cnc}\cdot\mathfrak{t}''\cdot i \geqslant \mathrm{cnc}\cdot\mathfrak{t}_1\cdot i$ and $\mathrm{cnc}\cdot\mathfrak{t}''\cdot i \geqslant \mathrm{cnc}\cdot\mathfrak{t}_2\cdot i$ so that $\mathrm{cnc}\cdot\mathfrak{t}''\cdot i \geqslant \max\{\mathrm{cnc}\cdot\mathfrak{t}_1\cdot i, \mathrm{cnc}\cdot\mathfrak{t}_2\cdot i\}$. Since, by definition of $\vee$, $\mathrm{cnc}\cdot\mathfrak{t}'\cdot i = \max\{\mathrm{cnc}\cdot\mathfrak{t}_1\cdot i, \mathrm{cnc}\cdot\mathfrak{t}_2\cdot i\}$, we have $\mathfrak{t}' \preccurlyeq \mathfrak{t}''$. This shows that $\vee$ is the join operation of $(\downarrow\cdot\mathfrak{t}, \preccurlyeq)$. Finally, since any join-semilattice with a unique minimal element is a lattice [Sta11], the stated property holds. □

A lattice $\mathcal{L}$ with meet operation $\wedge$ and join operation $\vee$ is *join semi-distributive* [FJN95] if for any $x, x_1, x_2 \in \mathcal{L}$, $x_1 \vee x = x \vee x_2$ implies $x \vee (x_1 \wedge x_2) = x_1 \vee x$.

For some $\mathcal{S}$-terms $\mathfrak{t}$, the $\mathcal{S}$-easterly wind posets of $\mathfrak{t}$ are not join semi-distributive lattices. Indeed, let us consider the $\mathcal{S}_{\mathrm{e}}$-terms

- $\mathfrak{t} := \mathsf{a}_4 \llcorner \mathsf{a}_1 \circ \lrcorner \llcorner \mathsf{a}_1 \circ \lrcorner \llcorner \mathsf{a}_1 \circ \lrcorner \llcorner \mathsf{a}_1 \circ \lrcorner$;
- $\mathfrak{s} := \mathsf{a}_4 \llcorner \mathsf{a}_1 \circ \lrcorner \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \circ \lrcorner \lrcorner \lrcorner \circ\circ$;
- $\mathfrak{s}_1 := \mathsf{a}_4 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \circ \lrcorner \lrcorner \lrcorner \circ \llcorner \mathsf{a}_1 \circ \lrcorner \circ$;
- $\mathfrak{s}_2 := \mathsf{a}_4 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \circ \lrcorner \lrcorner \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \circ \lrcorner \lrcorner \circ\circ$;
- $\mathfrak{s}' := \mathsf{a}_4 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \llcorner \mathsf{a}_1 \circ \lrcorner \lrcorner \lrcorner \lrcorner \circ\circ\circ$.

It is easy to check that $\mathfrak{s}$, $\mathfrak{s}_1$, and $\mathfrak{s}_2$ belong to the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathfrak{t}$ and that we have $\mathfrak{s}_1 \vee \mathfrak{s} = \mathfrak{s}' = \mathfrak{s} \vee \mathfrak{s}_2$. Now, since we have $\mathfrak{s} \vee (\mathfrak{s}_1 \wedge \mathfrak{s}_2) = \mathfrak{s} \vee \mathfrak{t} = \mathfrak{s} \neq \mathfrak{s}'$, this yields a contradiction with the required relation to be join semi-distributive.

An equivalence relation $\equiv$ on a lattice $\mathcal{L}$ is a *lattice congruence* [CS98; Rea04] of $\mathcal{L}$ if each $\equiv$-equivalence class is an interval of $\mathcal{L}$ and both the maps sending each $x \in \mathcal{L}$ to the smallest or greatest element of $[x]_{\equiv}$ are order-preserving. Observe that despite the fact that by Proposition 2.3.3.A, for any set $\mathcal{X}$ of positive integers, each $\equiv_{\mathrm{tlt}\cdot\mathcal{X}}$-equivalence class is an interval of the $\mathcal{S}$-easterly wind poset, by the remark stated at the end of Section 2.3.2, $\mathrm{tlt}^{\mathrm{r}}\cdot\mathcal{X}$ is not order-preserving. For this reason, for any $\mathcal{S}$-term $\mathfrak{t}$, the restriction of the equivalence relation $\equiv_{\mathrm{tlt}\cdot\mathcal{X}}$ on the lattice $\downarrow\cdot\mathfrak{t}$ is not in general a lattice congruence of this lattice.

3.2.3 Lattice structure on tilted terms. Let $\mathfrak{t}$ be an $\mathcal{S}$-term and $\mathcal{X}$ be a set of positive integers. Let on the set $\downarrow\cdot\mathcal{X}\cdot\mathfrak{t}$ the binary operation $\vee\cdot\mathcal{X}$ defined for any $\mathcal{X}$-tilted $\mathcal{S}$-terms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by $\mathfrak{t}_1 \vee\cdot\mathcal{X}\ \mathfrak{t}_2 := \mathrm{tlt}\cdot\mathcal{X}\cdot\llcorner \mathfrak{t}_1 \vee \mathfrak{t}_2 \lrcorner$ where $\vee$ is the operation defined in Section 3.2.1.

▶ **Proposition 3.2.3.A** — *For any signature $\mathcal{S}$, any set $\mathcal{X}$ of positive integers, and any $\mathcal{X}$-tilted $\mathcal{S}$-term $\mathfrak{t}$, the subposet $\downarrow\cdot\mathcal{X}\cdot\mathfrak{t}$ of the $\mathcal{S}$-easterly wind poset is a lattice. Moreover, this lattice admits $\vee\cdot\mathcal{X}$ as join operation.*

◀ **Proof** — This is a consequence of the fact that by Theorem 2.3.2.B, $\mathrm{tlt}\cdot\mathcal{X}$ is a closure operator of $\downarrow\cdot\mathfrak{t}$ and the fact that, by Theorem 3.2.2.A, $\downarrow\cdot\mathfrak{t}$ is a lattice. Indeed, as exposed in [DP02], if ϕ is a closure operator of a lattice $\mathcal{L}$, then $\phi\cdot\mathcal{L}$ is also a lattice. This lattice has the same meet operation as the one of $\mathcal{L}$, and admits the join operation $\vee'$ satisfying $x_1 \vee' x_2 = \phi\cdot\llcorner x_1 \vee x_2 \lrcorner$ where $\vee$ is the join operation of $\mathcal{L}$, for any $x_1, x_2 \in \mathcal{L}$. □

Remark that even if, as provided by Proposition 3.2.3.A, $\downarrow\cdot\mathcal{X}\cdot\mathfrak{t}$ is a lattice, this lattice is not a sublattice of $\downarrow\cdot\mathfrak{t}$. Indeed, in the $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathsf{a}_2 \llcorner \mathsf{a}_2 \llcorner \mathsf{a}_3 \llcorner \mathsf{a}_2 \circ\circ \lrcorner \llcorner \mathsf{a}_2 \circ\circ \lrcorner \circ \lrcorner \llcorner \mathsf{a}_2 \circ\circ \lrcorner \lrcorner \circ$, we have for instance

$$\mathsf{a}_2 \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_2 \circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ \rfloor \circ \vee \mathsf{a}_2 \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ = \mathsf{a}_2 \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ \rfloor \circ, \tag{3.2.3.A}$$

but even if the two operands are $\{1,3\}$-tilted, the result is not.

4 Easterly wind lattices of forests

Easterly-wind posets are sufficiently large to contain, as subposets, several notable structures. In particular, we introduce a notion of forests regarded as specific $\mathcal{S}$-terms and study their posets. We then obtain, on the one hand, a family of lattices on Fuss–Catalan objects and, on the other hand, an alternative construction of the Tamari lattices. The section concludes with lattice structures on leaning forests and several key notions that will later be used to describe the natural Hopf algebras of free nonsymmetric operads.

4.1 Forests and maximal intervals

We introduce the notion of $\mathcal{S}$-forests, which are particular $\mathcal{S}$-terms. We also study certain maximal intervals involving such forests in the easterly wind posets.

4.1.1 Forests. By seeing $\mathbb{N}$ as the signature such that for any $n \in \mathbb{N}$, $\mathrm{ar}{\cdot}n := n$, let $\mathcal{S}_{\mathbb{N}}$ be the signature $\mathcal{S} \sqcup \mathbb{N}$. An *$\mathcal{S}$-forest* is an $\mathcal{S}_{\mathbb{N}}$-term $\mathfrak{f}$ of the form $\mathfrak{f} = n\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ where $n \in \mathbb{N}$ and for any $i \in [n]$, $\mathfrak{t}_i$ is an $\mathcal{S}$-term. For instance, $4\, \mathsf{a}_0 \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \rfloor$ is an $\mathcal{S}_{\mathrm{e}}$-forest. Moreover, the *concatenation* of two $\mathcal{S}$-forests $n\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ and $n'\, \mathfrak{t}'_1 \dots \mathfrak{t}'_{n'}$ is the $\mathcal{S}$-forest

$$n\, \mathfrak{t}_1 \dots \mathfrak{t}_n \;\centerdot\; n'\, \mathfrak{t}'_1 \dots \mathfrak{t}'_{n'} := \lfloor n + n' \rfloor\, \mathfrak{t}_1 \dots \mathfrak{t}_n \mathfrak{t}'_1 \dots \mathfrak{t}'_{n'}. \tag{4.1.1.A}$$

An $\mathcal{S}$-forest $\mathfrak{f} = n\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ is *balanced* if $\mathrm{dg}{\cdot}\mathfrak{f} = n + 1$. For instance, 0 and $3 \lfloor \mathsf{a}_2 \circ \mathsf{a}_0 \rfloor \circ \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor$ are balanced $\mathcal{S}_{\mathrm{e}}$-forests. On the contrary, $2 \lfloor \mathsf{a}_2 \mathsf{a}_0 \circ \rfloor \mathsf{a}_0$ is an $\mathcal{S}_{\mathrm{e}}$-forest which is not balanced. The *size* of a balanced $\mathcal{S}$-forest $\mathfrak{f}$ is the decoration of the root of $\mathfrak{f}$ (or, equivalently, the degree of $\mathfrak{f}$ minus one). Observe that the concatenation of two balanced $\mathcal{S}$-forests is a balanced $\mathcal{S}$-forest.

4.1.2 Maximal intervals. For any $\mathcal{S}$-forest $\mathfrak{f}$ of arity 1 or more and any $\mathcal{S}$-term $\mathfrak{t}$, let $\mathfrak{f} \bullet \mathfrak{t}$ be the $\mathcal{S}$-forest obtained by replacing the leftmost leaf of $\mathfrak{f}$ by $\mathfrak{t}$. For instance,

$$3 \lfloor \mathsf{a}_2 \mathsf{a}_0 \mathsf{a}_0 \rfloor \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \circ \;\bullet\; \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ = 3 \lfloor \mathsf{a}_2 \mathsf{a}_0 \mathsf{a}_0 \rfloor \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \rfloor \circ \rfloor \circ. \tag{4.1.2.A}$$

Now, given a word w on $\mathcal{S}$ of length $n \in \mathbb{N}$, let the $\mathcal{S}$-forests

$$\mathbf{f}^{\Uparrow}{\cdot}w := n \lfloor \iota{\cdot}\lfloor w{\cdot}1 \rfloor \rfloor \lfloor \iota{\cdot}\lfloor w{\cdot}2 \rfloor \rfloor \dots \lfloor \iota{\cdot}\lfloor w{\cdot}n \rfloor \rfloor \tag{4.1.2.B}$$

and

$$\mathbf{f}^{\Downarrow}{\cdot}w := (\dots((\lfloor \iota{\cdot}n \rfloor \bullet \iota{\cdot}\lfloor w{\cdot}1 \rfloor) \bullet \iota{\cdot}\lfloor w{\cdot}2 \rfloor) \dots) \bullet \iota{\cdot}\lfloor w{\cdot}n \rfloor. \tag{4.1.2.C}$$

For instance, for $w := \mathsf{a}_2\mathsf{a}_1\mathsf{a}_0\mathsf{a}_0\mathsf{a}_0\mathsf{a}_3\mathsf{a}_2 \in \mathcal{S}_{\mathrm{e}}^*$, we have

$$\mathbf{f}^{\Uparrow}{\cdot}w = 7 \lfloor \mathsf{a}_2 \circ\circ \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \mathsf{a}_0 \mathsf{a}_0 \mathsf{a}_0 \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \tag{4.1.2.D}$$

and

$$\mathbf{f}^{\Downarrow}\cdot w = 7 \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \mathsf{a}_0 \rfloor \mathsf{a}_0 \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ\circ \rfloor \circ\circ\circ\circ. \tag{4.1.2.E}$$

Note that $\mathbf{f}^{\Uparrow}\cdot\epsilon = 0 = \mathbf{f}^{\Downarrow}\cdot\epsilon$. It is straightforward to prove that both $\mathbf{f}^{\Uparrow}\cdot w$ and $\mathbf{f}^{\Downarrow}\cdot w$ are well-defined balanced $\mathcal{S}$-forests of size $\ell\cdot w$.

▶ **Theorem** 4.1.2.A — *For any signature $\mathcal{S}$, any set $\mathcal{X}$ of positive integers, and any word w on $\mathcal{S}$ of length $n \in \mathbb{N}$,*

(i) in the $\mathcal{X}$-tilted $\mathcal{S}_{\mathbb{N}}$-easterly wind poset, $\mathbf{f}^{\Uparrow}\cdot w \preccurlyeq \mathbf{f}^{\Downarrow}\cdot w$;

(ii) the $\mathcal{S}$-forest $\mathbf{f}^{\Uparrow}\cdot w$ (resp. $\mathbf{f}^{\Downarrow}\cdot w$) is a minimal (resp. maximal) element of the $\mathcal{X}$-tilted $\mathcal{S}_{\mathbb{N}}$-easterly wind poset.

◀ **Proof** — Observe first that neither $\mathbf{f}^{\Uparrow}\cdot w$ nor $\mathbf{f}^{\Downarrow}\cdot w$ depend on the set $\mathcal{X}$. Indeed, directly from their definitions, it follows that these two $\mathcal{S}$-forests are $\mathcal{X}$-tilted for any set $\mathcal{X}$ of positive integers. For this reason, in this proof, we consider simply that $\mathcal{X} = \emptyset$.

To prove (i), let us proceed by induction on n. When $n = 0$, $\mathbf{f}^{\Uparrow} = 0 = \mathbf{f}^{\Downarrow}$ so that the property is satisfied. Assume that the property holds for any word w on $\mathcal{S}$ of length $n \geqslant 0$ and let $a \in \mathcal{S}$. By definition of $\mathbf{f}^{\Uparrow}$, we have $\mathrm{E}\cdot\lfloor \mathbf{f}^{\Uparrow}\cdot\lfloor w \centerdot a\rfloor\rfloor = \mathrm{E}\cdot\lfloor \mathbf{f}^{\Uparrow}\cdot w\rfloor \cup \{(1, n+1, n+2)\}$. Moreover, by definition of $\mathbf{f}^{\Downarrow}$, we have $\mathrm{E}\cdot\lfloor \mathbf{f}^{\Downarrow}\cdot\lfloor w \centerdot a\rfloor\rfloor = \mathrm{E}\cdot\lfloor \mathbf{f}^{\Downarrow}\cdot w\rfloor \cup \{(i, j, n+2)\}$ where i is the internal node of $\mathbf{f}^{\Downarrow}\cdot w$ which is the parent of the leftmost leaf and j is the position of this leaf in its siblings. By induction hypothesis, we have $\mathbf{f}^{\Uparrow}\cdot w \preccurlyeq \mathbf{f}^{\Downarrow}\cdot w$. Thus, by Lemma 2.2.3.A, for any $i' \in [n+1]$, the parent edge of i' in $\mathbf{f}^{\Uparrow}\cdot w$ is dominated by that of i' in $\mathbf{f}^{\Downarrow}\cdot w$. Moreover, the edge $(1, n+1, n+2)$ is dominated by the edge $(i, j, n+2)$. Indeed, otherwise, we would have $i = 1$ and $j > n+1$, which is absurd since the internal node 1 of both $\mathbf{f}^{\Uparrow}\cdot\lfloor w \centerdot a\rfloor$ and $\mathbf{f}^{\Downarrow}\cdot\lfloor w \centerdot a\rfloor$ has arity $n+1$. Therefore, again by Lemma 2.2.3.A, we have $\mathbf{f}^{\Uparrow}\cdot\lfloor w \centerdot a\rfloor \preccurlyeq \mathbf{f}^{\Downarrow}\cdot\lfloor w \centerdot a\rfloor$ as expected.

To prove (ii), assume first that $\mathfrak{t}$ is an $\mathcal{S}$-forest such that $\mathfrak{t} \preccurlyeq \mathbf{f}^{\Uparrow}\cdot w$. Hence, by Lemma 2.2.3.A, for any $i \in [2, n]$, the parent edge $(\mathrm{pa}\cdot\mathfrak{t}\cdot i, \mathrm{lp}\cdot\mathfrak{t}\cdot i, i)$ of i in $\mathfrak{t}$ is dominated by the parent edge $(1, i-1, i)$ of i in $\mathbf{f}^{\Uparrow}\cdot w$. By definition of the notion of edge domination, we have necessarily $\mathrm{pa}\cdot\mathfrak{t}\cdot i = 1$ and $\mathrm{lp}\cdot\mathfrak{t}\cdot i \geqslant i - 1$. It follows that $\mathrm{lp}\cdot\mathfrak{t}\cdot i = i - 1$ so that $\mathfrak{t} = \mathbf{f}^{\Uparrow}\cdot w$. This shows that $\mathbf{f}^{\Uparrow}\cdot w$ is a minimal element of the $\mathcal{S}_{\mathbb{N}}$-easterly wind poset. Finally, observe that in $\mathbf{f}^{\Downarrow}\cdot w$, all leaves are visited after all internal internal nodes in its preorder traversal. Therefore, there is no $\mathcal{S}$-forest $\mathfrak{t}$ such that $\mathbf{f}^{\Downarrow} \rightharpoonup \mathfrak{t}$. Therefore, by Lemma 2.2.3.E, this implies that $\mathbf{f}^{\Downarrow}\cdot w$ is a maximal element of the $\mathcal{S}_{\mathbb{N}}$-easterly wind poset. □

By Theorem 4.1.2.A and Proposition 3.2.3.A, the interval $\left[\mathbf{f}^{\Uparrow}\cdot w, \mathbf{f}^{\Downarrow}\cdot w\right]$ of the $\mathcal{X}$-tilted $\mathcal{S}_{\mathbb{N}}$-easterly wind poset is a maximal interval and a lattice. Let us call it the *balanced forest $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind poset of w*. When $\mathcal{X} = \emptyset$, this poset is the *balanced forest $\mathcal{S}$-easterly wind poset of w*. For instance, by replacing all decorations a_3 of the roots of the terms of Figure 1 by the decoration $3 \in \mathbb{N}$, the resulting Hasse diagram of this figure is the balanced forest $\emptyset$-tilted $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathsf{a}_3\mathsf{a}_1\mathsf{a}_2$. With the same change, Figure 2 shows the Hasse diagram of the balanced forest $\{1, 2\}$-tilted $\mathcal{S}_{\mathrm{e}}$-easterly wind lattice of $\mathsf{a}_3\mathsf{a}_1\mathsf{a}_2$.

4.2 Catalan lattices and Fuss-Catalan lattices

The purpose of this section is to build, as a particular balanced forest $\mathcal{X}$-tilted easterly wind posets of some words, partial order structures on the combinatorial set of Catalan or Fuss-Catalan objects. We begin by constructing such structures by introducing a nontrivial bijection between the set of some balanced forests and the set of terms whose internal node all have a fixed arity. Using this bijection, we prove that our first family of posets admits as underlying set the set of Fuss-Catalan objects. Independently, we consider a balanced forest easterly wind poset on fully tilted terms and show that this construction yields an alternative description of the well-known Tamari partial order.

4.2.1 Fuss-Catalan lattices. For any $n, m \in \mathbb{N}$, the *m-Fuss-Catalan easterly wind poset of order n* is the balanced forest $\mathbb{N}$-easterly wind poset of the word m^n, as defined in Section 4.1. Figure 4 shows the Hasse diagrams of some such posets.

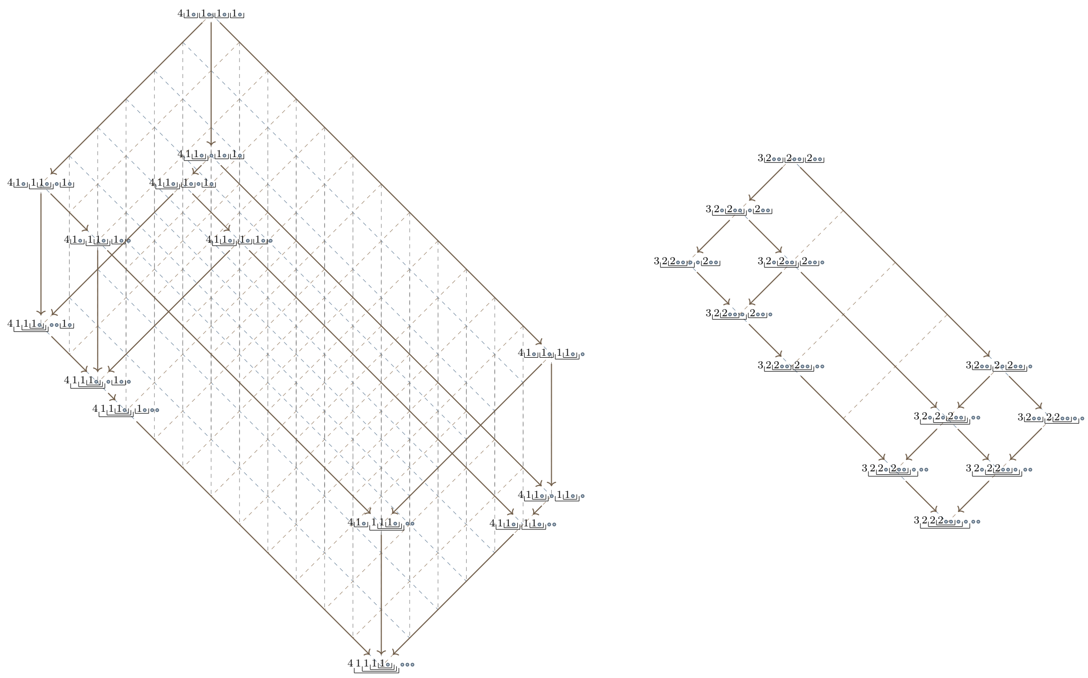


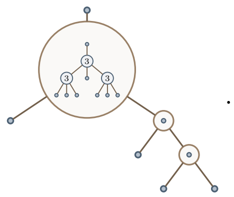
Figure 4: The Hasse diagram of the 1-Fuss-Catalan easterly wind poset of order 4 on the left, and the Hasse diagram of the 2-Fuss-Catalan easterly wind poset of order 3 on the right. Observe that these graphs are not regular (that is, not all vertices have the same degree).

▶ **Lemma** 4.2.1.A — *For any $m, n \in \mathbb{N}$, $\downarrow\cdot\lfloor \mathbf{f}^{\Uparrow}\cdot m^n \rfloor$ is the set of $\mathcal{S}$-forests of the form $\mathfrak{f} = n\,\mathfrak{t}_1 \dots \mathfrak{t}_n$ such that $\mathfrak{f}$ is balanced and for any $\ell \in [n]$, $\mathrm{dg}\cdot\mathfrak{t}_{n-\ell+1} + \dots + \mathrm{dg}\cdot\mathfrak{t}_n \leqslant \ell$.*

◀ **Proof** — Assume first that $\mathfrak{f} = n\,\mathfrak{t}_1 \dots \mathfrak{t}_n$ and $\mathfrak{f}' = n\,\mathfrak{t}'_1 \dots \mathfrak{t}'_n$ are two $\mathbb{N}$-forests such that $\mathfrak{f}$ satisfies the condition of the statement and $\mathfrak{f} \rightharpoonup \mathfrak{f}'$. From the definition of $\rightharpoonup$, we have either that $\mathrm{dg}\cdot\mathfrak{t}_j = \mathrm{dg}\cdot\mathfrak{t}'_j$ for all $j \in [n]$, or that there exists $j \in [n-1]$ such that $\mathrm{dg}\cdot\mathfrak{t}_j = \mathrm{dg}\cdot\mathfrak{t}'_{j+1} = 0$ and $\mathrm{dg}\cdot\mathfrak{t}_{j+1} = \mathrm{dg}\cdot\mathfrak{t}'_j \neq 0$. In both cases, $\mathfrak{f}'$ satisfies also the condition of the statement. Since $\mathbf{f}^{\Uparrow}\cdot m^n$ satisfies the condition of the statement and, by Theorem 2.2.3.F, $\rightharpoonup$ is the covering relation of $\preccurlyeq$, this shows that for any $\mathbb{N}$-forest $\mathfrak{f}$ such that $\mathbf{f}^{\Uparrow}\cdot m^n \preccurlyeq \mathfrak{f}$, $\mathfrak{f}$ satisfies the condition of the statement.

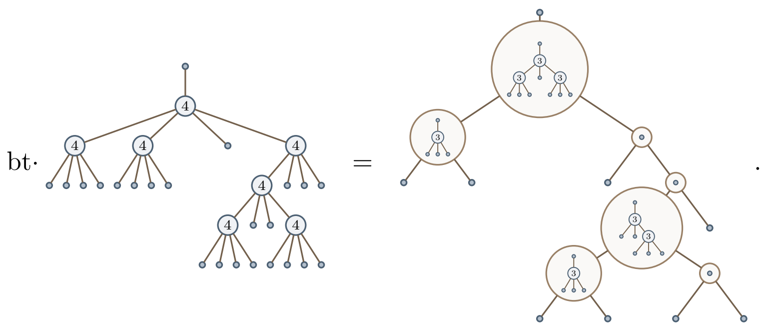
Conversely, assume that $\mathfrak{f} = n\,\mathfrak{t}_1 \dots \mathfrak{t}_n$ is an $\mathbb{N}$-forest satisfying the condition of the statement and let i be an internal node of $\mathfrak{f}$. First, if $\mathrm{pa}\cdot\mathfrak{f}\cdot i \geqslant 2$, since $\mathrm{pa}\cdot\lfloor \mathbf{f}^{\Uparrow}\cdot m^n \rfloor\cdot i = 1$, the parent edge of i in $\mathfrak{f}$ dominates the parent edge of i in $\mathbf{f}^{\Uparrow}\cdot m^n$. Otherwise, when $\mathrm{pa}\cdot\mathfrak{f}\cdot i = 1$, let us set $j := \mathrm{lp}\cdot\mathfrak{f}\cdot i$. Since $\mathfrak{f}$ satisfies the condition of the statement, $\mathrm{dg}\cdot\mathfrak{t}_j + \dots + \mathrm{dg}\cdot\mathfrak{t}_n \leqslant n - j + 1$. Moreover, as the subterm of $\mathfrak{f}$ rooted at i is $\mathfrak{t}_j$ and the set of internal nodes of $\mathfrak{f}$ which are greater than of equal to i is $\{i, \dots, n+1\}$, we have $\mathrm{dg}\cdot\mathfrak{t}_j + \dots + \mathrm{dg}\cdot\mathfrak{t}_n = n - i + 2$. This implies that $j \leqslant i - 1$, so that the parent edge of i in $\mathfrak{f}$ dominates the parent edge $(1, i-1, i)$ of i in $\mathbf{f}^{\Uparrow}\cdot m^n$. Hence, by Lemma 2.2.3.A, we have in both cases $\mathbf{f}^{\Uparrow}\cdot m^n \preccurlyeq \mathfrak{f}$, as expected. □

For any $m \in \mathbb{N}$, an *m-tree* is an element of $\mathfrak{T}\cdot\{m+1\}$. Besides, an *m-binary tree* is an element of $\mathfrak{T}\cdot\lfloor\mathfrak{T}\cdot\{m\}\rfloor$ where, here, $\mathfrak{T}\cdot\{m\}$ is seen as a signature whose all elements are of arity 2. In other words, an m-binary tree is a binary tree whose internal nodes are decorated by $m-1$-trees. Given an $m-1$-tree $\mathfrak{s}$ of degree $k \geqslant 1$, the *right comb of $\mathfrak{s}$* is the m-binary tree such that its root is decorated by $\mathfrak{s}$, the first child of the root is a leaf, and the second child is a right comb binary tree consisting in $k-1$ internal nodes decorated by $\circ$. For instance, the right comb of the 2-tree $\mathfrak{s} := 3\lfloor\iota\cdot 3\rfloor\circ\lfloor\iota\cdot 3\rfloor$ is the 3-binary tree



(4.2.1.A)

Let the map bt from the set of m-trees to the set of m-binary trees defined recursively, for any m-tree $\mathfrak{t}$, as follows. First, if $\mathfrak{t} = \circ$, then $\mathrm{bt}\cdot\mathfrak{t} = \circ$. Otherwise, let $\mathfrak{s}$ be the $m-1$-tree obtained by keeping the root of $\mathfrak{t}$ and by deleting recursively all first subterms of the kept internal nodes. In this process, let us denote by $\mathfrak{t}_1, \dots, \mathfrak{t}_k$ the forgotten subterms from left to right, with $k := \mathrm{dg}\cdot\mathfrak{s}$. Now, let $\mathfrak{r}$ be the right comb of $\mathfrak{s}$. With these definitions, $\mathrm{bt}\cdot\mathfrak{t}$ is obtained by replacing, for any $i \in [k]$, the i-th leaf of $\mathfrak{r}$ by $\mathrm{bt}\cdot\mathfrak{t}_i$. The last leaf of $\mathfrak{r}$ is left as is. For instance, for $m := 3$, we have



(4.2.1.B)

Finally, let $\mathfrak{B}\cdot m$ be the set of m-binary trees $\mathfrak{r}$ such that from any internal node of $\mathfrak{r}$ decorated by an $m-1$-tree $\mathfrak{s}$ of degree $k \geqslant 1$, there is a right branch consisting in $k-1$ internal nodes decorated by $\circ$, and each internal node decorated by $\circ$ in $\mathfrak{r}$ is a part of such right branch.

▶ **Lemma** 4.2.1.B — *For any $m \in \mathbb{N}$, from the domain consisting in the set of m-trees and on the codomain $\mathfrak{B}\cdot m$, the map* bt *is a bijection.*

◀ **Proof** — Let $\phi : \mathfrak{B}\cdot m \to \mathfrak{T}\cdot\{m+1\}$ be the map defined recursively, for any $\mathfrak{r} \in \mathfrak{B}\cdot m$ of degree n, as follows. First, if $n = 0$, then $\mathfrak{r} = \circ$. In this case, set $\phi\cdot\mathfrak{r} := \circ$. Otherwise, we have $n \geqslant 1$ and, from the description of $\mathfrak{B}\cdot m$, $\mathfrak{r}$ is the right comb $\mathfrak{r}'$ of an $m-1$-tree $\mathfrak{s}$ of degree $k \geqslant 1$ such that for any $i \in [k]$, the i-th leaf of $\mathfrak{r}'$ is attached to a subterm $\mathfrak{r}_i$ of $\mathfrak{r}$. Since for any $i \in [k]$, $\mathfrak{r}_i$ belongs to $\mathfrak{B}\cdot m$, the m-tree $\mathfrak{t}_i := \phi\cdot\mathfrak{r}_i$ is, by induction, well-defined. Let also $\mathfrak{t}'$ be the m-tree obtained by adding to each internal node of $\mathfrak{s}$ a leaf as first child. We define $\phi\cdot\mathfrak{r}$ as the m-tree obtained by replacing, for any $i \in [k]$, the first leaf of the internal node i of $\mathfrak{t}'$ by $\mathfrak{t}_i$. It follows by induction on n that ϕ is a well-defined map. Again by induction on n, it is straightforward to show that ϕ is the inverse map of bt. □

The *inorder traversal* of an m-binary tree $\mathfrak{r}$ is defined recursively as follows. It $\mathfrak{r} = \circ$, then the inorder traversal of $\mathfrak{r}$ is empty. Otherwise, we have $\mathfrak{r} = \mathfrak{s}\, \mathfrak{r}_1 \mathfrak{r}_2$ where $\mathfrak{s}$ is an $m-1$-tree, and $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are two m-binary trees. In this case, $\mathfrak{r}_1$ is visited according to the inorder traversal, then the root of $\mathfrak{r}$, and finally, $\mathfrak{r}_2$ is visited according to the inorder traversal. This procedure induces a total order on the internal nodes of $\mathfrak{r}$ where the first visited internal node is the smallest one. Now, let the map if from the set of m-binary trees to the set of balanced $\mathbb{N}$-forests such that, for any m-binary tree $\mathfrak{r}$ of degree n, $\mathrm{if}\cdot\mathfrak{r}$ is the $\mathbb{N}$-forest $n\, \mathfrak{s}_1 \dots \mathfrak{s}_n$ where for any $i \in [n]$, $\mathfrak{s}_i$ is the decoration of the i-th visited internal node of $\mathfrak{r}$ w.r.t. the inorder traversal of $\mathfrak{r}$. For instance, by considering the 3-binary tree $\mathfrak{r}$ of the right-hand side of (4.2.1.B),

$$\mathrm{if}\cdot\mathfrak{r} = 7 \lfloor \iota\cdot 3 \rfloor \lfloor 3 \lfloor \iota\cdot 3 \rfloor \circ \lfloor \iota\cdot 3 \rfloor \rfloor \circ \lfloor \iota\cdot 3 \rfloor \lfloor 3 \circ\circ \lfloor \iota\cdot 3 \rfloor \rfloor \circ\circ. \tag{4.2.1.C}$$

▶ **Lemma** 4.2.1.C — *For any $m \in \mathbb{N}$, from the domain $\mathfrak{B}\cdot m$ and on the codomain $\bigcup_{n\in\mathbb{N}} \downarrow\cdot\left(\mathbf{f}^{\Uparrow}\cdot m^n\right)$, the map* if *is a bijection.*

◀ **Proof** — Let $\phi : \bigcup_{n\in\mathbb{N}} \downarrow\cdot\left(\mathbf{f}^{\Uparrow}\cdot m^n\right) \to \mathfrak{B}\cdot m$ be the map defined recursively, for any $\mathfrak{f} \in \downarrow\cdot\left(\mathbf{f}^{\Uparrow}\cdot m^n\right)$, $n \in \mathbb{N}$, as follows. First, if $n = 0$, then $\mathfrak{f} = 0$. In this case, set $\phi\cdot\mathfrak{f} := \circ$. Otherwise, we have $n \geqslant 1$ and by Lemma 4.2.1.A, it follows by induction on n that $\mathfrak{f}$ decomposes as

$$\mathfrak{f} = \mathfrak{f}_1 \;\centerdot\; 1\,\mathfrak{s} \;\centerdot\; k-1\, \underbrace{\circ \dots \circ}_{k-1} \centerdot\, \mathfrak{f}_2 \tag{4.2.1.D}$$

where $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are two $\mathbb{N}$-forests and $\mathfrak{s}$ is an $m-1$-tree $\mathfrak{s}$ of degree $k \geqslant 1$. Let us consider this decomposition when the size $\ell \geqslant 0$ of $\mathfrak{f}_1$ is minimal. Since $\mathfrak{f}_1 \centerdot \mathfrak{f}_2$ satisfies the conditions described in Lemma 4.2.1.A, the m-binary tree $\mathfrak{r} := \phi\cdot\lfloor \mathfrak{f}_1 \centerdot \mathfrak{f}_2 \rfloor$ is, by induction, well-defined. Let also $\mathfrak{r}'$ be the right comb of $\mathfrak{s}$. We define $\phi\cdot\mathfrak{f}$ as the m-binary tree obtained by inserting the root of $\mathfrak{r}'$ onto the unique edge of $\mathfrak{r}$ such that, w.r.t. the inorder traversal, the internal nodes coming from $\mathfrak{f}_1$ are visited first, then the ones of $\mathfrak{r}'$ are visited, and finally the ones coming from $\mathfrak{f}_2$ are visited. It follows by induction on n that ϕ is a well-defined map. Again by induction on n, it is straightforward to show that ϕ is the inverse map of if. □

▶ **Theorem** 4.2.1.D — *For any $m, n \in \mathbb{N}$, the underlying set of the m-Fuss-Catalan easterly wind poset of order n is in one-to-one correspondence with the set of m-trees of degree n. The map* $\mathrm{if} \circ \mathrm{bt}$ *is such a one-to-one correspondence.*

◀ **Proof** — By Lemma 4.2.1.B, bt is a bijection between the set of m-trees and $\mathfrak{B}\cdot m$. Moreover, By Lemma 4.2.1.C, if is a bijection between $\mathfrak{B}\cdot m$ and $\bigcup_{n\in\mathbb{N}} \downarrow\cdot\lfloor \mathbf{f}^{\Uparrow}\cdot m^n \rfloor$. Since these two bijections preserve the degree, the composition $\mathrm{if} \circ \mathrm{bt}$ satisfies the property described in the statement of the theorem. □

By Theorem 4.2.1.D, the m-Fuss-Catalan easterly wind posets involve the combinatorial family of Fuss-Catalan objects. Hence, the cardinality of such posets of order n is

$$\frac{1}{mn+1}\binom{mn+n}{n}. \tag{4.2.1.E}$$

Many other posets involving this family of objects exist [BP12; CG22], and our posets differ from those presented in the cited works.

4.2.2 Rooted tree lattices. For any $n \in \mathbb{N}$, let $\mathrm{dw}{\cdot}n$ be the word on $\mathbb{N}$ of length n such that for any $i \in [n]$, $\mathrm{dw}{\cdot}n{\cdot}i = n - i$. The *rooted tree easterly wind poset of order n* is the fully tilted $\mathbb{N}$-easterly wind poset of $\mathrm{dw}{\cdot}n$, as defined in Section 2.3.5. Figure 5 shows the Hasse diagrams of such poset. By definition, all $\mathbb{N}$-terms of the rooted tree easterly wind posets are balanced

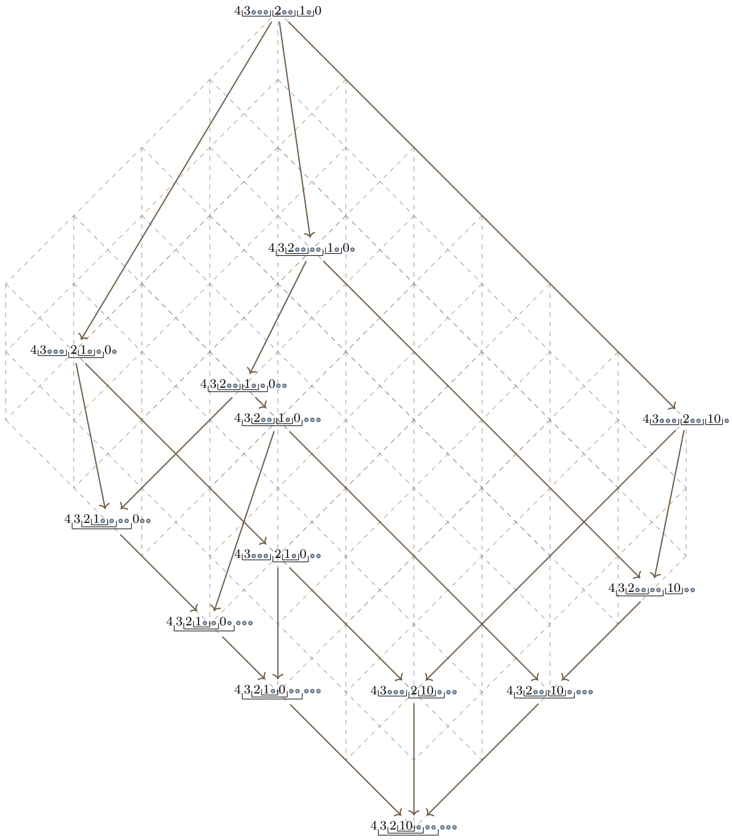
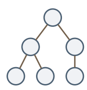


Figure 5: The Hasse diagram of the rooted tree easterly wind poset of order 4.

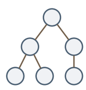
forests. Observe that the minimal element of the rooted tree easterly wind poset of order n is $\mathrm{dt}{\cdot}n := \mathbf{f}^{\Uparrow}{\cdot}\lfloor \mathrm{dw}{\cdot}n \rfloor = n \lfloor \iota{\cdot}\lfloor n-1 \rfloor \rfloor \dots \lfloor \iota{\cdot}0 \rfloor$. For instance, $\mathrm{dt}{\cdot}4 = 4 \lfloor 3 \circ\circ\circ \rfloor \lfloor 2 \circ\circ \rfloor \lfloor 1 \circ \rfloor 0$.

A *rooted tree* is recursively defined as a node together with a possibly empty list of rooted trees, each of which is attached to the node as a child. The *size* of a rooted tree $\mathfrak{r}$ is the number of nodes of $\mathfrak{r}$. The *underlying rooted tree* of an $\mathcal{S}$-term $\mathfrak{t}$ is the rooted tree $\mathrm{rt}{\cdot}\mathfrak{t}$ obtained by removing the leaves and their adjacent edges in $\mathfrak{t}$, as well as the decorations of its internal nodes. For instance,

$$\mathrm{rt}{\cdot}\lfloor \mathsf{a}_4 \circ \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \circ \rfloor \mathsf{a}_0 \rfloor \circ \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ\circ \rfloor \rfloor = \quad (4.2.2.A)$$



Moreover, the *fully tilted term* $\mathrm{ft}{\cdot}\mathfrak{r}$ of a rooted tree $\mathfrak{r}$ is the fully tilted $\mathbb{N}$-term obtained by labelling from $n-1$ to 0 the nodes of $\mathfrak{r}$ w.r.t. the preorder traversal and then, by grafting to each node some leaves as rightmost children such that each node labeled by k has k children. This ensures that $\mathrm{ft}{\cdot}\mathfrak{r}$ is fully tilted. For instance,

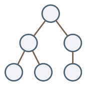


$$\mathrm{ft}{\cdot} \quad = 5 \lfloor 4 \lfloor 3 \circ\circ\circ \rfloor \lfloor 2 \circ\circ \rfloor \circ\circ \rfloor \lfloor 1\, 0 \rfloor \circ\circ\circ. \quad (4.2.2.B)$$

▶ **Lemma** 4.2.2.A — *For any $n \in \mathbb{N}$, the map* rt *is a one-to-one correspondence from the underlying* *set of the rooted tree easterly wind poset of order n and the set of rooted trees of size n.*

◀ **Proof** — Let T be the underlying set of the rooted tree easterly wind poset of order n. The fact that the map rt on the domain T is injective is a consequence of the fact that all $\mathbb{N}$-terms of T are fully tilted. Let us show that each rooted tree $\mathfrak{r}$ admits an antecedent in T for the map rt. By setting $\mathfrak{t} := \mathrm{ft} \cdot \mathfrak{r}$, from the definitions of the maps rt and ft, it follows immediately that $\mathrm{rt} \cdot \mathfrak{t} = \mathfrak{r}$. It remains to prove that $\mathfrak{t}$ belongs to T. By construction of the $\mathbb{N}$-term $\mathbf{f}^{\Uparrow} \cdot \lfloor \mathrm{dw} \cdot n \rfloor$, for any internal node i of $\mathfrak{t}$, the parent edge of i in $\mathbf{f}^{\Uparrow} \cdot \lfloor \mathrm{dw} \cdot n \rfloor$ is dominated by that of i in $\mathfrak{t}$. Hence, by Lemma 2.2.3.A, $\mathbf{f}^{\Uparrow} \cdot \lfloor \mathrm{dw} \cdot n \rfloor \preccurlyeq \mathfrak{t}$. This shows that $\mathfrak{t} \in T$ and implies the statement of the lemma. □

The *scope sequence* of a rooted tree $\mathfrak{r}$ of size n is the word $\mathrm{sc} \cdot \mathfrak{r}$ of length n such that for any $i \in [n]$, $\mathrm{sc} \cdot \mathfrak{r} \cdot i$ is the number descendants of the i-th visited node in the preorder traversal of $\mathfrak{r}$. For instance,

sc·  = 520010. (4.2.2.C)

Observe that the scope sequence of an $\mathcal{S}$-term $\mathfrak{t}$ as defined in Section 2.3.5 and the scope sequence of the rooted tree $\mathrm{rt} \cdot \mathfrak{t}$ coincide, that is $\mathrm{sc} \cdot \lfloor \mathrm{rt} \cdot \mathfrak{t} \rfloor = \mathrm{sc} \cdot \mathfrak{t}$.

The *Tamari partial order* [Tam62] is a partial order $\preccurlyeq_{\mathrm{T}}$ defined on the family of Catalan objects of a given size. We consider here the following description of this order involving rooted trees [Knu04]. Given two rooted trees $\mathfrak{r}_1$ and $\mathfrak{r}_2$ of the same size n, $\mathfrak{r}_1 \preccurlyeq_{\mathrm{T}} \mathfrak{r}_2$ holds if and only if for any $i \in [n]$, $\mathrm{sc} \cdot \mathfrak{r}_1 \cdot i \leqslant \mathrm{sc} \cdot \mathfrak{r}_2 \cdot i$.

▶ **Proposition** 4.2.2.B — *For any $n \in \mathbb{N}$, the map* rt *is a poset isomorphism between the rooted tree easterly wind poset of order n and the Tamari poset of order n.*

◀ **Proof** — Let T be the underlying set of the rooted tree easterly wind poset of order n. First of all, by Lemma 4.2.2.A, the map rt is a bijection between T and the underlying set of the Tamari poset of order n. It remains to prove that rt is an order embedding, that is, for any $\mathfrak{t}_1, \mathfrak{t}_2 \in T$, $\mathfrak{t}_1 \preccurlyeq \mathfrak{t}_2$ if and only if $\mathrm{rt} \cdot \mathfrak{t}_1 \preccurlyeq_{\mathrm{T}} \mathrm{rt} \cdot \mathfrak{t}_2$. This property is a consequence of Proposition 2.3.5.B and the fact that, as noticed above, for any $\mathbb{N}$-term $\mathfrak{t}$, $\mathrm{sc} \cdot \mathfrak{t} = \mathrm{sc} \cdot \lfloor \mathrm{rt} \cdot \mathfrak{t} \rfloor$. □

4.3 Leaning forest lattices

Leaning forest $\mathcal{S}$-easterly wind lattices will be used to construct bases of natural Hopf algebras of free nonsymmetric operads in the next section. Here we introduce these lattices and establish some of their properties. We also define two concatenation operations and a shuffle operation on leaning forests. These operations will subsequently be employed to describe the product of natural Hopf algebras of free nonsymmetric operads on alternative bases.

4.3.1 Leaning forests.

An $\mathcal{S}$-forest $\mathfrak{f} = n\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ is *leaning* if $\mathfrak{f}$ is balanced and is $\{1\}$-tilted. For instance, 0 and $4\, \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \rfloor \mathsf{a}_0 \circ \circ$ are leaning $\mathcal{S}_{\mathrm{e}}$-forests. On the contrary, $4\, \mathsf{a}_0 \circ \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_2 \circ \circ \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \circ$ is a balanced $\mathcal{S}_{\mathrm{e}}$-forest which is not leaning since it is not $\{1\}$-tilted, and $3\, \lfloor \mathsf{a}_1 \circ \rfloor \circ \circ$ is a $\{1\}$-tilted $\mathcal{S}$-forest which is not leaning since it is not balanced. Let us denote by $\mathfrak{L} \cdot \mathcal{S}$ the set of leaning $\mathcal{S}$-forests. The *length* of a leaning $\mathcal{S}$-forest $\mathfrak{f}$ is the number of children subterms of the root of $\mathfrak{f}$ which are not leaves. For instance, the length of $4\, \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \rfloor \mathsf{a}_0 \circ \circ$ is 2.

For any $w \in \mathcal{S}^*$, the *leaning forest $\mathcal{S}$-easterly wind poset of w* is the balanced forest $\{1\}$-tilted $\mathcal{S}$-easterly wind poset of w, as defined in Section 4.1. Figure 6 shows the Hasse diagram of such poset.

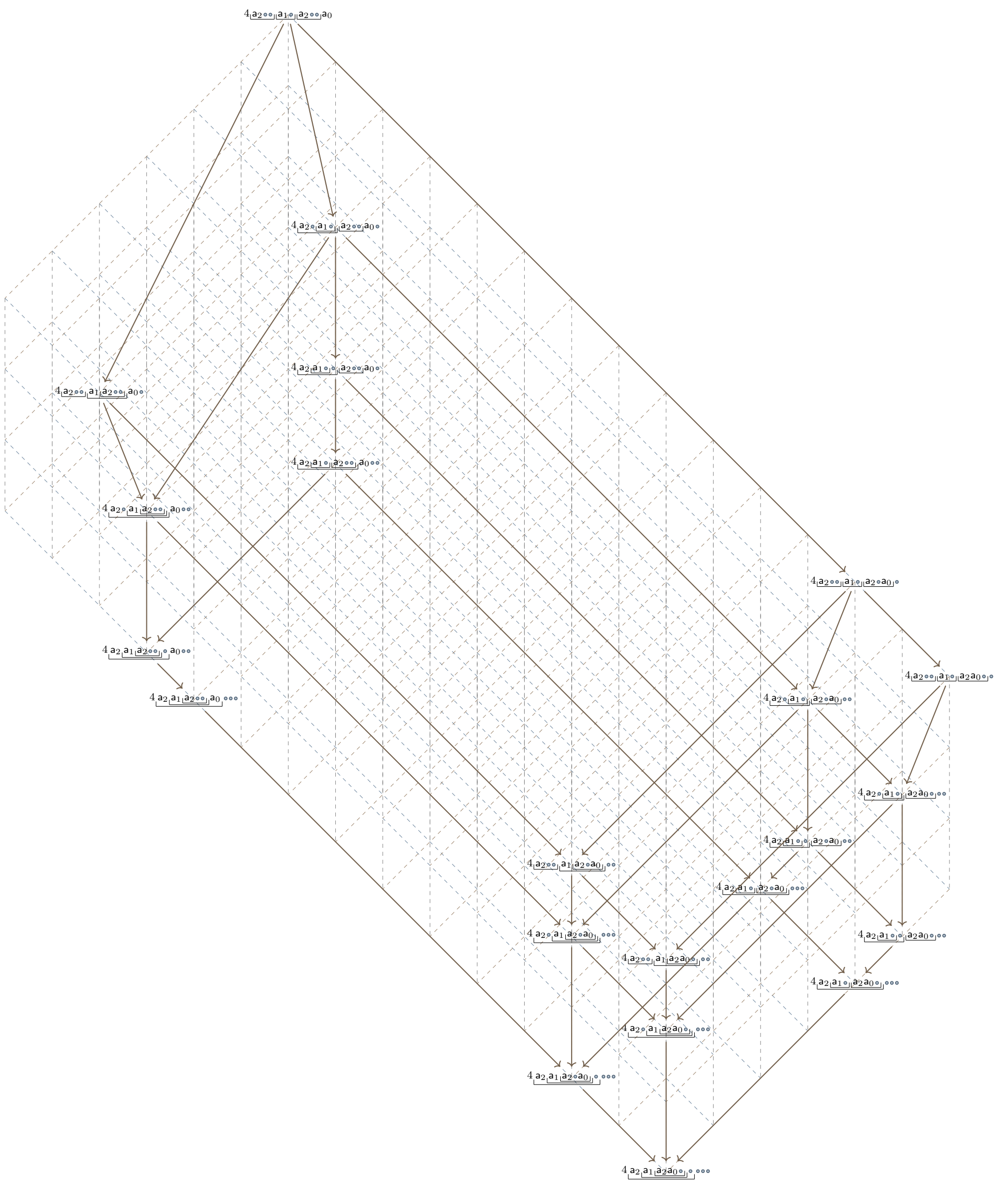


Figure 6: The Hasse diagram of the leaning forest $\mathcal{S}_{\mathrm{e}}$-easterly wind poset of $\mathsf{a}_4\mathsf{a}_2\mathsf{a}_1\mathsf{a}_2\mathsf{a}_0$.

▶ **Proposition** 4.3.1.A — *Let $\mathcal{S}$ be a signature and let w be a word on $\mathcal{S}$ of length $n \in \mathbb{N}$. The leaning forest $\mathcal{S}$-easterly wind poset of w contains all leaning $\mathcal{S}$-forests having $n \centerdot w$ as decoration word.*

◀ **Proof** — Let us prove that for any leaning forest $\mathfrak{f}$ having $n \centerdot w$ as decoration word, $\mathbf{f}^{\Uparrow} \cdot w \preccurlyeq \mathfrak{f}$. For this, let i be an internal node of $\mathfrak{f}$ different from the root, and let (i', j, i) be the parent edge of i in $\mathfrak{f}$. From the definition of $\mathbf{f}^{\Uparrow} \cdot w$, the parent edge of the internal node i in $\mathbf{f}^{\Uparrow} \cdot w$ is $(1, i-1, i)$. Now, if $i' \geqslant 2$, then the parent edge of i in $\mathfrak{f}$ dominates that of i in $\mathbf{f}^{\Uparrow} \cdot w$. Otherwise, we have $i' = 1$. Let us prove in this case that $j \leqslant i - 1$. Indeed, assume by contradiction that $j \geqslant i$. Under this assumption, among the first $j - 1$ children of the root of $\mathfrak{f}$, since they comprise only $i - 2$

internal nodes, there must be at least one leaf. Therefore, $\mathfrak{f}$ would not be $\{1\}$-tilted, contradicting our hypotheses. Hence, even in this case, the parent edge of i in $\mathfrak{f}$ dominates that of i in $\mathbf{f}^{\Uparrow} \cdot w$. It follows now from Lemma 2.2.3.A that $\mathbf{f}^{\Uparrow} \cdot w \preccurlyeq \mathfrak{f}$. □

Proposition 4.3.1.A provides the notable property that any leaning $\mathcal{S}$-forest belong to a leaning forest $\mathcal{S}$-easterly wind poset.

4.3.2 Restrictions. Let $\mathfrak{f}$ be a leaning $\mathcal{S}$-forest of size n. Given a subset I of $[n]$, let $\bar{I} := \{i + 1 : i \in I\}$. The *restriction* $\mathfrak{f} \cdot I$of $\mathfrak{f}$ on I is the leaning $\mathcal{S}$-forest obtained by keeping only the internal nodes from the set $\{1\} \cup \bar{I}$ of $\mathfrak{f}$ and their adjacent edges, and by setting $\# I$ as the decoration of the root. For instance, by considering the leaning $\mathcal{S}_{\mathrm{e}}$-forest

$$\mathfrak{f} := 5\, \mathsf{a}_0 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \rfloor \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \circ\circ, \tag{4.3.2.A}$$

we have $\mathfrak{f} \cdot \{1, 4\} = 2\, \mathsf{a}_0 \mathsf{a}_0$, $\mathfrak{f} \cdot \{2, 3\} = 2 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \circ$, and $\mathfrak{f} \cdot \{2, 4, 5\} = 3 \lfloor \mathsf{a}_2 \circ\circ \rfloor \mathsf{a}_0 \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor$.

When I is an interval of $[n]$, let $\theta_I : \bar{I} \to [\# I]$ be the map defined for any $i \in \bar{I}$ by $\theta_I \cdot i := i - \min I + 1$. Observe that for any $i \in \bar{I}$, the internal node i of $\mathfrak{f}$ gives rise to the internal node $\theta_I \cdot i$ in $\mathfrak{f} \cdot I$. This map will be used to lighten the notation during the proof of the following lemma.

▶ **Lemma** 4.3.2.A — *Let $\mathcal{S}$ be a signature, and let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two leaning $\mathcal{S}$-forests of the same size n. If $\mathfrak{f}_1 \preccurlyeq \mathfrak{f}_2$ and I is an interval of $[n]$, then $\mathfrak{f}_1 \cdot I \preccurlyeq \mathfrak{f}_2 \cdot I$.*

◀ **Proof** — Assume that $\mathfrak{f}_1 \preccurlyeq \mathfrak{f}_2$ and that I is an interval of $[n]$. As the case where I is empty is immediate, we assume that I is nonempty. Let i be an internal node of both $\mathfrak{f}_1$ and $\mathfrak{f}_2$ belonging to $\bar{I}$. Let also i_1 (resp. i_2) be the parent of i in $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$). We have several cases to explore depending on whether i_1 and i_2 belong to $\bar{I}$.

1. Assume first that $i_1 \in \bar{I}$. By Lemma 2.2.3.A, the parent edge of i in $\mathfrak{f}_1$ is dominated by that of i in $\mathfrak{f}_2$. In particular, this implies that $i_1 \leqslant i_2 < i$ so that, since I is an interval, $i_2 \in \bar{I}$. For this reason, we have also $\mathrm{lp} \cdot \lfloor \mathfrak{f}_2 \cdot I \rfloor \cdot \lfloor \theta_I \cdot i \rfloor = \mathrm{lp} \cdot \mathfrak{f}_2 \cdot i$. Similarly, since $i_1 \in \bar{I}$, $\mathrm{lp} \cdot \lfloor \mathfrak{f}_1 \cdot I \rfloor \cdot \lfloor \theta_I \cdot i \rfloor = \mathrm{lp} \cdot \mathfrak{f}_1 \cdot i$. These properties imply that the parent edge of $\theta_I \cdot i$ in $\mathfrak{f}_1 \cdot I$ is dominated by that of $\theta_I \cdot i$ in $\mathfrak{f}_2 \cdot I$.
2. Assume now that $i_1 \notin \bar{I}$ and $i_2 \notin \bar{I}$. By definition of the restriction operation, the parent of i is the internal node 1 in both $\mathfrak{f}_1 \cdot I$ and $\mathfrak{f}_2 \cdot I$. Moreover, since both $\mathfrak{f}_1 \cdot I$ and $\mathfrak{f}_2 \cdot I$ are leaning, $\mathrm{lp} \cdot \lfloor \mathfrak{f}_1 \cdot I \rfloor \cdot \lfloor \theta_I \cdot i \rfloor \geqslant \mathrm{lp} \cdot \lfloor \mathfrak{f}_2 \cdot I \rfloor \cdot \lfloor \theta_I \cdot i \rfloor$. Therefore, the parent edge of $\theta_I \cdot i$ in $\mathfrak{f}_1 \cdot I$ is dominated by that of $\theta_I \cdot i$ in $\mathfrak{f}_2 \cdot I$.
3. In the remaining case, $i_1 \notin \bar{I}$ and $i_2 \in \bar{I}$. By definition of the restriction operation, the parent of i is the internal node 1 in $\mathfrak{f}_1$. Moreover, since $i_2 \in \bar{I}$, $\theta_I \cdot i_2 \geqslant 2$. This shows that the parent edge of $\theta_I \cdot i$ in $\mathfrak{f}_1 \cdot I$ is dominated by that of $\theta_I \cdot i$ in $\mathfrak{f}_2 \cdot I$.

We have shown that for any internal node $\theta_I \cdot i$ of both $\mathfrak{f}_1 \cdot I$ and $\mathfrak{f}_2 \cdot I$, the parent edge of $\theta_I \cdot i$ in $\mathfrak{f}_1 \cdot I$ is dominated by that of $\theta_I \cdot i$ in $\mathfrak{f}_2 \cdot I$. Therefore, by Lemma 2.2.3.A, $\mathfrak{f}_1 \cdot I \preccurlyeq \mathfrak{f}_2 \cdot I$. □

Let us introduce two specific restrictions. Given a leaning $\mathcal{S}$-forest $\mathfrak{f}$ of size n, for any $k \in [\![n]\!]$, let $\Uparrow \cdot k \cdot \mathfrak{f} := \mathfrak{f} \cdot [1, k]$ and $\Downarrow \cdot k \cdot \mathfrak{f} := \mathfrak{f} \cdot [k + 1, n]$. We call the first restriction the *k-top restriction* of $\mathfrak{f}$ and the second, the *k-bottom restriction* of $\mathfrak{f}$. For instance, by considering the leaning $\mathcal{S}_{\mathrm{e}}$-forest $\mathfrak{f}$ of (4.3.2.A), we have

- $(\Uparrow\cdot 0\cdot\mathfrak{f}, \Downarrow\cdot 0\cdot\mathfrak{f}) = (0, \mathfrak{f})$;
- $(\Uparrow\cdot 1\cdot\mathfrak{f}, \Downarrow\cdot 1\cdot\mathfrak{f}) = \left(1\, \mathsf{a}_0, 4\, \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \rfloor \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \circ\circ\right)$;
- $(\Uparrow\cdot 2\cdot\mathfrak{f}, \Downarrow\cdot 2\cdot\mathfrak{f}) = (2\, \mathsf{a}_0 \lfloor \mathsf{a}_2 \circ\circ \rfloor, 3\, \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \circ)$;
- $(\Uparrow\cdot 3\cdot\mathfrak{f}, \Downarrow\cdot 3\cdot\mathfrak{f}) = \left(3\, \mathsf{a}_0 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor, 2\, \mathsf{a}_0 \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor\right)$;
- $(\Uparrow\cdot 4\cdot\mathfrak{f}, \Downarrow\cdot 4\cdot\mathfrak{f}) = \left(4\, \mathsf{a}_0 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \mathsf{a}_0 \rfloor \rfloor \circ\circ\circ, 1\, \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor\right)$;
- $(\Uparrow\cdot 5\cdot\mathfrak{f}\ \Downarrow\cdot 5\cdot\mathfrak{f}) = (\mathfrak{f}, 0)$.

4.3.3 Over and under operations. Let $\diagup$ be the *over* operation on leaning $\mathcal{S}$-forests defined, for any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, by $\mathfrak{f}_1 \diagup \mathfrak{f}_2 := \mathrm{tlt}\cdot\{1\}\cdot\lfloor \mathfrak{f}_1 \centerdot \mathfrak{f}_2 \rfloor$ where tlt is the tilting map defined in Section 2.3.1 and $\centerdot$ is the concatenation operation defined in Section 4.1.1. For instance, on leaning $\mathcal{S}_{\mathrm{e}}$-forests,

$$3\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ \diagup 5\, \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \circ = 8\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ. \tag{4.3.3.A}$$

Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two leaning $\mathcal{S}$-forests such that $\mathfrak{f}_1$ is of size n_1 and of length ℓ_1. From the definition of the over operation, each edge (i', j, i) of $\mathfrak{f}_1 \diagup \mathfrak{f}_2$ obeys to the following rules:

1. if $i' < i \leqslant n_1 + 1$ (that is, both i' and i come from internal nodes of $\mathfrak{f}_1$), then (i', j, i) is an edge of $\mathfrak{f}_1$;
2. if $n_1 + 2 \leqslant i' < i$ (that is, both i' and i come from internal nodes of $\mathfrak{f}_2$), then $(i' - n_1, j, i - n_1)$ is an edge of $\mathfrak{f}_2$;
3. otherwise, we have $i' \leqslant n_1 + 1$ (that is, i' comes from an internal node of $\mathfrak{f}_1$) and $n_1 + 2 \leqslant i$ (that is, i comes from an internal node of $\mathfrak{f}_2$). In this case, we have $i' = 1$ (that is, i' is the root of $\mathfrak{f}_1$) and that $(1, j - \ell_1, i - n_1)$ is an edge of $\mathfrak{f}_2$.

Similarly, let $\diagdown$ be the *under* operation on leaning $\mathcal{S}$-forests such that, for any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, by denoting by n_1 the size of $\mathfrak{f}_1$, by n_2 the size of $\mathfrak{f}_2$, and by r the number of extreme leaves of $\mathfrak{f}_1$, $\mathfrak{f}_1 \diagdown \mathfrak{f}_2$ is the leaning $\mathcal{S}$-forest built by grafting, for any $j \in [n_2 + r]$, the j-th subterm of the root of $\mathfrak{f}_2 \centerdot \lfloor \iota\cdot r \rfloor$ onto the j-th extreme leaf of $\mathfrak{f}_1 \centerdot \lfloor \iota\cdot n_2 \rfloor$. For instance, on leaning $\mathcal{S}_{\mathrm{e}}$-forests,

$$3\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ \diagdown 5\, \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \circ = 8\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \rfloor \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ\circ\circ. \tag{4.3.3.B}$$

Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two leaning $\mathcal{S}$-forests such that $\mathfrak{f}_1$ is of size n_1 and $\mathfrak{f}_2$ is of size n_2. From the definition of the under operation, each edge (i', j, i) of $\mathfrak{f}_1 \diagdown \mathfrak{f}_2$ obeys to the following rules:

1. if $i' < i \leqslant n_1 + 1$ (that is, both i' and i come from internal nodes of $\mathfrak{f}_1$), then (i', j, i) is an edge of $\mathfrak{f}_1$;
2. if $n_1 + 2 \leqslant i' < i$ (that is, both i' and i come from internal nodes of $\mathfrak{f}_2$), then $(i' - n_1, j, i - n_1)$ is an edge of $\mathfrak{f}_2$;
3. otherwise, we have $i' \leqslant n_1 + 1$ (that is, i' comes from an internal node of $\mathfrak{f}_1$) and $n_1 + 2 \leqslant i$ (that is, i comes from an internal node of $\mathfrak{f}_2$). In this case, $(1, j', i - n_1)$ is an edge of $\mathfrak{f}_2$ where in $\mathfrak{f}_1 \centerdot \lfloor \iota\cdot n_2 \rfloor$, the child of i' at position j is the j'-th extreme leaf of this $\mathcal{S}$-forest.

▶ **Lemma** 4.3.3.A — *Let $\mathcal{S}$ be a signature and let $\mathfrak{f}_1$, $\mathfrak{f}'_1$, $\mathfrak{f}_2$, and $\mathfrak{f}'_2$ be leaning $\mathcal{S}$-forests. If $\mathfrak{f}_1 \preccurlyeq \mathfrak{f}'_1$ and $\mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_2$, then*

(i) $\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_1 \diagup \mathfrak{f}'_2$; *(ii)* $\mathfrak{f}_1 \diagdown \mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_1 \diagdown \mathfrak{f}'_2$.

◀ **Proof** — Le us assume that $\mathfrak{f}_1 \preccurlyeq \mathfrak{f}'_1$ and $\mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_2$. By Lemma 2.2.3.A, for any internal node i of $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$), the parent edge of i in $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$) is dominated by that of i in $\mathfrak{f}'_1$ (resp. $\mathfrak{f}'_2$).

Now, by the previous description of the edges of a leaning $\mathcal{S}$-forest obtained from the over operation applied on two leaning $\mathcal{S}$-forests, together with the fact that for any leaning $\mathcal{S}$-forests $\mathfrak{g}$ and $\mathfrak{g}'$, $\mathfrak{g} \preccurlyeq \mathfrak{g}'$ implies that the length of $\mathfrak{g}$ is greater than or equal to the length of $\mathfrak{g}'$, for any internal node i of $\mathfrak{f}_1 \diagup \mathfrak{f}_2$ the parent edge of i in $\mathfrak{f}_1 \diagup \mathfrak{f}_2$ is dominated by that of i in $\mathfrak{f}'_1 \diagup \mathfrak{f}'_2$. Therefore, by Lemma 2.2.3.A, (i) holds.

Similarly, by the previous description of the edges of a leaning $\mathcal{S}$-forest obtained from the under operation applied on two leaning $\mathcal{S}$-forests, together with the fact that for any leaning $\mathcal{S}$-forests $\mathfrak{g}$ and $\mathfrak{g}'$, $\mathfrak{g} \preccurlyeq \mathfrak{g}'$ implies that the number of extreme leaves of $\mathfrak{g}$ is smaller than or equal to the number of extreme leaves of $\mathfrak{g}'$, for any internal node i of $\mathfrak{f}_1 \diagup \mathfrak{f}_2$ the parent edge of i in $\mathfrak{f}_1 \diagdown \mathfrak{f}_2$ is dominated by that of i in $\mathfrak{f}'_1 \diagdown \mathfrak{f}'_2$. Therefore, by Lemma 2.2.3.A, (ii) holds. □

For any leaning $\mathcal{S}$-forest $\mathfrak{f}$ of size n and any $k \in [\![0, n]\!]$, let

$$\diagup \cdot k \cdot \mathfrak{f} := \Uparrow \cdot k \cdot \mathfrak{f} \diagup \Downarrow \cdot k \cdot \mathfrak{f} \tag{4.3.3.C}$$

and

$$\diagdown \cdot k \cdot \mathfrak{f} := \Uparrow \cdot k \cdot \mathfrak{f} \diagdown \Downarrow \cdot k \cdot \mathfrak{f}. \tag{4.3.3.D}$$

▶ **Lemma** 4.3.3.B — *Let $\mathcal{S}$ be a signature and let $\mathfrak{f}$ be a leaning $\mathcal{S}$-forest of size n. For any $k \in [\![0, n]\!]$,*

$$\diagup \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f} \preccurlyeq \diagdown \cdot k \cdot \mathfrak{f}. \tag{4.3.3.E}$$

◀ **Proof** — From the definition of $\diagup \cdot k$ (resp. $\diagdown \cdot k$) and the previous description of the edges of a leaning $\mathcal{S}$-forest obtained from the over (resp. under) operation applied on two leaning $\mathcal{S}$-forests, the leaning $\mathcal{S}$-forests $\mathfrak{f}$ and $\diagup \cdot k \cdot \mathfrak{f}$ (resp. $\diagdown \cdot k \cdot \mathfrak{f}$) have the same edges, except possibly the parent edges of the internal nodes i with $i \geqslant k + 2$, which are of the form (i', j, i) in $\diagup \cdot k \cdot \mathfrak{f}$ (resp. $\diagdown \cdot k \cdot \mathfrak{f}$) where j is an integer, and of the form (i'', j', i) in $\mathfrak{f}$ with $i'' > i'$ (resp. $i'' < i'$), or both $i'' = i'$ and $j' = j$. By Lemma 2.2.3.A, this implies that $\diagup \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}$ (resp. $\mathfrak{f} \preccurlyeq \diagdown \cdot k \cdot \mathfrak{f}$). □

▶ **Lemma** 4.3.3.C — *Let $\mathcal{S}$ be a signature, and let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two leaning $\mathcal{S}$-forests. By denoting by k the size of $\mathfrak{f}_1$, for any leaning $\mathcal{S}$-forest $\mathfrak{f}$, the two following properties hold:*

(i) $\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \mathfrak{f}$ if and only if $\mathfrak{f}_1 \preccurlyeq \Uparrow \cdot k \cdot \mathfrak{f}$ and $\mathfrak{f}_2 \preccurlyeq \Downarrow \cdot k \cdot \mathfrak{f}$;

(ii) $\mathfrak{f} \preccurlyeq \mathfrak{f}_1 \diagdown \mathfrak{f}_2$ if and only if $\Uparrow \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_1$ and $\Downarrow \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_2$.

◀ **Proof** — Assume first that $\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \mathfrak{f}$ (resp. $\mathfrak{f} \preccurlyeq \mathfrak{f}_1 \diagdown \mathfrak{f}_2$). By Lemma 4.3.2.A, we have $\Uparrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagup \mathfrak{f}_2 \rfloor \preccurlyeq \Uparrow \cdot k \cdot \mathfrak{f}$ (resp. $\Uparrow \cdot k \cdot \mathfrak{f} \preccurlyeq \Uparrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagdown \mathfrak{f}_2 \rfloor$) and $\Downarrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagup \mathfrak{f}_2 \rfloor \preccurlyeq \Downarrow \cdot k \cdot \mathfrak{f}$ (resp. $\Downarrow \cdot k \cdot \mathfrak{f} \preccurlyeq \Downarrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagdown \mathfrak{f}_2 \rfloor$). Now, by definition of the over (resp. under) operation, we have $\Uparrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagup \mathfrak{f}_2 \rfloor = \mathfrak{f}_1$ (resp. $\Uparrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagdown \mathfrak{f}_2 \rfloor = \mathfrak{f}_1$) and $\Downarrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagup \mathfrak{f}_2 \rfloor = \mathfrak{f}_2$ (resp. $\Downarrow \cdot k \cdot \lfloor \mathfrak{f}_1 \diagdown \mathfrak{f}_2 \rfloor = \mathfrak{f}_2$). Hence, $\mathfrak{f}_1 \preccurlyeq \Uparrow \cdot k \cdot \mathfrak{f}$ (resp. $\Uparrow \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_1$) and $\mathfrak{f}_2 \preccurlyeq \Downarrow \cdot k \cdot \mathfrak{f}$ (resp. $\Downarrow \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_2$). This proves the direct implication of (i) (resp. (ii)).

Assume conversely that $\mathfrak{f}_1 \preccurlyeq \Uparrow \cdot k \cdot \mathfrak{f}$ (resp. $\Uparrow \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_1$) and $\mathfrak{f}_2 \preccurlyeq \Downarrow \cdot k \cdot \mathfrak{f}$ (resp. $\Downarrow \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_2$). By Lemma 4.3.3.B, we have $\diagup \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}$ (resp. $\mathfrak{f} \preccurlyeq \diagdown \cdot k \cdot \mathfrak{f}$). Given this, by Lemma 4.3.3.A, we have $\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \diagup \cdot k \cdot \mathfrak{f}$ (resp. $\diagdown \cdot k \cdot \mathfrak{f} \preccurlyeq \mathfrak{f}_1 \diagdown \mathfrak{f}_2$). Therefore, by using the transitivity of $\preccurlyeq$, this implies that $\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \mathfrak{f}$ (resp. $\mathfrak{f} \preccurlyeq \mathfrak{f}_1 \diagdown \mathfrak{f}_2$). The converse of (i) (resp. (ii)) has been established. □

4.3.4 Shuffle product. Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two leaning $\mathcal{S}$-forests of respective sizes n_1 and n_2. The *shifted shuffle* of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ is the set $\mathfrak{f}_1 \,\bar{\sqcup\!\sqcup}\, \mathfrak{f}_2$ of leaning $\mathcal{S}$-forests $\mathfrak{f}$ of size $n_1 + n_2$ such that $\Uparrow \cdot n_1 \cdot \mathfrak{f} = \mathfrak{f}_1$ and $\Downarrow \cdot n_1 \cdot \mathfrak{f} = \mathfrak{f}_2$.

For instance, let the leaning $\mathcal{S}_{\mathrm{e}}$-forests $\mathfrak{f}_1 := 4\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \circ \rfloor \circ \rfloor \circ\circ$ and $\mathfrak{f}_2 := 3\, \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \mathsf{a}_0 \circ$. The set $\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2$ contains exactly the four leaning $\mathcal{S}_{\mathrm{e}}$-forests

- $7\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \mathsf{a}_0 \circ\circ\circ$;
- $7\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \circ \rfloor \rfloor \mathsf{a}_0 \circ\circ\circ\circ$;
- $7\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \circ \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \rfloor \mathsf{a}_0 \circ\circ\circ\circ$;
- $7\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_3 \circ \mathsf{a}_0 \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \rfloor \mathsf{a}_0 \rfloor \circ\circ\circ\circ\circ$.

Observe that among these four $\mathcal{S}_{\mathrm{e}}$-forests, the first (resp. last) one is $\mathfrak{f}_1 \,/\, \mathfrak{f}_2$ (resp. $\mathfrak{f}_1 \setminus \mathfrak{f}_2$).

▶ **Proposition** 4.3.4.A — *For any signature $\mathcal{S}$ and any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, $\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2$ is an interval of the leaning forest $\mathcal{S}$-easterly wind poset of $w_1 \centerdot w_2$, where the decoration word of $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$) is $n_1 \centerdot w_1$ (resp. $n_2 \centerdot w_2$) and n_1 (resp. n_2) is the size of $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$). More precisely,*

$$\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2 = [\mathfrak{f}_1 \,/\, \mathfrak{f}_2, \mathfrak{f}_1 \setminus \mathfrak{f}_2]. \tag{4.3.4.A}$$

◀ **Proof** — First of all, by definition of $\overline{\sqcup\!\sqcup}$, all $\mathcal{S}$-forests of $\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2$ have $(n_1 + n_2) \centerdot w_1 \centerdot w_2$ as decoration word. Hence, by Proposition 4.3.1.A, $\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2$ is a subset of the leaning forest $\mathcal{S}$-easterly wind poset of $w_1 \centerdot w_2$.

Let $\mathfrak{f}$ be an $\mathcal{S}$-forest. Assume that $\mathfrak{f}$ belongs to the set $\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2$. By definition of $\overline{\sqcup\!\sqcup}$, we have $\Uparrow \cdot n_1 \cdot \mathfrak{f} = \mathfrak{f}_1$ and $\Downarrow \cdot n_1 \cdot \mathfrak{f} = \mathfrak{f}_2$. Therefore, by Lemma 4.3.3.B, $\mathfrak{f}_1 \,/\, \mathfrak{f}_2 \preccurlyeq \mathfrak{f} \preccurlyeq \mathfrak{f}_1 \setminus \mathfrak{f}_2$. Conversely, assume that $\mathfrak{f}_1 \,/\, \mathfrak{f}_2 \preccurlyeq \mathfrak{f} \preccurlyeq \mathfrak{f}_1 \setminus \mathfrak{f}_2$. By Lemma 4.3.2.A, we have $\Uparrow \cdot n_1 \cdot \lfloor \mathfrak{f}_1 \,/\, \mathfrak{f}_2 \rfloor \preccurlyeq \Uparrow \cdot n_1 \cdot \mathfrak{f} \preccurlyeq \Uparrow \cdot n_1 \lfloor \mathfrak{f}_1 \setminus \mathfrak{f}_2 \rfloor$ and $\Downarrow \cdot n_1 \cdot \lfloor \mathfrak{f}_1 \,/\, \mathfrak{f}_2 \rfloor \preccurlyeq \Downarrow \cdot n_1 \cdot \mathfrak{f} \preccurlyeq \Downarrow \cdot n_1 \lfloor \mathfrak{f}_1 \setminus \mathfrak{f}_2 \rfloor$. Directly from the definitions of the over and under operations, we have $\Uparrow \cdot n_1 \cdot \lfloor \mathfrak{f}_1 \,/\, \mathfrak{f}_2 \rfloor = \mathfrak{f}_1 = \Uparrow \cdot n_1 \cdot \lfloor \mathfrak{f}_1 \setminus \mathfrak{f}_2 \rfloor$ and $\Downarrow \cdot n_1 \cdot \lfloor \mathfrak{f}_1 \,/\, \mathfrak{f}_2 \rfloor = \mathfrak{f}_2 = \Downarrow \cdot n_1 \cdot \lfloor \mathfrak{f}_1 \setminus \mathfrak{f}_2 \rfloor = \mathfrak{f}_2$. Hence, we have $\Uparrow \cdot n_1 \cdot \mathfrak{f} = \mathfrak{f}_1$ and $\Downarrow \cdot n_1 \cdot \mathfrak{f} = \mathfrak{f}_2$, showing that $\mathfrak{f}$ belongs to $\mathfrak{f}_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}_2$. Therefore, (4.3.4.A) holds. □

5 Natural Hopf algebras of nonsymmetric operads

In this final section we build on the preceding material to accomplish one of the main objectives of this work: introducing new bases for the natural Hopf algebras of free nonsymmetric operads. Existing descriptions of these Hopf algebras are given with respect to an elementary basis, under which the product of two basis elements is expressed as a concatenation of leaning forests. We then construct two additional bases, the fundamental basis and the homogeneous basis, such that the product of two elements is, respectively, a shuffle or a specialized concatenation of leaning forests.

5.1 Nonsymmetric operads and natural Hopf algebras

In this preliminary section, we recall the main elementary concepts of operad theory and of the natural Hopf algebra construction.

5.1.1 Nonsymmetric operads. We use the standard definitions about nonsymmetric operads (called simply *operads* here), as found in [Gir18]. An operad $\mathcal{O}$ is above all considered to be a signature. We denote by

$$\gamma : \mathcal{O} \cdot n \to \mathcal{O} \cdot m_1 \to \cdots \to \mathcal{O} \cdot m_n \to \mathcal{O} \cdot \lfloor m_1 + \cdots + m_n \rfloor \tag{5.1.1.A}$$

the composition map of $\mathcal{O}$ defined for any $n, m_1, \ldots, m_n \in \mathbb{N}$, and by $\mathbb{1}$ the unit of $\mathcal{O}$.

Let $\mathcal{O}$ be an operad. When each $x \in \mathcal{O}$ admits finitely many factorizations of the form $x = \gamma \cdot y \cdot y_1 \cdot \ldots \cdot y_{\mathrm{ar} \cdot y}$ where $y, y_1, \ldots, y_{\mathrm{ar} \cdot y} \in \mathcal{O}$, $\mathcal{O}$ is *finitely factorizable*. When there exists a map $\mathrm{dg} : \mathcal{O} \to \mathbb{N}$ such that $\mathrm{dg}^{-1} \cdot 0 = \{\mathbb{1}\}$ and, for any $y, y_1, \ldots, y_{\mathrm{ar} \cdot y} \in \mathcal{O}$,

$$\mathrm{dg} \cdot \lfloor \gamma \cdot y \cdot y_1 \cdot \ldots \cdot y_{\mathrm{ar} \cdot y} \rfloor = \mathrm{dg} \cdot y + \mathrm{dg} \cdot y_1 + \cdots + \mathrm{dg} \cdot y_{\mathrm{ar} \cdot y}, \tag{5.1.1.B}$$

the map dg is a *grading* of $\mathcal{O}$.

5.1.2 Natural Hopf algebras of nonsymmetric operads. The *natural Hopf algebra* [Laa04; ML14; BG16; Gir24] of a finitely factorizable operad $\mathcal{O}$ admitting a grading dg is the Hopf algebra $\mathbf{N} \cdot \mathcal{O}$ defined as follows. Let $\mathrm{rd} : \mathcal{O}^* \to (\mathcal{O} \setminus \{\mathbb{1}\})^*$ be the map such that $\mathrm{rd} \cdot w$ is the subword of $w \in \mathcal{O}^*$ consisting of its letters different from $\mathbb{1}$. Any fixed point of rd is *reduced*. Let $\mathbf{N} \cdot \mathcal{O}$ be the $\mathbb{K}$-linear span of the set $\mathrm{rd} \cdot \mathcal{O}^*$. The bases of $\mathbf{N} \cdot \mathcal{O}$ are thus indexed by $\mathrm{rd} \cdot \mathcal{O}^*$, and the *elementary basis* (or *E-basis* for short) of $\mathbf{N} \cdot \mathcal{O}$ is the set $\{\mathsf{E}_w : w \in \mathrm{rd} \cdot \mathcal{O}^*\}$.

This vector space is endowed with an associative algebra structure through the product $\star$ satisfying, for any $w_1, w_2 \in \mathrm{rd} \cdot \mathcal{O}^*$,

$$\mathsf{E}_{w_1} \star \mathsf{E}_{w_2} = \mathsf{E}_{w_1 \centerdot w_2}. \tag{5.1.2.A}$$

The element E_ϵ is the identity w.r.t. the product $\star$.

Moreover, $\mathbf{N} \cdot \mathcal{O}$ is endowed with the coproduct Δ defined as the unique associative algebra morphism satisfying, for any $x \in \mathcal{O}$,

$$\Delta \cdot \mathsf{E}_x = \sum_{y \in \mathcal{O}} \sum_{w \in \mathcal{O}^{\mathrm{ar} \cdot y}} \left[x = \gamma \cdot y \cdot \lfloor w \cdot 1 \rfloor \cdot \ldots \cdot \lfloor w \cdot \lfloor \mathrm{ar} \cdot y \rfloor \rfloor \right] \mathsf{E}_{\mathrm{rd} \cdot y} \otimes \mathsf{E}_{\mathrm{rd} \cdot w}, \tag{5.1.2.B}$$

where $[\,-\,]$ is the Iverson bracket as defined at the end of Section 1. Due to the fact that $\mathcal{O}$ is finitely factorizable, (5.1.2.B) is a finite sum. This coproduct endows $\mathbf{N} \cdot \mathcal{O}$ with the structure of a bialgebra. By extending additively dg on $\mathcal{O}^*$, the map dg defines a grading of $\mathbf{N} \cdot \mathcal{O}$. Thus, $\mathbf{N} \cdot \mathcal{O}$ admits an antipode and becomes a Hopf algebra.

5.2 Natural Hopf algebras of free operads

Here, we begin by describing the free operads on terms and then describe the natural Hopf algebras of free operads in terms of leaning forests.

5.2.1 Free operads on terms. The *free operad on $\mathcal{S}$* is the set $\mathfrak{T} \cdot \mathcal{S}$ considered as a signature through the arity map ar, with the composition map such that for any $\mathfrak{t}, \mathfrak{t}_1, \ldots, \mathfrak{t}_{\mathrm{ar} \cdot \mathfrak{t}} \in \mathfrak{T} \cdot \mathcal{S}$, $\gamma \cdot \mathfrak{t} \cdot \mathfrak{t}_1 \cdot \ldots \cdot \mathfrak{t}_{\mathrm{ar} \cdot \mathfrak{t}}$ is the $\mathcal{S}$-term obtained by substituting each leaf of $\mathfrak{t}$ from left to right with $\mathfrak{t}_1, \ldots, \mathfrak{t}_{\mathrm{ar} \cdot \mathfrak{t}}$, and with $\circ$ as unit. For instance, in $\mathfrak{T} \cdot \mathcal{S}_{\mathrm{e}}$, we have

$$\gamma \cdot \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \rfloor \cdot \lfloor \mathsf{a}_2 \circ\circ \rfloor \cdot \circ \cdot \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \cdot \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \rfloor = \mathsf{a}_2 \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_1 \circ \rfloor \rfloor \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \rfloor \rfloor. \tag{5.2.1.A}$$

5.2.2 Hopf algebras on leaning forests. By construction, the Hopf algebra $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S} \rfloor$ is graded by dg and its bases are indexed by the set of words $(\mathfrak{t}_1, \ldots, \mathfrak{t}_k)$ such that $k \geqslant 0$ and for each $i \in [k]$, $\mathfrak{t}_i \in \mathfrak{T} \cdot \mathcal{S} \setminus \{\circ\}$. The map sending such a word $(\mathfrak{t}_1, \ldots, \mathfrak{t}_k)$ to the leaning $\mathcal{S}$-forest

$n\, \mathfrak{t}_1 \dots \mathfrak{t}_k \circ \dots \circ$ where $n := \mathrm{dg} \cdot \mathfrak{t}_1 + \dots + \mathrm{dg} \cdot \mathfrak{t}_k$ is a one-to-one correspondence between the set of words on $\mathfrak{T} \cdot \mathcal{S} \setminus \{\circ\}$ and the set of leaning $\mathcal{S}$-forests. For instance, the word $(\mathsf{a}_2 \circ \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor, \mathsf{a}_1 \circ, \mathsf{a}_3 \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ)$ on $\mathfrak{T} \cdot \mathcal{S}_{\mathrm{e}} \setminus \{\circ\}$ is sent to the leaning $\mathcal{S}_{\mathrm{e}}$-forest $5\, \lfloor \mathsf{a}_2 \circ \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \rfloor \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ \rfloor \circ\circ$. For this reason, through this correspondence, we shall identify words on $\mathfrak{T} \cdot \mathcal{S} \setminus \{\circ\}$ with leaning $\mathcal{S}$-forests.

On the E-basis, the product of $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S} \rfloor$ expresses, for any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, as

$$\mathsf{E}_{\mathfrak{f}_1} \star \mathsf{E}_{\mathfrak{f}_2} = \mathsf{E}_{\mathfrak{f}_1 \,/\, \mathfrak{f}_2}, \tag{5.2.2.A}$$

where $/$ is the over operation defined in Section 4.3.3. For instance, in $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S}_{\mathrm{e}} \rfloor$,

$$\mathsf{E}_{4\, \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ \rfloor \circ\circ} \star \mathsf{E}_{2\, \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \circ} = \mathsf{E}_{6\, \lfloor \mathsf{a}_1 \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \lfloor \mathsf{a}_3 \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ \rfloor \lfloor \mathsf{a}_2 \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \circ\circ\circ}. \tag{5.2.2.B}$$

Let us now describe the coproduct of $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S} \rfloor$ on the E-basis. Given a leaning $\mathcal{S}$-forest $\mathfrak{f}$ of size n, a pair (I_1, I_2) of sets is *$\mathfrak{f}$-admissible* if $I_1 \sqcup I_2 = [n]$, for any $i_1 \in I_1$, all ancestors of the internal node $i_1 + 1$ of $\mathfrak{f}$ except the root belong to I_1, and for any $i_2 \in I_2$, all descendants of the internal node $i_2 + 1$ of $\mathfrak{f}$ belong to I_2. This property is denoted by $(I_1, I_2) \vdash \mathfrak{f}$. For instance, by considering the leaning $\mathcal{S}_{\mathrm{e}}$-forest $3\, \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ$, the $\mathfrak{f}$-admissible pairs of sets are exactly $(\{1,2,3\}, \emptyset)$, $(\{1,2\}, \{3\})$, $(\{1,3\}, \{2\})$, $(\{1\}, \{2,3\})$, $(\{3\}, \{1,2\})$, and $(\emptyset, \{1,2,3\})$.

It follows from a description of [Gir24] of the coproduct of $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S} \rfloor$ that for any leaning $\mathcal{S}$-forest $\mathfrak{f}$ of size n,

$$\Delta \cdot \mathsf{E}_{\mathfrak{f}} = \sum_{I_1, I_2 \subseteq [n]} [(I_1, I_2) \vdash \mathfrak{f}]\, \mathsf{E}_{\mathfrak{f} \cdot I_1} \otimes \mathsf{E}_{\mathfrak{f} \cdot I_2}, \tag{5.2.2.C}$$

where, for any leaning $\mathcal{S}$-forest $\mathfrak{f}$ of size n and any subset I of $[n]$, the notation $\mathfrak{f} \cdot I$ refers to the restriction, as introduced in Section 4.3.2. For instance, in $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S}_{\mathrm{e}} \rfloor$,

$$\begin{aligned} \Delta \cdot \mathsf{E}_{3\, \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ} = {} & \mathsf{E}_0 \otimes \mathsf{E}_{3\, \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ} + \mathsf{E}_{1\, \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor} \otimes \mathsf{E}_{2\, \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor} \\ & + \mathsf{E}_{1\, \lfloor \mathsf{a}_2 \circ\circ \rfloor} \otimes \mathsf{E}_{2\, \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \circ} + \mathsf{E}_{2\, \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \circ} \otimes \mathsf{E}_{1\, \lfloor \mathsf{a}_2 \circ\circ \rfloor} \\ & + \mathsf{E}_{2\, \lfloor \mathsf{a}_3 \circ\circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor} \otimes \mathsf{E}_{1\, \lfloor \mathsf{a}_1 \circ \rfloor} + \mathsf{E}_{3\, \lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ} \otimes \mathsf{E}_0. \end{aligned} \tag{5.2.2.D}$$

5.2.3 Noncommutative symmetric functions. In particular, the Hopf algebra of noncommutative symmetric functions **Sym** [Gel+95] can be understood as a natural Hopf algebra of a free operad. Indeed, **Sym** is isomorphic to the natural Hopf algebra $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S} \rfloor$ of the free operad on the signature $\mathcal{S} := \{\mathsf{a}_1\}$ where $\mathrm{ar} \cdot \mathsf{a}_1 = 1$ (see [Gir24]). Indeed, by encoding a leaning $\mathcal{S}$-forest $\mathfrak{f}$ by the integer composition $(r_1, \dots, r_k)$, $k \geqslant 0$, such that for any $i \in [k]$, r_i is the degree of the i-th subterm of $\mathfrak{f}$, $\mathbf{N} \cdot \lfloor \mathfrak{T} \cdot \mathcal{S} \rfloor$ and **Sym** are defined on the same vector space. Moreover, for any integer compositions $(r_1, \dots, r_k)$, $k \geqslant 0$, and $(r'_1, \dots, r'_{k'})$, $k' \geqslant 0$, we have

$$\mathsf{E}_{(r_1, \dots, r_k)} \star \mathsf{E}_{(r'_1, \dots, r'_{k'})} = \mathsf{E}_{(r_1, \dots, r_k, r'_1, \dots, r'_{k'})} \tag{5.2.3.A}$$

and

$$\Delta \cdot \mathsf{E}_{(r_1)} = \sum_{i \in [\![r_1]\!]} \mathsf{E}_{(i)} \otimes \mathsf{E}_{(r_1 - i)}, \tag{5.2.3.B}$$

where (0) and the empty integer composition are identified. This product and coproduct are the ones of **Sym** through its elementary basis.

5.3 FUNDAMENTAL AND HOMOGENEOUS BASES

We introduce a two new bases of the natural Hopf algebra of a free operad. These bases mimic well-known constructions of bases in combinatorial Hopf algebras defined by summing on intervals of particular posets. Here, the posets intervening in this new basis are the leaning forest $\mathcal{S}$-easterly wind posets.

5.3.1 FUNDAMENTAL BASIS. Let us use the leaning forest $\mathcal{S}$-easterly wind posets to build a new basis of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$. For any leaning $\mathcal{S}$-forest $\mathfrak{f}$, let

$$\mathsf{F}_{\mathfrak{f}} := \sum_{\mathfrak{f}' \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f} \preccurlyeq \mathfrak{f}'] \, \lfloor \mu\cdot\{1\}\cdot\mathfrak{f}\cdot\mathfrak{f}' \rfloor \, \mathsf{E}_{\mathfrak{f}'}, \tag{5.3.1.A}$$

where $\mathfrak{L}\cdot\mathcal{S}$ is the set defined in Section 4.3.1, $\preccurlyeq$ is the partial order relation of the leaning forest $\mathcal{S}$-easterly wind posets, and, in accordance with the notations introduced at the end of Section 3.1.1, $\mu\cdot\{1\}$ is the Möbius function of these lattices. For instance, in $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}_{\mathrm{e}}\rfloor$,

$$\mathsf{F}_{3\,\lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ} = \mathsf{E}_{3\,\lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ} - \mathsf{E}_{3\,\lfloor \mathsf{a}_3 \circ \lfloor \mathsf{a}_1 \circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ\circ} - \mathsf{E}_{3\,\lfloor \mathsf{a}_3 \lfloor \mathsf{a}_1 \circ \rfloor \circ\circ \rfloor \lfloor \mathsf{a}_2 \circ\circ \rfloor \circ} - \mathsf{E}_{3\,\lfloor \mathsf{a}_3 \lfloor \mathsf{a}_1 \circ \rfloor \circ \lfloor \mathsf{a}_2 \circ\circ \rfloor \rfloor \circ\circ}. \tag{5.3.1.B}$$

By Möbius inversion and triangularity, for any leaning $\mathcal{S}$-forest $\mathfrak{f}$,

$$\mathsf{E}_{\mathfrak{f}} = \sum_{\mathfrak{f}' \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f} \preccurlyeq \mathfrak{f}'] \, \mathsf{F}_{\mathfrak{f}'}, \tag{5.3.1.C}$$

so that the set $\{\mathsf{F}_{\mathfrak{f}} : \mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}\}$ is a basis of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$, called the *fundamental basis* (or F-*basis* for short).

We can now state one of the main results of this work.

▶ **Theorem** 5.3.1.A — *For any signature $\mathcal{S}$ and any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$,*

$$\mathsf{F}_{\mathfrak{f}_1} \star \mathsf{F}_{\mathfrak{f}_2} = \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \mathfrak{f} \preccurlyeq \mathfrak{f}_1 \diagdown \mathfrak{f}_2] \, \mathsf{F}_{\mathfrak{f}}. \tag{5.3.1.D}$$

◀ **Proof** — Let $\star'$ be the product on $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$ such that for any $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, $\mathsf{F}_{\mathfrak{f}_1} \star' \mathsf{F}_{\mathfrak{f}_2}$ is the right-hand side of (5.3.1.D). For any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, by denoting by n_1 the size of $\mathfrak{f}_1$ and by using Proposition 4.3.4.A and Lemma 4.3.3.C, we have

$$\begin{aligned}
\mathsf{E}_{\mathfrak{f}_1} \star' \mathsf{E}_{\mathfrak{f}_2} &= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \preccurlyeq \mathfrak{f}'_1][\mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_2] \, \mathsf{F}_{\mathfrak{f}'_1} \star' \mathsf{F}_{\mathfrak{f}'_2} \\
&= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \preccurlyeq \mathfrak{f}'_1][\mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_2][\mathfrak{f}'_1 \diagup \mathfrak{f}'_2 \preccurlyeq \mathfrak{f} \preccurlyeq \mathfrak{f}'_1 \diagdown \mathfrak{f}'_2] \, \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \preccurlyeq \mathfrak{f}'_1][\mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_2] \sum_{\mathfrak{f} \in \mathfrak{f}'_1 \sqcup\!\sqcup \mathfrak{f}'_2} \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \preccurlyeq \mathfrak{f}'_1][\mathfrak{f}_2 \preccurlyeq \mathfrak{f}'_2] \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\Uparrow\cdot n_1\cdot\mathfrak{f} = \mathfrak{f}'_1][\Downarrow\cdot n_1\cdot\mathfrak{f} = \mathfrak{f}'_2] \, \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \preccurlyeq \Uparrow\cdot n_1\cdot\mathfrak{f}][\mathfrak{f}_2 \preccurlyeq \Downarrow\cdot n_1\cdot\mathfrak{f}] \, \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}_1 \diagup \mathfrak{f}_2 \preccurlyeq \mathfrak{f}] \, \mathsf{F}_{\mathfrak{f}}
\end{aligned} \tag{5.3.1.E}$$

$$= \mathsf{E}_{\mathfrak{f}_1 \diagup \mathfrak{f}_2}.$$

This shows that $\mathsf{E}_{\mathfrak{f}_1} \star' \mathsf{E}_{\mathfrak{f}_2} = \mathsf{E}_{\mathfrak{f}_1} \star \mathsf{E}_{\mathfrak{f}_2}$, so that $\star'$ and $\star$ are the same products. Therefore, (5.3.1.D) holds. □

The product of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$ on the F-basis is akin to a shuffle of reduced $\mathcal{S}$-forests. For instance, in $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}_{\mathrm{e}}\rfloor$, we have

$$\begin{aligned}
\mathsf{F}_{3\,\lfloor \mathsf{a}_3\circ\circ\circ\rfloor\lfloor \mathsf{a}_2\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\rfloor\circ} \star \mathsf{F}_{4\,\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\rfloor\lfloor \mathsf{a}_1\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ} &= \mathsf{F}_{7\,\lfloor \mathsf{a}_3\circ\circ\circ\rfloor\lfloor \mathsf{a}_2\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\rfloor\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\rfloor\lfloor \mathsf{a}_1\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\circ} \\
&+ \mathsf{F}_{7\,\lfloor \mathsf{a}_3\circ\circ\circ\rfloor\lfloor \mathsf{a}_2\circ\lfloor \mathsf{a}_2\circ\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\rfloor\rfloor\rfloor\lfloor \mathsf{a}_1\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\circ\circ} \\
&+ \mathsf{F}_{7\,\lfloor \mathsf{a}_3\circ\circ\circ\rfloor\lfloor \mathsf{a}_2\circ\lfloor \mathsf{a}_2\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\rfloor\circ\rfloor\rfloor\lfloor \mathsf{a}_1\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\circ\circ} \\
&+ \mathsf{F}_{7\,\lfloor \mathsf{a}_3\circ\circ\circ\rfloor\lfloor \mathsf{a}_2\circ\lfloor \mathsf{a}_2\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\rfloor\lfloor \mathsf{a}_1\circ\rfloor\rfloor\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ\circ\circ\circ}.
\end{aligned} \tag{5.3.1.F}$$

Observe that the fundamental basis of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$ coincides with the ribbon basis of **Sym** [Gel+95]. Indeed, by employing the notation introduced in Section 5.2.3, for any integer compositions $(r_1, \dots, r_k)$, $k \geqslant 0$, and $(r'_1, \dots, r'_{k'})$, $k' \geqslant 0$, we have

$$\mathsf{F}_{(r_1,\dots,r_k)} \star \mathsf{F}_{(r'_1,\dots,r'_{k'})} = \mathsf{F}_{(r_1,\dots,r_k,r'_1,\dots,r'_{k'})} + \mathsf{F}_{(r_1,\dots,r_k+r'_1,\dots,r'_{k'})}. \tag{5.3.1.G}$$

This product is the one of **Sym** through its ribbon basis.

5.3.2 Homogeneous basis. Let us use again the leaning forest $\mathcal{S}$-easterly wind posets to build a new basis of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$. For any leaning $\mathcal{S}$-forest $\mathfrak{f}$, let

$$\mathsf{H}_{\mathfrak{f}} := \sum_{\mathfrak{f}' \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}' \preccurlyeq \mathfrak{f}]\, \mathsf{F}_{\mathfrak{f}'}. \tag{5.3.2.A}$$

For instance, in $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}_{\mathrm{e}}\rfloor$,

$$\mathsf{H}_{3\,\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_1\circ\rfloor\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ} = \mathsf{F}_{3\,\lfloor \mathsf{a}_3\circ\circ\circ\rfloor\lfloor \mathsf{a}_1\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor} + \mathsf{F}_{3\,\lfloor \mathsf{a}_3\circ\circ\lfloor \mathsf{a}_1\circ\rfloor\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ} + \mathsf{F}_{3\,\lfloor \mathsf{a}_3\circ\lfloor \mathsf{a}_1\circ\rfloor\circ\rfloor\lfloor \mathsf{a}_2\circ\circ\rfloor\circ}. \tag{5.3.2.B}$$

By Möbius inversion and triangularity, for any leaning $\mathcal{S}$-forest $\mathfrak{f}$,

$$\mathsf{F}_{\mathfrak{f}} = \sum_{\mathfrak{f}' \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}' \preccurlyeq \mathfrak{f}]\, \lfloor \mu\cdot\{1\}\cdot\mathfrak{f}'\cdot\mathfrak{f} \rfloor\, \mathsf{H}_{\mathfrak{f}'}, \tag{5.3.2.C}$$

so that the set $\{\mathsf{H}_{\mathfrak{f}} : \mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}\}$ is a basis of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$, called the *homogeneous basis* (or H-*basis* for short).

We can now state one of the main results of this work.

▶ **Theorem** 5.3.2.A — *For any signature $\mathcal{S}$ and any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$,*

$$\mathsf{H}_{\mathfrak{f}_1} \star \mathsf{H}_{\mathfrak{f}_2} = \mathsf{H}_{\mathfrak{f}_1 \diagdown \mathfrak{f}_2}. \tag{5.3.2.D}$$

◀ **Proof** — For any leaning $\mathcal{S}$-forests $\mathfrak{f}_1$ and $\mathfrak{f}_2$, by denoting by n_1 the size of $\mathfrak{f}_1$ and by using

Theorem 5.3.1.A, Proposition 4.3.4.A, and Lemma 4.3.3.C, we have

$$\begin{aligned}
\mathsf{H}_{\mathfrak{f}_1} \star \mathsf{H}_{\mathfrak{f}_2} &= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}'_1 \preccurlyeq \mathfrak{f}_1][\mathfrak{f}'_2 \preccurlyeq \mathfrak{f}_2]\, \mathsf{F}_{\mathfrak{f}'_1} \star \mathsf{F}_{\mathfrak{f}'_2} \\
&= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}'_1 \preccurlyeq \mathfrak{f}_1][\mathfrak{f}'_2 \preccurlyeq \mathfrak{f}_2][\mathfrak{f}'_1 \diagup \mathfrak{f}'_2 \preccurlyeq \mathfrak{f} \preccurlyeq \mathfrak{f}'_1 \diagdown \mathfrak{f}'_2]\, \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}'_1 \preccurlyeq \mathfrak{f}_1][\mathfrak{f}'_2 \preccurlyeq \mathfrak{f}_2] \sum_{\mathfrak{f} \in \mathfrak{f}'_1 \,\overline{\sqcup\!\sqcup}\, \mathfrak{f}'_2} \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f}'_1 \in \mathfrak{L}\cdot\mathcal{S}} \sum_{\mathfrak{f}'_2 \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f}'_1 \preccurlyeq \mathfrak{f}_1][\mathfrak{f}'_2 \preccurlyeq \mathfrak{f}_2] \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\Uparrow\cdot n_1\cdot\mathfrak{f} = \mathfrak{f}'_1][\Downarrow\cdot n_1\cdot\mathfrak{f} = \mathfrak{f}'_2]\, \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\Uparrow\cdot n_1\cdot\mathfrak{f} \preccurlyeq \mathfrak{f}_1][\Downarrow\cdot n_1\cdot\mathfrak{f} \preccurlyeq \mathfrak{f}_2]\, \mathsf{F}_{\mathfrak{f}} \\
&= \sum_{\mathfrak{f} \in \mathfrak{L}\cdot\mathcal{S}} [\mathfrak{f} \preccurlyeq \mathfrak{f}_1 \diagdown \mathfrak{f}_2]\, \mathsf{F}_{\mathfrak{f}} \\
&= \mathsf{H}_{\mathfrak{f}_1 \diagdown \mathfrak{f}_2}.
\end{aligned} \tag{5.3.2.E}$$

This shows that (5.3.2.D) holds. □

By employing the notation introduced at the end of Section 5.2.2, the H-basis of **Sym** admits the following expression for its product. For any integer compositions $(r_1, \dots, r_k)$, $k \geqslant 0$, and $(r'_1, \dots, r'_{k'})$, $k' \geqslant 0$, we have

$$\mathsf{H}_{(r_1,\dots,r_k)} \star \mathsf{H}_{(r'_1,\dots,r'_{k'})} = \mathsf{H}_{(r_1,\dots,r_k+r'_1,\dots,r'_{k'})}. \tag{5.3.2.F}$$

6 Conclusion and open questions

We have introduced a new partial order relation $\preccurlyeq$ on the underlying set $\mathfrak{T}\cdot\mathcal{S}$ of free operads, namely the easterly wind partial order. As shown in this work, the resulting posets yield new bases of natural Hopf algebras $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$ of free operads, sharing key properties with a broad class of combinatorial Hopf algebras. We list here several open questions and directions for future research in this context.

At the general level of the easterly wind posets, many properties remain unknown, including an explicit expression for the Möbius function of $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind posets and the enumeration of the set $\downarrow\cdot\mathcal{X}\cdot\mathfrak{t}$ of terms greater than or equal to the $\mathcal{X}$-tilted $\mathcal{S}$-term $\mathfrak{t}$. This last question is linked with the enumeration of the intervals of such posets.

In Section 4.2.2, we have shown that certain $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind posets contain, as maximal intervals, the Tamari lattices. A natural question is whether one can similarly realize, as $\mathcal{X}$-tilted $\mathcal{S}$-easterly wind posets, other classical lattices involving treelike structures, such as the Kreweras lattices [Kre72], the Stanley lattices [Sta75; Knu04], the m-Tamari lattices [BP12], the m-canyon lattices [CG22], and the pruning-grafting lattices [BP08].

In Section 4.2.1, we have defined lattices on the combinatorial family of Fuss-Catalan objects. To the best of our knowledge, these lattices are new and warrant a detailed combinatorial study, including the enumeration of their intervals and their relationships with known structures on the same combinatorial family.

Besides, now linked with the natural Hopf algebra $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$ of a free operad $\mathfrak{T}\cdot\mathcal{S}$, one may ask how to express the coproduct and the antipode of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$ on the new F-basis and H-basis. A further question concerns using these two bases to investigate the cofreeness and self-duality of $\mathbf{N}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\rfloor$, and obtain necessary and sufficient conditions for these properties, depending on $\mathcal{S}$.

Finally, for now the easterly wind order is defined only at the level of terms. It would be valuable to generalize this order on the underlying set of any operad $\mathcal{O}$, possibly subject to certain restrictions, so that the analogous F-basis and H-basis of $\mathbf{N}\cdot\mathcal{O}$ satisfy generalizations of Theorems 5.3.1.A and 5.3.2.A. One approach to build such a partial order relation on $\mathcal{O}$ is to choose a generating set $\mathcal{S}_{\mathcal{O}}$ of $\mathcal{O}$ and consider the easterly wind order on treelike factorizations of the elements of $\mathcal{O}$ as elements of the free operad $\mathfrak{T}\cdot\mathcal{S}_{\mathcal{O}}$. The main challenge is to choose a canonical factorization for each $x \in \mathcal{O}$, since an element may admit, when $\mathcal{O}$ is not free, several factorizations.

7 References

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