Clones of pigmented words and realizations of special classes of monoids

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Abstract. Clones are generalizations of operads forming powerful instruments to describe varieties of algebras wherein repeating variables are allowed in their relations. They allow us in this way to realize and study a large range of algebraic structures. A functorial construction from the category of monoids to the category of clones is introduced. The obtained clones involve words on positive integers where letters are pigmented by elements of a monoid. By considering quotients of these structures, we construct a complete hierarchy of clones involving some families of combinatorial objects. This provides clone realizations of some known and some new special classes of monoids as among others the variety of left-regular bands, bounded semilattices, and regular band monoids.

Contents

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1 Introduction

Given a variety of algebras specified by a set of generating operations together with relations between the operations, an important question consists in deciding if two compound operations are equivalent. For instance, in the variety of groups, the two operations $(x_1, x_2) \mapsto (x_1 \cdot x_2)^{-1}$ and $(x_1, x_2) \mapsto x_2^{-1} \cdot x_1^{-1}$ compute always both the same value, where $(x_1, x_2) \mapsto x_1 \cdot x_2$ is the multiplication operation and $x_1 \mapsto x_1^{-1}$ is the inverse operation of groups. This general question is known as the word problem and in some cases, term rewrite systems [BN98; Bez+03] offer solutions by orienting in a suitable way the relations which define the variety in order to form a terminating and confluent rewrite system.

While the word problem is in general undecidable, this inherent undecidability does not obstruct the development of tools capable of resolving specific instances. Rather than focusing on finding the optimal orientation or completion of the relations within a variety, an alternative combinatorial approach involves encoding compound operations by using combinatorial objects. In this context, the functional composition can be interpreted as a relevant operation on these objects. Within this framework, operads [LV12; Mén15; Gir18] emerge as valuable instruments to facilitate these abstractions, called operad realizations of a variety. An illustrating example can be found in the realization of the variety of pre-Lie algebras in terms of rooted trees [CL01] and grafting operations on such trees. Besides, operads are also great tools to tackling problems originating from combinatorics. Indeed, by endowing a set of combinatorial objects with an operad structure, we obtain a framework for enumerating [Gir20b] and generating [Gir19] their elements. This is based on presentations by generators and relations of the operads to study and more precisely on their orientations in order to form, here again, terminating and confluent term rewrite systems.

Despite their broad utility, operads have limitations, particularly when dealing with varieties that are defined through relations with repeating variables. This issue arises for instance in the variety of groups, lattices, or flexible algebras, where natural descriptions of these varieties require relations involving repeated inputs. Although it is feasible to capture a certain part of such varieties by working with operads in the category of vector spaces on a field of zero characteristic and by considering some tricks to encode relations with repeating variables by linear combinations of linear terms (like in the case of the variety of flexible algebras [May72]), operads are not the ideal instrument in this context. Some other devices have been developed for these purposes. Examples include abstract clones [Coh65; Tay93], Lawvere theories [Law63; Adá+10], and monads with arities [EM65; HP07; BMW12]. The aim of this work is to create bridges between the theory of abstract clones —called simply "clones" here henceforth— and combinatorics. To our knowledge, contrary to what operad theory has experienced since its rebirth in the 1990s [Lod96], not many such connections have been established in the existing literature. We have opted to work with clones rather than with Lawvere theories or monads with arities because clones can be perceived as generalized operads with minor distinctions. Since as presented above, the connections between operads and combinatorics are now very clear and well-established (see also [CL01; Gir15; Gir18; Gir20a]), we anticipate that new significant connections between clones and combinatorics could be unearthed.

In an initial, humble, and modest first step in this direction, we introduce a new combinatorial recipe to build clones of combinatorial objects. More precisely, given a monoid $\mathcal{M}$, we construct a clone $\mathbf{P}(\mathcal{M})$ involving $\mathcal{M}$-pigmented words, that are some words of integers whose letters are pigmented by elements of $\mathcal{M}$. The variety of algebras described by $\mathbf{P}(\mathcal{M})$, called variety of $\mathcal{M}$-pigmented monoids, bears similarities to the variety of algebras described by the operad $\mathbf{T}(\mathcal{M})$, where $\mathbf{T}$ is a construction from monoids to operads introduced in [Gir15]. More specifically, the variety of $\mathcal{M}$-pigmented algebras has an extra generator (playing the role of a unit) and some relations involving it compared to the variety of algebras described by $\mathbf{T}(\mathcal{M})$. For this reason, the present work can be seen as a continuation and a generalization of [Gir15], but in the context of clones rather than of operads.

The clone $\mathbf{P}(\mathcal{M})$ is rich enough to contain some notable quotients. In order to construct quotients of $\mathbf{P}(\mathcal{M})$, we consider clone congruences $\equiv$ of $\mathbf{P}(\mathcal{M})$ each coming with a $\mathbb{P}$-symbol to decide whether two $\mathcal{M}$-pigmented words are $\equiv$-equivalent. A $\mathbb{P}$-symbol for a clone congruence $\equiv$ is a map sending an $\mathcal{M}$-pigmented word to a representative of its $\equiv$-equivalence class. Such maps enable us to obtain concrete realizations and presentations by generators and relations of quotients of $\mathbf{P}(\mathcal{M})$. The studied quotients of $\mathbf{P}(\mathcal{M})$ fit into a diagram of surjective clone morphisms generalizing some lattices of varieties of special classes of monoids (see [GLV22]) and of semigroups (see [Eva71; SVV09; KKP11]). In particular, we obtain as main results clone realizations of commutative monoids, left-regular bands, bounded semilattices, and regular bands. These clone realizations allow us to solve the word problem in these varieties by using algorithms akin to those developed in [SS82; NS00] for idempotent semigroups.

This paper is organized as follows. Section 2 contains preliminary notions about terms, clones and free clones, presentations of clones, and varieties of algebras. In particular, we show Proposition 2.3.2.A which is an important result to establish presentations of clones. Next, in Section 3, we introduce the varieties of $\mathcal{M}$-pigmented monoids and describe the construction $\mathbf{P}$. By Theorem 3.3.3.B, the main result of this section, we show that $\mathbf{P}(\mathcal{M})$ is a clone realization of the variety of $\mathcal{M}$-pigmented monoids. In Section 4 we introduce some tools to investigate quotient clones of $\mathbf{P}(\mathcal{M})$. In particular, we introduce the concept of $\mathbb{P}$-symbol specific to our context and its relationships with clone congruences by way of Propositions 4.1.1.A, 4.1.1.B, and 4.1.3.A. We show also with Proposition 4.1.2.A how to obtain a concrete description of a quotient of $\mathbf{P}(\mathcal{M})$ by a congruence $\equiv$ admitting a $\mathbb{P}$-symbol $\mathbb{P}_{\equiv}$. Continuing this, two clone congruences $\equiv_{\mathsf{sort}}$ and $\equiv_{\mathsf{first}_k}$, $k \geqslant 0$, are introduced. These congruences as well as some of their compositions are used to build the quotient clones $\mathsf{WInc}(\mathcal{M})$, $\mathsf{Arra}_k(\mathcal{M})$, $k \geqslant 0$, and Inc_k, $k \geqslant 0$. By Propositions 4.3.1.A, 4.3.2.A, and 4.3.3.A, we describe presentations of these clones. Finally, Section 5 contains the most technical results under a combinatorial point of view. Here, we construct three quotients of $\mathbf{P}(\mathcal{M})$ by clone congruences defined by intersecting some of the congruences $\equiv_{\mathsf{sort}}$ and $\equiv_{\mathsf{first}_k}$, $k \geqslant 0$. The main results contain Theorems 5.1.4.B, 5.2.4.B, and 5.3.4.B describing realizations of these clones, and Theorems 5.1.5.B, 5.2.5.B, and 5.3.5.B giving presentations for these clones. In particular, we obtain here a clone realization of the variety of regular bands which seems new at the best of our knowledge. This text ends with a list of open questions and future research directions.

General notations and conventions. For any integers i and j, $[i, j]$ denotes the set $\{i, i+1, \dots, j\}$. For any integer i, $[i]$ denotes the set $[1, i]$ and $[\![i]\!]$ denotes the set $[0, i]$. For any set A, A^* is the set of words on A. For any $w \in A^*$, $\ell(w)$ is the length of w, and for any $i \in [\ell(w)]$, $w(i)$ is the i-th letter of w. For any $a \in A$, $|w|_a$ is the number of occurrences of a in w. The only word of length 0 is the empty word ϵ. For any $i \leqslant j \in [\ell(w)]$, $w(i, j)$ is the word $w(i)w(i+1)\dots w(j)$. The word $\mathrm{r}(w)$ is the mirror image $w(\ell(w))\dots w(1)$ of w. Given two words w and w', the concatenation of w and w' is denoted by ww' or by $w.w'$.

2 Clones and realizations of varieties

This preliminary section contains the main definitions and notions about abstract clones, free abstract clones, presentations of abstract clones by generators and relations, varieties of algebras, and clone realizations of varieties of algebras.

2.1 Abstract clones

In this part, we set our notations and main notions about abstract clones. Let us begin with graded sets.

2.1.1 Graded sets. A *graded set* is a set $G := \bigsqcup_{n \geqslant 0} G(n)$. For any $x \in G$, the unique integer $n \geqslant 0$ such that $x \in G(n)$ is the *arity* of x, denoted by $|x|$. If for any $n \geqslant 0$, $G(n)$ is finite, then G is *combinatorial*. In this case, the *sequence of dimensions* of G is the sequence $(\# G(n))_{n \geqslant 0}$. Let G' be another graded set. A map $\phi : G \to G'$ is a *graded set morphism* if ϕ preserves the arities. Besides, if for any $n \geqslant 0$, $G'(n) \subseteq G(n)$, then G' is a *graded subset* of G. A binary relation $\mathfrak{R}$ on G is a *graded set binary relation* on G if $\mathfrak{R}$ preserve the arities. The *quotient* of G by a graded set equivalence relation $\equiv$ is the graded set $G/_{\equiv}$ defined for any $n \geqslant 0$ by $G/_{\equiv}(n) := \{[x]_{\equiv} : x \in G(n)\}$ where $[x]_{\equiv}$ is the $\equiv$-equivalence class of $x \in G$.

2.1.2 Abstract clones. Abstract clones are devices which can be used to describe algebraic structures [Coh65; Neu70; Tay93] (see also [Fuj20] for a point of view from universal algebra). An *abstract clone* (or *clone* for short) $\mathcal{C}$ is a graded set $\mathcal{C}$ endowed with maps

$$-[-, \dots, -]_{n,m} : \mathcal{C}(n) \times \mathcal{C}(m)^n \to \mathcal{C}(m), \tag{2.1.2.A}$$

where $n, m \geqslant 0$, called *superposition maps*, and with distinguished elements $\mathbb{1}_{i,n} \in \mathcal{C}(n)$, where $n \geqslant 1$ and $i \in [n]$, called *projections*. This data has to satisfy, for any $x \in \mathcal{C}(n)$, $n \geqslant 0$, $y_1, \dots, y_n \in \mathcal{C}(m)$, $m \geqslant 0$, $z_1, \dots, z_m \in \mathcal{C}(k)$, $k \geqslant 0$, and $i \in [n]$, the relations

$$\mathbb{1}_{i,n}[y_1, \dots, y_n]_{n,m} = y_i, \tag{2.1.2.B}$$

$$x[\mathbb{1}_1, \dots, \mathbb{1}_n]_{n,n} = x, \tag{2.1.2.C}$$

$$x[y_1, \dots, y_n]_{n,m}[z_1, \dots, z_m]_{m,k} = x\Big[y_1[z_1, \dots, z_m]_{m,k}, \dots, y_n[z_1, \dots, z_m]_{m,k}\Big]_{n,k}. \tag{2.1.2.D}$$

To lighten the notation when the context is clear, we shall drop the indices of the superposition maps in order to write $x[y_1, \dots, y_n]$ instead of $x[y_1, \dots, y_n]_{n,m}$ for any $x \in \mathcal{C}(n)$, $n \geqslant 0$ and $y_1, \dots, y_n \in \mathcal{C}(m)$, $m \geqslant 0$. In the same way, we shall write $\mathbb{1}_i$ instead of $\mathbb{1}_{i,n}$ for any $n \geqslant 1$ and $i \in [n]$ when the value of n is clear or not significant.

Observe that for any $0 \leqslant n \leqslant m$, there is a map $\iota_{n,m} : \mathcal{C}(n) \to \mathcal{C}(m)$ such that for any $x \in \mathcal{C}(n)$, $\iota_{n,m}(x) := x[\mathbb{1}_{1,m}, \dots, \mathbb{1}_{n,m}]$. It is easy to check that ι is an injection. Therefore, in each set $\mathcal{C}(m)$, there is a copy of the elements of $\mathcal{C}(n)$, seen in $\mathcal{C}(m)$ as elements of arity m. Observe also that for any $n \geqslant 0$, $\iota_{n,n}$ is the identity map on $\mathcal{C}(n)$, and that for any $0 \leqslant n \leqslant m \leqslant k$, the relation $\iota_{m,k} \circ \iota_{n,m} = \iota_{n,k}$ holds.

The *trivial clone* is the clone $\mathcal{T}$ such that for any $n \geqslant 0$, $\mathcal{T}(n)$ is a singleton. Observe that there is no choice for the definition of the superposition maps of $\mathcal{T}$. Let $\mathcal{C}'$ be another clone. A graded set morphism $\phi : \mathcal{C} \to \mathcal{C}'$ is a *clone morphism* if, for any $n \geqslant 1$ and $i \in [n]$, ϕ sends the projection $\mathbb{1}_{i,n}$ of $\mathcal{C}$ to the projection $\mathbb{1}'_{i,n}$ of $\mathcal{C}'$, and for any $x \in \mathcal{C}(n)$, $n \geqslant 0$, and any $y_1, \dots, y_n \in \mathcal{C}(m)$, $m \geqslant 0$,

$$\phi(x[y_1, \dots, y_n]) = \phi(x)[\phi(y_1), \dots, \phi(y_n)]. \tag{2.1.2.E}$$

Besides, if $\mathcal{C}'$ is a graded subset of $\mathcal{C}$ such that $\mathcal{C}'$ contains the projections of $\mathcal{C}$, and $\mathcal{C}'$ is closed under the superposition maps of $\mathcal{C}$, then $\mathcal{C}'$ is a *subclone* of $\mathcal{C}$. Given $S \subseteq \mathcal{C}$, the subclone of $\mathcal{C}$

generated by S is the smallest subclone $\mathcal{C}^S$ of $\mathcal{C}$ containing S. When $\mathcal{C}^S = \mathcal{C}$, S is a *generating set* of $\mathcal{C}$. A *clone congruence* of $\mathcal{C}$ is a graded set equivalence relation $\equiv$ on $\mathcal{C}$ such that for any $x, x' \in \mathcal{C}(n)$, $n \geqslant 0$, and any $y_1, y'_1, \dots, y_n, y'_n \in \mathcal{C}(m)$, $m \geqslant 0$, if $x \equiv x'$ and $y_1 \equiv y'_1$, …, $y_n \equiv y'_n$, then $x[y_1, \dots, y_n] \equiv x'[y'_1, \dots, y'_n]$. The *quotient* of $\mathcal{C}$ by $\equiv$ is the clone on the graded set $\mathcal{C}/_\equiv$ such that for any $x \in \mathcal{C}(n)$, $n \geqslant 0$, $y_1, \dots, y_n \in \mathcal{C}(m)$, $m \geqslant 0$, the superposition maps of $\mathcal{C}/_\equiv$ satisfy

$$[x]_\equiv[[y_1]_\equiv, \dots, [y_n]_\equiv] = [x[y_1, \dots, y_n]]_\equiv, \tag{2.1.2.F}$$

and for any $n \geqslant 1$ and $i \in [n]$, the projection $\mathbb{1}_{i,n}$ of $\mathcal{C}/_\equiv$ is the $\equiv$-equivalence class of the projection $\mathbb{1}_{i,n}$ of $\mathcal{C}$.

2.1.3 ALGEBRAS OVER CLONES. Let $\mathcal{C}$ be a clone. An *algebra* over $\mathcal{C}$ (or a *$\mathcal{C}$-algebra* for short) is a pair $(\mathcal{A}, \mathfrak{op}_n)$ where $\mathcal{A}$ is a set and for for any $n \geqslant 0$, each $\mathfrak{op}_n$ is a map

$$\mathfrak{op}_n : \mathcal{C}(n) \to (\mathcal{A}^n \to \mathcal{A}) \tag{2.1.3.A}$$

satisfying the following relations. For any $a_1, \dots, a_m \in \mathcal{A}$, $m \geqslant 0$, $i \in [m]$, $x \in \mathcal{C}(n)$, $n \geqslant 0$, and $y_1, \dots, y_n \in \mathcal{C}(m)$,

$$\mathfrak{op}(\mathbb{1}_{i,m})(a_1, \dots, a_m) = a_i, \tag{2.1.3.B}$$

$$\mathfrak{op}(x[y_1, \dots, y_n])(a_1, \dots, a_m) = \mathfrak{op}(x)(\mathfrak{op}(y_1)(a_1, \dots, a_m), \dots, \mathfrak{op}(y_n)(a_1, \dots, a_m)). \tag{2.1.3.C}$$

In other terms, each $x \in \mathcal{C}(n)$ gives rise to an operation $\mathfrak{op}(x)$ on $\mathcal{A}$ with n inputs and one output, and the functional composition of such operations is coherent with the superposition maps of $\mathcal{C}$.

2.2 TERMS AND FREE CLONES

In order to describe free clones, we need to introduce some notions and combinatorics about terms. The reason behind this is that the elements of free clones can be described as terms and their superposition maps as graftings in terms.

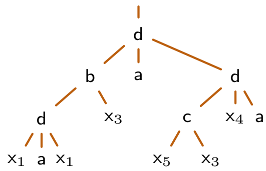
2.2.1 TERMS. A *signature* is a graded set $\mathfrak{G}$. Its elements are called *constants*. Any element of the set $\mathbb{X} := \bigsqcup_{n \geqslant 1} \mathbb{X}_n$, where $\mathbb{X}_n := \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$, is a *variable*. A *$\mathfrak{G}$-term* (or simply *term* when the context is clear) is recursively either a variable or a pair $(\mathsf{g}, (\mathsf{t}_1, \dots, \mathsf{t}_k))$, where $\mathsf{g} \in \mathfrak{G}(k)$, $k \geqslant 0$, and t_1, …, t_k are $\mathfrak{G}$-terms. For convenience, we shall write $\mathsf{g}[\mathsf{t}_1, \dots, \mathsf{t}_k]$ instead of $(\mathsf{g}, (\mathsf{t}_1, \dots, \mathsf{t}_k))$. From this definition, any $\mathfrak{G}$-term is a rooted planar tree where internal nodes are decorated by constants and leaves are decorated by variables. The graded set of $\mathfrak{G}$-terms is denoted by $\mathfrak{T}(\mathfrak{G})$ where, for any $n \geqslant 0$, $\mathfrak{T}(\mathfrak{G})(n)$ is a copy of the set of the $\mathfrak{G}$-terms having all variables belonging to $\mathbb{X}_n$.

Let t be a $\mathfrak{G}$-term. The *degree* $\mathrm{dg}(\mathsf{t})$ of t is the number of internal nodes of t seen as a tree. The *length* $\ell(\mathsf{t})$ of t is the number of variables of t. If $\mathfrak{G}'$ is a signature and $\phi : \mathfrak{G} \to \mathfrak{G}'$ is a graded set morphism, we denote by $\widehat{\phi} : \mathfrak{T}(\mathfrak{G}) \to \mathfrak{T}(\mathfrak{G}')$ the map such that, for any $\mathsf{t} \in \mathfrak{T}(\mathfrak{G})$, $\widehat{\phi}(\mathsf{t})$ is the $\mathfrak{G}'$-term obtained by replacing each decoration $\mathsf{g} \in \mathfrak{G}$ of an internal node of t by $\phi(\mathsf{g})$.

For instance, by setting $\mathfrak{G}$ as the signature satisfying $\mathfrak{G} = \mathfrak{G}(0) \sqcup \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ with $\mathfrak{G}(0) = \{\mathsf{a}\}$, $\mathfrak{G}(2) = \{\mathsf{b}, \mathsf{c}\}$, and $\mathfrak{G}(3) = \{\mathsf{d}\}$,

$$\mathsf{t} := \mathsf{d}[\mathsf{b}[\mathsf{d}[\mathsf{x}_1, \mathsf{a}, \mathsf{x}_1], \mathsf{x}_3], \mathsf{a}, \mathsf{d}[\mathsf{c}[\mathsf{x}_5, \mathsf{x}_3], \mathsf{x}_4, \mathsf{a}]] \tag{2.2.1.A}$$

is a $\mathfrak{G}$-term. The treelike representation of $\mathfrak{t}$ is

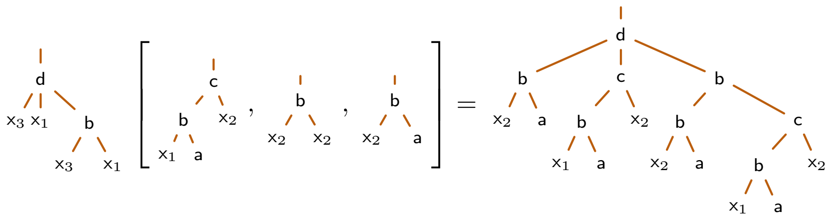


(2.2.1.B)

This term has 8 as degree and 6 as length.

There is at this stage a little subtlety to remark: a $\mathfrak{G}$-term $\mathfrak{t}$ gives rise to different elements of the graded set $\mathfrak{T}(\mathfrak{G})$ depending on the arity attributed to it. For instance, the term defined in (2.2.1.B) can among others be an element of $\mathfrak{T}(\mathfrak{G})(5)$ or of $\mathfrak{T}(\mathfrak{G})(6)$, both distinct from each other.

2.2.2 Free clones. Given a signature $\mathfrak{G}$, $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})(n)$, $n \geqslant 0$, and $\mathfrak{t}', \dots, \mathfrak{t}'_n \in \mathfrak{T}(\mathfrak{G})(m)$, $m \geqslant 0$, the *composition* of $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n$ in $\mathfrak{t}$ is the $\mathfrak{G}$-term $\mathfrak{t}[\mathfrak{t}'_1, \dots, \mathfrak{t}'_n]$ obtained by simultaneously replacing for all $i \in [n]$ all occurrences of the variables x_i in $\mathfrak{t}$ by $\mathfrak{t}'_i$. For instance, by considering the signature $\mathfrak{G}$ defined at the end of Section 2.2.1, we have the composition



(2.2.2.A)

of $\mathfrak{G}$-terms.

The *free clone* on $\mathfrak{G}$ is the clone $\mathfrak{T}(\mathfrak{G})$ on the graded set of the $\mathfrak{G}$-terms endowed with the following superposition maps and projections. Given $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})(n)$, $n \geqslant 0$, and $\mathfrak{t}', \dots, \mathfrak{t}'_n \in \mathfrak{T}(\mathfrak{G})(m)$, $m \geqslant 0$ the superposition $\mathfrak{t}[\mathfrak{t}'_1, \dots, \mathfrak{t}'_n]$ is the composition of $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n$ in $\mathfrak{t}$. Moreover, for any $n \geqslant 1$ and $i \in [n]$, the projection $\mathbb{1}_{i,n}$ is the $\mathfrak{G}$-term x_n.

2.3 Clone presentations and varieties

This preliminary section ends by setting up some notions about varieties of algebras and clone presentations.

2.3.1 Evaluation maps. If $\mathcal{C}$ is a clone, $\mathcal{C}$ is in particular a graded set and thus, a signature. Therefore, the free clone on $\mathcal{C}$ is a well-defined clone $\mathfrak{T}(\mathcal{C})$. The *evaluation map* of $\mathcal{C}$ is the map $\mathsf{ev}_{\mathcal{C}} : \mathfrak{T}(\mathcal{C}) \to \mathcal{C}$ recursively defined, for any $n \geqslant 1$ and $i \in [n]$ by

$$\mathsf{ev}_{\mathcal{C}}(\mathsf{x}_i) := \mathbb{1}_{i,n}, \tag{2.3.1.A}$$

and, for any $\mathsf{g} \in \mathcal{C}(k)$, $k \geqslant 0$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_k \in \mathfrak{T}(\mathcal{C})$, by

$$\mathsf{ev}_{\mathcal{C}}(\mathsf{g}[\mathfrak{t}_1, \dots, \mathfrak{t}_k]) := \mathsf{g}[\mathsf{ev}_{\mathcal{C}}(\mathfrak{t}_1), \dots, \mathsf{ev}_{\mathcal{C}}(\mathfrak{t}_k)], \tag{2.3.1.B}$$

where the superposition of the right-hand side of (2.3.1.B) is the one of $\mathcal{C}$.

2.3.2 Varieties and presentations. A *variety* is a pair $\mathcal{V} := (\mathfrak{G}, \mathfrak{R})$ such that $\mathfrak{G}$ is a signature and $\mathfrak{R}$ is an equivalence relation on $\mathfrak{T}(\mathfrak{G})$. Any pair $(\mathsf{t}, \mathsf{t}')$ of $\mathfrak{G}$-terms such that $\mathsf{t} \mathrel{\mathfrak{R}} \mathsf{t}'$ is an *equation* of $\mathcal{V}$. The *clone congruence generated* by $\mathfrak{R}$ is the smallest clone congruence $\equiv_{\mathfrak{R}}$ of $\mathfrak{T}(\mathfrak{G})$ containing $\mathfrak{R}$. If it exists, the *degree* of $\mathcal{V}$ is the smallest integer $d \geqslant 0$ such that all $\mathfrak{R}$-equivalence classes of $\mathfrak{G}$-terms of degrees $d' \geqslant d + 1$ are trivial.

A *presentation* of a clone $\mathcal{C}$ is a variety $\mathcal{V} := (\mathfrak{G}, \mathfrak{R})$ such that $\mathcal{C}$ is isomorphic as a clone to $\mathfrak{T}(\mathfrak{G})/_{\equiv_{\mathfrak{R}}}$. A presentation $\mathcal{V} := (\mathfrak{G}, \mathfrak{R})$ of $\mathcal{C}$ is *finitely related* if $\mathcal{V}$ admits a degree. An *algebra* over the variety $\mathcal{V}$ is an algebra over the clone admitting $\mathcal{V}$ as presentation.

The following statement is an important tool used in the sequel to establish clones presentations.

▶ Proposition 2.3.2.A — *Let $\mathcal{C}$ be a clone, $\mathcal{V} := (\mathfrak{G}, \mathfrak{R})$ be a variety, and $\phi : \mathfrak{G} \to \mathcal{C}$ be a graded set morphism. If $\phi(\mathfrak{G})$ is a generating set of $\mathcal{C}$ and, for any $\mathsf{t}, \mathsf{t}' \in \mathfrak{T}(\mathfrak{G})$, $\mathsf{t} \equiv_{\mathfrak{R}} \mathsf{t}'$ if and only if $\mathsf{ev}_{\mathcal{C}}\left(\widehat{\phi}(\mathsf{t})\right) = \mathsf{ev}_{\mathcal{C}}\left(\widehat{\phi}(\mathsf{t}')\right)$, then $\mathcal{V}$ is a presentation of $\mathcal{C}$.*

◀ Proof — Let us denote by $\theta : \mathfrak{T}(\mathfrak{G}) \to \mathcal{C}$ the map $\mathsf{ev}_{\mathcal{C}} \circ \widehat{\phi}$. Since $\mathsf{ev}_{\mathcal{C}} : \mathfrak{T}(\mathcal{C}) \to \mathcal{C}$ is a surjective clone morphism and $\phi(\mathfrak{G})$ is a generating set of $\mathcal{C}$, θ is a surjective clone morphism. Moreover, the fact that, by hypothesis, for any $\mathfrak{G}$-terms t and t' such that $\mathsf{t} \equiv_{\mathfrak{R}} \mathsf{t}'$, $\theta(\mathsf{t}) = \theta(\mathsf{t}')$ holds, θ induces a well-defined surjective clone morphism $\overline{\theta} : \mathfrak{T}(\mathfrak{G})/_{\equiv_{\mathfrak{R}}} \to \mathcal{C}$. Besides, if $[\mathsf{t}]_{\equiv_{\mathfrak{R}}}$ and $[\mathsf{t}']_{\equiv_{\mathfrak{R}}}$ are two $\equiv_{\mathfrak{R}}$-equivalence classes of $\mathfrak{G}$-terms such that $\overline{\theta}\left([\mathsf{t}]_{\equiv_{\mathfrak{R}}}\right) = \overline{\theta}\left([\mathsf{t}']_{\equiv_{\mathfrak{R}}}\right)$, then for any $\mathsf{t} \in [\mathsf{t}]_{\equiv_{\mathfrak{R}}}$ and $\mathsf{t}' \in [\mathsf{t}']_{\equiv_{\mathfrak{R}}}$, we have $\theta(\mathsf{t}) = \theta(\mathsf{t}')$. This implies by using the hypothesis of the statement of the proposition that $\mathsf{t} \equiv_{\mathfrak{R}} \mathsf{t}'$. Therefore, $[\mathsf{t}]_{\equiv_{\mathfrak{R}}} = [\mathsf{t}']_{\equiv_{\mathfrak{R}}}$, showing that $\overline{\theta}$ is injective. We have shown that $\overline{\theta}$ is a clone isomorphism between $\mathfrak{T}(\mathfrak{G})/_{\equiv_{\mathfrak{R}}}$ and $\mathcal{C}$, implying the statement of the proposition. □

2.3.3 Clone realizations of varieties. In the other direction, given a variety $\mathcal{V}$, any clone admitting $\mathcal{V}$ as presentation is a *clone realization* of $\mathcal{V}$ (see [Neu70]).

For instance, let the variety $\mathcal{V} := (\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G}$ is the signature satisfying $\mathfrak{G} = \mathfrak{G}(2) = \{\wedge\}$ and $\mathfrak{R}$ is the equivalence relation on $\mathfrak{T}(\mathfrak{G})$ satisfying

$$\wedge[\wedge[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3] \mathrel{\mathfrak{R}} \wedge[\mathsf{x}_1, \wedge[\mathsf{x}_2, \mathsf{x}_3]], \tag{2.3.3.A}$$

$$\wedge[\mathsf{x}_1, \mathsf{x}_2] \mathrel{\mathfrak{R}} \wedge[\mathsf{x}_2, \mathsf{x}_1], \tag{2.3.3.B}$$

$$\wedge[\mathsf{x}_1, \mathsf{x}_1] \mathrel{\mathfrak{R}} \mathsf{x}_1. \tag{2.3.3.C}$$

This is the variety of semilattices. The clone realization $\mathcal{C} := \mathfrak{T}(\mathfrak{G})/_{\equiv_{\mathfrak{R}}}$ admits the following concrete description. For any $n \geqslant 0$, $\mathcal{C}(n)$ is a copy of the set of nonempty subsets of $[n]$. The superposition maps of $\mathcal{C}$ satisfy, for any $n \geqslant 0$, $\mathfrak{U} \in \mathcal{C}(n)$, and $\mathfrak{U}'_1, \dots, \mathfrak{U}'_n \in \mathcal{C}(m)$, $m \geqslant 0$,

$$\mathfrak{U}[\mathfrak{U}'_1, \dots, \mathfrak{U}'_n] = \bigcup_{i \in \mathfrak{U}} \mathfrak{U}'_i, \tag{2.3.3.D}$$

and for any $n \geqslant 1$ and $i \in [n]$, the projection $\mathbb{1}_{i,n}$ is $\{i\}$. Any algebra over $\mathcal{C}$ is a semilattice.

3 Pigmented monoids and clones of pigmented words

We introduce here the variety of pigmented monoids which is roughly speaking a variety wherein algebras are monoids endowed with monoid endomorphisms indexed on another monoid $\mathcal{M}$ —the pigments— with some extra structure. A clone realization $\mathbf{P}(\mathcal{M})$ of this variety involving some particular words as main combinatorial objects is described.

3.1 Pigmented monoids

Let us describe the variety of pigmented monoids and browse some examples of such structures having some combinatorial interest.

3.1.1 Varieties of pigmented monoids. Let $(\mathcal{M}, \cdot, \mathrm{e})$ be a monoid. Recall that $\cdot$ is an associative binary operation and that e is the unit w.r.t. the operation $\cdot$. We denote by $\mathcal{E}$ the trivial monoid, that is the monoid having e as unique element.

The *variety of $\mathcal{M}$-pigmented monoids* (or simply *pigmented monoids* when the context is clear) is the variety $(\mathfrak{G}_\mathcal{M}, \mathfrak{R}_\mathcal{M})$ such that $\mathfrak{G}_\mathcal{M} := \mathfrak{G}_\mathcal{M}(0) \sqcup \mathfrak{G}_\mathcal{M}(1) \sqcup \mathfrak{G}_\mathcal{M}(2)$ where

$$\mathfrak{G}_\mathcal{M}(0) := \{\mathrm{u}\}, \quad \mathfrak{G}_\mathcal{M}(1) := \{\mathrm{p}_\alpha : \alpha \in \mathcal{M}\}, \quad \mathfrak{G}_\mathcal{M}(2) := \{\star\}, \tag{3.1.1.A}$$

and $\mathfrak{R}_\mathcal{M}$ is the equivalence relation on $\mathfrak{T}(\mathfrak{G}_\mathcal{M})$ satisfying

$$\star[\star[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3] \ \mathfrak{R}_\mathcal{M} \ \star[\mathsf{x}_1, \star[\mathsf{x}_2, \mathsf{x}_3]], \tag{3.1.1.B}$$

$$\star[\mathrm{u}, \mathsf{x}_1] \ \mathfrak{R}_\mathcal{M} \ \mathsf{x}_1 \ \mathfrak{R}_\mathcal{M} \ \star[\mathsf{x}_1, \mathrm{u}], \tag{3.1.1.C}$$

$$\mathrm{p}_\alpha[\star[\mathsf{x}_1, \mathsf{x}_2]] \ \mathfrak{R}_\mathcal{M} \ \star[\mathrm{p}_\alpha[\mathsf{x}_1], \mathrm{p}_\alpha[\mathsf{x}_2]], \tag{3.1.1.D}$$

$$\mathrm{p}_\alpha[\mathrm{u}] \ \mathfrak{R}_\mathcal{M} \ \mathrm{u}, \tag{3.1.1.E}$$

$$\mathrm{p}_{\alpha_1}[\mathrm{p}_{\alpha_2}[\mathsf{x}_1]] \ \mathfrak{R}_\mathcal{M} \ \mathrm{p}_{\alpha_1 \cdot \alpha_2}[\mathsf{x}_1], \tag{3.1.1.F}$$

$$\mathrm{p}_\mathrm{e}[\mathsf{x}_1] \ \mathfrak{R}_\mathcal{M} \ \mathsf{x}_1, \tag{3.1.1.G}$$

for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

Let $(\mathcal{A}, \mathfrak{op})$ be an algebra over the variety of $\mathcal{M}$-pigmented monoids. By denoting by $\star$ the binary product $\mathfrak{op}(\star)$, by u the constant $\mathfrak{op}(\mathrm{u})$, and for any $\alpha \in \mathcal{M}$, by p_α the unary product $\mathfrak{op}(\mathrm{p}_\alpha)$, the following properties hold.

(i) By (3.1.1.B) and (3.1.1.C), $(\mathcal{A}, \star, \mathrm{u})$ is a monoid.

(ii) By (3.1.1.D) and (3.1.1.E), each p_α, $\alpha \in \mathcal{M}$, is a monoid endomorphism of $(\mathcal{A}, \star, \mathrm{u})$.

(iii) By (3.1.1.F) and (3.1.1.G), for any $\alpha \in \mathcal{M}$, the map $\cdot : \mathcal{M} \times \mathcal{A} \to \mathcal{A}$ defined by $\alpha \cdot x := \mathrm{p}_\alpha(x)$ is a left monoid action of $\mathcal{M}$ on $\mathcal{A}$.

Any such quadruple $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ is an *$\mathcal{M}$-pigmented monoid* (or simply *pigmented monoid* when the context is clear).

For instance, any $\mathbb{Z}/2\mathbb{Z}$-pigmented monoid is a set $\mathcal{A}$ endowed with an associative product $\star$ and two unary operations p_0 and p_1 such that $\star$ admits a unit $\mathrm{u} \in \mathcal{A}$, p_0 is the identity map on $\mathcal{A}$, and for any $x, x_1, x_2 \in \mathcal{A}$, $\mathrm{p}_1(x_1 \star x_2) = \mathrm{p}_1(x_1) \star \mathrm{p}_1(x_2)$, $\mathrm{p}_1(\mathrm{u}) = \mathrm{u}$, and $\mathrm{p}_1(\mathrm{p}_1(x)) = x$. In other terms, a $\mathbb{Z}/2\mathbb{Z}$-pigmented monoid is a monoid endowed with an involutive monoid endomorphism. Similarly, a $(\{0, 1\}, \times, 1)$-pigmented monoid is a monoid endowed with an idempotent monoid endomorphism.

A variation of $\mathcal{M}$-pigmented monoids have been considered in [Gir15] (see also [Gir18, Chap. 4]) as algebras over some operads. In this cited work, the considered variety admits $\mathfrak{G}_\mathcal{M} \setminus \{\mathrm{u}\}$ as signature and $\mathfrak{R}_\mathcal{M}$ deprived of Relations (3.1.1.C) and (3.1.1.E) as equivalence relation.

3.1.2 Examples. Let us consider the following examples of pigmented monoids.

(E1) Let $\mathcal{A} := (\mathbb{N}^*, ., \epsilon, \mathrm{p}_\alpha)$ where . is the concatenation product and for any $\alpha \in \mathbb{N}$, p_α is the map sending any word to its subword made of the letters greater than or equal to α. This quadruple is an $\mathcal{M}$-pigmented monoid where $\mathcal{M} := (\mathbb{N}, \max, 0)$. For instance,

$$\mathrm{p}_2(0015213.41200) = 52342 = \mathrm{p}_2(0015213).\mathrm{p}_2(41200). \tag{3.1.2.A}$$

(E2) Let $\mathcal{A} := (\mathbb{Z}^*, ., \epsilon, \mathrm{p}_\alpha)$ where . is the concatenation product and for any $\alpha \in \mathbb{Z}$, p_α is the map sending any word to the word obtained by incrementing by α its letters. This quadruple is an $\mathcal{M}$-pigmented monoid where $\mathcal{M} := (\mathbb{Z}, +, 0)$. For instance, by denoting by $\overline{n}$ any negative integer having n as absolute value,

$$\mathrm{p}_{\overline{3}}(24\overline{3}0.\overline{2}64) = \overline{1}1\overline{6}\overline{3}531 = \mathrm{p}_{\overline{3}}(24\overline{3}0).\mathrm{p}_{\overline{3}}(\overline{2}64). \tag{3.1.2.B}$$

(E3) Let $\mathcal{A} := (\mathbb{K}\langle\langle \mathsf{z} \rangle\rangle, +, 0, \mathrm{p}_\alpha)$ where $\mathbb{K}$ is a field of zero characteristic with multiplication denoted by $\cdot$, $\mathbb{K}\langle\langle \mathsf{z} \rangle\rangle$ is the space of formal power series on the parameter z, and for any $\alpha \in \mathbb{K}$, p_α is the map sending any series to the series obtained by multiplying its coefficients by α. This quadruple is an $\mathcal{M}$-pigmented monoid where $\mathcal{M} := (\mathbb{K}, \cdot, 1)$.

(E4) Generalizing the previous example, let $\mathcal{A} := (V, +, 0, \mathrm{p}_\alpha)$ where V is a vector space on a field $\mathbb{K}$ with multiplication denoted by $\cdot$, and for any $\alpha \in \mathbb{K}$ and $v \in V$, $\mathrm{p}_\alpha(v) = \alpha \cdot v$. This quadruple is an $\mathcal{M}$-pigmented monoid where $\mathcal{M} := (\mathbb{K}, \cdot, 1)$.

3.2 Clone of pigmented words

We describe now a construction taking at input a monoid $\mathcal{M}$ and outputting a clone $\mathbf{P}(\mathcal{M})$ on the graded set of $\mathcal{M}$-pigmented words. We show some first properties of this construction $\mathbf{P}$, as the fact that it is a functor from the category of monoids to the category of clones and describe a generating set of $\mathbf{P}(\mathcal{M})$.

3.2.1 Pigmented words. Let S be an nonempty set. An *S-pigmented letter* (or *pigmented letter* when the context is clear) is a pair (i, α), denoted by i^α, where $\alpha \in S$ and i is a positive integer. We call i (resp. α) the *value* (resp. the *pigment*) of i^α. Let $\mathcal{L}_\mathcal{M}$ be the set of $\mathcal{M}$-pigmented letters. An *S-pigmented word* (or *pigmented word* when the context is clear) of *arity* n, $n \geqslant 0$, is a word $\mathfrak{p}$ on $\mathcal{L}_\mathcal{M}$ such that all values of the pigmented letters of $\mathfrak{p}$ belong to $[n]$. The only S-pigmented word of arity 0 is the empty word, denoted by $\boldsymbol{\epsilon}$ in this context. For instance, $\mathfrak{p} := 2^a 1^a 1^b 6^a$ is an $\{a, b, c, d\}$-pigmented word of arity 17.

3.2.2 Construction. Let $(\mathcal{M}, \cdot, \mathrm{e})$ be a monoid. Let $\mathbf{P}(\mathcal{M})$ be the graded set of $\mathcal{M}$-pigmented words. Let $\bar{\cdot} : \mathcal{M} \times \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map defined for any $\alpha \in \mathcal{M}$ and any $\mathcal{M}$-pigmented word $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}$ by

$$\alpha \,\bar{\cdot}\, i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} := i_1^{\alpha \cdot \alpha_1} \dots i_\ell^{\alpha \cdot \alpha_\ell}. \tag{3.2.2.A}$$

Let us moreover endow $\mathbf{P}(\mathcal{M})$ with the superposition maps defined for any $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, by

$$i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}[\mathfrak{p}_1, \dots, \mathfrak{p}_n] := (\alpha_1 \,\bar{\cdot}\, \mathfrak{p}_{i_1}) \;.\; \dots \;.\; (\alpha_\ell \,\bar{\cdot}\, \mathfrak{p}_{i_\ell}). \tag{3.2.2.B}$$

For instance, by denoting by A^* the free monoid $(A^*, ., \epsilon)$ generated by $A := \{a, b, c\}$, we have in $\mathbf{P}(A^*)$,

$$\begin{aligned} 2^{ba} 2^{aa} 4^{baa} 3^{\epsilon} \left[2^b 1^{aa}, 1^{bbb} 1^{\epsilon} 2^b, 2^{aa} 2^a, \boldsymbol{\epsilon}\right] &= \left(ba \,\bar{\cdot}\, 1^{bbb} 1^{\epsilon} 2^b\right).\left(aa \,\bar{\cdot}\, 1^{bbb} 1^{\epsilon} 2^b\right).\left(baa \,\bar{\cdot}\, \boldsymbol{\epsilon}\right).\left(\epsilon \,\bar{\cdot}\, 2^{aa} 2^a\right) \\ &= 1^{babbb} 1^{ba} 2^{bab} \;.\; 1^{aabbb} 1^{aa} 2^{aab} \;.\; \boldsymbol{\epsilon} \;.\; 2^{aa} 2^a \\ &= 1^{babbb} 1^{ba} 2^{bab} 1^{aabbb} 1^{aa} 2^{aab} 2^{aa} 2^a. \end{aligned} \tag{3.2.2.C}$$

We also set, for any $n \geqslant 1$ and $i \in [n]$, $\mathbb{1}_{i,n}$ as the pigmented word i^{e} of length 1. For instance, by considering the monoid $\mathcal{M}$ of the previous example, $\mathbb{1}_{2,4}$ is the pigmented word 2^{ϵ}.

Besides, given a monoid morphism $\phi : \mathcal{M} \to \mathcal{M}'$ between two monoids $\mathcal{M}$ and $\mathcal{M}'$, let $\mathbf{P}(\phi) : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M}')$ be the map defined for any $\mathcal{M}$-pigmented word $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}$ by

$$\mathbf{P}(\phi)(i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}) := i_1^{\phi(\alpha_1)} \dots i_\ell^{\phi(\alpha_\ell)}. \tag{3.2.2.D}$$

For instance, by denoting by $\mathbb{N}$ the additive monoid $(\mathbb{N}, +, 0)$, the map $\phi : A^* \to \mathbb{N}$ sending each $w \in A^*$ to its length is a monoid morphism. We have in this context

$$\mathbf{P}(\phi)\left(2^{ba} 2^{aa} 3^{\epsilon}\right) = 2^2 2^2 3^0. \tag{3.2.2.E}$$

▶ **Theorem** 3.2.2.A — *The construction* $\mathbf{P}$ *is a functor from the category of monoids to the category of clones. Moreover, this functor preserves injections and surjections.*

◀ **Proof** — In this proof, we consider two monoids $(\mathcal{M}, \cdot, \mathrm{e})$ and $(\mathcal{M}', \cdot', \mathrm{e}')$.

Let us first prove that $\mathbf{P}(\mathcal{M})$ is a clone. For any $n \geqslant 1$, $i \in [n]$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathbf{P}(\mathcal{M})(n)$, since e is the unit of $\mathcal{M}$, we have $i^{\mathrm{e}}[\mathfrak{p}_1, \dots, \mathfrak{p}_n] = \mathfrak{p}_i$ so that Relation (2.1.2.B) is satisfied. Moreover, for any $n \geqslant 0$ and $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, again since e is the unit of $\mathcal{M}$, we have $\mathfrak{p}[1^{\mathrm{e}}, \dots, n^{\mathrm{e}}] = \mathfrak{p}$ so that Relation (2.1.2.C) is satisfied. Finally, for any $n \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$, $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathbf{P}(\mathcal{M})(n)$, $j_{1,1}^{\beta_{1,1}} \dots j_{1,k_1}^{\beta_{1,k_1}}, \dots, j_{n,1}^{\beta_{n,1}} \dots j_{n,k_n}^{\beta_{n,k_n}} \in \mathbf{P}(\mathcal{M})(m)$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_m \in \mathbf{P}(\mathcal{M})(k)$, since $\cdot$ is associative, we have

$$\begin{aligned}
& i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \left[j_{1,1}^{\beta_{1,1}} \dots j_{1,k_1}^{\beta_{1,k_1}}, \dots, j_{n,1}^{\beta_{n,1}} \dots j_{n,k_n}^{\beta_{n,k_n}}\right] [\mathfrak{p}_1, \dots, \mathfrak{p}_m] \\
& = \left(\alpha_1 \mathbin{\bar{\cdot}} j_{i_1,1}^{\beta_{i_1,1}} \dots j_{i_1,k_{i_1}}^{\beta_{i_1,k_{i_1}}}\right) \dots \left(\alpha_\ell \mathbin{\bar{\cdot}} j_{i_\ell,1}^{\beta_{i_\ell,1}} \dots j_{i_\ell,k_{i_\ell}}^{\beta_{i_\ell,k_{i_\ell}}}\right) [\mathfrak{p}_1, \dots, \mathfrak{p}_m] \\
& = j_{i_1,1}^{\alpha_1 \cdot \beta_{i_1,1}} \dots j_{i_1,k_{i_1}}^{\alpha_1 \cdot \beta_{i_1,k_{i_1}}} \dots j_{i_\ell,1}^{\alpha_\ell \cdot \beta_{i_\ell,1}} \dots j_{i_\ell,k_{i_\ell}}^{\alpha_\ell \cdot \beta_{i_\ell,k_{i_\ell}}} [\mathfrak{p}_1, \dots, \mathfrak{p}_m] \\
& = \left((\alpha_1 \cdot \beta_{i_1,1}) \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_1,1}}\right) \dots \left(\left(\alpha_1 \cdot \beta_{i_1,k_{i_1}}\right) \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_1,k_{i_1}}}\right) \\
& \qquad \dots \left((\alpha_\ell \cdot \beta_{i_\ell,1}) \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_\ell,1}}\right) \dots \left(\left(\alpha_\ell \cdot \beta_{i_\ell,k_{i_\ell}}\right) \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_\ell,k_{i_\ell}}}\right) \\
& = \alpha_1 \mathbin{\bar{\cdot}} \left(\left(\beta_{i_1,1} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_1,1}}\right) \dots \left(\beta_{i_1,k_{i_1}} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_1,k_{i_1}}}\right)\right) \dots \alpha_\ell \mathbin{\bar{\cdot}} \left(\left(\beta_{i_\ell,1} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_\ell,1}}\right) \dots \left(\beta_{i_\ell,k_{i_\ell}} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{i_\ell,k_{i_\ell}}}\right)\right) \\
& = i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \left[\left(\beta_{1,1} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{1,1}}\right) \dots \left(\beta_{1,k_1} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{1,k_1}}\right), \dots, \left(\beta_{n,1} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{n,1}}\right) \dots \left(\beta_{n,k_n} \mathbin{\bar{\cdot}} \mathfrak{p}_{j_{n,k_n}}\right)\right] \\
& = i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \left[j_{1,1}^{\beta_{1,1}} \dots j_{1,k_1}^{\beta_{1,k_1}} [\mathfrak{p}_1, \dots, \mathfrak{p}_m], \dots, j_{n,1}^{\beta_{n,1}} \dots j_{n,k_n}^{\beta_{n,k_n}} [\mathfrak{p}_1, \dots, \mathfrak{p}_m]\right]
\end{aligned} \tag{3.2.2.F}$$

so that Relation (2.1.2.D) is satisfied. Therefore, $\mathbf{P}(\mathcal{M})$ is a clone.

Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a monoid morphism. Let us show that $\mathbf{P}(\phi)$ is a clone morphism. First, $\mathbf{P}(\phi)$ is a graded set morphism. Moreover, for any $n \geqslant 1$ and $i \in [n]$, since ϕ sends the unit of $\mathcal{M}$ to the unit of $\mathcal{M}'$, we have $\mathbf{P}(\phi)(i^{\mathrm{e}}) = i^{\phi(\mathrm{e})} = i^{\mathrm{e}'}$. Finally, for any $n \geqslant 0$, $m \geqslant 0$, $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathbf{P}(\mathcal{M})(n)$, and $j_{1,1}^{\beta_{1,1}} \dots j_{1,k_1}^{\beta_{1,k_1}}, \dots, j_{n,1}^{\beta_{n,1}} \dots j_{n,k_n}^{\beta_{n,k_n}} \in \mathbf{P}(\mathcal{M})(m)$, since ϕ is a monoid morphism, we have

$$\begin{aligned}
& \mathbf{P}(\phi)\left(i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \left[j_{1,1}^{\beta_{1,1}} \dots j_{1,k_1}^{\beta_{1,k_1}}, \dots, j_{n,1}^{\beta_{n,1}} \dots j_{n,k_n}^{\beta_{n,k_n}}\right]\right) \\
& = \mathbf{P}(\phi)\left(\left(\alpha_1 \mathbin{\bar{\cdot}} j_{i_1,1}^{\beta_{i_1,1}} \dots j_{i_1,k_{i_1}}^{\beta_{i_1,k_{i_1}}}\right) \dots \left(\alpha_\ell \mathbin{\bar{\cdot}} j_{i_\ell,1}^{\beta_{i_\ell,1}} \dots j_{i_\ell,k_{i_\ell}}^{\beta_{i_\ell,k_{i_\ell}}}\right)\right) \\
& = \mathbf{P}(\phi)\left(j_{i_1,1}^{\alpha_1 \cdot \beta_{i_1,1}} \dots j_{i_1,k_{i_1}}^{\alpha_1 \cdot \beta_{i_1,k_{i_1}}} \dots j_{i_\ell,1}^{\alpha_\ell \cdot \beta_{i_\ell,1}} \dots j_{i_\ell,k_{i_\ell}}^{\alpha_\ell \cdot \beta_{i_\ell,k_{i_\ell}}}\right) \\
& = j_{i_1,1}^{\phi(\alpha_1 \cdot \beta_{i_1,1})} \dots j_{i_1,k_{i_1}}^{\phi\left(\alpha_1 \cdot \beta_{i_1,k_{i_1}}\right)} \dots j_{i_\ell,1}^{\phi(\alpha_\ell \cdot \beta_{i_\ell,1})} \dots j_{i_\ell,k_{i_\ell}}^{\phi\left(\alpha_\ell \cdot \beta_{i_\ell,k_{i_\ell}}\right)} \\
& = j_{i_1,1}^{\phi(\alpha_1) \cdot' \phi\left(\beta_{i_1,1}\right)} \dots j_{i_1,k_{i_1}}^{\phi(\alpha_1) \cdot' \phi\left(\beta_{i_1,k_{i_1}}\right)} \dots j_{i_\ell,1}^{\phi(\alpha_\ell) \cdot' \phi\left(\beta_{i_\ell,1}\right)} \dots j_{i_\ell,k_{i_\ell}}^{\phi(\alpha_\ell) \cdot' \phi\left(\beta_{i_\ell,k_{i_\ell}}\right)}
\end{aligned} \tag{3.2.2.G}$$

$$= \left(\phi(\alpha_1) \,\bar{\cdot}'\, j_{i_1,1}^{\phi\left(\beta_{i_1,1}\right)} \dots j_{i_1,k_{i_1}}^{\phi\left(\beta_{i_1,k_{i_1}}\right)}\right) \dots \left(\phi(\alpha_\ell) \,\bar{\cdot}'\, j_{i_\ell,1}^{\phi\left(\beta_{i_\ell,1}\right)} \dots j_{i_\ell,k_{i_\ell}}^{\phi\left(\beta_{i_\ell,k_{i_\ell}}\right)}\right)$$
$$= i_1^{\phi(\alpha_1)} \dots i_\ell^{\phi(\alpha_\ell)} \left[j_{1,1}^{\phi(\beta_{1,1})} \dots j_{1,k_1}^{\phi(\beta_{1,k_1})}, \dots, j_{n,1}^{\phi(\beta_{n,1})} \dots j_{n,k_n}^{\phi(\beta_{n,k_n})}\right]$$
$$= \mathbf{P}(\phi)(i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell})\left[\mathbf{P}(\phi)\left(j_{1,1}^{\beta_{1,1}} \dots j_{1,k_1}^{\beta_{1,k_1}}\right), \dots, \mathbf{P}(\phi)\left(j_{n,1}^{\beta_{n,1}} \dots j_{n,k_n}^{\beta_{n,k_n}}\right)\right].$$

Therefore, $\mathbf{P}(\phi)$ is a clone morphism. Moreover, it is immediate, for any monoid $\mathcal{M}''$ and monoid morphism $\phi' : \mathcal{M}' \to \mathcal{M}''$, that $\mathbf{P}(\phi' \circ \phi) = \mathbf{P}(\phi') \circ \mathbf{P}(\phi)$. It is also immediate that if $\mathrm{I} : \mathcal{M} \to \mathcal{M}$ is the identity map, then $\mathbf{P}(\mathrm{I})$ is the identity map on $\mathbf{P}(\mathcal{M})$. For these reasons, $\mathbf{P}$ is a functor from the category of monoids to the category of clones.

Let us finally prove that $\mathbf{P}$ preserves injections and surjections. Assume that ϕ is injective. If $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}$ and $j_1^{\beta_1} \dots j_k^{\beta_k}$ are two elements of $\mathbf{P}(\mathcal{M})$ such that $\mathbf{P}(\phi)(i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}) = \mathbf{P}(\phi)\left(j_1^{\beta_1} \dots j_k^{\beta_k}\right)$, then $i_1^{\phi(\alpha_1)} \dots i_\ell^{\phi(\alpha_\ell)} = j_1^{\phi(\beta_1)} \dots j_k^{\phi(\beta_k)}$. Thus, $\ell = k$, $i_1 = j_1$, ..., $i_\ell = j_\ell$, $\phi(\alpha_1) = \phi(\beta_1)$, ..., $\phi(\alpha_\ell) = \phi(\beta_\ell)$. Since ϕ is injective, we have $\alpha_1 = \beta_1$, ..., $\alpha_\ell = \beta_\ell$, showing that $\mathbf{P}(\phi)$ is injective. Assume that ϕ is surjective. Let $j_1^{\beta_1} \dots j_k^{\beta_k} \in \mathbf{P}(\mathcal{M}')$. Since ϕ is surjective, there are $\alpha_1, \dots, \alpha_k \in \mathcal{M}$ such that $\phi(\alpha_1) = \beta_1$, ..., $\phi(\alpha_k) = \beta_k$. Therefore, we have $\mathbf{P}(\phi)(j_1^{\alpha_1} \dots j_k^{\alpha_k}) = j_1^{\beta_1} \dots j_k^{\beta_k}$, showing that $\mathbf{P}(\phi)$ is surjective. □

3.2.3 First properties. We describe now a generating set of $\mathbf{P}(\mathcal{M})$ and show that the map sending any $\mathcal{M}$-pigmented word to its mirror image is an involutive clone automorphism of $\mathbf{P}(\mathcal{M})$.

▶ **Proposition** 3.2.3.A — *For any monoid $\mathcal{M}$, the graded set $G_\mathcal{M} := G_\mathcal{M}(0) \sqcup G_\mathcal{M}(1) \sqcup G_\mathcal{M}(2)$ defined by $G_\mathcal{M}(0) := \{\epsilon\}$, $G_\mathcal{M}(1) := \{1^\alpha : \alpha \in \mathcal{M}\}$, and $G_\mathcal{M}(2) := \{1^{\mathrm{e}}2^{\mathrm{e}}\}$ where e is the unit of $\mathcal{M}$ is a generating set of the clone $\mathbf{P}(\mathcal{M})$.*

◀ **Proof** — Let us prove by induction on the length ℓ of $\mathfrak{p} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, that $\mathfrak{p} \in \mathbf{P}(\mathcal{M})^{G_\mathcal{M}}$. First, if $\ell = 0$, then $\mathfrak{p} = \epsilon$ and since $\epsilon \in G_\mathcal{M}$, the property holds. If $\ell \geqslant 1$, then $\mathfrak{p}$ decomposes as $\mathfrak{p} = \mathfrak{p}'.i^\alpha$ where $\mathfrak{p}' \in \mathbf{P}(\mathcal{M})(n)$ and $i^\alpha \in \mathcal{L}_\mathcal{M}$. By definition of the superposition maps of $\mathbf{P}(\mathcal{M})$, $\mathfrak{p}$ expresses as $\mathfrak{p} = 1^{\mathrm{e}}2^{\mathrm{e}}[\mathfrak{p}', 1^\alpha[\mathbb{1}_{i,n}]]$. Now, since $\ell(\mathfrak{p}') = \ell - 1$, by induction hypothesis, $\mathfrak{p}' \in \mathbf{P}(\mathcal{M})^{G_\mathcal{M}}$. Moreover, since $1^{\mathrm{e}}2^{\mathrm{e}} \in G_\mathcal{M}$ and $1^\alpha \in G_\mathcal{M}$, this shows the previously stated property. □

By considering the graded set $G_\mathcal{M}$ introduced by Proposition 3.2.3.A, let $\mathsf{int}_\mathcal{M} : \mathfrak{G}_\mathcal{M} \to G_\mathcal{M}$ be the graded set morphism defined by $\mathsf{int}_\mathcal{M}(\mathrm{u}) := \epsilon$, $\mathsf{int}_\mathcal{M}(\mathrm{p}_\alpha) := 1^\alpha$, $\alpha \in \mathcal{M}$, and $\mathsf{int}_\mathcal{M}(\star) := 1^{\mathrm{e}}2^{\mathrm{e}}$. This bijective map will be used together with Proposition 2.3.2.A in order to establish a presentation of $\mathbf{P}(\mathcal{M})$.

The map r sending any word to its mirror image is in particular a well-defined graded set morphism from $\mathbf{P}(\mathcal{M})$ to $\mathbf{P}(\mathcal{M})$. As stated by the following result, this map has an additional property.

▶ **Proposition** 3.2.3.B — *For any monoid $\mathcal{M}$, the map $\mathrm{r} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ is an involutive clone automorphism.*

◀ **Proof** — Let $\cdot$ be the operation of $\mathcal{M}$ and e its unit. It is first immediate that the projections $i^{\mathrm{e}} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 1$, $i \in [n]$, are fixed-points of r. Moreover, as a consequence of the fact that for any words u and v on any alphabet, $\mathrm{r}(u.v) = \mathrm{r}(v).\mathrm{r}(u)$, for any $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathbf{P}(\mathcal{M})$, $n \geqslant 0$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, we have

$$\mathrm{r}(i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}[\mathfrak{p}_1, \dots, \mathfrak{p}_n]) = \mathrm{r}(\alpha_\ell \,\bar{\cdot}\, \mathfrak{p}_{i_\ell}) \dots \mathrm{r}(\alpha_1 \,\bar{\cdot}\, \mathfrak{p}_{i_1}) = \mathrm{r}(i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell})[\mathrm{r}(\mathfrak{p}_1), \dots, \mathrm{r}(\mathfrak{p}_n)]. \qquad (3.2.3.A)$$

Therefore, r is a clone morphism. Finally, since the r is an involution, the statement of the proposition follows. □

3.3 CLONE REALIZATION

This last part of the present section is devoted to establish its main result, namely the fact that $\mathbf{P}(\mathcal{M})$ is a clone realization of the variety of $\mathcal{M}$-pigmented monoids. For this, we shall use a method consisting in building a specific system of representatives for the quotient $\mathfrak{T}(\mathfrak{G}_\mathcal{M})/_{\equiv_{\mathfrak{R}_\mathcal{M}}}$ which is in one-to-one correspondence with the graded set of $\mathcal{M}$-pigmented words. Other approaches are possible as well including those using term rewrite systems [BN98; Bez+03] and proofs for their termination and confluence.

3.3.1 PROPERTIES OF THE EQUATION SET. We begin with two elementary properties satisfied by the equivalence relation $\mathfrak{R}_\mathcal{M}$.

▶ **Lemma** 3.3.1.A — *For any monoid $\mathcal{M}$ and any $\mathsf{t}, \mathsf{t}' \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, $\mathsf{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathsf{t}'$ implies that t and t' are of equal length.*

◀ **Proof** — For any equation $(\mathsf{t}, \mathsf{t}')$ of the variety $(\mathfrak{G}_\mathcal{M}, \mathfrak{R}_\mathcal{M})$ (see Relations (3.1.1.B)—(3.1.1.G)), we can observe that $\ell(\mathsf{t}) = \ell(\mathsf{t}')$. Since by definition, $\equiv_{\mathfrak{R}_\mathcal{M}}$ is the smallest clone congruence containing $\mathfrak{R}_\mathcal{M}$, the statement of the lemma follows. □

The *frontier map* is the map $\mathsf{fr}_\mathcal{M} : \mathfrak{T}(\mathfrak{G}_\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ defined by $\mathsf{fr}_\mathcal{M} := \mathsf{ev}_{\mathbf{P}(\mathcal{M})} \circ \widehat{\mathsf{int}_\mathcal{M}}$, where $\mathsf{int}_\mathcal{M}$ is the graded set morphism defined in Section 3.2.3. For instance, by considering the free monoid $(A^*, ., \epsilon)$ generated by $A := \{a, b\}$, we have in $\mathbf{P}(A^*)$,

$$\begin{aligned}
&\mathsf{fr}_{\mathbf{P}(A^*)}(\star[\mathrm{p}_a[\star[\mathsf{x}_3, \mathrm{p}_b[\mathsf{x}_2]]], \star[\mathsf{x}_1, \mathrm{p}_b[\mathsf{x}_2]]]) \\
&\quad = \mathsf{ev}_{\mathbf{P}(A^*)}\left(\widehat{\mathsf{int}_{A^*}}(\star[\mathrm{p}_a[\star[\mathsf{x}_3, \mathrm{p}_b[\mathsf{x}_2]]], \star[\mathsf{x}_1, \mathrm{p}_b[\mathsf{x}_2]]])\right) \\
&\quad = \mathsf{ev}_{\mathbf{P}(A^*)}\left(1^\epsilon 2^\epsilon \left[1^a \left[1^\epsilon 2^\epsilon \left[3^\epsilon, 1^b[2^\epsilon]\right]\right], 1^\epsilon 2^\epsilon \left[1^\epsilon, 1^b[2^\epsilon]\right]\right]\right) \\
&\quad = 3^a 2^{ab} 1^\epsilon 2^b.
\end{aligned} \tag{3.3.1.A}$$

▶ **Lemma** 3.3.1.B — *For any monoid $\mathcal{M}$ and any $\mathsf{t}, \mathsf{t}' \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, $\mathsf{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathsf{t}'$ implies $\mathsf{fr}_\mathcal{M}(\mathsf{t}) = \mathsf{fr}_\mathcal{M}(\mathsf{t}')$.*

◀ **Proof** — Let $\cdot$ be the operation of $\mathcal{M}$ and e is its unit. For any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$, we have

$$\mathsf{fr}_\mathcal{M}(\star[\star[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3]) = 1^\mathrm{e} 2^\mathrm{e} 3^\mathrm{e} = \mathsf{fr}_\mathcal{M}(\star[\mathsf{x}_1, \star[\mathsf{x}_2, \mathsf{x}_3]]), \tag{3.3.1.B}$$

$$\mathsf{fr}_\mathcal{M}(\star[\mathrm{u}, \mathsf{x}_1]) = 1^\mathrm{e} = \mathsf{fr}_\mathcal{M}(\mathsf{x}_1) = 1^\mathrm{e} = \mathsf{fr}_\mathcal{M}(\star[\mathsf{x}_1, \mathrm{u}]), \tag{3.3.1.C}$$

$$\mathsf{fr}_\mathcal{M}(\mathrm{p}_\alpha[\star[\mathsf{x}_1, \mathsf{x}_2]]) = 1^\alpha 2^\alpha = \mathsf{fr}_\mathcal{M}(\star[\mathrm{p}_\alpha[\mathsf{x}_1], \mathrm{p}_\alpha[\mathsf{x}_2]]), \tag{3.3.1.D}$$

$$\mathsf{fr}_\mathcal{M}(\mathrm{p}_\alpha[\mathrm{u}]) = \epsilon = \mathsf{fr}_\mathcal{M}(\mathrm{u}), \tag{3.3.1.E}$$

$$\mathsf{fr}_\mathcal{M}(\mathrm{p}_{\alpha_1}[\mathrm{p}_{\alpha_2}[\mathsf{x}_1]]) = 1^{\alpha_1 \cdot \alpha_2} = \mathsf{fr}_\mathcal{M}((\mathrm{p}_{\alpha_1 \cdot \alpha_2})[\mathsf{x}_1]), \tag{3.3.1.F}$$

$$\mathsf{fr}_\mathcal{M}(\mathrm{p}_\mathrm{e}[\mathsf{x}_1]) = 1^\mathrm{e} = \mathsf{fr}_\mathcal{M}(\mathsf{x}_1). \tag{3.3.1.G}$$

Since by definition, $\equiv_{\mathfrak{R}_\mathcal{M}}$ is the smallest clone congruence containing $\mathfrak{R}_\mathcal{M}$ and, as we have seen here, for any $(\mathsf{t}, \mathsf{t}') \in \mathfrak{R}_\mathcal{M}$, we have $\mathsf{fr}_\mathcal{M}(\mathsf{t}) = \mathsf{fr}_\mathcal{M}(\mathsf{t}')$, the statement of the lemma follows. □

3.3.2 RIGHT COMB FACTORIZATION. We describe now a way to encode any $\mathcal{M}$-pigmented word as a particular $\mathfrak{G}_\mathcal{M}$-term having some important properties.

The *right comb factorization map* is the map $\mathsf{rc}_\mathcal{M} : \mathbf{P}(\mathcal{M}) \to \mathfrak{T}(\mathfrak{G}_\mathcal{M})$ recursively defined, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, by

$$\mathsf{rc}_\mathcal{M}(\mathfrak{p}) := \begin{cases} \mathrm{u} & \text{if } \mathfrak{p} = \epsilon, \\ \star[\mathrm{p}_\alpha[\mathsf{x}_i], \mathsf{rc}_\mathcal{M}(\mathfrak{p}')] & \text{otherwise, where } \mathfrak{p} = i^\alpha.\mathfrak{p}', \end{cases} \tag{3.3.2.A}$$

where $i^\alpha \in \mathcal{L}_\mathcal{M}$, $\mathsf{x}_i \in \mathbb{X}$, and $\mathfrak{p}' \in \mathbf{P}(\mathcal{M})$. For instance,

$$\mathsf{rc}_\mathcal{M}\left(1^{ab}3^{aa}2^\epsilon 2^b\right) = \star[\mathrm{p}_{ab}[\mathsf{x}_1], \star[\mathrm{p}_{aa}[\mathsf{x}_3], \star[\mathrm{p}_\epsilon[\mathsf{x}_2], \star[\mathrm{p}_b[\mathsf{x}_2], \mathrm{u}]]]]. \tag{3.3.2.B}$$

▶ **Lemma** 3.3.2.A — *For any monoid $\mathcal{M}$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(\mathfrak{p})) = \mathfrak{p}$.*

◀ **Proof** — Let $\cdot$ be the operation of $\mathcal{M}$ and e is its unit. We proceed by induction on the length ℓ of $\mathfrak{p}$. If $\ell = 0$, then $\mathfrak{p} = \epsilon$ and since $\mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(\epsilon)) = \mathsf{fr}_\mathcal{M}(\mathrm{u}) = \epsilon$, the property holds. If $\ell \geqslant 1$, $\mathfrak{p}$ decomposes as $\mathfrak{p} = i^\alpha.\mathfrak{p}'$ where $i^\alpha \in \mathcal{L}_\mathcal{M}$ and $\mathfrak{p}' \in \mathbf{P}(\mathcal{M})$. By definition of $\mathsf{rc}_\mathcal{M}$ and by induction hypothesis,

$$\begin{aligned}\mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(\mathfrak{p})) = \mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(i^\alpha.\mathfrak{p}')) &= \mathsf{fr}_\mathcal{M}(\star[\mathrm{p}_\alpha[\mathsf{x}_i], \mathsf{rc}_\mathcal{M}(\mathfrak{p}')]) \\ &= 1^\mathrm{e}2^\mathrm{e}[1^\alpha[i^\mathrm{e}], \mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(\mathfrak{p}'))] = 1^\mathrm{e}2^\mathrm{e}[1^\alpha[i^\mathrm{e}], \mathfrak{p}'] = i^\alpha.\mathfrak{p}' = \mathfrak{p}.\end{aligned} \tag{3.3.2.C}$$

Therefore, the stated property holds. □

▶ **Lemma** 3.3.2.B — *For any monoid $\mathcal{M}$ and any $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, there exists $\mathfrak{t}' \in \mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$ such that $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{t}'$.*

◀ **Proof** — Let $\cdot$ be the operation of $\mathcal{M}$ and e is its unit. We proceed by induction on the pairs (ℓ, d) ordered lexicographically, where ℓ is the length of $\mathfrak{t}$ and d is the degree of $\mathfrak{t}$.

(I) If $\ell = 0$, then $\mathfrak{t}$ has no variable. By (3.1.1.C) and (3.1.1.E), $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathrm{u}$. Since u belongs to $\mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$, the stated property is satisfied.

(II) If $\ell \geqslant 1$, we have three sub-cases to explore depending on the general form of $\mathfrak{t}$.

- *(a)* If $\mathfrak{t} = \mathsf{x}_i$ where $\mathsf{x}_i \in \mathbb{X}$, by (3.1.1.C), $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \star[\mathsf{x}_i, \mathrm{u}]$. By (3.1.1.G), $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \star[\mathrm{p}_\mathrm{e}(\mathsf{x}_i), \mathrm{u}]$. Since $\star[\mathrm{p}_\mathrm{e}(\mathsf{x}_i), \mathrm{u}]$ belongs to $\mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$, the stated property is satisfied.
- *(b)* If $\mathfrak{t} = \mathrm{p}_\alpha[\mathfrak{s}]$ where $\alpha \in \mathcal{M}$ and $\mathfrak{s} \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, since $\ell(\mathfrak{s}) = \ell(\mathfrak{t})$ and $\mathrm{dg}(\mathfrak{s}) < \mathrm{dg}(\mathfrak{t})$, by induction hypothesis, there exists $\mathfrak{s}' \in \mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$ such that $\mathfrak{s} \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{s}'$. By definition of $\mathsf{rc}_\mathcal{M}$, $\mathfrak{s}'$ can have two different forms.
  - *(i)* If $\mathfrak{s}' = \mathrm{u}$, we have $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathrm{p}_\alpha[\mathrm{u}]$. By (3.1.1.E), $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathrm{u}$. Since u belongs to $\mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$, the stated property is satisfied.
  - *(ii)* Otherwise, $\mathfrak{s}' = \star[\mathrm{p}_{\alpha'}[\mathsf{x}_i], \mathfrak{r}]$ where $\alpha' \in \mathcal{M}$, $\mathsf{x}_i \in \mathbb{X}$, and $\mathfrak{r} \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$. We have $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathrm{p}_\alpha[\star[\mathrm{p}_{\alpha'}[\mathsf{x}_i], \mathfrak{r}]]$. By (3.1.1.D), we have $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \star[\mathrm{p}_\alpha[\mathrm{p}_{\alpha'}[\mathsf{x}_i]], \mathrm{p}_\alpha[\mathfrak{r}]]$ and by (3.1.1.F), we have $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \star[\mathrm{p}_{\alpha \cdot \alpha'}[\mathsf{x}_i], \mathrm{p}_\alpha[\mathfrak{r}]]$. Now, by Lemma 3.3.1.A, $\ell(\mathrm{p}_\alpha[\mathfrak{r}]) < \ell(\mathfrak{t})$. Thus, by induction hypothesis, there exists $\mathfrak{r}' \in \mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$ such that $\mathrm{p}_\alpha[\mathfrak{r}] \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{r}'$. Therefore, $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \star[\mathrm{p}_{\alpha \cdot \alpha'}[\mathsf{x}_i], \mathfrak{r}']$. By definition of $\mathsf{rc}_\mathcal{M}$, $\star[\mathrm{p}_{\alpha \cdot \alpha'}[\mathsf{x}_i], \mathfrak{r}']$ belongs to $\mathbf{P}(\mathcal{M})$ so that the stated property is satisfied.
- *(c)* Otherwise, $\mathfrak{t} = \star[\mathfrak{s}_1, \mathfrak{s}_2]$ where $\mathfrak{s}_1 \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$ and $\mathfrak{s}_2 \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$. Since $\ell(\mathfrak{s}_1) \leqslant \ell(\mathfrak{t})$, $\mathrm{dg}(\mathfrak{s}_1) < \mathrm{dg}(\mathfrak{t})$, $\ell(\mathfrak{s}_2) \leqslant \ell(\mathfrak{t})$, and $\mathrm{dg}(\mathfrak{s}_2) < \mathrm{dg}(\mathfrak{t})$, by induction hypothesis, there exist $\mathfrak{s}_1', \mathfrak{s}_2' \in \mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$ such that $\mathfrak{s}_1 \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{s}_1'$ and $\mathfrak{s}_2 \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{s}_2'$. To simplify the notations, let us treat the constant $\star$ as an infix operator which associates from right to left. This means that for any $\mathfrak{r}_1, \dots, \mathfrak{r}_k \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, $k \geqslant 1$, $\mathfrak{r}_1 \star \mathfrak{r}_2 \star \dots \star \mathfrak{t}_{k-1} \star \mathfrak{t}_k$ specifies the $\mathfrak{G}_\mathcal{M}$-term $\star[\mathfrak{r}_1, \star[\mathfrak{r}_2, \star[\dots \star [\mathfrak{t}_{k-1}, \mathfrak{t}_k] \dots]]]$. By definition of $\mathsf{rc}_\mathcal{M}$, $\mathfrak{s}_1'$ and $\mathfrak{s}_2'$ decompose respectively as $\mathfrak{s}_1' = \mathrm{p}_{\alpha_{1,1}}\left[\mathsf{x}_{i_{1,1}}\right] \star \dots \star \mathrm{p}_{\alpha_{1,k_1}}\left[\mathsf{x}_{i_{1,k_1}}\right] \star \mathrm{u}$ and $\mathfrak{s}_2' = \mathrm{p}_{\alpha_{2,1}}\left[\mathsf{x}_{i_{2,1}}\right] \star \dots \star \mathrm{p}_{\alpha_{2,k_2}}\left[\mathsf{x}_{i_{2,k_2}}\right] \star \mathrm{u}$ for some $\alpha_{1,1}, \dots, \alpha_{1,k_1}, \alpha_{2,1}, \dots, \alpha_{2,k_2} \in \mathcal{M}$, $\mathsf{x}_{i_{1,1}}, \dots, \mathsf{x}_{i_{1,k_1}}, \mathsf{x}_{i_{2,1}}, \dots, \mathsf{x}_{i_{2,k_2}} \in \mathbb{X}$, $k_1 \geqslant 0$, and $k_2 \geqslant 0$. Now, by (3.1.1.B) and (3.1.1.C), we have

$$\begin{aligned}\mathfrak{t} &\equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{s}_1' \star \mathfrak{s}_2' \\ &= \left(\mathrm{p}_{\alpha_{1,1}}\left[\mathsf{x}_{i_{1,1}}\right] \star \dots \star \mathrm{p}_{\alpha_{1,k_1}}\left[\mathsf{x}_{i_{1,k_1}}\right] \star \mathrm{u}\right) \star \left(\mathrm{p}_{\alpha_{2,1}}\left[\mathsf{x}_{i_{2,1}}\right] \star \dots \star \mathrm{p}_{\alpha_{2,k_2}}\left[\mathsf{x}_{i_{2,k_2}}\right] \star \mathrm{u}\right)\end{aligned} \tag{3.3.2.D}$$

$$\equiv_{\mathfrak{R}_\mathcal{M}} \left(\mathrm{p}_{\alpha_{1,1}}\left[\mathsf{x}_{i_{1,1}}\right] \star \ldots \star \mathrm{p}_{\alpha_{1,k_1}}\left[\mathsf{x}_{i_{1,k_1}}\right]\right) \star \left(\mathrm{p}_{\alpha_{2,1}}\left[\mathsf{x}_{i_{2,1}}\right] \star \ldots \star \mathrm{p}_{\alpha_{2,k_2}}\left[\mathsf{x}_{i_{2,k_2}}\right] \star \mathrm{u}\right)$$
$$\equiv_{\mathfrak{R}_\mathcal{M}} \mathrm{p}_{\alpha_{1,1}}\left[\mathsf{x}_{i_{1,1}}\right] \star \ldots \star \mathrm{p}_{\alpha_{1,k_1}}\left[\mathsf{x}_{i_{1,k_1}}\right] \star \mathrm{p}_{\alpha_{2,1}}\left[\mathsf{x}_{i_{2,1}}\right] \star \ldots \star \mathrm{p}_{\alpha_{2,k_2}}\left[\mathsf{x}_{i_{2,k_2}}\right] \star \mathrm{u}.$$

By definition of $\mathsf{rc}_\mathcal{M}$, the last term of (3.3.2.D) belongs to $\mathsf{rc}_\mathcal{M}(\mathbf{P}(\mathcal{M}))$ so that the stated property is satisfied.

□

3.3.3 Clone presentation. We use now the tools developed in the previous sections to prove that $\mathbf{P}(\mathcal{M})$ is a clone realization of the variety of $\mathcal{M}$-pigmented monoids.

▶ **Lemma** 3.3.3.A — *For any monoid $\mathcal{M}$ and any $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) = \mathsf{fr}_\mathcal{M}(\mathfrak{t}')$ implies $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{t}'$.*

◀ **Proof** — Assume that $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) = \mathsf{fr}_\mathcal{M}(\mathfrak{t}')$. By Lemma 3.3.2.B, there exist $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathsf{rc}_\mathcal{M}(\mathfrak{p})$ and $\mathfrak{t}' \equiv_{\mathfrak{R}_\mathcal{M}} \mathsf{rc}_\mathcal{M}(\mathfrak{p}')$. By Lemma 3.3.1.B, $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) = \mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(\mathfrak{p}))$ and $\mathsf{fr}_\mathcal{M}(\mathfrak{t}') = \mathsf{fr}_\mathcal{M}(\mathsf{rc}_\mathcal{M}(\mathfrak{p}'))$. By Lemma 3.3.2.A, $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) = \mathfrak{p}$ and $\mathsf{fr}_\mathcal{M}(\mathfrak{t}') = \mathfrak{p}'$. Since $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) = \mathsf{fr}_\mathcal{M}(\mathfrak{t}')$, we have $\mathfrak{p} = \mathfrak{p}'$. This shows that $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathsf{rc}_\mathcal{M}(\mathfrak{p}) = \mathsf{rc}_\mathcal{M}(\mathfrak{p}') \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{t}'$, so that $\mathfrak{t} \equiv_{\mathfrak{R}_\mathcal{M}} \mathfrak{t}'$. □

Here is the main result of the section.

▶ **Theorem** 3.3.3.B — *For any monoid $\mathcal{M}$, the clone $\mathbf{P}(\mathcal{M})$ is a clone realization of the variety of $\mathcal{M}$-pigmented monoids.*

◀ **Proof** — By Lemmas 3.3.3.A and 3.3.1.B, for any $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, $\mathfrak{t} \equiv_{\mathfrak{R}} \mathfrak{t}'$ if and only if $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) = \mathsf{fr}_\mathcal{M}(\mathfrak{t}')$. Moreover, by Proposition 3.2.3.A, $G_\mathcal{M} = \mathsf{int}_\mathcal{M}(\mathfrak{G}_\mathcal{M})$ is a generating set of $\mathbf{P}(\mathcal{M})$. Therefore, by Proposition 2.3.2.A, these two properties imply that the variety $(\mathfrak{G}_\mathcal{M}, \mathfrak{R}_\mathcal{M})$ of $\mathcal{M}$-pigmented monoids is a presentation of $\mathbf{P}(\mathcal{M})$. □

By Theorem 3.3.3.B, for any monoid $\mathcal{M}$, all algebras over $\mathbf{P}(\mathcal{M})$ are $\mathcal{M}$-pigmented monoids. Since all algebras over the operad $\mathbf{T}(\mathcal{M})$ can be seen as specialized versions of $\mathcal{M}$-pigmented monoids [Gir15], we can see the construction $\mathbf{P}$ as a generalization of the construction $\mathbf{T}$ at the level of clones.

Let us end this section by giving a tool to establish presentations of quotients of $\mathbf{P}(\mathcal{M})$.

▶ **Proposition** 3.3.3.C — *Let $\mathcal{M}$ be a monoid and $\mathfrak{R}'_\mathcal{M}$ be an equivalence relation on $\mathfrak{T}(\mathfrak{G}_\mathcal{M})$ containing $\mathfrak{R}_\mathcal{M}$. If $\equiv'$ is the clone congruence of $\mathbf{P}(\mathcal{M})$ generated by $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) \equiv' \mathsf{fr}_\mathcal{M}(\mathfrak{t}')$ whenever $\mathfrak{t} \, \mathfrak{R}'_\mathcal{M} \, \mathfrak{t}'$, then $\left(\mathfrak{G}_\mathcal{M}, \mathfrak{R}'_\mathcal{M}\right)$ is a presentation of the clone $\mathbf{P}(\mathcal{M})/_{\equiv'}$.*

◀ **Proof** — First of all since by Theorem 3.3.3.B, $\mathbf{P}(\mathcal{M})$ admits $(\mathfrak{G}_\mathcal{M}, \mathfrak{R}_\mathcal{M})$ as presentation and since $\mathfrak{R}_\mathcal{M} \subseteq \mathfrak{R}'_\mathcal{M}$, the clone admitting $\left(\mathfrak{G}_\mathcal{M}, \mathfrak{R}'_\mathcal{M}\right)$ as presentation is a quotient of $\mathbf{P}(\mathcal{M})$.

Let $\theta : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})/_{\equiv'}$ be the canonical projection map. Since $\equiv_{\mathfrak{R}'_\mathcal{M}}$ is the clone congruence generated by $\mathfrak{R}'_\mathcal{M}$, by definition of $\equiv'$, for any $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}(\mathfrak{G}_\mathcal{M})$, $\mathfrak{t} \equiv_{\mathfrak{R}'_\mathcal{M}} \mathfrak{t}'$ is equivalent to $\mathsf{fr}_\mathcal{M}(\mathfrak{t}) \equiv' \mathsf{fr}_\mathcal{M}(\mathfrak{t}')$. This in turn is equivalent to $\theta(\mathsf{fr}_\mathcal{M}(\mathfrak{t})) = \theta(\mathsf{fr}_\mathcal{M}(\mathfrak{t}'))$, which is finally equivalent to $\mathsf{ev}_{\mathbf{P}(\mathcal{M})/_{\equiv'}}(\mathfrak{t}) = \mathsf{ev}_{\mathbf{P}(\mathcal{M})/_{\equiv'}}(\mathfrak{t}')$. By Proposition 2.3.2.A, the clone $\mathbf{P}(\mathcal{M})/_{\equiv'}$ admits the stated presentation. □

4 Construction of quotients

The clones $\mathbf{P}(\mathcal{M})$ are very large and contain a lot of subclones and quotients worth investigating. We present here some tools to construct quotients of $\mathbf{P}(\mathcal{M})$ through $\mathbb{P}$-symbols which are here particular maps from $\mathbf{P}(\mathcal{M})$ to itself. Results about the description of the elements of such quotients are provided. As a direct application, we construct in this section the quotients clones $\mathsf{WInc}(\mathcal{M})$, $\mathsf{Arra}_k(\mathcal{M})$, and Inc_k of $\mathbf{P}(\mathcal{M})$.

4.1 ℙ-symbols and realizations of quotient clones

Let us present here ℙ-symbols and how to use these to build quotients of $\mathbf{P}(\mathcal{M})$.

4.1.1 ℙ-symbols and clone congruences. Let $\mathcal{M}$ be a monoid and $\equiv$ be an equivalence relation on $\mathbf{P}(\mathcal{M})$ which is not necessarily at this stage a clone congruence. A *ℙ-symbol* for $\equiv$ is a map $\mathbb{P}_\equiv : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ such that

(i) for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv \mathbb{P}_\equiv(\mathfrak{p})$;

(ii) for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv \mathfrak{p}'$ implies $\mathbb{P}_\equiv(\mathfrak{p}) = \mathbb{P}_\equiv(\mathfrak{p}')$.

By extension, given an $\mathcal{M}$-pigmented word $\mathfrak{p}$, $\mathbb{P}_\equiv(\mathfrak{p})$ is the *ℙ-symbol* of $\mathfrak{p}$. Besides, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, by (i), $\mathfrak{p} \equiv \mathbb{P}_\equiv(\mathfrak{p})$, and by (ii), this implies that $\mathbb{P}_\equiv(\mathfrak{p}) = \mathbb{P}_\equiv(\mathbb{P}_\equiv(\mathfrak{p}))$. For this reason, $\mathbb{P}_\equiv$ is idempotent. Moreover, observe that for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, if $\mathbb{P}_\equiv(\mathfrak{p}) = \mathbb{P}_\equiv(\mathfrak{p}')$, then by (i), $\mathfrak{p} \equiv \mathbb{P}_\equiv(\mathfrak{p}) = \mathbb{P}_\equiv(\mathfrak{p}') \equiv \mathfrak{p}'$, which implies $\mathfrak{p} \equiv \mathfrak{p}'$. Therefore, the converse of (ii) holds.

As it is usually the case in the description of ℙ-symbols, it is always possible to provide an iterative description of such maps through algorithms by setting $\mathbb{P}(\boldsymbol{\epsilon}) := \boldsymbol{\epsilon}$ and by computing $\mathbb{P}(\mathfrak{p}.i^\alpha)$ as the insertion of the $\mathcal{M}$-pigmented letter i^α into the $\mathcal{M}$-pigmented word $\mathbb{P}(\mathfrak{p})$. As a side remark, most ℙ-symbols appearing in the literature map words to other combinatorial objects (like Young tableaux [Lot02, Chap. 5], binary trees [HNT05], or pairs of twin binary trees [Gir12]). Here, our notion of ℙ-symbol is very specific to our purposes.

In the other direction, given a map $\mathbb{P} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$, the *fiber equivalence relation* of $\mathbb{P}$ is the equivalence relation $\equiv_\mathbb{P}$ on $\mathbf{P}(\mathcal{M})$ such that for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv_\mathbb{P} \mathfrak{p}'$ whenever $\mathbb{P}(\mathfrak{p}) = \mathbb{P}(\mathfrak{p}')$.

▶ **Proposition** 4.1.1.A — *Let $\mathcal{M}$ be a monoid and $\mathbb{P} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be a map. If $\mathbb{P}$ is idempotent, then the map $\mathbb{P}$ is a ℙ-symbol for the fiber equivalence relation $\equiv_\mathbb{P}$ of $\mathbb{P}$.*

◀ **Proof** — The map $\mathbb{P}$ satisfies Condition (ii) immediately by construction of $\equiv_\mathbb{P}$. Besides, since $\mathbb{P}$ is idempotent, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, we have $\mathbb{P}(\mathfrak{p}) = \mathbb{P}(\mathbb{P}(\mathfrak{p}))$ so that $\mathfrak{p} \equiv_\mathfrak{p} \mathbb{P}(\mathfrak{p})$. Therefore, Condition (i) holds. □

▶ **Proposition** 4.1.1.B — *Let $\mathcal{M}$ be a monoid, $\equiv$ be an equivalence relation on $\mathbf{P}(\mathcal{M})$, and $\mathbb{P}_\equiv$ be a ℙ-symbol for $\equiv$. The equivalence relation $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$ if and only if for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$,*

$$\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n] \equiv \mathbb{P}_\equiv(\mathfrak{p})[\mathbb{P}_\equiv(\mathfrak{p}'_1), \dots, \mathbb{P}_\equiv(\mathfrak{p}'_n)]. \tag{4.1.1.A}$$

◀ **Proof** — If $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$, (4.1.1.A) holds by the fact that since $\mathbb{P}_\equiv$ is a ℙ-symbol for $\equiv$, $\mathbb{P}_\equiv$ satisfies Condition (i) of ℙ-symbols.

Conversely, let us assume that (4.1.1.A) holds. Let $\mathfrak{p} \in \mathbf{P}(\mathcal{M})(n)$, $\mathfrak{q} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n \in \mathbf{P}(\mathcal{M})(m)$, $\mathfrak{q}'_1, \dots, \mathfrak{q}'_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, such that $\mathfrak{p} \equiv \mathfrak{q}$ and $\mathfrak{p}'_i \equiv \mathfrak{q}'_i$ for all $i \in [n]$. Therefore, by Condition (ii) of ℙ-symbols, $\mathbb{P}_\equiv(\mathfrak{p}) = \mathbb{P}_\equiv(\mathfrak{q})$ and $\mathbb{P}_\equiv(\mathfrak{p}'_i) = \mathbb{P}_\equiv(\mathfrak{q}'_i)$ for all $i \in [n]$, so that $\mathbb{P}_\equiv(\mathfrak{p})[\mathbb{P}_\equiv(\mathfrak{p}'_1), \dots, \mathbb{P}_\equiv(\mathfrak{p}'_n)] = \mathbb{P}_\equiv(\mathfrak{q})[\mathbb{P}_\equiv(\mathfrak{q}'_1), \dots, \mathbb{P}_\equiv(\mathfrak{q}'_n)]$. By (4.1.1.A), this implies that $\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n] \equiv \mathfrak{q}[\mathfrak{q}'_1, \dots, \mathfrak{q}'_n]$ and shows as expected that $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$. □

4.1.2 Realizations of quotient clones. The next result uses $\mathbb{P}$-symbols for clone congruences $\equiv$ of $\mathbf{P}(\mathcal{M})$ in order to build realizations of the quotients $\mathbf{P}(\mathcal{M})/_{\equiv}$.

▶ **Proposition** 4.1.2.A — *Let $\mathcal{M}$ be a monoid, $\equiv$ be a clone congruence of $\mathbf{P}(\mathcal{M})$, and $\mathbb{P}_{\equiv}$ be a $\mathbb{P}$-symbol for $\equiv$. The clone $\mathbf{P}(\mathcal{M})/_{\equiv}$ is isomorphic to the clone on $\mathbb{P}_{\equiv}(\mathbf{P}(\mathcal{M}))$ with superposition maps defined, for any $\mathfrak{p} \in \mathbb{P}_{\equiv}(\mathbf{P}(\mathcal{M}))(n)$, $n \geqslant 0$, and $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n \in \mathbb{P}_{\equiv}(\mathbf{P}(\mathcal{M}))(m)$, $m \geqslant 0$, by*

$$\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n] := \mathbb{P}_{\equiv}(\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n]), \tag{4.1.2.A}$$

where the superposition map of the right-hand side of (4.1.2.A) is the one of $\mathbf{P}(\mathcal{M})$.

◀ **Proof** — This is a direct consequence of the fact that by (i) and (ii), $\mathbb{P}_{\equiv}$ sends each $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ to the representative of its $\equiv$-equivalence class and thus, that $\mathbb{P}_{\equiv}(\mathbf{P}(\mathcal{M}))$ is a system of representatives of the quotient graded set $\mathbf{P}(\mathcal{M})/_{\equiv}$. □

4.1.3 Composition of ℙ-symbols. Let us focus now on the compositions of $\mathbb{P}$-symbols and on the properties of the resulting maps.

▶ **Proposition** 4.1.3.A — *Let $\mathcal{M}$ be a monoid, $\equiv_1$ and $\equiv_2$ be two clone congruences of $\mathbf{P}(\mathcal{M})$, and $\mathbb{P}_{\equiv_1}$ and $\mathbb{P}_{\equiv_2}$ be two $\mathbb{P}$-symbols, respectively for $\equiv_1$ and $\equiv_2$. If $\mathbb{P}_{\equiv_1}$ and $\mathbb{P}_{\equiv_2}$ commute for the composition of maps, then by setting $\mathbb{P}_{12}$ as the map $\mathbb{P}_{\equiv_1} \circ \mathbb{P}_{\equiv_2} = \mathbb{P}_{\equiv_2} \circ \mathbb{P}_{\equiv_1}$ and $\equiv$ as the fiber equivalence relation of $\mathbb{P}_{12}$,*

(i) the map $\mathbb{P}_{12}$ is a $\mathbb{P}$-symbol for $\equiv$;

(ii) the equivalence relation $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$;

(iii) the clone $\mathbf{P}(\mathcal{M})/_{\equiv}$ is a quotient of both $\mathbf{P}(\mathcal{M})/_{\equiv_1}$ and $\mathbf{P}(\mathcal{M})/_{\equiv_2}$.

◀ **Proof** — In this proof, in order to lighten the notation, for any word $w \in [2]^*$, we denote by $\mathbb{P}_w$ the map $\mathbb{P}_{\equiv_{w(1)}} \circ \dots \circ \mathbb{P}_{\equiv_{w(\ell(w))}}$.

Let us first show (i). Since $\mathbb{P}_1$ and $\mathbb{P}_2$ are $\mathbb{P}$-symbols, they are idempotent. Moreover, by hypothesis, they commute for the composition of maps. Thus, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, we have $\mathbb{P}_{12}(\mathbb{P}_{12}(\mathfrak{p})) = \mathbb{P}_{1212}(\mathfrak{p}) = \mathbb{P}_{1122}(\mathfrak{p}) = \mathbb{P}_{12}(\mathfrak{p})$. Therefore, $\mathbb{P}_{12}$ is idempotent, implying by Proposition 4.1.1.A that $\mathbb{P}_{12}$ is a $\mathbb{P}$-symbol for $\equiv$.

Let us prove (ii). Since $\mathbb{P}_1$ and $\mathbb{P}_2$ are respectively $\mathbb{P}$-symbols for the congruences $\equiv_1$ and $\equiv_2$ of $\mathbf{P}(\mathcal{M})$, and $\mathbb{P}_1$ and $\mathbb{P}_2$ commute for the composition of maps, by Proposition 4.1.1.B, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, we have

$$\begin{aligned}
\mathbb{P}_{12}(\mathfrak{p}[\mathfrak{p}'_1, \dots \mathfrak{p}'_n]) &= \mathbb{P}_{12}(\mathbb{P}_2(\mathfrak{p})[\mathbb{P}_2(\mathfrak{p}'_1), \dots, \mathbb{P}_2(\mathfrak{p}'_n)]) \\
&= \mathbb{P}_{21}(\mathbb{P}_2(\mathfrak{p})[\mathbb{P}_2(\mathfrak{p}'_1), \dots, \mathbb{P}_2(\mathfrak{p}'_n)]) \\
&= \mathbb{P}_{21}(\mathbb{P}_{12}(\mathfrak{p})[\mathbb{P}_{12}(\mathfrak{p}'_1), \dots, \mathbb{P}_{12}(\mathfrak{p}'_n)]) \\
&= \mathbb{P}_{12}(\mathbb{P}_{12}(\mathfrak{p})[\mathbb{P}_{12}(\mathfrak{p}'_1), \dots, \mathbb{P}_{12}(\mathfrak{p}'_n)]).
\end{aligned} \tag{4.1.3.A}$$

By Proposition 4.1.1.B, $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$.

To show (iii), let $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, such that $\mathfrak{p} \equiv_1 \mathfrak{p}'$. Since $\mathbb{P}_1$ is a $\mathbb{P}$-symbol for $\equiv_1$, we have $\mathbb{P}_1(\mathfrak{p}) = \mathbb{P}_1(\mathfrak{p}')$, so that $\mathbb{P}_{21}(\mathfrak{p}) = \mathbb{P}_{21}(\mathfrak{p}')$. Since $\mathbb{P}_1$ and $\mathbb{P}_2$ commute, this shows that $\mathbb{P}_{12}(\mathfrak{p}) = \mathbb{P}_{12}(\mathfrak{p}')$. Hence, we have $\mathfrak{p} \equiv \mathfrak{p}'$. The same argument shows that $\mathfrak{p} \equiv_2 \mathfrak{p}'$ implies $\mathfrak{p} \equiv \mathfrak{p}'$. Therefore, as equivalence relations, $\equiv$ is coarser than both $\equiv_1$ and $\equiv_2$. By (ii), $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$ so that $\mathbf{P}(\mathcal{M})/_{\equiv}$ is a well-defined quotient of $\mathbf{P}(\mathcal{M})$. The statement follows. □

4.2 Congruences of the clone of pigmented words

Two maps $\mathsf{sort}_{\preccurlyeq}$ and first_k from $\mathbf{P}(\mathcal{M})$ to $\mathbf{P}(\mathcal{M})$ are introduced. These maps and some of their compositions lead through their fiber equivalence relations to clone congruences of $\mathbf{P}(\mathcal{M})$.

In this section, $\mathcal{M}$ is any monoid but in order to give concrete examples here, we shall consider $\mathcal{M}$ as the free monoid $(A^*, ., \epsilon)$ where A is the alphabet $\{a, b, c\}$.

4.2.1 Reversions of congruences. We start by introducing an involutive transformation on clone congruences of $\mathbf{P}(\mathcal{M})$. For any clone congruence $\equiv$ of $\mathbf{P}(\mathcal{M})$, the *reversion* of $\equiv$ is the equivalence relation $\equiv^{\mathrm{r}}$ on $\mathbf{P}(\mathcal{M})$ satisfying, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv^{\mathrm{r}} \mathfrak{p}'$ if $\mathrm{r}(\mathfrak{p}) \equiv \mathrm{r}(\mathfrak{p}')$.

▶ **Proposition** 4.2.1.A — *Let $\mathcal{M}$ be a monoid. If $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$, then*

(i) the equivalence relation $\equiv^{\mathrm{r}}$ is a clone congruence of $\mathbf{P}(\mathcal{M})$;

(ii) the map $\mathrm{r} : \mathbf{P}(\mathcal{M})/_{\equiv} \to \mathbf{P}(\mathcal{M})/_{\equiv^{\mathrm{r}}}$ is a clone isomorphism.

◀ **Proof** — Let $\mathfrak{p}, \mathfrak{q} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n, \mathfrak{q}'_1, \dots, \mathfrak{q}'_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, such that $\mathfrak{p} \equiv^{\mathrm{r}} \mathfrak{q}$ and $\mathfrak{p}'_i \equiv^{\mathrm{r}} \mathfrak{q}'_i$ for all $i \in [n]$. By definition of $\equiv^{\mathrm{r}}$ and since r is an involution, we have $\mathrm{r}(\mathfrak{p}) \equiv \mathrm{r}(\mathfrak{q})$ and $\mathrm{r}(\mathfrak{p}'_i) \equiv \mathrm{r}(\mathfrak{q}'_i)$ for all $i \in [n]$. Now, since $\equiv$ is a clone of congruence of $\mathbf{P}(\mathcal{M})$,

$$\mathrm{r}(\mathfrak{p})[\mathrm{r}(\mathfrak{p}'_1), \dots, \mathrm{r}(\mathfrak{p}'_n)] \equiv \mathrm{r}(\mathfrak{q})[\mathrm{r}(\mathfrak{q}'_1), \dots, \mathrm{r}(\mathfrak{q}'_n)]. \tag{4.2.1.A}$$

This implies, since by Proposition 3.2.3.B, r is a clone isomorphism of $\mathbf{P}(\mathcal{M})$, that

$$\mathrm{r}(\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n]) \equiv \mathrm{r}(\mathfrak{q}[\mathfrak{q}'_1, \dots, \mathfrak{q}'_n]). \tag{4.2.1.B}$$

Therefore, by definition of $\equiv^{\mathrm{r}}$, this shows that $\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n]$ is $\equiv^{\mathrm{r}}$-equivalent to $\mathfrak{q}[\mathfrak{q}'_1, \dots, \mathfrak{q}'_n]$, establishing (i).

To prove (ii), observe first that since r is an involution of $\mathbf{P}(\mathcal{M})$, by definition of $\equiv^{\mathrm{r}}$, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$,

$$\mathrm{r}([\mathfrak{p}]_{\equiv}) = \{\mathrm{r}(\mathfrak{p}') : \mathfrak{p} \equiv \mathfrak{p}'\} = \{\mathrm{r}(\mathfrak{p}') : \mathrm{r}(\mathfrak{p}) \equiv^{\mathrm{r}} \mathrm{r}(\mathfrak{p}')\} = \{\mathfrak{p}' : \mathrm{r}(\mathfrak{p}) \equiv^{\mathrm{r}} \mathfrak{p}'\} = [\mathrm{r}(\mathfrak{p})]_{\equiv^{\mathrm{r}}}. \tag{4.2.1.C}$$

Therefore, the map r from $\mathbf{P}(\mathcal{M})/_{\equiv}$ to $\mathbf{P}(\mathcal{M})/_{\equiv^{\mathrm{r}}}$ is well-defined and is bijective. Now, by using consecutively the fact that $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$, Relation (4.2.1.C), the fact that by Proposition 3.2.3.B, r is an endomorphism of $\mathbf{P}(\mathcal{M})$, and the fact that by (i), $\equiv^{\mathrm{r}}$ is a clone congruence of $\mathbf{P}(\mathcal{M})$, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, we have

$$\begin{aligned} \mathrm{r}([\mathfrak{p}]_{\equiv}[[\mathfrak{p}'_1]_{\equiv}, \dots, [\mathfrak{p}'_n]_{\equiv}]) &= \mathrm{r}([\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n]]_{\equiv}) \\ &= [\mathrm{r}(\mathfrak{p}[\mathfrak{p}'_1, \dots, \mathfrak{p}'_n])]_{\equiv^{\mathrm{r}}} \\ &= [\mathrm{r}(\mathfrak{p})[\mathrm{r}(\mathfrak{p}'_1), \dots, \mathrm{r}(\mathfrak{p}'_n)]]_{\equiv^{\mathrm{r}}} \\ &= [\mathrm{r}(\mathfrak{p})]_{\equiv^{\mathrm{r}}}[[\mathrm{r}(\mathfrak{p}'_1)]_{\equiv^{\mathrm{r}}}, \dots, [\mathrm{r}(\mathfrak{p}'_n)]_{\equiv^{\mathrm{r}}}]. \end{aligned} \tag{4.2.1.D}$$

Observe also that, by denoting by e the unit of $\mathcal{M}$, for any $i^{\mathrm{e}} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 1$, $i \in [n]$, $\mathrm{r}([i^{\mathrm{e}}]_{\equiv}) = [i^{\mathrm{e}}]_{\equiv^{\mathrm{r}}}$. Therefore, r is a clone isomorphism from $\mathbf{P}(\mathcal{M})/_{\equiv}$ to $\mathbf{P}(\mathcal{M})/_{\equiv^{\mathrm{r}}}$. □

For any clone $\mathcal{C} := \mathbf{P}(\mathcal{M})/_{\equiv}$ where $\equiv$ is a clone congruence of $\mathbf{P}(\mathcal{M})$, we denote by $\mathcal{C}^{\mathrm{r}}$ the clone $\mathbf{P}(\mathcal{M})/_{\equiv^{\mathrm{r}}}$. This clone is, by Proposition 4.2.1.A, well-defined and isomorphic to $\mathcal{C}$.

4.2.2 Sorting congruence. For any total order relation $\preccurlyeq$ on $\mathcal{M}$, let $\mathsf{sort}_{\preccurlyeq} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map sending any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ to the $\mathcal{M}$-pigmented word obtained by rearranging the values of $\mathfrak{p}$ in weakly increasing way w.r.t. the total order relation $\preccurlyeq$ on the set of the $\mathcal{M}$-pigmented letters satisfying $i_1^{\alpha_1} \preccurlyeq i_2^{\alpha_2}$ if $i_1 < i_2$, or $i_1 = i_2$ and $\alpha_1 \preccurlyeq \alpha_2$. For instance, in $\mathbf{P}(A^*)$, where $\preccurlyeq$ is the lexicographic order on A^* satisfying $a \preccurlyeq b \preccurlyeq c$, we have

$$\mathsf{sort}_{\preccurlyeq}\left(3^\epsilon 1^b 3^a 1^a 4^{ab} 2^b 3^\epsilon 1^\epsilon\right) = 1^\epsilon 1^a 1^b 2^b 3^\epsilon 3^\epsilon 3^a 4^{ab}. \tag{4.2.2.A}$$

Let $\equiv_{\mathsf{sort}_{\preccurlyeq}}$ be the fiber equivalence relation of $\mathsf{sort}_{\preccurlyeq}$. By Proposition 4.1.1.A, since $\mathsf{sort}_{\preccurlyeq}$ is idempotent, $\mathsf{sort}_{\preccurlyeq}$ is a $\mathbb{P}$-symbol for $\equiv_{\mathsf{sort}_{\preccurlyeq}}$. Observe moreover that for any total order relations $\preccurlyeq$ and $\preccurlyeq'$ on $\mathcal{M}$ and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, we have $\mathfrak{p} \equiv_{\mathsf{sort}_{\preccurlyeq}} \mathfrak{p}'$ if and only if $\mathfrak{p} \equiv_{\mathsf{sort}_{\preccurlyeq'}} \mathfrak{p}'$. For this reason, the equivalence relation $\equiv_{\mathsf{sort}_{\preccurlyeq}}$ does not depend on the total order relation $\preccurlyeq$. Therefore, we denote simply by $\equiv_{\mathsf{sort}}$ this equivalence relation.

▶ **Proposition** 4.2.2.A — *For any monoid $\mathcal{M}$, the equivalence relation $\equiv_{\mathsf{sort}}$ is a clone congruence of $\mathbf{P}(\mathcal{M})$.*

◀ **Proof** — Let $\preccurlyeq$ be any total order relation on $\mathcal{M}$ and $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$. For any $i^\alpha \in \mathcal{L}_{\mathcal{M}}$, $\mathfrak{p}$ and $\mathsf{sort}_{\preccurlyeq}(\mathfrak{p})$ admit the same number of occurrences of i^α. For this reason and by the definition of the superposition maps of $\mathbf{P}(\mathcal{M})$, the $\mathbb{P}$-symbol $\mathsf{sort}_{\preccurlyeq}$ for $\equiv_{\mathsf{sort}}$ satisfies the prerequisites of Proposition 4.1.1.B. This implies the statement of the proposition. □

4.2.3 First occurrences congruence. For any $k \geqslant 0$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, a position $j \in [\ell(\mathfrak{p})]$ is a *left k-witness* of $\mathfrak{p}$ if in $\mathfrak{p}(1, j-1)$, there are at most $k-1$ $\mathcal{M}$-pigmented letters having as value the one of $\mathfrak{p}(j)$. Similarly, a position $j \in [\ell(\mathfrak{p})]$ is a *right k-witness* of $\mathfrak{p}$ if in $\mathfrak{p}(j+1, \ell(\mathfrak{p}))$, there are at most $k-1$ $\mathcal{M}$-pigmented letters having as value the one of $\mathfrak{p}(j)$.

We shall highlight these properties by putting a segment with a circle on the left (resp. right) under each $\mathcal{M}$-pigmented letter such that its position is a left (resp. right) k-witness. In the opposite case, we shall put a cross on the left (resp. right) edge of the segment to highlight the fact that this position is not a left (resp. right) k-witness when it is the case. For instance, by setting $\mathfrak{p} := 2^{aa} 2^b 3^a 1^a 3^{ba} 2^b 3^\epsilon$, the left and right 1-witnesses of $\mathfrak{p}$ are highlighted as

$$\underset{\bullet\!-\!\times}{2^{aa}}\ \underset{\times\!-\!\times}{2^{b}}\ \underset{\bullet\!-\!\times}{3^{a}}\ \underset{\bullet\!-\!\bullet}{1^{a}}\ \underset{\times\!-\!\times}{3^{ba}}\ \underset{\times\!-\!\bullet}{2^{b}}\ \underset{\times\!-\!\bullet}{3^{\epsilon}} \tag{4.2.3.A}$$

and the left and right 2-witnesses of $\mathfrak{p}$ are highlighted as

$$\underset{\bullet\!-\!\times}{2^{aa}}\ \underset{\bullet\!-\!\bullet}{2^{b}}\ \underset{\bullet\!-\!\times}{3^{a}}\ \underset{\bullet\!-\!\bullet}{1^{a}}\ \underset{\bullet\!-\!\bullet}{3^{ba}}\ \underset{\times\!-\!\bullet}{2^{b}}\ \underset{\times\!-\!\bullet}{3^{\epsilon}}. \tag{4.2.3.B}$$

Moreover, a left (resp. right) edge of a segment having neither a circle nor a cross specifies the fact that the status of this position is unknown. For instance, for a fixed $k \geqslant 0$, the notation

$$\mathfrak{p}_1 . \underset{\bullet\!-}{1^{ba}} . \mathfrak{p}_2 . \underset{-\!\times}{1^{ab}}\ \underset{-\!\bullet}{1^{b}} . \mathfrak{p}_3 \tag{4.2.3.C}$$

where $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are some A^*-pigmented words specifies an A^*-pigmented word such that the position of the shown $\mathcal{M}$-pigmented letter 1^{ba} is a left k-witness and may or may not be a right k-witness, that the position of the shown $\mathcal{M}$-pigmented letter 1^{ab} may or may not be a left k-witness and is not a right k-witness, and that the position of the shown $\mathcal{M}$-pigmented letter 1^b may or may not be a left k-witness and is a right k-witness.

Now, let $\mathsf{first}_k : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map sending any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ to the $\mathcal{M}$-pigmented word defined as the subword of $\mathfrak{p}$ consisting in the letters whose positions are left k-witnesses. For instance,

$$\mathsf{first}_1\left(\underset{\bullet\!-}{1^{\epsilon}}\ \underset{\bullet\!-}{3^{ab}}\ \underset{\times\!-}{1^{b}}\ \underset{\times\!-}{3^{b}}\ \underset{\times\!-}{1^{aa}}\ \underset{\times\!-}{3^{\epsilon}}\ \underset{\bullet\!-}{2^{aa}}\ \underset{\times\!-}{3^{bba}}\right) = \underset{\bullet\!-}{1^{\epsilon}}\ \underset{\bullet\!-}{3^{ab}}\ \underset{\bullet\!-}{2^{aa}}, \tag{4.2.3.D}$$

$$\mathsf{first}_2\left(\underset{\bullet\!-}{1^{\epsilon}}\ \underset{\bullet\!-}{3^{ab}}\ \underset{\bullet\!-}{1^{b}}\ \underset{\bullet\!-}{3^{b}}\ \underset{\times\!-}{1^{aa}}\ \underset{\times\!-}{3^{\epsilon}}\ \underset{\bullet\!-}{2^{aa}}\ \underset{\times\!-}{3^{bba}}\right) = \underset{\bullet\!-}{1^{\epsilon}}\ \underset{\bullet\!-}{3^{ab}}\ \underset{\bullet\!-}{1^{b}}\ \underset{\bullet\!-}{3^{b}}\ \underset{\bullet\!-}{2^{aa}}. \tag{4.2.3.E}$$

Let $\equiv_{\mathsf{first}_k}$ be the fiber equivalence relation of first_k. By Proposition 4.1.1.A, since first_k is idempotent, first_k is a $\mathbb{P}$-symbol for $\equiv_{\mathsf{first}_k}$.

Observe that for any $0 \leqslant k \leqslant k'$ and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv_{\mathsf{first}_{k'}} \mathfrak{p}'$ implies $\mathfrak{p} \equiv_{\mathsf{first}_k} \mathfrak{p}'$. Hence, the equivalence relation $\equiv_{\mathsf{first}_{k'}}$ is a refinement of $\equiv_{\mathsf{first}_k}$.

▶ **Proposition** 4.2.3.A — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, the equivalence relation $\equiv_{\mathsf{first}_k}$ is a clone congruence of $\mathbf{P}(\mathcal{M})$.*

◀ **Proof** — From the definitions of first_k and of the superposition maps of $\mathbf{P}(\mathcal{M})$, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})(n)$, $n \geqslant 0$, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathbf{P}(\mathcal{M})(m)$, $m \geqslant 0$, we have

$$\mathsf{first}_k(\mathfrak{p}[\mathfrak{p}_1, \dots, \mathfrak{p}_n]) = \mathsf{first}_k(\mathsf{first}_k(\mathfrak{p})[\mathfrak{p}_1, \dots, \mathfrak{p}_n]) \tag{4.2.3.F}$$

and, for any $j \in [n]$,

$$\mathsf{first}_k(\mathfrak{p}[\mathfrak{p}_1, \dots, \mathfrak{p}_n]) = \mathsf{first}_k(\mathfrak{p}[\mathfrak{p}_1, \dots, \mathfrak{p}_{j-1}, \mathsf{first}_k(\mathfrak{p}_j), \mathfrak{p}_{j+1}, \dots, \mathfrak{p}_n]). \tag{4.2.3.G}$$

These two properties imply that the $\mathbb{P}$-symbol first_k for $\equiv_{\mathsf{first}_k}$ satisfies the prerequisites of Proposition 4.1.1.B. This establishes the statement of the proposition. □

For any $k \geqslant 0$, let us denote by $\mathsf{first}^{\mathrm{r}}_k : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ the map defined for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ by $\mathsf{first}^{\mathrm{r}}_k(\mathfrak{p}) := \mathrm{r}(\mathsf{first}_k(\mathrm{r}(\mathfrak{p})))$. In this way, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathsf{first}^{\mathrm{r}}_k(\mathfrak{p})$ is the subword of $\mathfrak{p}$ consisting in the letters whose positions are right k-witnesses. It is straightforward to prove that $\mathsf{first}^{\mathrm{r}}_k$ is idempotent and that the fiber equivalence relation of $\mathsf{first}^{\mathrm{r}}_k$ is the equivalence relation $\equiv^{\mathrm{r}}_{\mathsf{first}_k}$. By Propositions 4.2.3.A and 4.2.1.A, $\equiv^{\mathrm{r}}_{\mathsf{first}_k}$ is a clone congruence of $\mathbf{P}(\mathcal{M})$.

4.2.4 Compositions. We consider here some compositions of the maps $\mathsf{sort}_{\preccurlyeq}$, first_k, and $\mathsf{first}^{\mathrm{r}}_{k'}$, $k, k' \geqslant 0$. Directly from the definition of the map first_k, for any $k, k' \geqslant 0$, $\mathsf{first}_k \circ \mathsf{first}_{k'} = \mathsf{first}_{\min\{k,k'\}}$. Moreover, for any $k, k' \geqslant 0$ such that $k \leqslant k'$, $\mathsf{first}_{k'} \circ \mathsf{first}^{\mathrm{r}}_k = \mathsf{first}^{\mathrm{r}}_k$ and $\mathsf{first}^{\mathrm{r}}_{k'} \circ \mathsf{first}_k = \mathsf{first}_k$. Observe also that the maps first_k and $\mathsf{first}_{k'}$, $k, k' \geqslant 0$ do not commute. Indeed, in $\mathbf{P}(\mathcal{E})$, we have

$$\mathsf{first}_1(\mathsf{first}^{\mathrm{r}}_2(2^{\mathrm{e}}1^{\mathrm{e}}2^{\mathrm{e}}1^{\mathrm{e}}2^{\mathrm{e}})) = 1^{\mathrm{e}}2^{\mathrm{e}} \neq 2^{\mathrm{e}}1^{\mathrm{e}} = \mathsf{first}^{\mathrm{r}}_2(\mathsf{first}_1(2^{\mathrm{e}}1^{\mathrm{e}}2^{\mathrm{e}}1^{\mathrm{e}}2^{\mathrm{e}})). \tag{4.2.4.A}$$

▶ **Proposition** 4.2.4.A — *For any monoid $\mathcal{M}$, any $k \geqslant 0$, and any total order relation $\preccurlyeq$ on $\mathcal{M}$, the maps $\mathsf{sort}_{\preccurlyeq}$ and first_k on $\mathbf{P}(\mathcal{M})$ commute if and only if $\mathcal{M}$ is the trivial monoid $\mathcal{E}$.*

◀ **Proof** — Let $\mathfrak{p} \in \mathbf{P}(\mathcal{E})(n)$, $n \geqslant 0$. By definition of sort and of first_k, $\mathsf{sort}_{\preccurlyeq}(\mathsf{first}_k(\mathfrak{p}))$ is the $\mathcal{E}$-pigmented word $\mathfrak{q}$ such that for any $j \in [\ell(\mathfrak{q}) - 1]$, $\mathfrak{q}(j) \preccurlyeq \mathfrak{q}(j+1)$, and for any $i^{\mathrm{e}} \in \mathcal{L}_{\mathcal{E}}$, $\mathfrak{q}$ has exactly $\min\{|\mathfrak{p}|_{i^{\mathrm{e}}}, k\}$ occurrences of i^{e}. Since $\mathsf{first}_k(\mathsf{sort}_{\preccurlyeq}(\mathfrak{p}))$ satisfies the same property, we have $\mathsf{sort}_{\preccurlyeq}(\mathsf{first}_k(\mathfrak{p})) = \mathsf{first}_k(\mathsf{sort}_{\preccurlyeq}(\mathfrak{p}))$.

Conversely, assume that $\mathcal{M}$ is not trivial. Thus, $\mathcal{M}$ contains two distinct elements α_1 and α_2. By considering without loss of generality that $\alpha_1 \preccurlyeq \alpha_2$, we have in particular $\mathsf{sort}_{\preccurlyeq}(\mathsf{first}_1(1^{\alpha_2}1^{\alpha_1})) = 1^{\alpha_2} \neq 1^{\alpha_1} = \mathsf{first}_1(\mathsf{sort}_{\preccurlyeq}(1^{\alpha_2}1^{\alpha_1}))$. This shows that $\mathsf{sort}_{\preccurlyeq}$ and first_k do not commute. □

4.3 Three simple quotients

We use the clone congruences introduced in the previous section to build three quotients $\mathsf{WInc}(\mathcal{M})$, $\mathsf{Arra}_k(\mathcal{M})$, and Inc_k of $\mathbf{P}(\mathcal{M})$. Each of these clones admits finitely related presentations: the first clone is a clone realization of a generalization of the variety of commutative monoids, the second one is a clone realization of a generalization of the variety of left-regular bands, and the last one is a clone realization of a generalization of the variety of bounded semilattices.

4.3.1 On pigmented weakly increasing words. Let

$$\mathsf{WInc}(\mathcal{M}) := \mathbf{P}(\mathcal{M})/_{\equiv_{\mathsf{sort}}}. \tag{4.3.1.A}$$

By Proposition 4.2.2.A, $\mathsf{WInc}(\mathcal{M})$ is a well-defined quotient clone of $\mathbf{P}(\mathcal{M})$.

Since $\mathsf{sort}_{\preccurlyeq}$ is a $\mathbb{P}$-symbol for $\equiv_{\mathsf{sort}}$ where $\preccurlyeq$ is any total order relation on $\mathcal{M}$, $\mathsf{WInc}(\mathcal{M})$ admits as realization the clone described by Proposition 4.1.2.A. Hence, by definition of $\mathsf{sort}_{\preccurlyeq}$, $\mathsf{WInc}(\mathcal{M})$ is a clone on the graded set of *weakly $\preccurlyeq$-increasing* $\mathcal{M}$-pigmented words, which are the $\mathcal{M}$-pigmented words $\mathfrak{p}$ such that, for any $j \in [\ell(\mathfrak{p}) - 1]$, $\mathfrak{p}(j) \preccurlyeq \mathfrak{p}(j+1)$. Equivalently, the elements of $\mathsf{WInc}(\mathcal{M})$ can be seen as multisets of $\mathcal{M}$-pigmented letters. For instance, in $\mathsf{WInc}(A^*)$, we have

$$\begin{aligned} 2^{ab}3^{\epsilon}3^{a}4^{b}4^{b}\left[1^{ab}2^{ba}, 1^{b}2^{ba}3^{\epsilon}3^{b}, 1^{\epsilon}2^{b}, 3^{b}\right] &= \mathsf{sort}_{\preccurlyeq}\left(1^{abb}2^{abba}3^{ab}3^{abb}1^{\epsilon}2^{b}1^{a}2^{ab}3^{bb}3^{bb}\right) \\ &= 1^{\epsilon}1^{a}1^{abb}2^{ab}2^{abba}2^{b}3^{ab}3^{abb}3^{bb}3^{bb}. \end{aligned} \tag{4.3.1.B}$$

Besides, the clone $\mathsf{WInc}(\mathcal{M})$ is not combinatorial because $\{\epsilon, 1^{\mathrm{e}}, 1^{\mathrm{e}}1^{\mathrm{e}}, \dots\} \subseteq \mathsf{WInc}(\mathcal{M})(1)$ where e is the unit of $\mathcal{M}$.

▶ **Proposition** 4.3.1.A — *For any monoid $\mathcal{M}$, the clone $\mathsf{WInc}(\mathcal{M})$ admits the presentation $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}'_{\mathcal{M}})$ where $\mathfrak{R}'_{\mathcal{M}}$ is the set $\mathfrak{R}_{\mathcal{M}}$ augmented with the $\mathfrak{G}_{\mathcal{M}}$-equation*

$$\mathsf{rc}_{\mathcal{M}}(1^{\mathrm{e}}2^{\mathrm{e}}) \;\mathfrak{R}'_{\mathcal{M}}\; \mathsf{rc}_{\mathcal{M}}(2^{\mathrm{e}}1^{\mathrm{e}}) \tag{4.3.1.C}$$

where e is the unit of $\mathcal{M}$.

◀ **Proof** — Let $\equiv'$ be the clone congruence of $\mathbf{P}(\mathcal{M})$ generated by

$$1^{\mathrm{e}}2^{\mathrm{e}} \equiv' 2^{\mathrm{e}}1^{\mathrm{e}}. \tag{4.3.1.D}$$

Let us show that the clone congruences $\equiv'$ and $\equiv_{\mathsf{sort}}$ of $\mathbf{P}(\mathcal{M})$ are equal. This will imply, by Proposition 3.3.3.C, that $\mathsf{WInc}(\mathcal{M})$ admits the stated presentation.

For this, let us introduce some intermediate binary relations on $\mathbf{P}(\mathcal{M})$. Let $\preccurlyeq$ be any total order on $\mathcal{M}$ and $\leadsto$ be the binary relation on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.i_1^{\alpha_1} i_2^{\alpha_2}.\mathfrak{p}' \;\leadsto\; \mathfrak{p}.i_2^{\alpha_2} i_1^{\alpha_1}.\mathfrak{p}' \qquad \text{if } i_1^{\alpha_1} \neq i_2^{\alpha_2} \text{ and } i_2^{\alpha_2} \preccurlyeq i_1^{\alpha_1}, \tag{4.3.1.E}$$

where $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ and $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$. Let $\sim$ be the reflexive, symmetric, and transitive closure of $\leadsto$ and let us show that $\sim$ is equal to $\equiv_{\mathsf{sort}}$. First, observe that directly from the definition of $\leadsto$, for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{r} \leadsto \mathfrak{r}'$ implies $\mathsf{sort}_{\preccurlyeq}(\mathfrak{r}) = \mathsf{sort}_{\preccurlyeq}(\mathfrak{r}')$. Hence, we have $\mathfrak{r} \equiv_{\mathsf{sort}} \mathfrak{r}'$, and since $\sim$ is the smallest equivalence relation containing $\leadsto$, $\mathfrak{r} \sim \mathfrak{r}'$ implies $\mathfrak{r} \equiv_{\mathsf{sort}} \mathfrak{r}'$. Conversely, assume that $\mathfrak{r} \equiv_{\mathsf{sort}} \mathfrak{r}'$ for $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$. By definition of $\mathsf{sort}_{\preccurlyeq}$, for any $\mathfrak{q} \in \mathbf{P}(\mathcal{M})$, the process consisting in swapping iteratively and as long as possible two adjacent $\mathcal{M}$-pigmented letters $i_1^{\alpha_1}$ and $i_2^{\alpha_2}$ of $\mathfrak{q}$ such that $i_1^{\alpha_1} \neq i_2^{\alpha_2}$ and $i_2^{\alpha_2} \preccurlyeq i_1^{\alpha_1}$ finally produces the $\mathcal{M}$-pigmented word $\mathsf{sort}_{\preccurlyeq}(\mathfrak{q})$. Moreover, observe that by definition of $\leadsto$, for any $\mathfrak{q}', \mathfrak{q}'' \in \mathbf{P}(\mathcal{M})$, the property $\mathfrak{q}' \leadsto \mathfrak{q}''$ is equivalent to the fact that $\mathfrak{q}''$ is obtained from $\mathfrak{q}'$ by swapping two adjacent $\mathcal{M}$-pigmented letters $i_1^{\alpha_1}$ and $i_2^{\alpha_2}$ such that $i_1^{\alpha_1} \neq i_2^{\alpha_2}$ and $i_2^{\alpha_2} \preccurlyeq i_1^{\alpha_1}$. Due to the fact that $\sim$ is the smallest equivalence relation containing $\leadsto$, $\mathfrak{r} \sim \mathfrak{r}'$ holds.

Now, let us show that $\equiv'$ is equal to $\sim$. First, since the left-hand and the right-hand sides of (4.3.1.D) are $\sim$-equivalent, $\equiv'$ is contained into $\sim$. Conversely, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ and $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$, we have

$$\mathfrak{p}.i_1^{\alpha_1} i_2^{\alpha_2}.\mathfrak{p}' = 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 1^{\mathrm{e}}2^{\mathrm{e}}[i_1^{\alpha_1}i_2^{\alpha_2}], \mathfrak{p}'] \equiv' 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 2^{\mathrm{e}}1^{\mathrm{e}}[i_1^{\alpha_1}i_2^{\alpha_2}], \mathfrak{p}'] = \mathfrak{p}.i_2^{\alpha_2} i_1^{\alpha_1}.\mathfrak{p}'. \tag{4.3.1.F}$$

This shows that for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$ implies $\mathfrak{r} \equiv' \mathfrak{r}'$. Since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\sim$ is contained into $\equiv'$. This establishes the statement of the proposition. □

By Proposition 4.3.1.A, any $\mathsf{WInc}(\mathcal{M})$-algebra is an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ where $\star$ is commutative. In particular, $\mathsf{WInc}(\mathcal{E})$ is a clone realization of the variety of commutative monoids.

4.3.2 On pigmented arrangements. For any $k \geqslant 0$, let

$$\mathsf{Arra}_k(\mathcal{M}) := \mathbf{P}(\mathcal{M})/_{\equiv_{\mathsf{first}_k}}. \tag{4.3.2.A}$$

By Proposition 4.2.3.A, $\mathsf{Arra}_k(\mathcal{M})$ is a well-defined quotient clone of $\mathbf{P}(\mathcal{M})$. Since for any $0 \leqslant k \leqslant k'$, $\equiv_{\mathsf{first}_{k'}}$ is a refinement of $\equiv_{\mathsf{first}_k}$, $\mathsf{Arra}_k(\mathcal{M})$ is a quotient of $\mathsf{Arra}_{k'}(\mathcal{M})$. Moreover, since $\equiv_{\mathsf{first}_0}$ is the coarsest clone congruence of $\mathbf{P}(\mathcal{M})$, $\mathsf{Arra}_0(\mathcal{M})$ is the trivial clone $\mathcal{T}$. Besides, the clone $\mathsf{Arra}_k^{\mathrm{r}}(\mathcal{M}) := \mathsf{Arra}_k(\mathcal{M})^{\mathrm{r}}$ is by Proposition 4.2.1.A isomorphic to $\mathsf{Arra}_k(\mathcal{M})$.

Since first_k is a $\mathbb{P}$-symbol for $\equiv_{\mathsf{first}_k}$, $\mathsf{Arra}_k(\mathcal{M})$ admits as realization the clone described by Proposition 4.1.2.A. Hence, by definition of first_k, $\mathsf{Arra}_k(\mathcal{M})$ is a clone on the graded set of *$\mathcal{M}$-pigmented arrangements*, which are the $\mathcal{M}$-pigmented words $\mathfrak{p}$ such that for any value i, there are at most k $\mathcal{M}$-pigmented letters of $\mathfrak{p}$ having i as value. For instance, in $\mathsf{Arra}_1(A^*)$,

$$\begin{aligned} 2^\epsilon 3^{aa} 1^b 4^{ca}\left[3^\epsilon 1^a, 2^{bb}, 2^b 1^a 3^a, 1^c 2^c\right] &= \mathsf{first}_1\left(2^{bb} 2^{aab} 1^{aaa} 3^{aaa} 3^b 1^{ba} 1^{cac} 2^{cac}\right) \\ &= 2^{bb} 1^{aaa} 3^{aaa}, \end{aligned} \tag{4.3.2.B}$$

and in $\mathsf{Arra}_2(A^*)$,

$$\begin{aligned} 2^\epsilon 3^{aa} 1^b 4^{ca}\left[3^\epsilon 1^a, 2^{bb}, 2^b 1^a 3^a, 1^c 2^c\right] &= \mathsf{first}_2\left(2^{bb} 2^{aab} 1^{aaa} 3^{aaa} 3^b 1^{ba} 1^{cac} 2^{cac}\right) \\ &= 2^{bb} 2^{aab} 1^{aaa} 3^{aaa} 3^b 1^{ba}. \end{aligned} \tag{4.3.2.C}$$

Besides, when $\mathcal{M}$ is finite, $\mathsf{Arra}_k(\mathcal{M})$ is combinatorial and for any $n \geqslant 0$,

$$\#\mathsf{Arra}_k(\mathcal{M})(n) = \sum_{u \in [\![k]\!]^n} \frac{(u(1) + \cdots + u(n))!}{u(1)! \ldots u(n)!} (\#\mathcal{M})^{u(1)+\cdots+u(n)} \tag{4.3.2.D}$$

In particular, we have

$$\#\mathsf{Arra}_1(\mathcal{M})(n) = \sum_{i \in [\![n]\!]} \binom{n}{i} i! (\#\mathcal{M})^i. \tag{4.3.2.E}$$

The sequences of dimensions of $\mathsf{Arra}_k(\mathcal{E})$ for $k \in [\![2]\!]$ start by

$$1, 1, 1, 1, 1, 1, 1, 1, 1, \qquad k = 0, \tag{4.3.2.F}$$

$$1, 2, 5, 16, 65, 326, 1957, 13700, 109601, \qquad k = 1, \tag{4.3.2.G}$$

$$1, 3, 19, 271, 7365, 326011, 21295783, 1924223799, 229714292041, \qquad k = 2. \tag{4.3.2.H}$$

The second and third ones are respectively Sequences **A000522** and **A003011** of [Slo].

▶ **Proposition** 4.3.2.A — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, the clone $\mathsf{Arra}_k(\mathcal{M})$ admits the presentation $\left(\mathfrak{G}_\mathcal{M}, \mathfrak{R}'_\mathcal{M}\right)$ where $\mathfrak{R}'_\mathcal{M}$ is the set $\mathfrak{R}_\mathcal{M}$ augmented with the $\mathfrak{G}_\mathcal{M}$-equation*

$$\mathsf{rc}_\mathcal{M}(1^{\alpha_1} 2^{\mathrm{e}} 1^{\alpha_2} 3^{\mathrm{e}} \ldots 1^{\alpha_k} (k+1)^{\mathrm{e}} 1^{\alpha_{k+1}}) \;\mathfrak{R}'_\mathcal{M}\; \mathsf{rc}_\mathcal{M}(1^{\alpha_1} 2^{\mathrm{e}} 1^{\alpha_2} 3^{\mathrm{e}} \ldots 1^{\alpha_k} (k+1)^{\mathrm{e}}) \tag{4.3.2.I}$$

with $\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1} \in \mathcal{M}$ where e is the unit of $\mathcal{M}$.

◄ **Proof** — Let $\equiv'$ be the clone congruence of $\mathbf{P}(\mathcal{M})$ generated by

$$1^{\alpha_1} 2^{\mathrm{e}} 1^{\alpha_2} 3^{\mathrm{e}} \dots 1^{\alpha_k} (k+1)^{\mathrm{e}} 1^{\alpha_{k+1}} \equiv' 1^{\alpha_1} 2^{\mathrm{e}} 1^{\alpha_2} 3^{\mathrm{e}} \dots 1^{\alpha_k} (k+1)^{\mathrm{e}} \tag{4.3.2.J}$$

with $\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1} \in \mathcal{M}$. Let us show that the clone congruences $\equiv'$ and $\equiv_{\mathsf{first}_k}$ of $\mathbf{P}(\mathcal{M})$ are equal. This will imply, by Proposition 3.3.3.C, that $\mathsf{Arra}_k(\mathcal{M})$ admits the stated presentation.

For this, let us introduce some intermediate binary relations on $\mathbf{P}(\mathcal{M})$. Let $\rightsquigarrow$ be the binary relation on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.i^{\alpha_1}.\mathfrak{q}_1.i^{\alpha_2}.\mathfrak{q}_2. \;\dots\; .i^{\alpha_k}.\mathfrak{q}_k.i^{\alpha_{k+1}}.\mathfrak{p}' \;\rightsquigarrow\; \mathfrak{p}.i^{\alpha_1}.\mathfrak{q}_1.i^{\alpha_2}.\mathfrak{q}_2. \;\dots\; .i^{\alpha_k}.\mathfrak{q}_k.\mathfrak{p}' \tag{4.3.2.K}$$

where $\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_k, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ and $i^{\alpha_1}, i^{\alpha_2}, \dots, i^{\alpha_k}, i^{\alpha_{k+1}} \in \mathcal{L}_{\mathcal{M}}$. Let $\sim$ be the reflexive, symmetric, and transitive closure of $\rightsquigarrow$ and let us show that $\sim$ is equal to $\equiv_{\mathsf{first}_k}$. First, observe that directly from the definition of $\rightsquigarrow$, for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$ implies $\mathsf{first}_k(\mathfrak{r}) = \mathsf{first}_k(\mathfrak{r}')$. Hence, we have $\mathfrak{r} \equiv_{\mathsf{first}_k} \mathfrak{r}'$, and since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\mathfrak{r} \sim \mathfrak{r}'$ implies $\mathfrak{r} \equiv_{\mathsf{first}_k} \mathfrak{r}'$. Conversely, assume that $\mathfrak{r} \equiv_{\mathsf{first}_k} \mathfrak{r}'$ for $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$. By definition of first_k, for any $\mathfrak{q} \in \mathbf{P}(\mathcal{M})$, the process consisting in deleting iteratively and as long as possible each letter of $\mathfrak{q}$ which is not a left k-witness finally produces the $\mathcal{M}$-pigmented word $\mathsf{first}_k(\mathfrak{q})$. Moreover, observe that by definition of $\rightsquigarrow$, for any $\mathfrak{q}', \mathfrak{q}'' \in \mathbf{P}(\mathcal{M})$, the property $\mathfrak{q}' \rightsquigarrow \mathfrak{q}''$ is equivalent to the fact that $\mathfrak{q}''$ is obtained from $\mathfrak{q}'$ by deleting a letter which is not a left k-witness. Due to the fact that $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\mathfrak{r} \sim \mathfrak{r}'$ holds.

Now, let us show that $\equiv'$ is equal to $\sim$. First, since the left-hand and right-hand sides of (4.3.2.J) are $\sim$-equivalent, $\equiv'$ is contained into $\sim$. Conversely, for any $\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_k, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ and $i^{\alpha_1}, i^{\alpha_2}, \dots, i^{\alpha_k}, i^{\alpha_{k+1}} \in \mathcal{L}_{\mathcal{M}}$, we have

$$\begin{aligned}
&\mathfrak{p}.i^{\alpha_1}.\mathfrak{q}_1.i^{\alpha_2}.\mathfrak{q}_2. \;\dots\; .i^{\alpha_k}.\mathfrak{q}_k.i^{\alpha_{k+1}}.\mathfrak{p}' \\
&\qquad = 1^{\mathrm{e}} 2^{\mathrm{e}} 3^{\mathrm{e}} [\mathfrak{p}, 1^{\alpha_1} 2^{\mathrm{e}} 1^{\alpha_2} 3^{\mathrm{e}} \dots 1^{\alpha_k} (k+1)^{\mathrm{e}} 1^{\alpha_{k+1}} [i^{\mathrm{e}}, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_k], \mathfrak{p}'] \\
&\qquad \equiv' 1^{\mathrm{e}} 2^{\mathrm{e}} 3^{\mathrm{e}} [\mathfrak{p}, 1^{\alpha_1} 2^{\mathrm{e}} 1^{\alpha_2} 3^{\mathrm{e}} \dots 1^{\alpha_k} (k+1)^{\mathrm{e}} [i^{\mathrm{e}}, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_k], \mathfrak{p}'] \\
&\qquad = \mathfrak{p}.i^{\alpha_1}.\mathfrak{q}_1.i^{\alpha_2}.\mathfrak{q}_2. \;\dots\; .i^{\alpha_k}.\mathfrak{q}_k.\mathfrak{p}'.
\end{aligned} \tag{4.3.2.L}$$

This shows that for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$ implies $\mathfrak{r} \equiv' \mathfrak{r}'$. Since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\sim$ is contained into $\equiv'$. This establishes the statement of the proposition. □

By Proposition 4.3.2.A, any $\mathsf{Arra}_k(\mathcal{M})$-algebra is an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ where $\star$ and p_α satisfy

$$\begin{aligned}
&\mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\alpha_2}(x_1) \star x_3 \star \dots \star \mathrm{p}_{\alpha_k}(x_1) \star x_{k+1} \star \mathrm{p}_{\alpha_{k+1}}(x_1) \\
&\qquad = \mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\alpha_2}(x_1) \star x_3 \star \dots \star \mathrm{p}_{\alpha_k}(x_1) \star x_{k+1}
\end{aligned} \tag{4.3.2.M}$$

for any $x_1, \dots, x_{k+1} \in \mathcal{A}$ and $\alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \mathcal{M}$. In particular, $\mathsf{Arra}_1(\mathcal{E})$ is a clone realization of the variety of left-regular bands, that are monoids $(\mathcal{A}, \star, \mathrm{u})$ such that $\star$ satisfies $x_1 \star x_2 \star x_1 = x_1 \star x_2$ for any $x_1, x_2 \in \mathcal{A}$.

4.3.3 On increasing monochrome words. Let us denote by $\preccurlyeq$ be the unique order relation on the trivial monoid $\mathcal{E}$. By Proposition 4.2.2.A (resp. 4.2.3.A), $\equiv_{\mathsf{sort}}$ (resp. $\equiv_{\mathsf{first}_k}$) is a clone congruence of $\mathbf{P}(\mathcal{E})$ and $\mathsf{sort}_{\preccurlyeq}$ (resp. first_k) is a $\mathbb{P}$-symbol for $\equiv_{\mathsf{sort}}$ (resp. $\equiv_{\mathsf{first}_k}$). Therefore, by Propositions 4.2.4.A and 4.1.3.A, the map $\phi_k := \mathsf{sort}_{\preccurlyeq} \circ \mathsf{first}_k = \mathsf{first}_k \circ \mathsf{sort}_{\preccurlyeq}$ is a $\mathbb{P}$-symbol for the fiber equivalence relation $\equiv_{\phi_k}$ of ϕ_k and $\equiv_{\phi_k}$ is a clone congruence of $\mathbf{P}(\mathcal{E})$.

For any $k \geqslant 0$, let

$$\mathsf{Inc}_k := \mathbf{P}(\mathcal{E})/_{\equiv_{\phi_k}}. \tag{4.3.3.A}$$

For the previous reasons, Inc_k is a well-defined quotient of $\mathbf{P}(\mathcal{M})$. Moreover, since for any $0 \leqslant k \leqslant k'$ and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{E})$, $\mathfrak{p} \equiv_{\phi_{k'}} \mathfrak{p}'$ implies $\mathfrak{p} \equiv_{\phi_k} \mathfrak{p}'$, the equivalence relation $\equiv_{\phi_{k'}}$ is a refinement of $\equiv_{\phi_k}$. Therefore, Inc_k is a quotient of $\mathsf{Inc}_{k'}$. Besides, since $\equiv_{\phi_0}$ is the coarsest clone congruence of $\mathbf{P}(\mathcal{E})$, Inc_0 is the trivial clone $\mathcal{T}$.

Since ϕ_k is a $\mathbb{P}$-symbol for $\equiv_{\phi_k}$, Inc_k admits the realization described by Proposition 4.1.2.A. Hence, by definition of ϕ_k, Inc_k is a clone on the set of *monochrome k-increasing words*, which are the $\mathcal{E}$-pigmented words $\mathfrak{p}$ such that $\mathfrak{p}$ are weakly $\preccurlyeq$-increasing and for any value i, $\mathfrak{p}$ have at most k occurrences of i^{e}. Equivalently, the elements of Inc_k can be seen as multisets of positive integers where each element has multiplicity at most k. For instance, in Inc_1,

$$1^{\mathrm{e}}3^{\mathrm{e}}[2^{\mathrm{e}}4^{\mathrm{e}}, 1^{\mathrm{e}}3^{\mathrm{e}}4^{\mathrm{e}}, 2^{\mathrm{e}}] = 2^{\mathrm{e}}4^{\mathrm{e}}, \tag{4.3.3.B}$$

and in Inc_2,

$$1^{\mathrm{e}}3^{\mathrm{e}}[2^{\mathrm{e}}4^{\mathrm{e}}, 1^{\mathrm{e}}3^{\mathrm{e}}4^{\mathrm{e}}, 2^{\mathrm{e}}] = 2^{\mathrm{e}}2^{\mathrm{e}}4^{\mathrm{e}}. \tag{4.3.3.C}$$

Besides, Inc_k is combinatorial and for any $n \geqslant 0$, $\#\mathsf{Inc}_k(n) = (k+1)^n$.

The clone Inc_k is not parameterized by a monoid since, as shown by Proposition 4.2.4.A, $\equiv_{\phi_k}$ is a clone congruence of $\mathbf{P}(\mathcal{M})$ only when $\mathcal{M} = \mathcal{E}$.

▶ **Proposition** 4.3.3.A — *For any $k \geqslant 0$, the clone Inc_k admits the presentation $\left(\mathfrak{G}_{\mathcal{E}}, \mathfrak{R}'_{\mathcal{E}}\right)$ where $\mathfrak{R}'_{\mathcal{E}}$ is the set $\mathfrak{R}_{\mathcal{E}}$ augmented with the $\mathfrak{G}_{\mathcal{E}}$-equations*

$$\mathsf{rc}_{\mathcal{M}}(1^{\mathrm{e}}2^{\mathrm{e}}) \; \mathfrak{R}'_{\mathcal{E}} \; \mathsf{rc}_{\mathcal{M}}(2^{\mathrm{e}}1^{\mathrm{e}}), \tag{4.3.3.D}$$

$$\mathsf{rc}_{\mathcal{M}}\left(\left(1^{\mathrm{e}}\right)^{k+1}\right) \; \mathfrak{R}'_{\mathcal{E}} \; \mathsf{rc}_{\mathcal{M}}\left(\left(1^{\mathrm{e}}\right)^{k}\right) \tag{4.3.3.E}$$

where e *is the unique element of $\mathcal{E}$.*

◀ **Proof** — Let $\equiv'$ be the clone congruence of $\mathbf{P}(\mathcal{E})$ generated by

$$1^{\mathrm{e}}2^{\mathrm{e}} \equiv' 2^{\mathrm{e}}1^{\mathrm{e}}, \tag{4.3.3.F}$$

$$\left(1^{\mathrm{e}}\right)^{k+1} \equiv' \left(1^{\mathrm{e}}\right)^{k}. \tag{4.3.3.G}$$

Let us show that the clone congruences $\equiv'$ and $\equiv_{\phi_k}$ of $\mathbf{P}(\mathcal{E})$ are equal. This will imply, by Proposition 3.3.3.C, that Inc_k admits the stated presentation.

For this, let us introduce some intermediate binary relations on $\mathbf{P}(\mathcal{E})$. Let $\rightsquigarrow$ be the binary relation on $\mathbf{P}(\mathcal{E})$ satisfying

$$\mathfrak{p}.i_1^{\mathrm{e}}i_2^{\mathrm{e}}.\mathfrak{p}' \;\rightsquigarrow\; \mathfrak{p}.i_2^{\mathrm{e}}i_1^{\mathrm{e}}.\mathfrak{p}' \qquad \text{if } i_2 < i_1, \tag{4.3.3.H}$$

$$\mathfrak{p}.\left(i^{\mathrm{e}}\right)^{k+1}.\mathfrak{p}' \;\rightsquigarrow\; \mathfrak{p}.\left(i^{\mathrm{e}}\right)^{k}.\mathfrak{p}', \tag{4.3.3.I}$$

where $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{E})$ and $i^{\mathrm{e}}, i_1^{\mathrm{e}}, i_2^{\mathrm{e}} \in \mathcal{L}_{\mathcal{E}}$. Let $\sim$ be the reflexive, symmetric, and transitive closure of $\rightsquigarrow$ and let us show that $\sim$ is equal to $\equiv_{\phi_k}$. First, observe that directly from the definition of $\rightsquigarrow$, for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{E})$, $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$ implies $\phi_k(\mathfrak{r}) = \phi_k(\mathfrak{r}')$. Hence, we have $\mathfrak{r} \equiv_{\phi_k} \mathfrak{r}'$, and since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\mathfrak{r} \sim \mathfrak{r}'$ implies $\mathfrak{r} \equiv_{\phi_k} \mathfrak{r}'$. Conversely, assume that $\mathfrak{r} \equiv_{\phi_k} \mathfrak{r}'$ for $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{E})$. By definition of ϕ_k, for any $\mathfrak{q} \in \mathbf{P}(\mathcal{M})$, the process consisting in swapping iteratively and as long as possible two adjacent $\mathcal{E}$-pigmented letters i_1^{e} and i_2^{e} of $\mathfrak{q}$ such that $i_2 < i_1$ and then by deleting iteratively and as long as possible each $\mathcal{E}$-pigmented letter i^{e} having on its left k occurrences of i^{e} finally produces the $\mathcal{E}$-pigmented word $\phi_k(\mathfrak{q})$. Moreover, observe that by definition of $\rightsquigarrow$, for any $\mathfrak{q}', \mathfrak{q}'' \in \mathbf{P}(\mathcal{E})$, the property $\mathfrak{q}' \rightsquigarrow \mathfrak{q}''$ is equivalent to the fact that $\mathfrak{q}''$ is obtained from $\mathfrak{q}'$ swapping two adjacent $\mathcal{E}$-pigmented letters i_1^{e} and i_2^{e} such that $i_2 < i_1$ or by deleting iteratively each $\mathcal{E}$-pigmented letter i^{e} having on its left k occurrences of i^{e}. Due to the fact that $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\mathfrak{r} \sim \mathfrak{r}'$ holds.

Now, let us show that $\equiv'$ is equal to $\sim$. First, since the left-hand and right-hand sides of (4.3.3.F) (resp. (4.3.3.G)) are $\sim$-equivalent, $\equiv'$ is contained into $\sim$. Conversely, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{E})$ and $i^{\mathrm{e}}, i_1^{\mathrm{e}}, i_2^{\mathrm{e}} \in \mathcal{L}_{\mathcal{E}}$, we have

$$\mathfrak{p}.i_1^{\mathrm{e}} i_2^{\mathrm{e}}.\mathfrak{p}' = 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 1^{\mathrm{e}}2^{\mathrm{e}}[i_1^{\mathrm{e}}, i_2^{\mathrm{e}}], \mathfrak{p}'] \equiv' 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 2^{\mathrm{e}}1^{\mathrm{e}}[i_1^{\mathrm{e}}, i_2^{\mathrm{e}}], \mathfrak{p}'] = \mathfrak{p}.i_2^{\mathrm{e}} i_1^{\mathrm{e}}.\mathfrak{p}' \tag{4.3.3.J}$$

and

$$\mathfrak{p}.(i^{\mathrm{e}})^k i^{\mathrm{e}}.\mathfrak{p}' = 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}\left[\mathfrak{p}, (1^{\mathrm{e}})^k 1^{\mathrm{e}}[i^{\mathrm{e}}], \mathfrak{p}'\right] \equiv' 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}\left[\mathfrak{p}, (1^{\mathrm{e}})^k [i^{\mathrm{e}}], \mathfrak{p}'\right] = \mathfrak{p}.(i^{\mathrm{e}})^k.\mathfrak{p}'. \tag{4.3.3.K}$$

This shows that for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{E})$, $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$ implies $\mathfrak{r} \equiv' \mathfrak{r}'$. Since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow$, $\sim$ is contained into $\equiv'$. This establishes the statement of the proposition. □

By Proposition 4.3.3.A, any Inc_k-algebra is a monoid $(\mathcal{A}, \star, \mathrm{u})$ where $\star$ is commutative and satisfies

$$\underbrace{x_1 \star \cdots \star x_1}_{k+1} = \underbrace{x_1 \star \cdots \star x_1}_{k}. \tag{4.3.3.L}$$

In particular, Inc_1 is a clone realization of the variety of meet-semilattices admitting a greatest element (also known as bounded semilattices).

5 A hierarchy of clones

We use the construction $\mathbf{P}$ and intersections of the clone congruences $\equiv_{\mathsf{sort}}$, $\equiv_{\mathsf{first}_k}$, and $\equiv_{\mathsf{first}^{\mathrm{r}}_{k'}}$ introduced in the previous section to build a hierarchy of clones quotients of $\mathbf{P}(\mathcal{M})$. Figure 1 contains the full diagram the constructed clones. The clones located on the bottom three lines of the diagram have been constructed and studied in Section 4. The clones constructed in the following sections are clone realizations of varieties generalizing some special classes of monoids, including regular bands. These structures allow us to solve the word problem in the corresponding varieties. The algorithms are described in terms of $\mathbb{P}$-symbols and are similar to the ones solving the word problem in idempotent semigroups by using conditional string rewrite systems [SS82; NS00].

In this section, $\mathcal{M}$ is a (finite or infinite) monoid endowed a total order relation $\preccurlyeq$. To give concrete examples, we shall consider $\mathcal{M}$ as the free monoid $(A^*, ., \epsilon)$ where A is the alphabet $\{a, b, c\}$ and $\preccurlyeq$ is the lexicographic order on A^* satisfying $a \preccurlyeq b \preccurlyeq c$.

5.1 On pigmented magnets

By considering the intersection of the clone congruences $\equiv_{\mathsf{first}_k}$, $k \geqslant 0$, and their reversions $\equiv^{\mathrm{r}}_{\mathsf{first}_{k'}}$, $k' \geqslant 0$, we construct a quotient clone $\mathsf{Magn}_{k,k'}(\mathcal{M})$ of $\mathbf{P}(\mathcal{M})$. This clone is studied in detail for the case $k = 1 = k'$. A realization through new combinatorial objects named $\mathcal{M}$-pigmented magnets is introduced and a finitely related presentation is described. The algebras over this clone are generalizations of regular bands. These results are based on the introduction of a $\mathbb{P}$-symbol for the underlying equivalence relation.

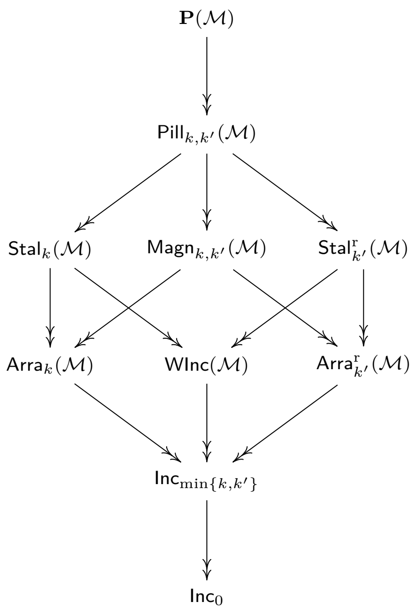


Figure 1: The full diagram of the considered quotients of the clone $\mathbf{P}(\mathcal{M})$ where $\mathcal{M}$ is a monoid and $k, k' \geqslant 0$. The arrows are clone surjections.

5.1.1 Clone construction. For any $k, k' \geqslant 0$, let $\equiv_{k,k'}$ be the equivalence relation $\equiv_{\mathsf{first}_k} \cap \equiv^{\mathrm{r}}_{\mathsf{first}_{k'}}$ and

$$\mathsf{Magn}_{k,k'}(\mathcal{M}) := \mathbf{P}(\mathcal{M})/_{\equiv_{k,k'}}. \tag{5.1.1.A}$$

By Propositions 4.2.3.A and 4.2.1.A, $\mathsf{Magn}_{k,k'}(\mathcal{M})$ is a well-defined clone, and $\mathsf{Arra}_k(\mathcal{M})$ and $\mathsf{Arra}^{\mathrm{r}}_{k'}(\mathcal{M})$ are both quotients of $\mathsf{Magn}_{k,k'}$. Since for any $0 \leqslant k \leqslant k''$ and $0 \leqslant k' \leqslant k'''$, $\equiv_{k'',k'''}$ is a refinement of $\equiv_{k,k'}$, $\mathsf{Magn}_{k,k'}(\mathcal{M})$ is a quotient of $\mathsf{Magn}_{k'',k'''}(\mathcal{M})$. Moreover, since $\equiv_{0,0}$ is the coarsest clone congruence of $\mathbf{P}(\mathcal{M})$, $\mathsf{Magn}_{0,0}$ is the trivial clone $\mathcal{T}$. Besides, the clone $\mathsf{Magn}^{\mathrm{r}}_{k,k'}(\mathcal{M}) := \mathsf{Magn}_{k,k'}(\mathcal{M})^{\mathrm{r}}$ is by Proposition 4.2.1.A isomorphic to $\mathsf{Magn}_{k,k'}(\mathcal{M})$. Since the reversion operation on congruences is involutive, the clones $\mathsf{Magn}^{\mathrm{r}}_{k,k'}(\mathcal{M})$ and $\mathsf{Magn}_{k',k}(\mathcal{M})$ are isomorphic.

5.1.2 Equivalence relation. To lighten the notation, we denote by $\equiv$ the equivalence relation $\equiv_{1,1}$ on $\mathbf{P}(\mathcal{M})$. By definition, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv \mathfrak{p}'$ holds if and only if $(\mathsf{first}_1(\mathfrak{p}), \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p})) = (\mathsf{first}_1(\mathfrak{p}'), \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}'))$.

In order to obtain properties about the clone $\mathsf{Magn}_{1,1}(\mathcal{M})$, we introduce an alternative equivalence relation $\sim$ for which we will show that it is equal to $\equiv$. Let $\rightsquigarrow_1$, $\rightsquigarrow_2$, and $\rightsquigarrow_3$ be the three binary relations on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.\underline{i^{\alpha}}.\mathfrak{p}' \rightsquigarrow_1 \mathfrak{p}.\mathfrak{p}', \tag{5.1.2.A}$$

$$\mathfrak{p}.\underline{i_1^{\alpha_1}}\,\underline{i_2^{\alpha_2}}.\mathfrak{p}' \rightsquigarrow_2 \mathfrak{p}.\underline{i_2^{\alpha_2}}\,\underline{i_1^{\alpha_1}}.\mathfrak{p}' \quad \text{where } i_1 \neq i_2, \tag{5.1.2.B}$$

$$\mathfrak{p}.\underline{i^{\alpha}}\,\underline{i^{\alpha}}.\mathfrak{p}' \rightsquigarrow_3 \mathfrak{p}.\underline{i^{\alpha}}.\mathfrak{p}', \tag{5.1.2.C}$$

where $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ and $i^{\alpha}, i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$. Let $\rightsquigarrow$ be the union $\rightsquigarrow_1 \cup \rightsquigarrow_2 \cup \rightsquigarrow_3$ and let also $\sim$ be the reflexive, symmetric, and transitive closure of $\rightsquigarrow$.

As a side remark, let up emphasize the fact that, despite appearances, $\rightsquigarrow_1$, $\rightsquigarrow_2$, and $\rightsquigarrow_3$ cannot be studied as rewrite rules of string rewrite systems [BF93; BN98; Bez+03]. Indeed, since we could have for instance $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}'$ but not $\mathfrak{p}.\mathfrak{q} \rightsquigarrow_2 \mathfrak{p}'.\mathfrak{q}$ for some $\mathfrak{p}, \mathfrak{p}', \mathfrak{q} \in \mathbf{P}(\mathcal{M})$, the compatibility with the context required by string rewrite systems is not satisfied.

▶ **Lemma** 5.1.2.A — *For any monoid $\mathcal{M}$, the equivalence relation $\sim$ is a monoid congruence of the monoid $(\mathbf{P}(\mathcal{M}), ., \epsilon)$.*

◀ **Proof** — To prove this statement, since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow_1$, $\rightsquigarrow_2$, and $\rightsquigarrow_3$, it is enough to prove that for any $j \in [3]$ and $\mathfrak{q}, \mathfrak{q}', \mathfrak{r} \in \mathbf{P}(\mathcal{M})$, if $\mathfrak{q} \rightsquigarrow_j \mathfrak{q}'$ then $\mathfrak{q}.\mathfrak{r} \sim \mathfrak{q}'.\mathfrak{r}$ and $\mathfrak{r}.\mathfrak{q} \sim \mathfrak{r}.\mathfrak{q}'$.

Directly from the definitions of $\rightsquigarrow_1$, $\rightsquigarrow_2$, and $\rightsquigarrow_3$, for any $j \in [3]$, $\mathfrak{q} \rightsquigarrow_j \mathfrak{q}'$ implies $\mathfrak{r}.\mathfrak{q} \rightsquigarrow_j \mathfrak{r}.\mathfrak{q}'$. This is due to the fact that (5.1.2.A), (5.1.2.B), and (5.1.2.C) do not require that some positions of the involved $\mathcal{M}$-pigmented words are left 1-witnesses. Moreover, directly from the definitions of $\rightsquigarrow_1$ and $\rightsquigarrow_3$, for any $j \in \{1, 3\}$, $\mathfrak{q} \rightsquigarrow_j \mathfrak{q}'$ implies $\mathfrak{q}.\mathfrak{r} \rightsquigarrow_j \mathfrak{q}'.\mathfrak{r}$. This is due to the fact that (5.1.2.A) and (5.1.2.C) do not require that some positions of the involved $\mathcal{M}$-pigmented words are right 1-witnesses. The remaining case to explore happens when $\mathfrak{q} \rightsquigarrow_2 \mathfrak{q}'$. In this case, $\mathfrak{q}$ and $\mathfrak{q}'$ decompose as $\mathfrak{q} = \mathfrak{p}.\, i_1^{\alpha_1}\, i_2^{\alpha_2} .\mathfrak{p}'$ and $\mathfrak{q}' = \mathfrak{p}.\, i_2^{\alpha_2}\, i_1^{\alpha_1} .\mathfrak{p}'$ where $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$, and $i_2 \neq i_1$. As the position $\ell(\mathfrak{p}) + 2$ of $\mathfrak{q}.\mathfrak{r}$ is a right 1-witness if and only if there is no $\mathcal{M}$-pigmented letter of value i_2 in $\mathfrak{r}$, we have two cases to explore. If this position is a right 1-witness, then

$$\mathfrak{q}.\mathfrak{r} = \mathfrak{p}.\, i_1^{\alpha_1}\, i_2^{\alpha_2} .\mathfrak{p}'.\mathfrak{r} \rightsquigarrow_2 \mathfrak{p}.\, i_2^{\alpha_2}\, i_1^{\alpha_1} .\mathfrak{p}'.\mathfrak{r} = \mathfrak{q}'.\mathfrak{r}. \tag{5.1.2.D}$$

Otherwise, we have

$$\mathfrak{q}.\mathfrak{r} = \mathfrak{p}.\, i_1^{\alpha_1}\, i_2^{\alpha_2} .\mathfrak{p}'.\mathfrak{r} \rightsquigarrow_1 \mathfrak{p}.\, i_1^{\alpha_1} .\mathfrak{p}'.\mathfrak{r} \tag{5.1.2.E}$$

and

$$\mathfrak{q}'.\mathfrak{r} = \mathfrak{p}.\, i_2^{\alpha_2}\, i_1^{\alpha_1} .\mathfrak{p}'.\mathfrak{r} \rightsquigarrow_1 \mathfrak{p}.\, i_1^{\alpha_1} .\mathfrak{p}'.\mathfrak{r}. \tag{5.1.2.F}$$

This shows that $\mathfrak{q}.\mathfrak{r} \sim \mathfrak{q}'.\mathfrak{r}$. □

▶ **Lemma** 5.1.2.B — *For any monoid $\mathcal{M}$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$,*

$$\mathfrak{p} \sim \mathsf{first}_1(\mathfrak{p}).\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}). \tag{5.1.2.G}$$

◀ **Proof** — Let us first show that $\mathfrak{p} \sim \mathfrak{p}.\mathfrak{p}$ by induction on $\ell := \ell(\mathfrak{p})$. If $\ell = 0$, then $\mathfrak{p} = \epsilon$ and since $\mathfrak{p}.\mathfrak{p} = \epsilon$, the stated property holds. Assume now that $\ell \geqslant 1$. In this case, $\mathfrak{p}$ decomposes as $\mathfrak{p} = i^\alpha .\mathfrak{p}'$ where $i^\alpha \in \mathcal{L}_{\mathcal{M}}$ and $\mathfrak{p}' \in \mathbf{P}(\mathcal{M})$. We have now $\mathfrak{p}.\mathfrak{p} = i^\alpha .\mathfrak{p}'.\, i^\alpha .\mathfrak{p}'$ and two cases to explore depending on whether the position $\ell + 1$ in $\mathfrak{p}.\mathfrak{p}$ is a right 1-witness.

(I) If it is the case, then $\mathfrak{p}.\mathfrak{p} = i^\alpha .\mathfrak{p}'.\, i^\alpha .\mathfrak{p}'$. Since there is no occurrence of any $\mathcal{M}$-pigmented letter having i as value in $\mathfrak{p}'$, and additionally, there is no position $j \in [\ell]$ in $\mathfrak{p}.\mathfrak{p}$ which is a right 1-witness, we have $i^\alpha .\mathfrak{p}'.\, i^\alpha .\mathfrak{p}' \rightsquigarrow_2 \dots \rightsquigarrow_2 i^\alpha\, i^\alpha .\mathfrak{p}'.\mathfrak{p}'$.

(II) Otherwise, $\mathfrak{p}.\mathfrak{p} = i^\alpha .\mathfrak{p}'.\, i^\alpha .\mathfrak{p}'$. Since there are occurrences of letters having i as value in $\mathfrak{p}'$, we have $i^\alpha .\mathfrak{p}'.\, i^\alpha .\mathfrak{p}' \rightsquigarrow_1 i^\alpha .\mathfrak{p}'.\mathfrak{p}'$ and $i^\alpha\, i^\alpha .\mathfrak{p}'.\mathfrak{p}' \rightsquigarrow_1 i^\alpha .\mathfrak{p}'.\mathfrak{p}'$.

In both cases, by induction hypothesis and by using the fact that by Lemma 5.1.2.A, $\sim$ is a monoid congruence, we obtain $\mathfrak{p}.\mathfrak{p} \sim i^\alpha\, i^\alpha .\mathfrak{p}'.\mathfrak{p}' \sim i^\alpha\, i^\alpha .\mathfrak{p}'$. Finally, since $i^\alpha\, i^\alpha .\mathfrak{p}' \rightsquigarrow_3 i^\alpha .\mathfrak{p}' = \mathfrak{p}$, the stated property is established.

Let us now show that $\mathfrak{p}.\mathfrak{p} \sim \mathsf{first}_1(\mathfrak{p}).\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p})$. By assuming that $\mathfrak{p}$ writes as $\mathfrak{p} = i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}$, there exists a unique pair $(r_1 \dots r_k, s_1 \dots s_k)$ of subwords of $1 \dots \ell$ such that $\mathsf{first}_1(\mathfrak{p}) = i_{r_1}^{\alpha_{r_1}} \dots i_{r_k}^{\alpha_{r_k}}$ and $\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}) = i_{s_1}^{\alpha_{s_1}} \dots i_{s_k}^{\alpha_{s_k}}$. Therefore, we have $i_{r_1}^{\alpha_{r_1}} .\mathfrak{p}_1. \dots .\, i_{r_k}^{\alpha_{r_k}} .\mathfrak{p}_k = \mathfrak{p} = \mathfrak{p}'_k.\, i_{s_1}^{\alpha_{s_1}}. \dots .\mathfrak{p}'_1.\, i_{s_k}^{\alpha_{s_k}}$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_k, \mathfrak{p}'_k, \dots, \mathfrak{p}'_1 \in \mathbf{P}(\mathcal{M})$. Hence,

$$\mathfrak{p}.\mathfrak{p} = i_{r_1}^{\alpha_{r_1}} .\mathfrak{p}_1. \dots .\, i_{r_k}^{\alpha_{r_k}} .\mathfrak{p}_k.\mathfrak{p}'_k.\, i_{s_1}^{\alpha_{s_1}}. \dots .\mathfrak{p}'_1.\, i_{s_k}^{\alpha_{s_k}}, \tag{5.1.2.H}$$

and since the positions in $\mathfrak{p}.\mathfrak{p}$ of the letters of its factors $\mathfrak{p}_1, \dots, \mathfrak{p}_k, \mathfrak{p}'_k, \dots, \mathfrak{p}'_1$ are neither left 1-witnesses nor right 1-witnesses, we have

$$\mathfrak{p}.\mathfrak{p} \rightsquigarrow_1 \dots \rightsquigarrow_1 \underline{i_{r_1}^{\alpha_{r_1}}} \;\dots\; \underline{i_{r_k}^{\alpha_{r_k}}}\; \underline{i_{s_1}^{\alpha_{s_1}}} \;\dots\; \underline{i_{s_k}^{\alpha_{s_k}}} = \mathsf{first}_1(\mathfrak{p}).\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}). \tag{5.1.2.I}$$

By putting these $\sim$-equivalences together, we obtain $\mathfrak{p} \sim \mathfrak{p}.\mathfrak{p} \sim \mathsf{first}_1(\mathfrak{p}).\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p})$ establishing the stated $\sim$-equivalence. □

▶ **Proposition** 5.1.2.C — *For any monoid $\mathcal{M}$, the binary relations $\equiv$ and $\sim$ on $\mathbf{P}(\mathcal{M})$ are equal.*

◀ **Proof** — First, observe that for any $j \in [3]$ and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, if $\mathfrak{p} \rightsquigarrow_j \mathfrak{p}'$, then $\mathsf{first}_1(\mathfrak{p}) = \mathsf{first}_1(\mathfrak{p}')$ and $\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}) = \mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}')$. Hence, and since $\sim$ is the smallest equivalence relation containing $\rightsquigarrow_1$, $\rightsquigarrow_2$, and $\rightsquigarrow_3$, we have $\mathfrak{p} \equiv \mathfrak{p}'$. Therefore, $\sim$ is contained into $\equiv$. Conversely, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{p} \equiv \mathfrak{p}'$, we have $\mathsf{first}_1(\mathfrak{p}) = \mathsf{first}_1(\mathfrak{p}')$ and $\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}) = \mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}')$. By Lemma 5.1.2.B, $\mathfrak{p} \sim \mathsf{first}_1(\mathfrak{p}).\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}) = \mathsf{first}_1(\mathfrak{p}').\mathsf{first}_1^{\mathrm{r}}(\mathfrak{p}') \sim \mathfrak{p}'$. For this reason, we have $\mathfrak{p} \sim \mathfrak{p}'$, showing that $\equiv$ is contained into $\sim$. □

5.1.3 $\mathbb{P}$-symbol algorithm. With the aim of describing a realization of $\mathsf{Magn}_{1,1}(\mathcal{M})$, we propose now a $\mathbb{P}$-symbol for $\equiv$. For any $j \in [3]$, let $\preccurlyeq_j$ be the reflexive and transitive closure of $\rightsquigarrow_j$.

▶ **Lemma** 5.1.3.A — *For any monoid $\mathcal{M}$, the binary relation $\preccurlyeq_j$, $j \in [3]$, is a partial order relation on $\mathbf{P}(\mathcal{M})$. Moreover, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, there is exactly one maximal element $\mathfrak{q}$ of the poset $(\mathbf{P}(\mathcal{M}), \preccurlyeq_j)$ such that $\mathfrak{p} \preccurlyeq_j \mathfrak{q}$.*

◀ **Proof** — Let us consider each binary relation $\preccurlyeq_j$, $j \in [3]$ one by one.

(I) For any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, we have $\mathfrak{p} \preccurlyeq_1 \mathfrak{p}'$ if and only if $\mathfrak{p}'$ can be obtained from $\mathfrak{p}$ by deleting some $\mathcal{M}$-pigmented letters whose positions are neither left 1-witnesses nor right 1-witnesses. This implies immediately the properties of the statement of lemma for $\preccurlyeq_1$.

(II) For any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, if $\mathfrak{p} \preccurlyeq_2 \mathfrak{p}'$, then by denoting by $\tau(\mathfrak{p})$ (resp. $\tau(\mathfrak{p}')$) the sum of the positions of $\mathfrak{p}$ (resp. $\mathfrak{p}'$) of the $\mathcal{M}$-pigmented letters which are right 1-witnesses, we have $\tau(\mathfrak{p}) = \tau(\mathfrak{p}') + 1$. Since $\preccurlyeq_2$ is the reflexive and transitive closure of $\rightsquigarrow_2$, this shows that $\preccurlyeq_2$ is antisymmetric. The second property is a consequence of the fact that for any $\mathfrak{p}, \mathfrak{p}', \mathfrak{p}'' \in \mathbf{P}(\mathcal{M})$, if $\mathfrak{p}' \neq \mathfrak{p}''$, $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}'$, and $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}''$, then there exists $\mathfrak{p}''' \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{p}' \rightsquigarrow_2 \mathfrak{p}'''$ and $\mathfrak{p}'' \rightsquigarrow_2 \mathfrak{p}'''$. This property is due to the fact that for any $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$ and $i_1^{\alpha_1}, i_2^{\alpha_2}, i_3^{\alpha_3} \in \mathcal{L}_{\mathcal{M}}$, it is not possible to have both $\mathfrak{r}.\, \underline{i_1^{\alpha_1}}\ \underline{i_2^{\alpha_2}}\ \underline{i_3^{\alpha_3}}\, .\mathfrak{r}' \rightsquigarrow_2 \mathfrak{r}.\, \underline{i_2^{\alpha_2}}\ \underline{i_1^{\alpha_1}}\ \underline{i_3^{\alpha_3}}\, .\mathfrak{r}'$ and $\mathfrak{r}.\, \underline{i_1^{\alpha_1}}\ \underline{i_2^{\alpha_2}}\ \underline{i_3^{\alpha_3}}\, .\mathfrak{r}' \rightsquigarrow_2 \mathfrak{r}.\, \underline{i_1^{\alpha_1}}\ \underline{i_3^{\alpha_3}}\ \underline{i_2^{\alpha_2}}\, .\mathfrak{r}'$. Indeed, these two properties would lead to the fact that the position $\ell(\mathfrak{r}) + 2$ of $\mathfrak{r}.i_1^{\alpha_1} i_2^{\alpha_2} i_3^{\alpha_3}.\mathfrak{r}'$ is a right 1-witness and, at the same time, is not a right 1-witness.

(III) For any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, we have $\mathfrak{p} \preccurlyeq_3 \mathfrak{p}'$ if and only if $\mathfrak{p}'$ can be obtained from $\mathfrak{p}$ by deleting some $\mathcal{M}$-pigmented letters which have a same $\mathcal{M}$-pigmented letter as neighbor. In the same way as the first case, this implies immediately the properties of the statement of lemma for $\preccurlyeq_3$.

□

Let, for any $j \in [3]$, $\downarrow_j : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map such that for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p}\downarrow_j$ is the maximal element of the poset $(\mathbf{P}(\mathcal{M}), \preccurlyeq_j)$ comparable with $\mathfrak{p}$. By Lemma 5.1.3.A, this map is well-defined.

Let $\mathbb{P}_{\equiv} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map defined for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ by

$$\mathbb{P}_{\equiv}(\mathfrak{p}) := \mathfrak{p}\downarrow_1\downarrow_2\downarrow_3 . \tag{5.1.3.A}$$

For instance, we have

$$\mathbb{P}_{\equiv}\left(2^{\epsilon}\, 1^{b}\, 2^{\epsilon}\, 3^{a}\, 1^{ba}\, 1^{b}\, 3^{\epsilon}\right) = 2^{\epsilon}\, 1^{b}\, 2^{\epsilon}\, 3^{a}\, 1^{ba}\, 1^{b}\, 3^{\epsilon} \downarrow_1 \downarrow_2 \downarrow_3$$
$$= 2^{\epsilon}\, 1^{b}\, 2^{\epsilon}\, 3^{a}\, 1^{b}\, 3^{\epsilon} \downarrow_2 \downarrow_3$$
$$= 2^{\epsilon}\, 2^{\epsilon}\, 1^{b}\, 1^{b}\, 3^{a}\, 3^{\epsilon} \downarrow_3$$
$$= 2^{\epsilon}\, 1^{b}\, 3^{a}\, 3^{\epsilon} \tag{5.1.3.B}$$

and

$$\mathbb{P}_{\equiv}\left(4^{a}\, 2^{b}\, 1^{c}\, 1^{c}\, 4^{b}\, 3^{b}\, 3^{a}\, 2^{a}\, 2^{a}\, 4^{a}\, 2^{c}\, 4^{a}\, 2^{c}\right)$$
$$= 4^{a}\, 2^{b}\, 1^{c}\, 1^{c}\, 4^{b}\, 3^{b}\, 3^{a}\, 2^{a}\, 2^{a}\, 4^{a}\, 2^{c}\, 4^{a}\, 2^{c} \downarrow_1 \downarrow_2 \downarrow_3$$
$$= 4^{a}\, 2^{b}\, 1^{c}\, 1^{c}\, 3^{b}\, 3^{a}\, 4^{a}\, 2^{c} \downarrow_2 \downarrow_3$$
$$= 4^{a}\, 2^{b}\, 1^{c}\, 1^{c}\, 3^{b}\, 3^{a}\, 4^{a}\, 2^{c} \downarrow_3$$
$$= 4^{a}\, 2^{b}\, 1^{c}\, 3^{b}\, 3^{a}\, 4^{a}\, 2^{c}. \tag{5.1.3.C}$$

Let us emphasize the fact that the maps $\downarrow_1$, $\downarrow_2$, and $\downarrow_3$ do not commute. Indeed, we have for instance

$$\mathbb{P}_{\equiv}\left(1^{\epsilon}\, 1^{a}\, 1^{\epsilon}\right) = 1^{\epsilon} \quad \neq \quad 1^{\epsilon}\, 1^{\epsilon} = 1^{\epsilon}\, 1^{a}\, 1^{\epsilon} \downarrow_2 \downarrow_3 \downarrow_1 . \tag{5.1.3.D}$$

▶ **Lemma** 5.1.3.B — *For any monoid $\mathcal{M}$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \sim \mathbb{P}_{\equiv}(\mathfrak{p})$.*

◀ **Proof** — First, since for any $j \in [3]$, $\sim$ contains $\preccurlyeq_j$, we have $\mathfrak{p} \downarrow_j \sim \mathfrak{p}$. Moreover, as $\mathbb{P}_{\equiv}$ is by definition the map composition $\downarrow_3 \circ \downarrow_2 \circ \downarrow_1$, $\mathbb{P}_{\equiv}(\mathfrak{p})$ is $\sim$-equivalent to $\mathfrak{p}$. □

▶ **Lemma** 5.1.3.C — *For any monoid $\mathcal{M}$ and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \sim \mathfrak{p}'$ implies $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$.*

◀ **Proof** — Let us consider the following three cases depending whether $\mathfrak{p} \rightsquigarrow_1 \mathfrak{p}'$, $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}'$, or $\mathfrak{p} \rightsquigarrow_3 \mathfrak{p}'$.

(I) Assume that $\mathfrak{p} \rightsquigarrow_1 \mathfrak{p}'$. By Lemma 5.1.3.A, $\mathfrak{p} \downarrow_1 = \mathfrak{p}' \downarrow_1$. Therefore, by definition of $\mathbb{P}_{\equiv}$, $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$.

(II) Assume that $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}'$. Hence, $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{q}.\, i_1^{\alpha_1}\, i_2^{\alpha_2} .\mathfrak{r}$ and $\mathfrak{p}' = \mathfrak{q}.\, i_2^{\alpha_2}\, i_1^{\alpha_1} .\mathfrak{r}$ where $\mathfrak{q}, \mathfrak{r} \in \mathbf{P}(\mathcal{M})$, $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$, and $i_1 \neq i_2$. If the letter at position $\ell(\mathfrak{q}) + 1$ of $\mathfrak{p}$ is not a left 1-witness, then the letter at position $\ell(\mathfrak{q}) + 2$ of $\mathfrak{p}'$ is not a left 1-witness and

$$\mathfrak{p} \downarrow_1 = (\mathfrak{q} \downarrow_1).\, i_2^{\alpha_2} .(\mathfrak{r} \downarrow_1) = \mathfrak{p}' \downarrow_1 . \tag{5.1.3.E}$$

Therefore, $\mathfrak{p} \downarrow_1 = \mathfrak{p}' \downarrow_1$ and $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$. Otherwise, when the letter at position $\ell(\mathfrak{q}) + 1$ of $\mathfrak{p}$ is a left 1-witness, the letter at position $\ell(\mathfrak{q}) + 2$ of $\mathfrak{p}'$ is also a left 1-witness and we have

$$\mathfrak{p} \downarrow_1 = (\mathfrak{q} \downarrow_1).\, i_1^{\alpha_1}\, i_2^{\alpha_2} .(\mathfrak{r} \downarrow_1) \rightsquigarrow_2 (\mathfrak{q} \downarrow_1).\, i_2^{\alpha_2}\, i_1^{\alpha_1} .(\mathfrak{r} \downarrow_1) = \mathfrak{p}' \downarrow_1 . \tag{5.1.3.F}$$

By Lemma 5.1.3.A, $\mathfrak{p} \downarrow_1 \downarrow_2 = \mathfrak{p}' \downarrow_1 \downarrow_2$. Therefore, by definition of $\mathbb{P}_{\equiv}$, $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$.

(III) Assume that $\mathfrak{p} \rightsquigarrow_3 \mathfrak{p}'$. Hence, $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{q}.\, i^{\alpha}\, i^{\alpha} .\mathfrak{r}$ and $\mathfrak{p}' = \mathfrak{q}.\, i^{\alpha} .\mathfrak{r}$ where $\mathfrak{q}, \mathfrak{r} \in \mathbf{P}(\mathcal{M})$ and $i^{\alpha} \in \mathcal{L}_{\mathcal{M}}$. If the letters at positions $\ell(\mathfrak{q}) + 1$ and $\ell(\mathfrak{q}) + 2$ of $\mathfrak{p}$ are neither left 1-witnesses nor right 1-witnesses, then the letter at position $\ell(\mathfrak{q}) + 1$ of $\mathfrak{p}'$ is neither a left 1-witnesses nor a right 1-witness and

$$\mathfrak{p} \downarrow_1 = (\mathfrak{q} \downarrow_1).(\mathfrak{r} \downarrow_1) = \mathfrak{p}' \downarrow_1 . \tag{5.1.3.G}$$

Therefore, $\mathfrak{p}\downarrow_1 = \mathfrak{p}'\downarrow_1$ and $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$. Otherwise, if there is exactly one position among $\ell(\mathfrak{q})+1$ and $\ell(\mathfrak{q})+2$ of $\mathfrak{p}$ which is neither a left 1-witness nor a right 1-witness, then, since the letter at position $\ell(\mathfrak{q})+1$ of $\mathfrak{p}$ cannot be a right 1-witness, we have

$$\mathfrak{p}\downarrow_1 = (\mathfrak{q}\downarrow_1).\, \underline{i^\alpha} \,.(\mathfrak{r}\downarrow_1) = \mathfrak{p}'\downarrow_1 . \tag{5.1.3.H}$$

Therefore, $\mathfrak{p}\downarrow_1 = \mathfrak{p}'\downarrow_1$ and $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$. The last possibility happens when the letter at position $\ell(\mathfrak{q})+1$ of $\mathfrak{p}$ is a left 1-witness and the letter at position $\ell(\mathfrak{q})+2$ of $\mathfrak{p}$ is a right 1-witness. In this case,

$$\begin{aligned} \mathfrak{p}\downarrow_1\downarrow_2 &= (\mathfrak{q}\downarrow_1\downarrow_2).\, i^\alpha\, i^\alpha .(\mathfrak{r}\downarrow_1\downarrow_2) \\ &\rightsquigarrow_3 (\mathfrak{q}\downarrow_1\downarrow_2).\, i^\alpha .(\mathfrak{r}\downarrow_1\downarrow_2) = \mathfrak{p}'\downarrow_1\downarrow_2 . \end{aligned} \tag{5.1.3.I}$$

By Lemma 5.1.3.A, $\mathfrak{p}\downarrow_1\downarrow_2\downarrow_3 = \mathfrak{p}'\downarrow_1\downarrow_2\downarrow_3$. Therefore, by definition of $\mathbb{P}_{\equiv}$, $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$. The statement of the lemma is now implied by the fact that $\sim$ is generated by $\rightsquigarrow$. □

By Proposition 5.1.2.C and Lemmas 5.1.3.B and 5.1.3.C, $\mathbb{P}_{\equiv}$ is a $\mathbb{P}$-symbol for $\equiv$.

5.1.4 Realization. An *$\mathcal{M}$-pigmented magnet* (or simply *pigmented magnet* when the context is clear) of arity $n \geqslant 0$ is an $\mathcal{M}$-pigmented word $\mathfrak{p}$ of arity n which is a maximal element at the same time in the posets $(\mathbf{P}(\mathcal{M}), \preccurlyeq_1)$, $(\mathbf{P}(\mathcal{M}), \preccurlyeq_2)$, and $(\mathbf{P}(\mathcal{M}), \preccurlyeq_3)$. For instance,

$$1^{aa}\, 1^{b}\, 2^{ab}\, 1^{b} \quad \text{and} \quad 2^{ba}\, 3^{\epsilon}\, 2^{ab}\, 1^{a}\, 3^{ba} \tag{5.1.4.A}$$

are not A^*-pigmented magnets. In contrast,

$$3^{b}\, 2^{ba}\, 4^{ba}\, 1^{a}\, 2^{ab} \quad \text{and} \quad 2^{bb}\, 1^{a}\, 1^{aa}\, 2^{bb} \tag{5.1.4.B}$$

are A^*-pigmented magnets.

▶ **Lemma 5.1.4.A** — *For any monoid $\mathcal{M}$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathbb{P}_{\equiv}(\mathfrak{p})$ is an $\mathcal{M}$-pigmented magnet.*

◀ **Proof** — Let $\mathfrak{p}_1 := \mathfrak{p}\downarrow_1$, $\mathfrak{p}_2 := \mathfrak{p}_1\downarrow_2$, and $\mathfrak{p}_3 := \mathfrak{p}_2\downarrow_3$. By definition of $\mathbb{P}_{\equiv}$, $\mathfrak{p}_3 = \mathbb{P}_{\equiv}(\mathfrak{p})$. Let us show that $\mathfrak{p}_3$ is a maximal element w.r.t. the partial order relations $\preccurlyeq_1$, $\preccurlyeq_2$, and $\preccurlyeq_3$ at the same time.

(I) By Lemma 5.1.3.A, $\mathfrak{p}_3$ is a maximal element w.r.t. $\preccurlyeq_3$.

(II) Assume by contraction that there is $\mathfrak{q} \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{p}_3 \preccurlyeq_2 \mathfrak{q}$ and $\mathfrak{p}_3 \neq \mathfrak{q}$. Recall that $\mathfrak{p}_3 = \mathfrak{p}_2\downarrow_3$ and that $\mathfrak{p}_2$ is by Lemma 5.1.3.A a maximal element w.r.t. $\preccurlyeq_2$. Therefore, $\mathfrak{p}_2$ admits no decomposition of the form $\mathfrak{p}_2 = \mathfrak{r}.\, i_1^{\alpha_1}\, i_2^{\alpha_2} .\mathfrak{r}'$ where $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$, $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$, and $i_1 \neq i_2$. Since $\mathfrak{p}_3$ is obtained from $\mathfrak{p}_2$ by deleting some $\mathcal{M}$-pigmented letters having as neighbor a same $\mathcal{M}$-pigmented letter, $\mathfrak{p}_3$ still does not admit any similar decomposition. This contradicts the existence of $\mathfrak{q}$ and shows that $\mathfrak{p}_3$ is a maximal element w.r.t. $\preccurlyeq_2$.

(III) Assume by contraction that there is $\mathfrak{q} \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{p}_3 \preccurlyeq_1 \mathfrak{q}$ and $\mathfrak{p}_3 \neq \mathfrak{q}$. Recall that $\mathfrak{p}_3 = \mathfrak{p}_1\downarrow_2\downarrow_3$ and that $\mathfrak{p}_1$ is by Lemma 5.1.3.A a maximal element w.r.t. $\preccurlyeq_1$. Therefore, $\mathfrak{p}_1$ admits no decomposition of the form $\mathfrak{p}_1 = \mathfrak{r}.\, i^\alpha .\mathfrak{r}'$ where $\mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$ and $i^\alpha \in \mathcal{L}_{\mathcal{M}}$. Since $\mathfrak{p}_2$ is obtained from $\mathfrak{p}_1$ by swapping some $\mathcal{M}$-pigmented letters, $\mathfrak{p}_2$ still does not admit any similar decomposition. Moreover, since $\mathfrak{p}_3$ is obtained from $\mathfrak{p}_2$ by deleting some $\mathcal{M}$-pigmented letters having as neighbor a same $\mathcal{M}$-pigmented letter, $\mathfrak{p}_3$ still does not admit any similar decomposition. This contradicts the existence of $\mathfrak{q}$ and shows that $\mathfrak{p}_3$ is a maximal element w.r.t. $\preccurlyeq_1$.

□

▶ **Theorem** 5.1.4.B — *For any monoid $\mathcal{M}$, $\mathbb{P}_\equiv$ is a $\mathbb{P}$-symbol for $\equiv$ and $\mathbb{P}_\equiv(\mathbf{P}(\mathcal{M}))$ is the set of $\mathcal{M}$-pigmented magnets. Moreover, the graded set $\mathsf{Magn}_{1,1}(\mathcal{M})$ is isomorphic to the graded set of $\mathcal{M}$-pigmented magnets.*

◀ **Proof** — By Proposition 5.1.2.C and Lemmas 5.1.3.B and 5.1.3.C, $\mathbb{P}_\equiv$ is a $\mathbb{P}$-symbol for $\equiv$. Therefore, $\mathbb{P}_\equiv$ is idempotent, which implies together with Lemma 5.1.4.A that $\mathbb{P}_\equiv(\mathbf{P}(\mathcal{M}))$ is the set of $\mathcal{M}$-pigmented magnets. The last part of the statement is a direct implication of Proposition 4.1.2.A and the fact that $\mathbb{P}_\equiv$ is, as we have just shown, a $\mathbb{P}$-symbol for $\equiv$. □

By Proposition 4.1.2.A and Theorem 5.1.4.B, $\mathsf{Magn}_{1,1}(\mathcal{M})$ can be seen as a clone on $\mathcal{M}$-pigmented magnets with superposition maps satisfying (4.1.2.A). For instance, in $\mathsf{Magn}_{1,1}(A^*)$,

$$\begin{aligned} 1^a 1^b 4^b 3^{ba} 2^b \left[3^b 3^a, 1^\epsilon 1^{ba} 3^\epsilon 2^\epsilon 2^{ab} 3^{ab}, 1^\epsilon 1^a, 2^\epsilon 3^a 3^b 1^a\right] \\ = \mathbb{P}_\equiv\left(3^{ab} 3^{aa} 3^{bb} 3^{ba} 2^b 3^{ba} 3^{bb} 1^{ba} 1^{ba} 1^{baa} 1^b 1^{bba} 3^b 2^b 2^{bab} 3^{bab}\right) \\ = 3^{ab} 2^b 1^{ba} 1^{bba} 2^{bab} 3^{bab}. \end{aligned} \tag{5.1.4.C}$$

Moreover, by Lemma 5.1.2.B and Theorem 5.1.4.B, the map $\phi : \mathsf{Arra}_1(\mathcal{M})^2 \to \mathsf{Magn}_{1,1}(\mathcal{M})$ defined for any $(\mathfrak{p}, \mathfrak{p}') \in \mathsf{Arra}_1(\mathcal{M})^2$ by $\phi((\mathfrak{p}, \mathfrak{p}')) := \mathbb{P}_\equiv(\mathfrak{p}.\mathfrak{p}')$ is a graded set isomorphism. Therefore, when $\mathcal{M}$ is finite, $\mathsf{Magn}_{1,1}(\mathcal{M})$ is combinatorial. Moreover, by (4.3.2.E), for any $n \geqslant 0$,

$$\#\mathsf{Magn}_{1,1}(\mathcal{M})(n) = \sum_{i \in [\![n]\!]} \binom{n}{i} i!^2 (\#\mathcal{M})^{2i}. \tag{5.1.4.D}$$

In particular, the sequence of dimensions of $\mathsf{Magn}_{1,1}(\mathcal{E})$ starts by

$$1, 2, 7, 52, 749, 17686, 614227, 29354312, 1844279257, \tag{5.1.4.E}$$

and forms Sequence **A046662** of [Slo].

5.1.5 Presentation. In order to establish a presentation of $\mathsf{Magn}_{1,1}(\mathcal{M})$, we introduce an alternative description of the clone congruence $\equiv$ through a new equivalence relation $\equiv'$. For this, let us define $\equiv'$ as the equivalence relation on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.\mathfrak{q}.\mathfrak{q}.\mathfrak{p}' \equiv' \mathfrak{p}.\mathfrak{q}.\mathfrak{p}', \tag{5.1.5.A}$$

$$\mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.(\alpha_2 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}' \equiv' \mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}', \tag{5.1.5.B}$$

where $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$.

▶ **Lemma** 5.1.5.A — *For any monoid $\mathcal{M}$, the binary relations $\equiv$ and $\equiv'$ on $\mathbf{P}(\mathcal{M})$ are equal.*

◀ **Proof** — Let $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{p} \equiv' \mathfrak{p}'$. Since $\equiv'$ is generated by (5.1.5.A) and (5.1.5.B), we have to cases to consider.

(I) If $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{p}''.\mathfrak{q}.\mathfrak{q}.\mathfrak{p}'''$ and $\mathfrak{p}' = \mathfrak{p}''.\mathfrak{q}.\mathfrak{p}'''$ where $\mathfrak{p}'', \mathfrak{p}''' \in \mathbf{P}(\mathcal{M})$, then $\mathsf{first}_1(\mathfrak{p}) = \mathsf{first}_1(\mathfrak{p}''.\mathfrak{q}.\mathfrak{p}''') = \mathsf{first}_1(\mathfrak{p}')$ and $\mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}) = \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}''.\mathfrak{q}.\mathfrak{p}''') = \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}')$. Therefore, $\mathfrak{p} \equiv \mathfrak{p}'$.

(II) If $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{p}''.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.(\alpha_2 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}'''$ and $\mathfrak{p}' = \mathfrak{p}''.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}'''$ where $\mathfrak{p}'', \mathfrak{p}''', \mathfrak{q}, \mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$, then $\mathsf{first}_1(\mathfrak{p}) = \mathsf{first}_1(\mathfrak{p}''.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.\mathfrak{r}'.\mathfrak{p}''') = \mathsf{first}_1(\mathfrak{p}')$ and $\mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}) = \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}''.\mathfrak{r}.\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}''') = \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}')$. Therefore, $\mathfrak{p} \equiv \mathfrak{p}'$.

This shows that $\mathfrak{p} \equiv' \mathfrak{p}'$ implies $\mathfrak{p} \equiv \mathfrak{p}'$.

Conversely, let $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ such that $\mathfrak{p} \equiv \mathfrak{p}'$. By Proposition 5.1.2.C, this is equivalent to the fact that $\mathfrak{p} \sim \mathfrak{p}'$. Since $\sim$ is generated by $\rightsquigarrow$, we have three cases to explore depending whether $\mathfrak{p} \rightsquigarrow_1 \mathfrak{p}'$, $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}'$, or $\mathfrak{p} \rightsquigarrow_3 \mathfrak{p}'$.

(I) If $\mathfrak{p} \rightsquigarrow_1 \mathfrak{p}'$, then $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{q}.\underline{i^{\alpha}}.\mathfrak{q}'$ and $\mathfrak{p}' = \mathfrak{q}.\mathfrak{q}'$ where $\mathfrak{q}, \mathfrak{q}' \in \mathbf{P}(\mathcal{M})$ and $i^{\alpha} \in \mathcal{L}_{\mathcal{M}}$. Since the position $\ell(\mathfrak{q})+1$ of $\mathfrak{p}$ is neither a left 1-witness nor a right 1-witness, there is necessarily an occurrence of an $\mathcal{M}$-pigmented letter having i as value both in $\mathfrak{q}$ and in $\mathfrak{q}'$. Hence, $\mathfrak{p} = \mathfrak{r}.\underline{i^{\alpha_1}}.\mathfrak{r}'.\underline{i^{\alpha}}.\mathfrak{r}''.\underline{i^{\alpha_2}}.\mathfrak{r}'''$ and $\mathfrak{p}' = \mathfrak{r}.\underline{i^{\alpha_1}}.\mathfrak{r}'.\mathfrak{r}''.\underline{i^{\alpha_2}}.\mathfrak{r}'''$ where $\mathfrak{r}, \mathfrak{r}', \mathfrak{r}'', \mathfrak{r}''' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{q} = \mathfrak{r}.i^{\alpha_1}.\mathfrak{r}'$, $\mathfrak{q}' = \mathfrak{r}''.i^{\alpha_2}.\mathfrak{r}'''$, and $i^{\alpha_1}, i^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$. By (5.1.5.B), we have $\mathfrak{p} \equiv' \mathfrak{p}'$.

(II) If $\mathfrak{p} \rightsquigarrow_2 \mathfrak{p}'$, then $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{q}.\underline{i_1^{\alpha_1}}\,\underline{i_2^{\alpha_2}}.\mathfrak{q}'$ and $\mathfrak{p}' = \mathfrak{q}.\underline{i_2^{\alpha_2}}\,\underline{i_1^{\alpha_1}}.\mathfrak{q}'$ where $\mathfrak{q}, \mathfrak{q}' \in \mathbf{P}(\mathcal{M})$, $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$, and $i_1 \neq i_2$. Since the position $\ell(\mathfrak{q})+1$ of $\mathfrak{p}$ is not a right 1-witness and the position $\ell(\mathfrak{q})+2$ of $\mathfrak{p}$ is not a left 1-witness, there is necessarily an occurrence of an $\mathcal{M}$-pigmented letter having i_2 as value in $\mathfrak{q}$ and an occurrence of an $\mathcal{M}$-pigmented letter having i_1 as value in $\mathfrak{q}'$. Hence,

$$\begin{aligned} \mathfrak{p} &= \mathfrak{r}.\underline{i_2^{\beta_2}}.\mathfrak{r}'.\underline{i_1^{\alpha_1}}\,\underline{i_2^{\alpha_2}}.\mathfrak{r}''.\underline{i_i^{\beta_1}}.\mathfrak{r}''' \\ &\equiv' \mathfrak{r}.\underline{i_2^{\beta_2}}.\mathfrak{r}'.\underline{i_2^{\alpha_2}}\,\underline{i_1^{\alpha_1}}\,\underline{i_2^{\alpha_2}}.\mathfrak{r}''.\underline{i_i^{\beta_1}}.\mathfrak{r}''' \\ &\equiv' \mathfrak{r}.\underline{i_2^{\beta_2}}.\mathfrak{r}'.\underline{i_2^{\alpha_2}}\,\underline{i_1^{\alpha_1}}\,\underline{i_2^{\alpha_2}}\,\underline{i_1^{\alpha_1}}.\mathfrak{r}''.\underline{i_i^{\beta_1}}.\mathfrak{r}''' \\ &\equiv' \mathfrak{r}.\underline{i_2^{\beta_2}}.\mathfrak{r}'.\underline{i_2^{\alpha_2}}\,\underline{i_1^{\alpha_1}}.\mathfrak{r}''.\underline{i_i^{\beta_1}}.\mathfrak{r}''' = \mathfrak{p}' \end{aligned} \tag{5.1.5.C}$$

where $\mathfrak{r}, \mathfrak{r}', \mathfrak{r}'', \mathfrak{r}''' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{q} = \mathfrak{r}.i_2^{\beta_2}.\mathfrak{r}'$, $\mathfrak{q}' = \mathfrak{r}''.i_1^{\beta_1}.\mathfrak{r}'''$, and $i_1^{\beta_1}, i_2^{\beta_2} \in \mathcal{L}_{\mathcal{M}}$. The first and second $\equiv'$-equivalences of (5.1.5.C) are consequences of (5.1.5.B) considered from right to left and the third $\equiv'$-equivalence of (5.1.5.C) is a consequence of (5.1.5.A) considered from left to right.

(III) If $\mathfrak{p} \rightsquigarrow_3 \mathfrak{p}'$, then $\mathfrak{p}$ and $\mathfrak{p}'$ decompose as $\mathfrak{p} = \mathfrak{q}.\underline{i^{\alpha}}\,\underline{i^{\alpha}}.\mathfrak{q}'$ and $\mathfrak{p}' = \mathfrak{q}.\underline{i^{\alpha}}.\mathfrak{q}'$ where $\mathfrak{q}, \mathfrak{q}' \in \mathbf{P}(\mathcal{M})$ and $i^{\alpha} \in \mathcal{L}_{\mathcal{M}}$. By (5.1.5.A), we have $\mathfrak{p} \equiv' \mathfrak{p}'$.

This shows that $\mathfrak{p} \equiv \mathfrak{p}'$ implies $\mathfrak{p} \equiv' \mathfrak{p}'$ and establishes the statement of the lemma. □

▸ **Theorem** 5.1.5.B — *For any monoid $\mathcal{M}$, the clone $\mathsf{Magn}_{1,1}(\mathcal{M})$ admits the presentation $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}'_{\mathcal{M}})$ where $\mathfrak{R}'_{\mathcal{M}}$ is the set $\mathfrak{R}_{\mathcal{M}}$ augmented with the $\mathfrak{G}_{\mathcal{M}}$-equations*

$$\mathsf{rc}_{\mathcal{M}}(1^{\mathrm{e}}1^{\mathrm{e}}) \; \mathfrak{R}'_{\mathcal{M}} \; \mathsf{rc}_{\mathcal{M}}(1^{\mathrm{e}}), \tag{5.1.5.D}$$

$$\mathsf{rc}_{\mathcal{M}}(1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3}) \; \mathfrak{R}'_{\mathcal{M}} \; \mathsf{rc}_{\mathcal{M}}(1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3}), \tag{5.1.5.E}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$ and e is the unit of $\mathcal{M}$.

◂ **Proof** — Let $\equiv''$ be the clone congruence of $\mathbf{P}(\mathcal{M})$ generated by

$$1^{\mathrm{e}}1^{\mathrm{e}} \equiv'' 1^{\mathrm{e}}, \tag{5.1.5.F}$$

$$1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3} \equiv'' 1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3} \tag{5.1.5.G}$$

with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$. Let us show that the clone congruences $\equiv$ and $\equiv''$ of $\mathbf{P}(\mathcal{M})$ are equal. This will imply, by using Proposition 3.3.3.C, that $\mathsf{Magn}_{1,1}$ admits the stated presentation.

First, since $\mathsf{first}_1(1^{\mathrm{e}}1^{\mathrm{e}}) = 1^{\mathrm{e}} = \mathsf{first}_1(1^{\mathrm{e}})$ and $\mathsf{first}^{\mathrm{r}}_1(1^{\mathrm{e}}1^{\mathrm{e}}) = 1^{\mathrm{e}} = \mathsf{first}^{\mathrm{r}}_1(1^{\mathrm{e}})$, we have $1^{\mathrm{e}}1^{\mathrm{e}} \equiv 1^{\mathrm{e}}$. Moreover, since for any $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$, $\mathsf{first}_1(1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3}) = 1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}} = \mathsf{first}_1(1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3})$ and $\mathsf{first}^{\mathrm{r}}_1(1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3}) = 2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3} = \mathsf{first}^{\mathrm{r}}_1(1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3})$, we have $1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3} \equiv 1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3}$. This shows that $\equiv''$ is contained into $\equiv$.

To prove that $\equiv$ is contained into $\equiv''$, let us show that $\equiv'$ is contained into $\equiv''$. By Lemma 5.1.5.A, the targeted property will follow. For any $\mathfrak{p}, \mathfrak{p}', \mathfrak{q} \in \mathbf{P}(\mathcal{M})$, we have

$$\mathfrak{p}.\mathfrak{q}.\mathfrak{q}.\mathfrak{p} = 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 1^{\mathrm{e}}1^{\mathrm{e}}[\mathfrak{q}], \mathfrak{p}'] \equiv'' 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 1^{\mathrm{e}}[\mathfrak{q}], \mathfrak{p}'] = \mathfrak{p}.\mathfrak{q}.\mathfrak{p} \tag{5.1.5.H}$$

so that the first and the last members of (5.1.5.H) are $\equiv''$-equivalent. Moreover, for any $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$, we have

$$\begin{aligned} \mathfrak{p}.(\alpha_1 \mathbin{\bar{\cdot}} \mathfrak{q}).\mathfrak{r}.(\alpha_2 \mathbin{\bar{\cdot}} \mathfrak{q}).\mathfrak{r}'.(\alpha_3 \mathbin{\bar{\cdot}} \mathfrak{q}).\mathfrak{p}' &= 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3}[\mathfrak{q}, \mathfrak{r}, \mathfrak{r}'], \mathfrak{p}'] \\ &\equiv'' 1^{\mathrm{e}}2^{\mathrm{e}}3^{\mathrm{e}}[\mathfrak{p}, 1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3}[\mathfrak{q}, \mathfrak{r}, \mathfrak{r}'], \mathfrak{p}'] = \mathfrak{p}.(\alpha_1 \mathbin{\bar{\cdot}} \mathfrak{q}).\mathfrak{r}.\mathfrak{r}'.(\alpha_3 \mathbin{\bar{\cdot}} \mathfrak{q}).\mathfrak{p}' \end{aligned} \tag{5.1.5.I}$$

so that the first and the last members of (5.1.5.I) are $\equiv''$-equivalent. Since $\equiv'$ is the equivalence relation generated by (5.1.5.A) and (5.1.5.B), the targeted property is shown. This establishes the statement of the theorem. □

By Theorem 5.1.5.B, any $\mathsf{Magn}_{1,1}(\mathcal{M})$-algebra is an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ where $\star$ is idempotent, and $\star$ and p_α satisfy

$$\mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\alpha_2}(x_1) \star x_3 \star \mathrm{p}_{\alpha_3}(x_1) = \mathrm{p}_{\alpha_1}(x_1) \star x_2 \star x_3 \star \mathrm{p}_{\alpha_3}(x_1) \tag{5.1.5.J}$$

for any $x_1, x_2, x_3 \in \mathcal{A}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}$. In particular, $\mathsf{Magn}_{1,1}(\mathcal{E})$ is a clone realization of the variety of regular bands.

5.2 On pigmented stalactites

By considering the intersection of the clone congruences $\equiv_{\mathsf{sort}}$ and $\equiv_{\mathsf{first}_k}$, $k \geqslant 0$, we construct a quotient clone $\mathsf{Stal}_k(\mathcal{M})$ of $\mathbf{P}(\mathcal{M})$. A realization through new combinatorial objects named $\mathcal{M}$-pigmented stalactites is introduced and a finitely related presentation is described. These results are based on the introduction of a $\mathbb{P}$-symbol for the underlying equivalence relation. The proofs of the results and intermediate lemmas of this section are very similar to the ones of Section 5.1 and are omitted for this reason.

5.2.1 Clone construction. For any $k \geqslant 0$, let $\equiv_k$ be the equivalence relation $\equiv_{\mathsf{sort}} \cap \equiv_{\mathsf{first}_k}$ and

$$\mathsf{Stal}_k(\mathcal{M}) := \mathbf{P}(\mathcal{M})/_{\equiv_k}. \tag{5.2.1.A}$$

By Propositions 4.2.2.A and 4.2.3.A, $\mathsf{Stal}_k(\mathcal{M})$ is a well-defined clone, and $\mathsf{WInc}(\mathcal{M})$ and $\mathsf{Arra}_k(\mathcal{M})$ are both quotients of $\mathsf{Stal}_k(\mathcal{M})$. Since for any $0 \leqslant k \leqslant k'$, $\equiv_{k'}$ is a refinement of $\equiv_k$, $\mathsf{Stal}_k(\mathcal{M})$ is a quotient of $\mathsf{Stal}_{k'}(\mathcal{M})$. Moreover, since $\equiv_0$ and $\equiv_{\mathsf{sort}}$ are the same equivalence relations, $\mathsf{Stal}_0(\mathcal{M})$ is isomorphic to $\mathsf{WInc}(\mathcal{M})$. Besides, the clone $\mathsf{Stal}_k^{\mathrm{r}}(\mathcal{M}) := \mathsf{Stal}_k(\mathcal{M})^{\mathrm{r}}$ is by Proposition 4.2.1.A isomorphic to $\mathsf{Stal}_k(\mathcal{M})$.

5.2.2 Equivalence relation. By definition, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv_k \mathfrak{p}'$ holds if and only $(\mathsf{first}_k(\mathfrak{p}), \mathsf{sort}_{\preccurlyeq}(\mathfrak{p})) = (\mathsf{first}_k(\mathfrak{p}'), \mathsf{sort}_{\preccurlyeq}(\mathfrak{p}'))$ where $\preccurlyeq$ is any total order relation on $\mathcal{M}$.

In order to obtain properties about the clone $\mathsf{Stal}_k(\mathcal{M})$, $k \geqslant 0$, we introduce an alternative equivalence relation $\sim^{(k)}$ for which it appears that it is equal to $\equiv_k$. Let $\rightsquigarrow_1^{(k)}$ and $\rightsquigarrow_2^{(k)}$ be the two binary relations on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.\underset{\times\!-}{i_1^{\alpha_1}}\;\underset{\bullet\!-}{i_2^{\alpha_2}}.\mathfrak{p}' \rightsquigarrow_1^{(k)} \mathfrak{p}.\underset{\bullet\!-}{i_2^{\alpha_2}}\;\underset{\times\!-}{i_1^{\alpha_1}}.\mathfrak{p}' \qquad \text{where } i_1 \neq i_2, \tag{5.2.2.A}$$

$$\mathfrak{p}.\underset{\times\!-}{i_1^{\alpha_1}}\;\underset{\times\!-}{i_2^{\alpha_2}}.\mathfrak{p}' \rightsquigarrow_2^{(k)} \mathfrak{p}.\underset{\times\!-}{i_2^{\alpha_2}}\;\underset{\times\!-}{i_1^{\alpha_1}}.\mathfrak{p}' \qquad \text{where } i_1^{\alpha_1} \neq i_2^{\alpha_2}, \text{ and } i_2^{\alpha_2} \preccurlyeq i_1^{\alpha_1}, \tag{5.2.2.B}$$

where $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$ and $i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$. Note that these definitions depend on k because the properties of being a left k-witness or a right k-witness of the shown pigmented letters in (5.2.2.A) and (5.2.2.B) depend themselves on k. Let $\rightsquigarrow^{(k)}$ be the union $\rightsquigarrow_1^{(k)} \cup \rightsquigarrow_2^{(k)}$ and let also $\sim^{(k)}$ be the reflexive, symmetric, and transitive closure of $\rightsquigarrow^{(k)}$.

▶ **Proposition** 5.2.2.A — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, the binary relations $\equiv_k$ and $\sim^{(k)}$ on $\mathbf{P}(\mathcal{M})$ are equal.*

5.2.3 $\mathbb{P}$-SYMBOL ALGORITHM. With the aim of describing a realization of $\mathsf{Stal}_k(\mathcal{M})$, we propose now a $\mathbb{P}$-symbol for $\equiv_k$. For any $j \in [2]$, let $\preccurlyeq_j^{(k)}$ be the reflexive and transitive closure of $\rightsquigarrow_j^{(k)}$.

▶ **Lemma** 5.2.3.A — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, the binary relation $\preccurlyeq_j^{(k)}$, $j \in [2]$, is a partial order relation on $\mathbf{P}(\mathcal{M})$. Moreover, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, there is exactly one maximal element $\mathfrak{q}$ of the poset $\left(\mathbf{P}(\mathcal{M}), \preccurlyeq_j^{(k)}\right)$ such that $\mathfrak{p} \preccurlyeq_j^{(k)} \mathfrak{q}$.*

Let, for any $j \in [2]$, $\downarrow_j^{(k)} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map such that for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p}\downarrow_j^{(k)}$ is the maximal element of the poset $\left(\mathbf{P}(\mathcal{M}), \preccurlyeq_j^{(k)}\right)$ comparable with $\mathfrak{p}$. By Lemma 5.2.3.A, this map is well-defined.

Let $\mathbb{P}_{\equiv_k} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map defined for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ by

$$\mathbb{P}_{\equiv_k}(\mathfrak{p}) := \mathfrak{p} \downarrow_1^{(k)} \downarrow_2^{(k)} . \tag{5.2.3.A}$$

For instance, we have

$$\begin{aligned} \mathbb{P}_{\equiv_1}\left(\underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\times}{1^b}\ \underset{\times}{1^{ba}}\ \underset{\times}{2^\epsilon}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^\epsilon}\ \underset{\times}{2^a}\ \underset{\bullet}{4^a}\ \underset{\times}{4^b}\right) & = \underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\times}{1^b}\ \underset{\times}{1^{ba}}\ \underset{\times}{2^\epsilon}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^\epsilon}\ \underset{\times}{2^a}\ \underset{\bullet}{4^a}\ \underset{\times}{4^b} \downarrow_1^{(1)} \downarrow_2^{(1)} \\ & = \underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\bullet}{4^a}\ \underset{\times}{1^b}\ \underset{\times}{1^{ba}}\ \underset{\times}{2^\epsilon}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^\epsilon}\ \underset{\times}{2^a}\ \underset{\times}{4^b} \downarrow_2^{(1)} \\ & = \underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\bullet}{4^a}\ \underset{\times}{1^\epsilon}\ \underset{\times}{1^b}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^{ba}}\ \underset{\times}{2^\epsilon}\ \underset{\times}{2^a}\ \underset{\times}{4^b} \end{aligned} \tag{5.2.3.B}$$

and

$$\begin{aligned} \mathbb{P}_{\equiv_2}\left(\underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\bullet}{1^b}\ 1^{ba}\ \underset{\bullet}{2^\epsilon}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^\epsilon}\ \underset{\times}{2^a}\ \underset{\bullet}{4^a}\ \underset{\bullet}{4^b}\right) & = \underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\bullet}{1^b}\ 1^{ba}\ \underset{\bullet}{2^\epsilon}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^\epsilon}\ \underset{\times}{2^a}\ \underset{\bullet}{4^a}\ \underset{\bullet}{4^b} \downarrow_1^{(2)} \downarrow_2^{(2)} \\ & = \underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\bullet}{1^b}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{4^a}\ \underset{\bullet}{4^b}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^\epsilon}\ \underset{\times}{2^a} \downarrow_2^{(2)} \\ & = \underset{\bullet}{3^a}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{1^a}\ \underset{\bullet}{1^b}\ \underset{\bullet}{2^\epsilon}\ \underset{\bullet}{4^a}\ \underset{\bullet}{4^b}\ \underset{\times}{1^\epsilon}\ \underset{\times}{1^{ba}}\ \underset{\times}{1^{ba}}\ \underset{\times}{2^a} . \end{aligned} \tag{5.2.3.C}$$

▶ **Lemma** 5.2.3.B — *For any monoid $\mathcal{M}$, any $k \geqslant 0$, and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \sim^{(k)} \mathbb{P}_{\equiv_k}(\mathfrak{p})$.*

▶ **Lemma** 5.2.3.C — *For any monoid $\mathcal{M}$, any $k \geqslant 0$, and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \sim^{(k)} \mathfrak{p}'$ implies $\mathbb{P}_{\equiv_k}(\mathfrak{p}) = \mathbb{P}_{\equiv_k}(\mathfrak{p}')$.*

By Proposition 5.2.2.A and Lemmas 5.2.3.B and 5.2.3.C, $\mathbb{P}_{\equiv_k}$ is a $\mathbb{P}$-symbol for $\equiv_k$.

5.2.4 REALIZATION. An *$\mathcal{M}$-pigmented k-stalactite* (or simply *pigmented k-stalactite* when the context is clear) of arity $n \geqslant 0$ is an $\mathcal{M}$-pigmented word $\mathfrak{p}$ of arity n which is a maximal element at the same time in the posets $\left(\mathbf{P}(\mathcal{M}), \preccurlyeq_1^{(k)}\right)$ and $\left(\mathbf{P}(\mathcal{M}), \preccurlyeq_2^{(k)}\right)$. For instance,

$$\underset{\bullet}{3^b}\ \underset{\bullet}{2^a}\ \underset{\bullet}{2^a}\ \underset{\times}{2^b}\ \underset{\bullet}{3^b}\ \underset{\bullet}{1^a}\ \underset{\times}{3^b}\ \underset{\bullet}{1^b}\ \underset{\times}{3^a} \tag{5.2.4.A}$$

is not an A^*-pigmented 2-stalactite. In contrast,

$$\underset{\bullet}{2^b}\ \underset{\bullet}{2^a}\ \underset{\bullet}{1^a}\ \underset{\bullet}{3^a}\ \underset{\bullet}{4^b}\ \underset{\bullet}{1^a}\ \underset{\bullet}{3^b}\ \underset{\times}{1^a}\ \underset{\times}{2^b}\ \underset{\times}{2^b}\ \underset{\times}{3^a}\ \underset{\times}{3^b} \tag{5.2.4.B}$$

is an A^*-pigmented 2-stalactite but not an A^*-pigmented 1-stalactite.

▶ **Lemma** 5.2.4.A — *For any monoid $\mathcal{M}$, any $k \geqslant 0$, and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathbb{P}_{\equiv_k}(\mathfrak{p})$ is an $\mathcal{M}$-pigmented k-stalactite.*

▶ **Theorem** 5.2.4.B — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, $\mathbb{P}_{\equiv_k}$ is a $\mathbb{P}$-symbol for $\equiv_k$ and $\mathbb{P}_{\equiv_k}(\mathbf{P}(\mathcal{M}))$ is the set of $\mathcal{M}$-pigmented k-stalactites. Moreover, the graded set $\mathsf{Stal}_k(\mathcal{M})$ is isomorphic to the graded set of $\mathcal{M}$-pigmented k-stalactites.*

By Proposition 4.1.2.A and Theorem 5.2.4.B, $\mathsf{Stal}_k(\mathcal{M})$ can be seen as a clone on $\mathcal{M}$-pigmented k-stalactites with superposition maps satisfying (4.1.2.A). For instance, in $\mathsf{Stal}_1(A^*)$,

$$\begin{aligned}4^{ab}1^a2^{ab}3^a3^\epsilon&\left[2^{ba}3^b, 3^{ba}1^b1^b3^\epsilon, 2^\epsilon 3^{ab}2^{ba}3^b, 2^a\right]\\&= \mathbb{P}_{\equiv_1}\left(2^{aba}2^{aba}3^{ab}3^{abba}1^{abb}1^{abb}3^{ab}2^a3^{aab}2^{aba}3^{ab}2^\epsilon 3^{ab}2^{ba}3^b\right)\\&= 2^{aba}3^{ab}1^{abb}1^{abb}2^\epsilon 2^a2^{aba}2^{aba}2^{ba}3^{aab}3^{ab}3^{ab}3^{ab}3^{abba}3^b\end{aligned} \tag{5.2.4.C}$$

and in $\mathsf{Stal}_2(A^*)$,

$$\begin{aligned}3^a2^a1^b3^{ba}3^\epsilon&\left[2^a1^{ab}, 3^b3^\epsilon 2^{ab}, 1^{ba}3^b, 1^a1^{ab}\right]\\&= \mathbb{P}_{\equiv_2}\left(1^{aba}3^{ab}3^{ab}3^a2^{aab}2^{ba}1^{bab}1^{baba}3^{bab}1^{ba}3^b\right)\\&= 1^{aba}3^{ab}3^{ab}2^{aab}2^{ba}1^{bab}1^{ba}1^{baba}3^a3^b3^{bab}.\end{aligned} \tag{5.2.4.D}$$

5.2.5 Presentation. In order to establish a presentation of $\mathsf{Stal}_k(\mathcal{M})$, we introduce an alternative description of the clone congruence $\equiv_k$ through a new equivalence relation $\equiv'_k$. For this, let us define $\equiv'_k$ as the equivalence relation on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.(\alpha_1 \bar{\cdot} \mathfrak{q}).\ \ldots\ .(\alpha_k \bar{\cdot} \mathfrak{q}).(\beta \bar{\cdot} \mathfrak{q}).\mathfrak{r}.\mathfrak{p}' \equiv'_k \mathfrak{p}.(\alpha_1 \bar{\cdot} \mathfrak{q}).\ \ldots\ .(\alpha_k \bar{\cdot} \mathfrak{q}).\mathfrak{r}.(\beta \bar{\cdot} \mathfrak{q}).\mathfrak{p}', \tag{5.2.5.A}$$

where $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{r} \in \mathbf{P}(\mathcal{M})$ and $\alpha_1, \ldots, \alpha_k, \beta \in \mathcal{M}$.

▶ **Lemma** 5.2.5.A — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, the binary relations $\equiv_k$ and $\equiv'_k$ on $\mathbf{P}(\mathcal{M})$ are equal.*

▶ **Theorem** 5.2.5.B — *For any monoid $\mathcal{M}$ and any $k \geqslant 0$, the clone $\mathsf{Stal}_k(\mathcal{M})$ admits the presentation $(\mathfrak{G}_\mathcal{M}, \mathfrak{R}'_\mathcal{M})$ where $\mathfrak{R}'_\mathcal{M}$ is the set $\mathfrak{R}_\mathcal{M}$ augmented with the $\mathfrak{G}_\mathcal{M}$-equation*

$$\mathsf{rc}_\mathcal{M}\left(1^{\alpha_1} \ldots 1^{\alpha_k}1^\beta 2^{\mathrm{e}}\right) \mathfrak{R}'_\mathcal{M} \ \mathsf{rc}_\mathcal{M}\left(1^{\alpha_1} \ldots 1^{\alpha_k}2^{\mathrm{e}}1^\beta\right), \tag{5.2.5.B}$$

where $\alpha_1, \ldots, \alpha_k, \beta \in \mathcal{M}$ and e is the unit of $\mathcal{M}$.

By Theorem 5.2.5.B, any $\mathsf{Stal}_k(\mathcal{M})$-algebra is an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ where $\star$ and p_α satisfy

$$\mathrm{p}_{\alpha_1}(x_1) \star \ldots \star \mathrm{p}_{\alpha_k}(x_1) \star \mathrm{p}_\beta(x_1) \star x_2 = \mathrm{p}_{\alpha_1}(x_1) \star \ldots \star \mathrm{p}_{\alpha_k}(x_1) \star x_2 \star \mathrm{p}_\beta(x_1) \tag{5.2.5.C}$$

for any $x_1, x_2 \in \mathcal{A}$ and $\alpha_1, \ldots, \alpha_k, \beta \in \mathcal{M}$.

As a side remark, the equivalence relation $\equiv_1$, as a monoid congruence, has been introduced in [HNT08] under the name of the "stalactic congruence". As a monoid congruence, $\equiv_k$, $k \geqslant 0$, is therefore a generalization of the previous one.

5.3 On pigmented pillars

By considering the intersection of the clone congruences $\equiv_{\mathsf{sort}}$, $\equiv_{\mathsf{first}_k}$, $k \geqslant 0$, and their reversions $\equiv^{\mathrm{r}}_{\mathsf{first}_{k'}}$, $k' \geqslant 0$, we construct a quotient clone $\mathsf{Pill}_{k,k'}(\mathcal{M})$ of $\mathbf{P}(\mathcal{M})$. This clone is studied in detail for the case $k = 1 = k'$. A realization through new combinatorial objects named $\mathcal{M}$-pigmented pillars is introduced and a finitely related presentation is described. These results are based on the introduction of a $\mathbb{P}$-symbol for the underlying equivalence relation. The proofs of the results and intermediate lemmas of this section are very similar to the ones of Section 5.1 and are omitted for this reason.

5.3.1 CLONE CONSTRUCTION. For any $k, k' \geqslant 0$, let $\equiv_{k,k'}$ be the equivalence relation $\equiv_{\mathsf{first}_k} \cap \equiv_{\mathsf{sort}} \cap \equiv^{\mathrm{r}}_{\mathsf{first}_{k'}}$ and

$$\mathsf{Pill}_{k,k'}(\mathcal{M}) := \mathbf{P}(\mathcal{M})/_{\equiv_{k,k'}}. \tag{5.3.1.A}$$

By Propositions 4.2.2.A, 4.2.3.A, and 4.2.1.A, $\mathsf{Pill}_{k,k'}(\mathcal{M})$ is a well-defined clone, and $\mathsf{Stal}_k(\mathcal{M})$, $\mathsf{Magn}_{k,k'}(\mathcal{M})$, and $\mathsf{Stal}^{\mathrm{r}}_{k'}(\mathcal{M})$ are quotients of $\mathsf{Pill}_{k,k'}$. Since for any $0 \leqslant k \leqslant k''$ and $0 \leqslant k' \leqslant k'''$, $\equiv_{k,k'}$ is a refinement of $\equiv_{k'',k'''}$, $\mathsf{Pill}_{k,k'}(\mathcal{M})$ is a quotient of $\mathsf{Pill}_{k'',k'''}(\mathcal{M})$. Moreover, since $\equiv_{0,0}$ and $\equiv_{\mathsf{sort}}$ are the same equivalence relations, $\mathsf{Pill}_{0,0}(\mathcal{M})$ is isomorphic to $\mathsf{WInc}(\mathcal{M})$. Besides, the clone $\mathsf{Pill}^{\mathrm{r}}_{k,k'}(\mathcal{M}) := \mathsf{Pill}_{k,k'}(\mathcal{M})^{\mathrm{r}}$ is by Proposition 4.2.1.A isomorphic to $\mathsf{Pill}_{k,k'}(\mathcal{M})$. Since the reversion operation on congruences is involutive, the clones $\mathsf{Pill}^{\mathrm{r}}_{k,k'}(\mathcal{M})$ and $\mathsf{Pill}_{k',k}(\mathcal{M})$ are isomorphic.

5.3.2 EQUIVALENCE RELATION. To lighten the notation, we denote by $\equiv$ the equivalence relation $\equiv_{1,1}$ on $\mathbf{P}(\mathcal{M})$. By definition, for any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \equiv \mathfrak{p}'$ holds if and only $(\mathsf{first}_1(\mathfrak{p}), \mathsf{sort}_{\preccurlyeq}(\mathfrak{p}), \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p})) = (\mathsf{first}_1(\mathfrak{p}'), \mathsf{sort}_{\preccurlyeq}(\mathfrak{p}'), \mathsf{first}^{\mathrm{r}}_1(\mathfrak{p}'))$ where $\preccurlyeq$ is any total order relation on $\mathcal{M}$.

In order to obtain properties about the clone $\mathsf{Pill}_{1,1}(\mathcal{M})$, we introduce an alternative equivalence relation $\sim$ for which we will show that it is equal to $\equiv$. Let $\rightsquigarrow_1$, $\rightsquigarrow_2$, and $\rightsquigarrow_3$ be the three binary relations on $\mathbf{P}(\mathcal{M})$ satisfying

$$\mathfrak{p}.\underline{i}^{\alpha_1}.\mathfrak{q}.\underline{i}^{\alpha_2}.\mathfrak{p}' \rightsquigarrow_1 \mathfrak{p}.\underline{i}^{\alpha_1}.\underline{i}^{\alpha_2}.\mathfrak{q}.\mathfrak{p}' \qquad \text{where } \mathfrak{q} \neq \epsilon, \text{ and } i \notin \mathfrak{q}, \tag{5.3.2.A}$$

$$\mathfrak{p}.\underline{i}^{\alpha_1}\,\underline{i}^{\alpha_2}.\mathfrak{p}' \rightsquigarrow_2 \mathfrak{p}.\underline{i}^{\alpha_2}\,\underline{i}^{\alpha_1}.\mathfrak{p}' \qquad \text{where } \alpha_1 \neq \alpha_2 \text{ and } \alpha_2 \prec \alpha_1, \tag{5.3.2.B}$$

$$\mathfrak{p}.\underline{i_1}^{\alpha_1}\,\underline{i_2}^{\alpha_2}.\mathfrak{p}' \rightsquigarrow_3 \mathfrak{p}.\underline{i_2}^{\alpha_2}\,\underline{i_1}^{\alpha_1}.\mathfrak{p}' \qquad \text{where } i_1 \neq i_2, \tag{5.3.2.C}$$

where $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $i^{\alpha_1}, i^{\alpha_2}, i_1^{\alpha_1}, i_2^{\alpha_2} \in \mathcal{L}_{\mathcal{M}}$, and where the notation $i \notin \mathfrak{r}$ means that the $\mathcal{M}$-pigmented word $\mathfrak{r}$ has no occurrence of any $\mathcal{M}$-pigmented letter having i as value. Let $\rightsquigarrow$ be the union $\rightsquigarrow_1 \cup \rightsquigarrow_2 \cup \rightsquigarrow_3$ and let also $\sim$ be the reflexive, symmetric, and transitive closure of $\rightsquigarrow$.

▶ **Proposition** 5.3.2.A — *For any monoid $\mathcal{M}$, the binary relations $\equiv$ and $\sim$ on $\mathbf{P}(\mathcal{M})$ are equal.*

5.3.3 $\mathbb{P}$-SYMBOL ALGORITHM. With the aim of describing a realization of $\mathsf{Pill}_{1,1}(\mathcal{M})$, we propose now a $\mathbb{P}$-symbol for $\equiv$. For any $j \in [3]$, let $\preccurlyeq_j$ be the reflexive and transitive closure of $\rightsquigarrow_j$.

▶ **Lemma** 5.3.3.A — *For any monoid $\mathcal{M}$, the binary relation $\preccurlyeq_j$, $j \in [3]$, is a partial order relation on $\mathbf{P}(\mathcal{M})$. Moreover, for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, there is exactly one maximal element $\mathfrak{q}$ of the poset $(\mathbf{P}(\mathcal{M}), \preccurlyeq_j)$ such that $\mathfrak{p} \preccurlyeq_j \mathfrak{q}$.*

Let, for any $j \in [3]$, $\downarrow_j : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map such that for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p}\downarrow_j$ is the maximal element of the poset $(\mathbf{P}(\mathcal{M}), \preccurlyeq_j)$ comparable with $\mathfrak{p}$. By Lemma 5.3.3.A, this map is well-defined.

Let $\mathbb{P}_{\equiv} : \mathbf{P}(\mathcal{M}) \to \mathbf{P}(\mathcal{M})$ be the map defined for any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$ by

$$\mathbb{P}_{\equiv}(\mathfrak{p}) := \mathfrak{p}\downarrow_1\downarrow_2\downarrow_3. \tag{5.3.3.A}$$

For instance, we have

$$\begin{aligned}\mathbb{P}_{\equiv}\left(\underline{2}^{ab}\,\underline{2}^{a}\,\underline{4}^{b}\,\underline{4}^{b}\,\underline{2}^{\epsilon}\,\underline{4}^{ab}\,\underline{4}^{\epsilon}\,\underline{3}^{a}\,\underline{3}^{a}\,\underline{3}^{ba}\,\underline{2}^{ab}\,\underline{5}^{b}\,\underline{3}^{ab}\right) &= \underline{2}^{ab}\,\underline{2}^{a}\,\underline{4}^{b}\,\underline{4}^{b}\,\underline{2}^{\epsilon}\,\underline{4}^{ab}\,\underline{4}^{\epsilon}\,\underline{3}^{a}\,\underline{3}^{a}\,\underline{3}^{ba}\,\underline{2}^{ab}\,\underline{5}^{b}\,\underline{3}^{ab}\downarrow_1\downarrow_2\downarrow_3 \\ &= \underline{2}^{ab}\,\underline{2}^{a}\,\underline{2}^{\epsilon}\,\underline{4}^{b}\,\underline{4}^{b}\,\underline{4}^{ab}\,\underline{4}^{\epsilon}\,\underline{3}^{a}\,\underline{3}^{a}\,\underline{3}^{ba}\,\underline{2}^{ab}\,\underline{5}^{b}\,\underline{3}^{ab}\downarrow_2\downarrow_3\end{aligned} \tag{5.3.3.B}$$

$$= 2^{ab}\, 2^{\epsilon}\, 2^{a}\, 4^{b}\, 4^{ab}\, 4^{b}\, 4^{\epsilon}\, 3^{a}\, 3^{a}\, 3^{ba}\, 2^{ab}\, 5^{b}\, 3^{ab} \downarrow_3$$
$$= 2^{ab}\, 2^{\epsilon}\, 2^{a}\, 4^{b}\, 4^{ab}\, 4^{b}\, 4^{\epsilon}\, 2^{ab}\, 3^{a}\, 3^{a}\, 3^{ba}\, 5^{b}\, 3^{ab}.$$

▶ **Lemma** 5.3.3.B — *For any monoid $\mathcal{M}$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \sim \mathbb{P}_{\equiv}(\mathfrak{p})$.*

▶ **Lemma** 5.3.3.C — *For any monoid $\mathcal{M}$ and any $\mathfrak{p}, \mathfrak{p}' \in \mathbf{P}(\mathcal{M})$, $\mathfrak{p} \sim \mathfrak{p}'$ implies $\mathbb{P}_{\equiv}(\mathfrak{p}) = \mathbb{P}_{\equiv}(\mathfrak{p}')$.*

By Proposition 5.3.2.A and Lemmas 5.3.3.B and 5.3.3.C, $\mathbb{P}_{\equiv}$ is a $\mathbb{P}$-symbol for $\equiv$.

5.3.4 Realization. An *$\mathcal{M}$-pigmented pillar* (or simply *pigmented pillar* when the context is clear) of arity $n \geqslant 0$ is an $\mathcal{M}$-pigmented word $\mathfrak{p}$ of arity n which is a maximal element at the same time in the posets $(\mathbf{P}(\mathcal{M}), \preccurlyeq_1)$, $(\mathbf{P}(\mathcal{M}), \preccurlyeq_2)$, and $(\mathbf{P}(\mathcal{M}), \preccurlyeq_3)$. For instance,

$$2^{b}\, 1^{a}\, 2^{ba}\, 4^{\epsilon}\, 1^{ba}\, 3^{a}\, 2^{a}\, 4^{b} \quad \text{and} \quad 1^{\epsilon}\, 2^{b}\, 6^{b}\, 6^{a}\, 2^{ba}\, 5^{ab}\, 2^{ab}\, 6^{\epsilon}\, 5^{a}\, 5^{ab} \tag{5.3.4.A}$$

are not A^*-pigmented pillars. In contrast,

$$1^{\epsilon}\, 2^{\epsilon}\, 2^{\epsilon}\, 4^{a}\, 4^{b}\, 3^{ab}\, 4^{\epsilon}\, 1^{\epsilon} \quad \text{and} \quad 4^{b}\, 4^{b}\, 5^{ba}\, 3^{ba}\, 3^{ab}\, 1^{ba}\, 5^{\epsilon}\, 3^{b}\, 6^{b}\, 4^{b} \tag{5.3.4.B}$$

are A^*-pigmented pillars.

▶ **Lemma** 5.3.4.A — *For any monoid $\mathcal{M}$ and any $\mathfrak{p} \in \mathbf{P}(\mathcal{M})$, $\mathbb{P}_{\equiv}(\mathfrak{p})$ is an $\mathcal{M}$-pigmented pillar.*

▶ **Theorem** 5.3.4.B — *For any monoid $\mathcal{M}$, $\mathbb{P}_{\equiv}$ is a $\mathbb{P}$-symbol for $\equiv$ and $\mathbb{P}_{\equiv}(\mathbf{P}(\mathcal{M}))$ is the set of $\mathcal{M}$-pigmented pillars. Moreover, the graded set $\mathsf{Pill}_{1,1}(\mathcal{M})$ is isomorphic to the graded set of $\mathcal{M}$-pigmented pillars.*

By Proposition 4.1.2.A and Theorem 5.3.4.B, $\mathsf{Pill}_{1,1}(\mathcal{M})$ can be seen as a clone on $\mathcal{M}$-pigmented pillars with superposition maps satisfying (4.1.2.A). For instance, in $\mathsf{Pill}_{1,1}(A^*)$,

$$\begin{aligned} & 3^{\epsilon}2^{ab}1^{b}1^{a}4^{a}\left[2^{ba}2^{ba}1^{ab}1^{\epsilon}, 2^{a}3^{a}, 1^{ba}, 3^{ba}3^{a}1^{ab}2^{ab}1^{b}\right] \\ & = \mathbb{P}_{\equiv}\left(1^{ba}2^{aba}3^{aba}2^{bba}2^{bba}1^{bab}1^{b}2^{aba}2^{aba}1^{aab}1^{a}3^{aba}3^{aa}1^{aab}2^{aab}1^{ab}\right) \\ & = 1^{ba}1^{a}1^{aab}1^{aab}1^{b}1^{bab}2^{aba}2^{aba}2^{aba}2^{bba}2^{bba}3^{aba}3^{aba}3^{aa}2^{aab}1^{ab}. \end{aligned} \tag{5.3.4.C}$$

5.3.5 Presentation. In order to establish a presentation of $\mathsf{Pill}_{1,1}(\mathcal{M})$, we introduce an alternative description of the clone congruence $\equiv$ through a new equivalence relation $\equiv'$. For this, let us define $\equiv'$ as the equivalence relation on $\mathbf{P}(\mathcal{M})$ satisfying

$$\begin{aligned} \mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).(\alpha_2 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}' & \equiv' \mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.(\alpha_2 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}'.(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}' \\ & \equiv' \mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}).\mathfrak{r}.\mathfrak{r}'.(\alpha_2 \bar{\cdot}\, \mathfrak{q}).(\alpha_3 \bar{\cdot}\, \mathfrak{q}).\mathfrak{p}', \end{aligned} \tag{5.3.5.A}$$

$$\mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}_1).\mathfrak{r}.(\beta_1 \bar{\cdot}\, \mathfrak{q}_2).(\alpha_2 \bar{\cdot}\, \mathfrak{q}_1).\mathfrak{r}'.(\beta_2 \bar{\cdot}\, \mathfrak{q}_2).\mathfrak{p}' \equiv' \mathfrak{p}.(\alpha_1 \bar{\cdot}\, \mathfrak{q}_1).\mathfrak{r}.(\alpha_2 \bar{\cdot}\, \mathfrak{q}_1).(\beta_1 \bar{\cdot}\, \mathfrak{q}_2).\mathfrak{r}'.(\beta_2 \bar{\cdot}\, \mathfrak{q}_2).\mathfrak{p}', \tag{5.3.5.B}$$

where $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{r}, \mathfrak{r}' \in \mathbf{P}(\mathcal{M})$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mathcal{M}$.

▶ **Lemma** 5.3.5.A — *For any monoid $\mathcal{M}$, the binary relations $\equiv$ and $\equiv'$ on $\mathbf{P}(\mathcal{M})$ are equal.*

▶ **Theorem** 5.3.5.B — *For any monoid $\mathcal{M}$, the clone $\mathsf{Pill}_{1,1}(\mathcal{M})$ admits the presentation $\left(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}'_{\mathcal{M}}\right)$ where $\mathfrak{R}'_{\mathcal{M}}$ is the set $\mathfrak{R}_{\mathcal{M}}$ augmented with the $\mathfrak{G}_{\mathcal{M}}$-equations*

$$\mathsf{rc}_{\mathcal{M}}(1^{\alpha_1}1^{\alpha_2}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_3}) \;\mathfrak{R}'_{\mathcal{M}}\; \mathsf{rc}_{\mathcal{M}}(1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\mathrm{e}}1^{\alpha_3}) \;\mathfrak{R}'_{\mathcal{M}}\; \mathsf{rc}_{\mathcal{M}}(1^{\alpha_1}2^{\mathrm{e}}3^{\mathrm{e}}1^{\alpha_2}1^{\alpha_3}), \tag{5.3.5.C}$$

$$\mathsf{rc}_{\mathcal{M}}\left(1^{\alpha_1}2^{\mathrm{e}}3^{\beta_1}1^{\alpha_2}4^{\mathrm{e}}3^{\beta_2}\right) \;\mathfrak{R}'_{\mathcal{M}}\; \mathsf{rc}_{\mathcal{M}}\left(1^{\alpha_1}2^{\mathrm{e}}1^{\alpha_2}3^{\beta_1}4^{\mathrm{e}}3^{\beta_2}\right), \tag{5.3.5.D}$$

where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mathcal{M}$ and e is the unit of $\mathcal{M}$.

By Theorem 5.3.5.B, any $\mathsf{Pill}_{1,1}(\mathcal{M})$-algebra is an $\mathcal{M}$-pigmented monoid $(\mathcal{A}, \star, \mathrm{u}, \mathrm{p}_\alpha)$ where $\star$ and p_α satisfy

$$\mathrm{p}_{\alpha_1}(x_1) \star \mathrm{p}_{\alpha_2}(x_1) \star x_2 \star x_3 \star \mathrm{p}_{\alpha_3}(x_1) = \mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\alpha_2}(x_1) \star x_3 \star \mathrm{p}_{\alpha_3}(x_1) = \mathrm{p}_{\alpha_1}(x_1) \star x_2 \star x_3 \star \mathrm{p}_{\alpha_2}(x_1) \star \mathrm{p}_{\alpha_3}(x_1), \tag{5.3.5.E}$$

$$\mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\beta_1}(x_3) \star \mathrm{p}_{\alpha_2}(x_1) \star x_4 \star \mathrm{p}_{\beta_2}(x_3) = \mathrm{p}_{\alpha_1}(x_1) \star x_2 \star \mathrm{p}_{\alpha_2}(x_1) \star \mathrm{p}_{\beta_1}(x_3) \star x_4 \star \mathrm{p}_{\beta_2}(x_3), \tag{5.3.5.F}$$

for any $x_1, x_2, x_3, x_4 \in \mathcal{A}$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mathcal{M}$.

6 Open questions and future work

We have introduced the construction $\mathbf{P}$ producing clones from monoids and studied a selection of quotient clones of $\mathbf{P}(\mathcal{M})$. This has resulted in a novel hierarchy of clone realizations of varieties of monoids. Here follow some open questions and future areas of investigation raised by this work.

Variations around the variety of pigmented monoids

As shown by Theorem 3.2.2.A, $\mathbf{P}(\mathcal{M})$ is a clone realization of the variety of $\mathcal{M}$-pigmented monoids. This variety stems from the six relations (3.1.1.B), (3.1.1.C), (3.1.1.D), (3.1.1.E), (3.1.1.F), and (3.1.1.G). A compelling question to consider involves the alternative varieties resulting from the omission of some of these relations, and proposing in this way variations of the construction $\mathbf{P}$ in order to describe the corresponding clone realizations. There are therefore $2^6 - 1 = 63$ such alternative varieties but only $2^3 - 1 = 7$ seem worth to study because these relations are naturally paired as outlined at the end of Section 3.1.1. Indeed, (3.1.1.B) pairs with (3.1.1.C), (3.1.1.D) with (3.1.1.E), and (3.1.1.F) with (3.1.1.G). In particular, in [Gir18] (see also [Gir17; Gir20a]), the variety that arises by omitting the pair consisting of Relations (3.1.1.D) and (3.1.1.E) (except for few detail) has been studied via operads and involves configurations of noncrossing and decorated diagonals in polygons. Such objects recur very frequently in combinatorics [CP92; FN99; DRS10; PR14] and considering clone structures on these objects could give an original point of view and lead to new questions and results in this domain.

Linearization of the construction and relations

The clones examined in this work are defined within the category of sets. It is of course possible to extend the construction $\mathbf{P}$ in order to see the produced clones as clones on the $\mathbb{K}$-linear span of the set of $\mathcal{M}$-pigmented words where $\mathbb{K}$ is any field of zero characteristic. This type of extension opens a myriad of new questions. Among these, the broad question of describing the nontrivial relations satisfied by certain linear combinations of terms of the variety of $\mathcal{M}$-pigmented monoids is worth considering. When translated into the language of clones, this equates to describe the presentations of certain subclones of the linearization of $\mathbf{P}(\mathcal{M})$ which are generated by some linear combinations of $\mathcal{M}$-pigmented words. More specifically, this question can be posed, given $\alpha_1, \alpha_2 \in \mathcal{M}$, for the commutator $1^{\alpha_1} 2^{\alpha_2} - 2^{\alpha_2} 1^{\alpha_1}$ and for the anti-commutator $1^{\alpha_1} 2^{\alpha_2} + 2^{\alpha_2} 1^{\alpha_1}$ in the linearization of $\mathbf{P}(\mathcal{M})$, as well as in the linearizations of some of its quotients constructed in Sections 4.2 and 5. Similar questions have been explored for different varieties of algebras: for instance for the anti-commutator of associative algebras [Gle70], for the commutator and anti-commutator of bicommutative algebras [DI18], and for the anti-commutator of pre-Lie algebras [BL11].

Finitely generated subclones

In the present work, the clone $\mathbf{P}(\mathcal{M})$ is studied along with some of its quotients. A potential next step in this research involves paying attention to subclones of $\mathbf{P}(\mathcal{M})$ and to some of its quotients generated by some finite sets of elements. This approach has been considered in [Gir15] where a construction $\mathbf{T}$ from monoids to operads has been introduced and numerous operads on combinatorial objects have been discovered (on several sorts of words, trees, and paths). Recall, as explained in Section 3.1.1, that the construction $\mathbf{P}$ can be seen as a generalization of the construction $\mathbf{T}$ at the level of clones. In this way, we could expect to develop a hierarchy of clones based on a large collection of sorts of combinatorial objects. As consequences, mainly by describing presentations of such derived clones, it may sometimes be feasible to establish a convergent rewrite system on the terms of the underlying variety. This could lead to new methods for the enumeration of the involved combinatorial objects and for their —exhaustive or random— generation (see [Gir19] and [Gir20b] in the context of operads rather than clones).

Plactic-like monoids and other constructions

As briefly highlighted in Section 4.1.1, many monoids hold a distinctive role in algebraic combinatorics. Examples include the plactic monoid [LS81; Lot02], the hypoplactic monoid [KT97], the sylvester monoid [HNT05], the Bell monoid [Rey07], the Baxter monoid [Gir12], the k-recoil monoid [NRT11], and the stalactic monoid [HNT08]. These monoids can be defined through congruences of free monoids on a totally ordered alphabet. The main observation here is that these monoids intervene in a crucial way to construct Hopf algebras generalizing the prototypical one of symmetric functions [Gel+95] (also refer to the previously cited works and [Gir11, Chap. 5] for a comprehensive description and properties of this construction). A key component here is formed by $\mathbb{P}$-symbols, which —akin to the present work— are maps sending words to some combinatorial objects encoding the equivalence classes. In the context of the present work, we are interested in clone congruences of $\mathbf{P}(\mathcal{M})$, which are in particular also monoid congruences on words on integers. As a matter of fact, most of the previously cited congruences do not define clone congruences of $\mathbf{P}(\mathcal{M})$. Nevertheless, instead of trying to use already existing monoids to propose new clone congruences of $\mathbf{P}(\mathcal{M})$ (which is a possible direction for future work that deserves to be explored), we can proceed in the opposite direction. This consists in trying to build Hopf algebras in the same manner by considering the clone congruences and monoids at the heart of the constructions of $\mathsf{Arra}_k(\mathcal{M})$, $\mathsf{Magn}_{k,k'}(\mathcal{M})$, $\mathsf{Stal}_k(\mathcal{M})$, and $\mathsf{Pill}_{k,k'}(\mathcal{M})$.

General case for pigmented magnets and pigmented pillars

The final question we ask here concerns the clones $\mathsf{Magn}_{k,k'}(\mathcal{M})$ and $\mathsf{Pill}_{k,k'}(\mathcal{M})$. These clone are well understood in the case $k = 1 = k'$. Indeed, both realizations and presentations are furnished for each clone in this case. The question here consists in establishing generalizations of these results working for any nonnegative integers k and k'.

7 References

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