Intervalles de Tamari généralisés et cartes planaires orientées

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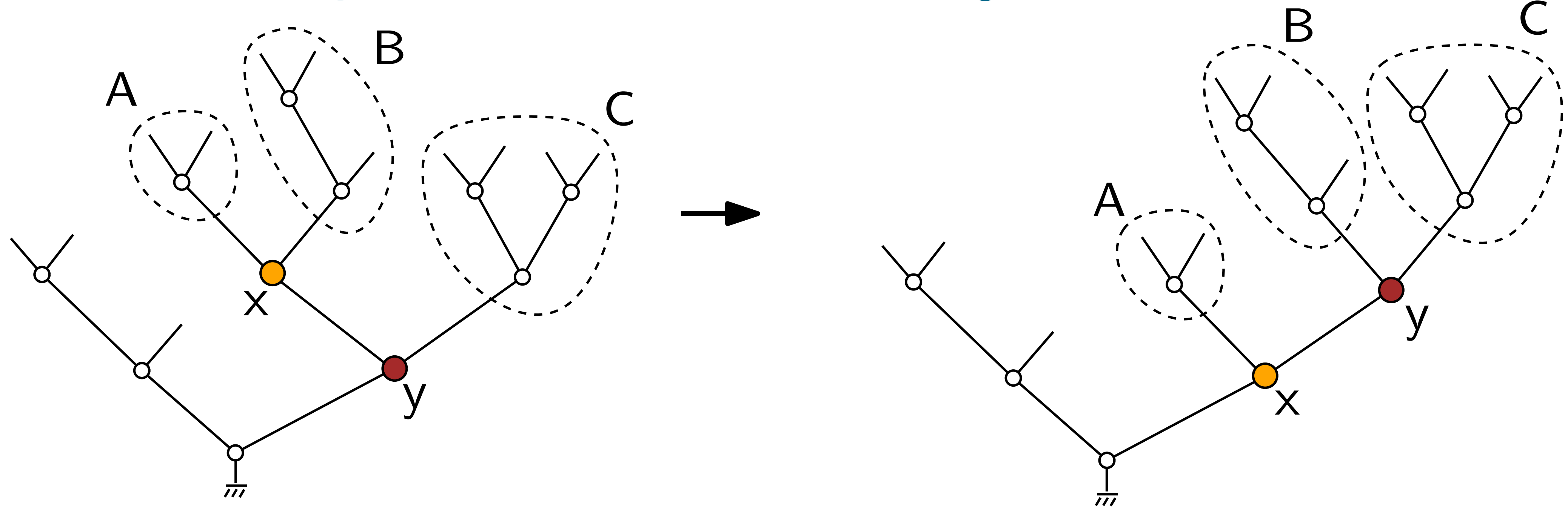
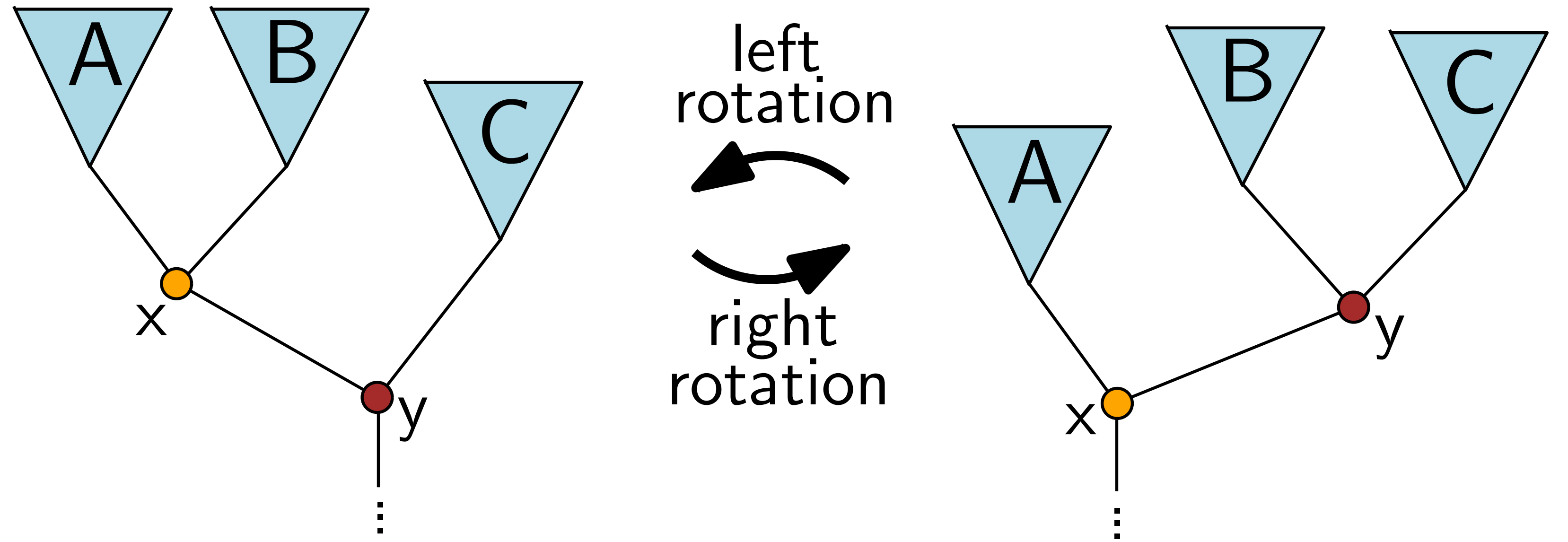
Travaux en commun avec Abel Humbert

Séminaire Flajolet, IHP, 3 octobre 2019

Rotation operations on binary trees

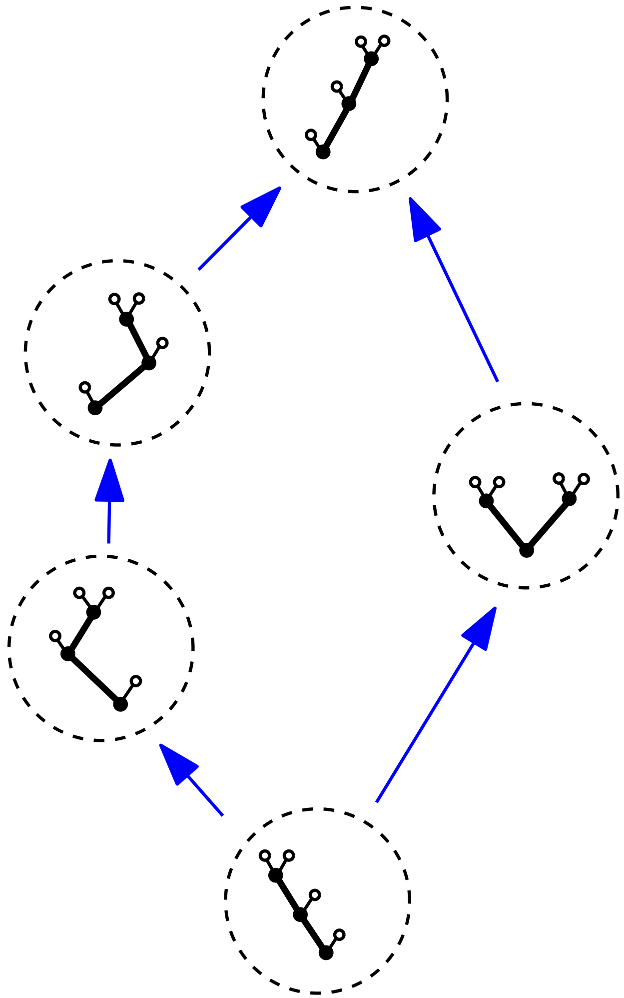




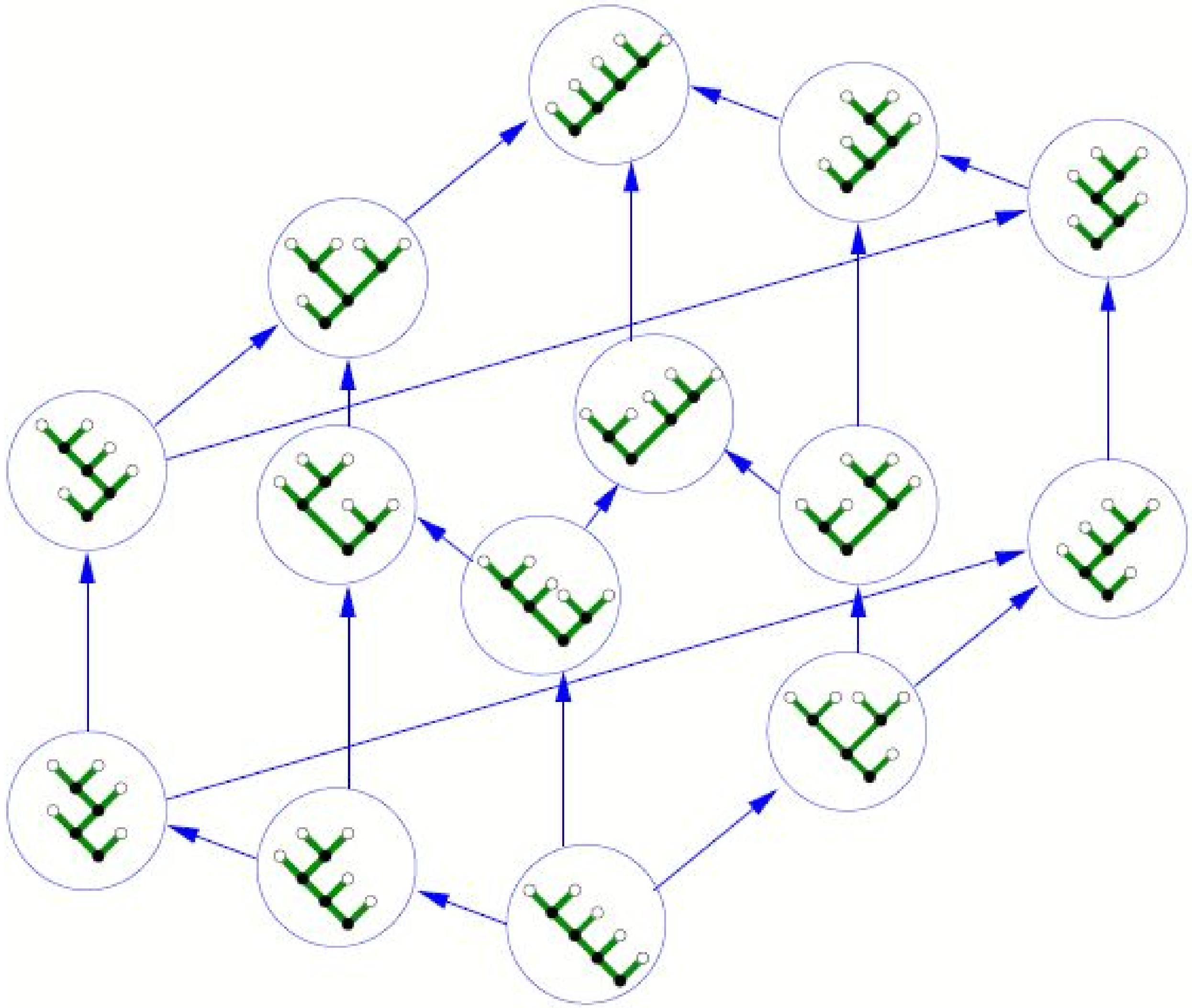
The Tamari lattice

$\mathcal{B}_n :=$ set of binary trees with n nodes

The Tamari lattice Tam_n is the partial order on $\mathcal{B}_n$ where the covering relation corresponds to right rotation

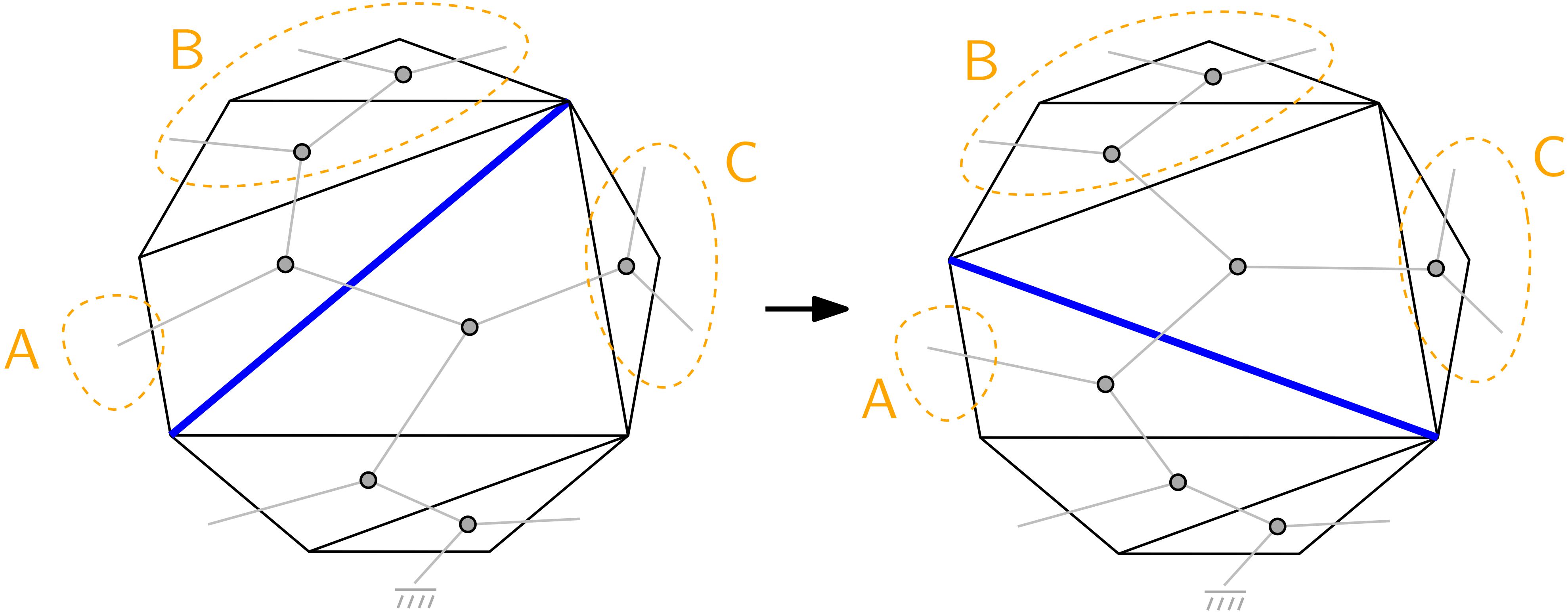


n=3

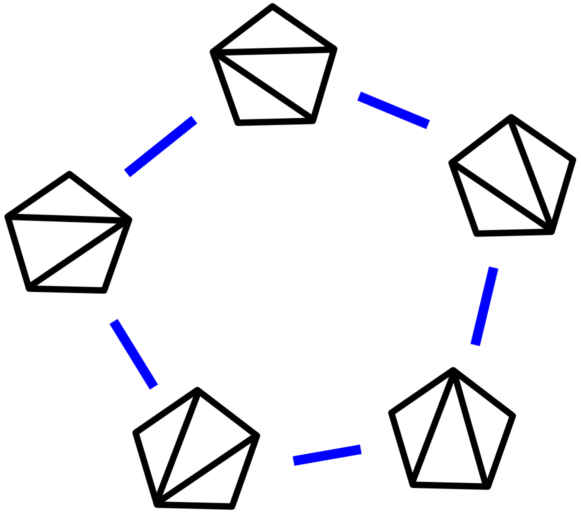


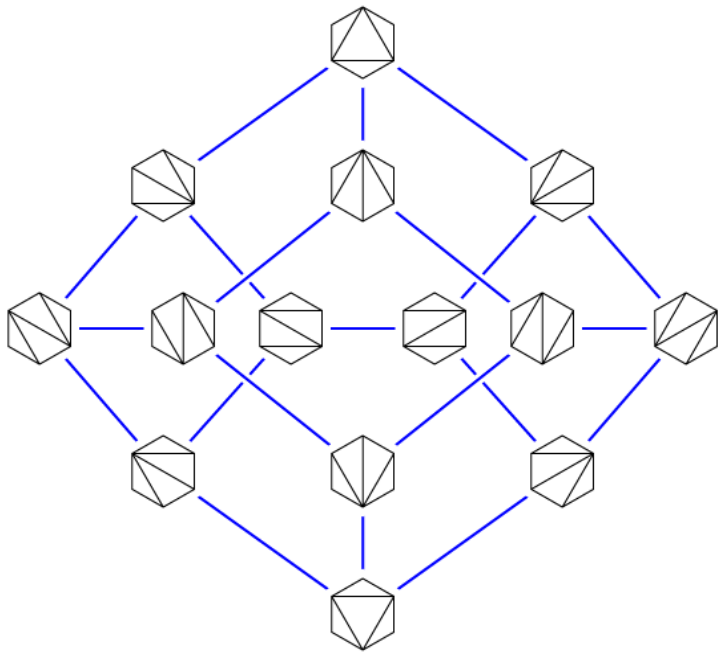
n=4

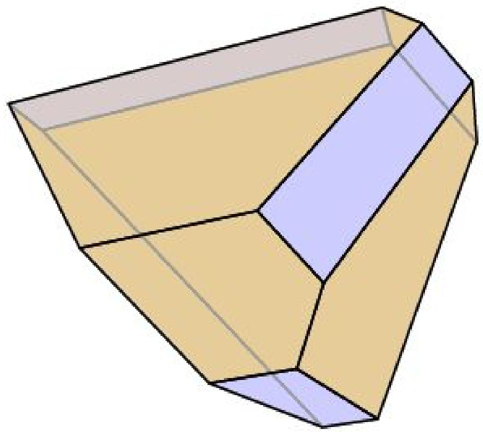
Rotation ⇔ flip on triangulated dissections



cf the associahedron

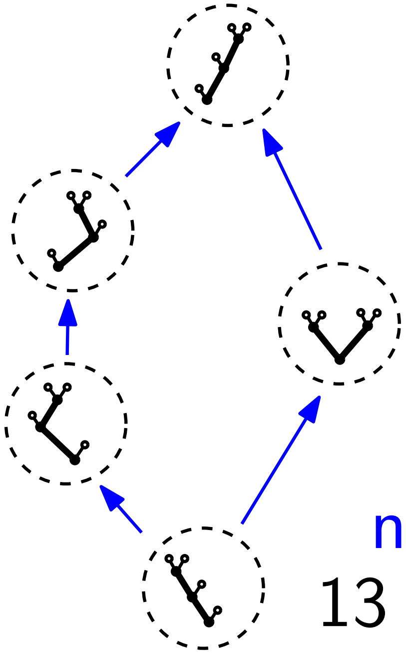






Tamari intervals

An **interval** in a poset $(E, \leq)$ is a pair $x, x' \in E$ such that $x \leq x'$



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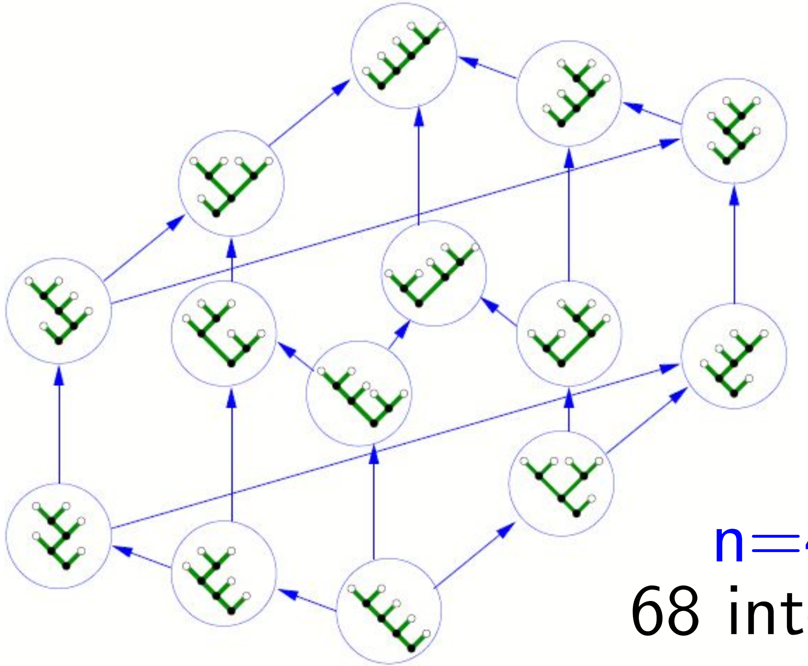
13 intervals



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Let $\mathcal{I}_n$:=set of intervals in Tam_n

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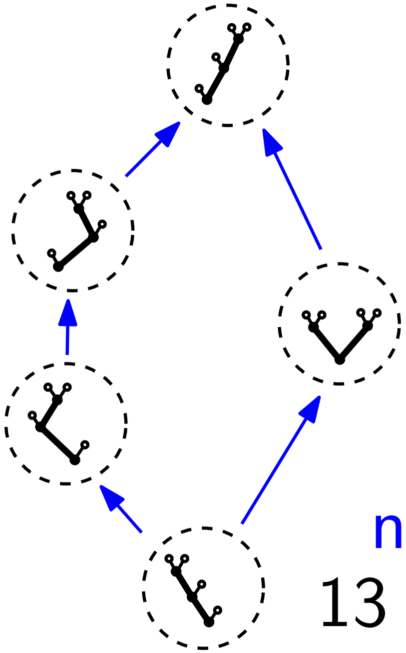

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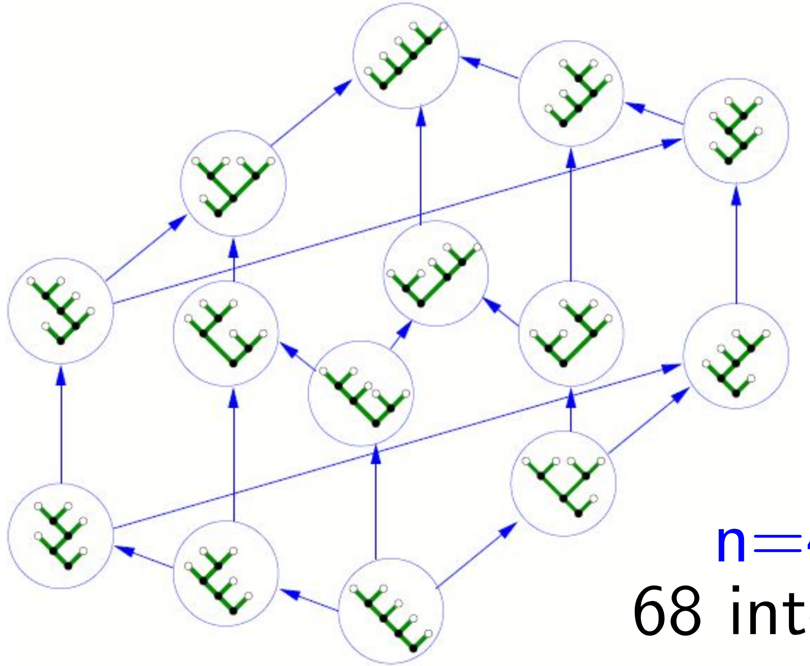

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Let $\mathcal{I}_n$:=set of intervals in Tam_n

Theorem [Chapoton'06]: $|\mathcal{I}_n| = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$

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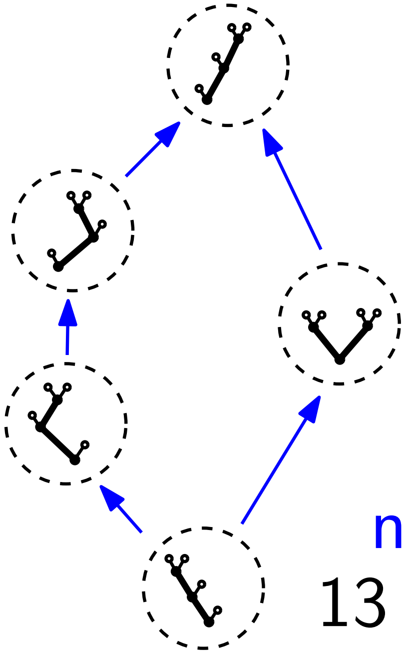

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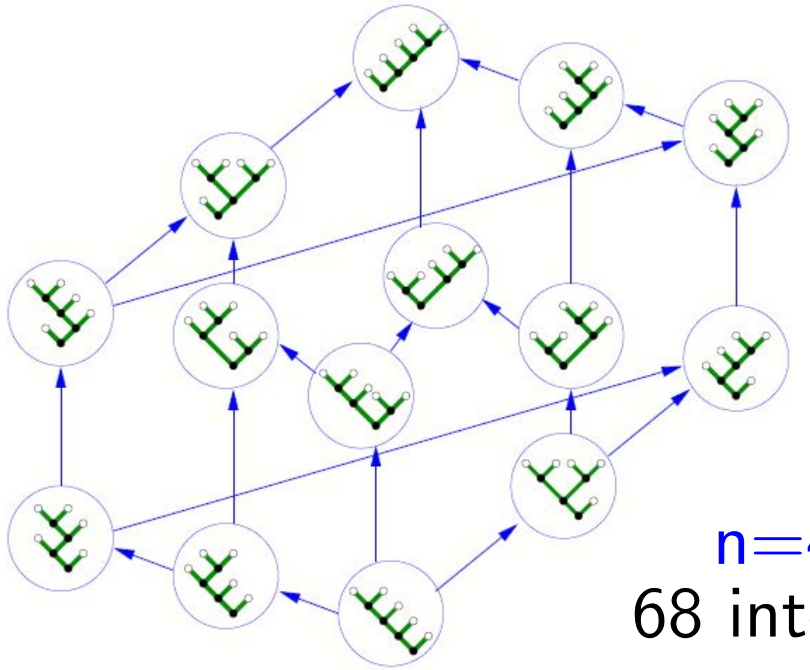

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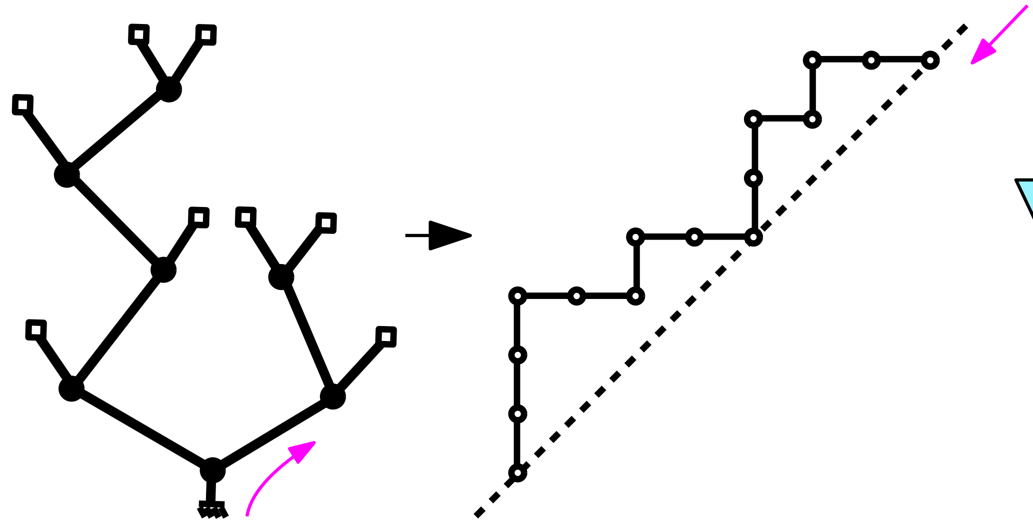
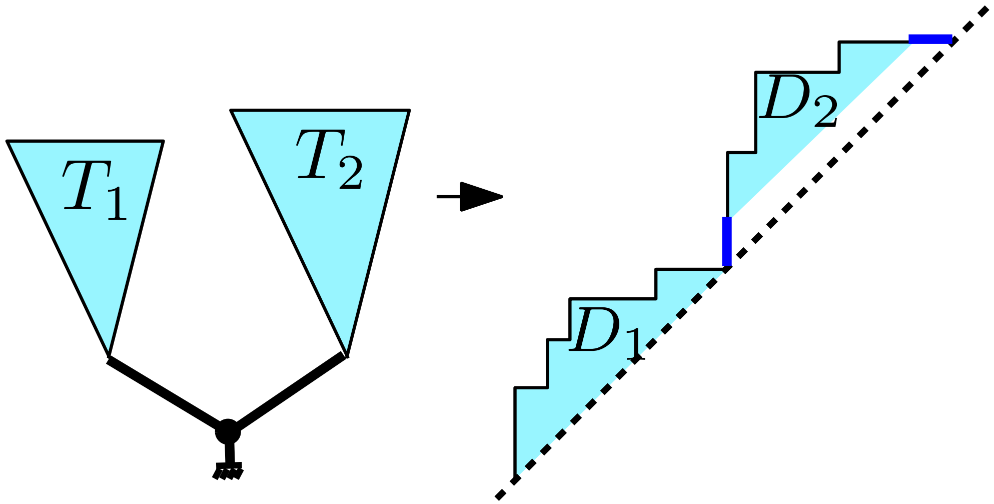
Very active research domain over last 10 years:

- various extensions with nice counting formulas
  - m-Tamari [Bousquet-Mélou,F,Préville-Ratelle'11]
  - labelled m-Tamari [Bousquet-Mélou,Chapuy,Préville-Ratelle'12]
  - v-Tamari [Préville-Ratelle, Viennot'14]
- connections to algebra [Bergeron, Préville-Ratelle'11]
- bijective links: planar maps [Bernardi,Bonichon'07] [Fang, Préville-Ratelle'16]
  interval posets [Chatel,Pons'13]

The covering relation for Dyck walks

- Encoding by left-to-right postfix order (⇔ right-to-left prefix order)





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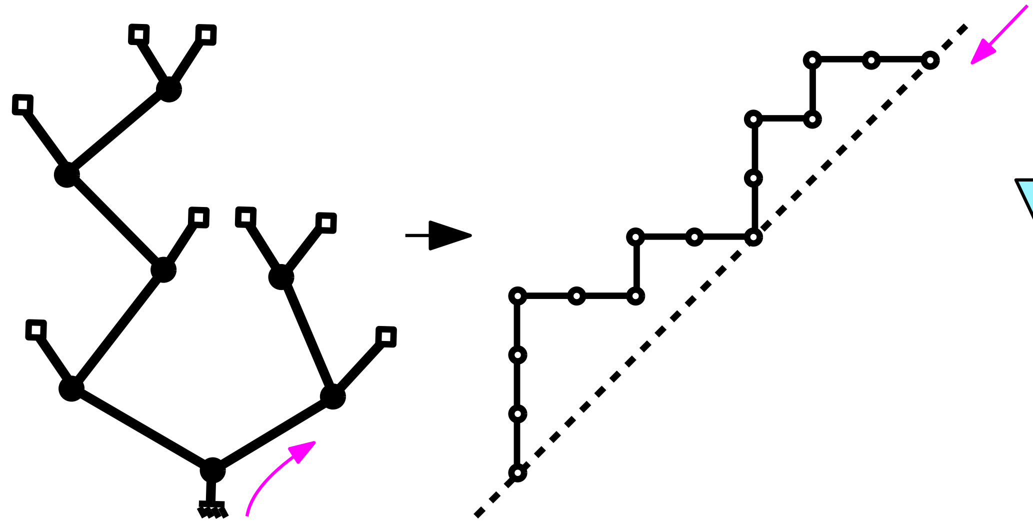
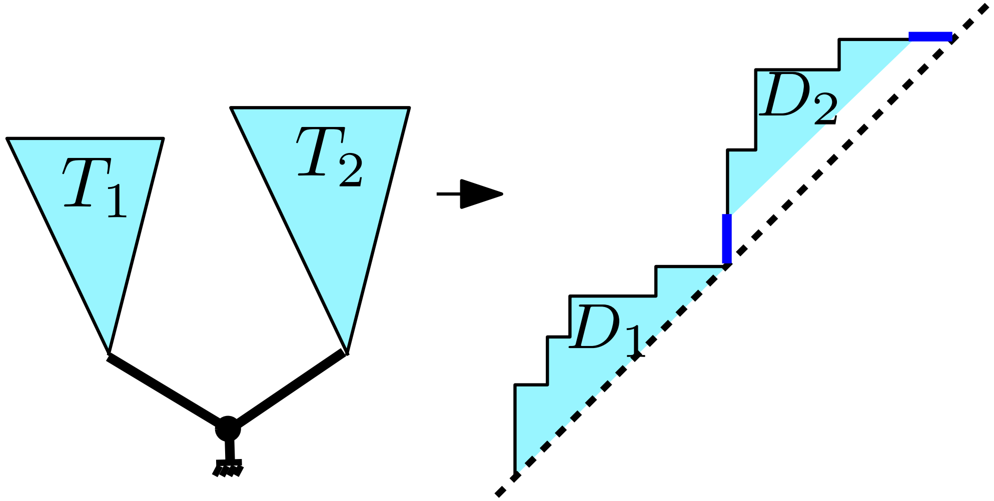
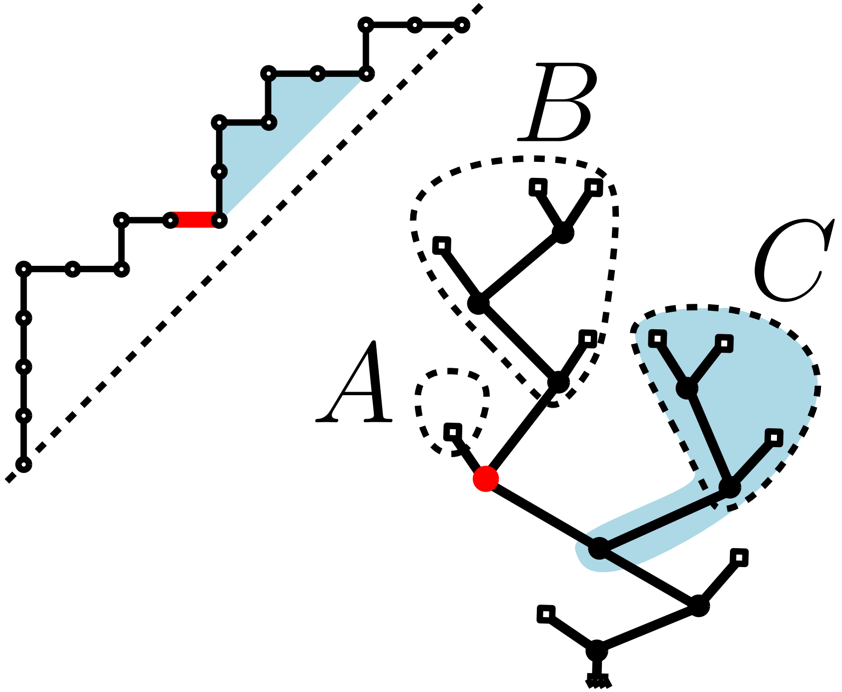




- Effect of a rotation on the associated Dyck walk:



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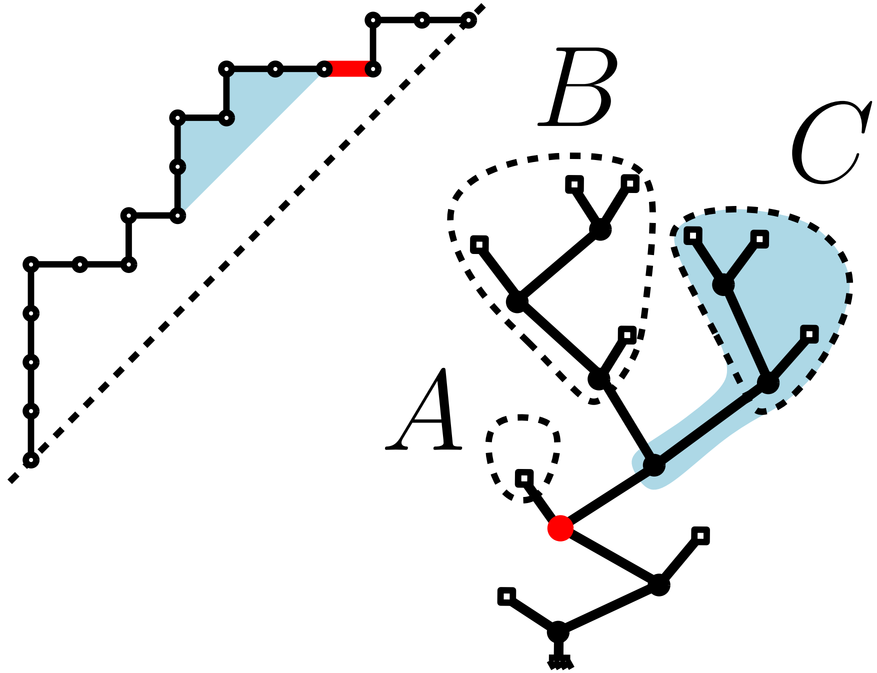

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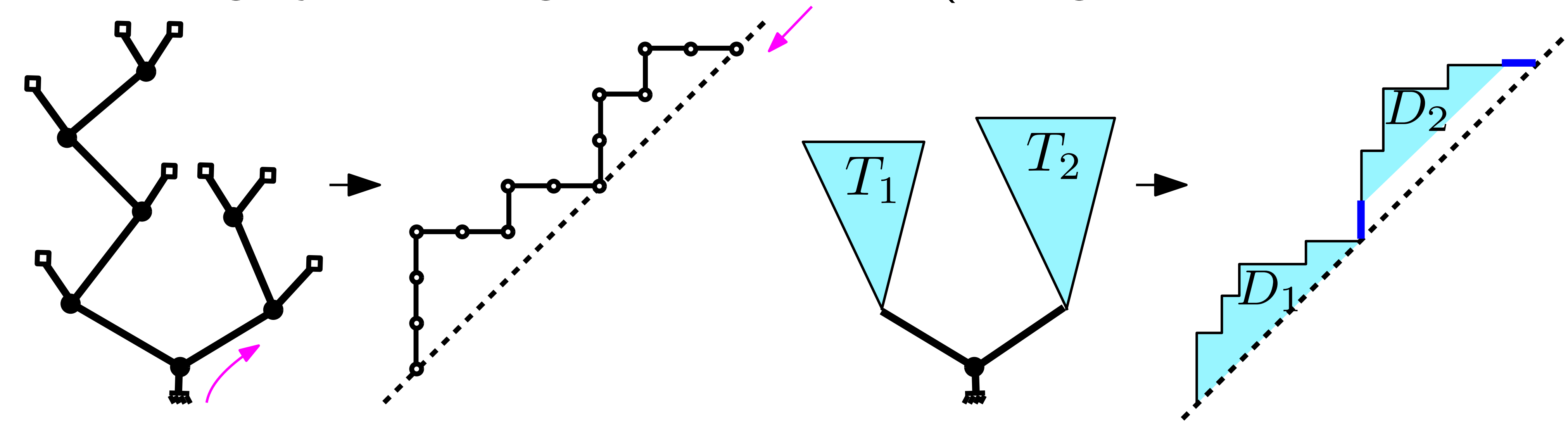
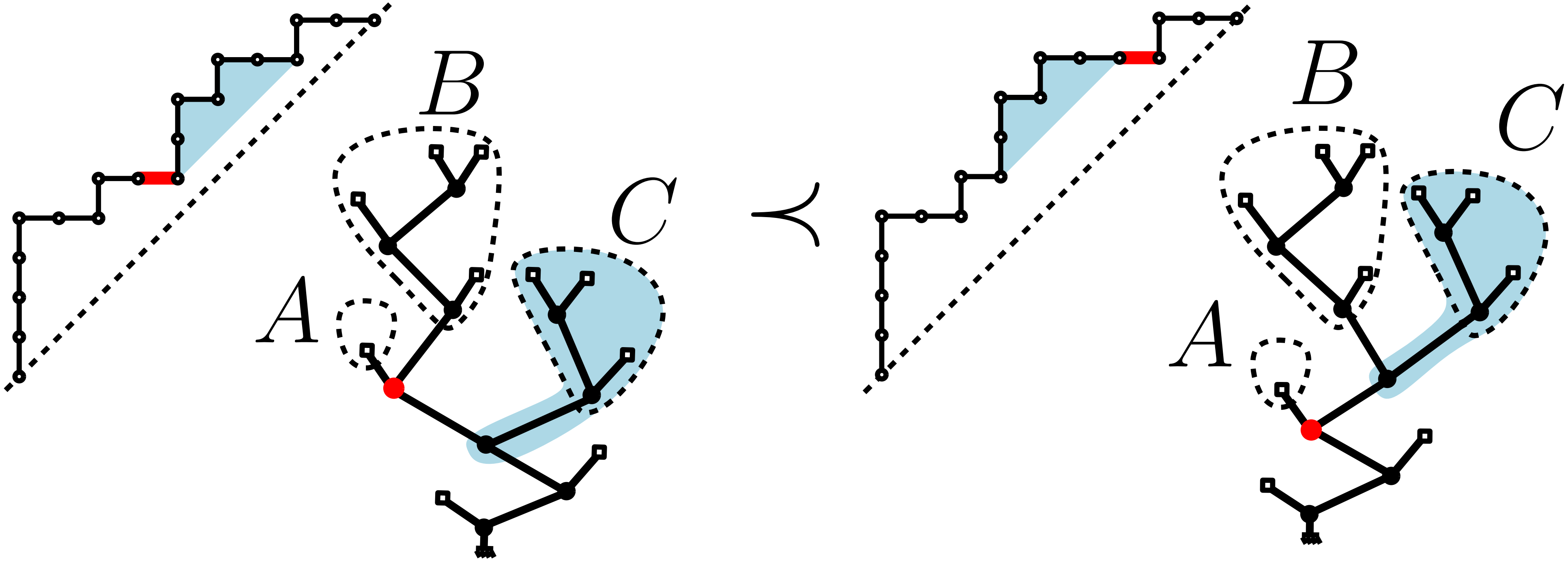


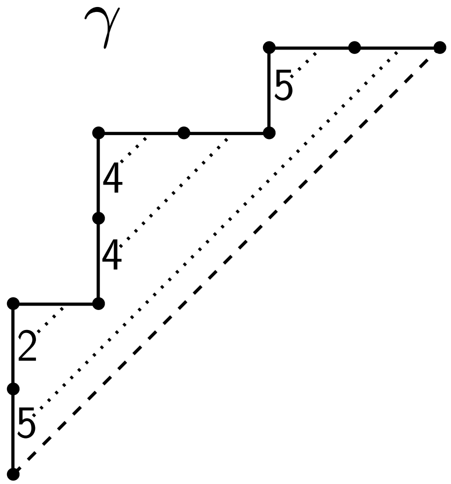
- Effect of a rotation on the associated Dyck walk:



Rk: If $\gamma \leq \gamma'$ in Tam_n then γ is below γ'

Bracket-vectors

Bracket-vector of a Dyck walk

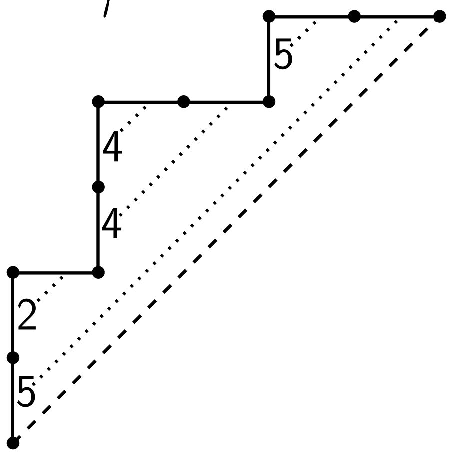


$$V(\gamma) = (5, 2, 4, 4, 5)$$

Bracket-vectors

Bracket-vector of a Dyck walk

γ



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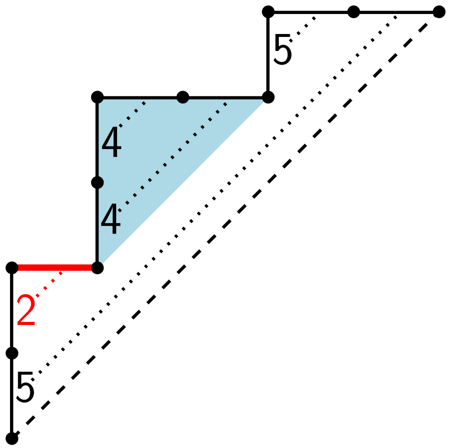

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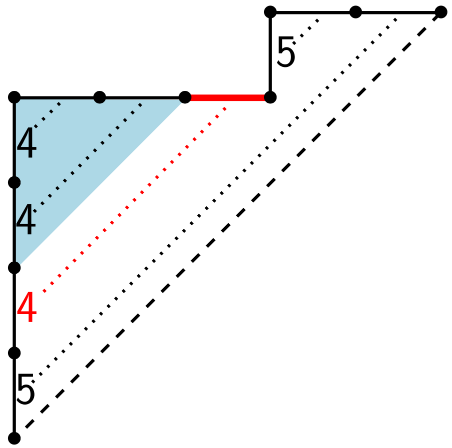
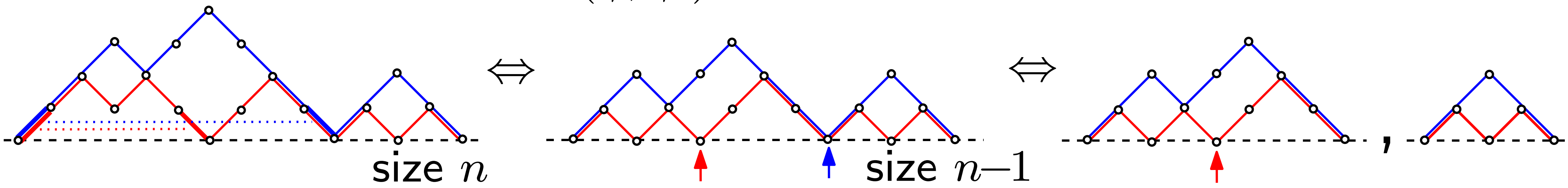


$(5, 2, 4, 4, 5)$ $(5, 4, 4, 4, 5)$

Property: $\gamma \leq \gamma'$ in Tam_n iff $V(\gamma) \leq V(\gamma')$ **[Huang,Tamari'72]**

Recursive decomposition of intervals [Chapoton'06]

- Reduction of an interval $(\gamma, \gamma') \in \mathcal{I}_n$:



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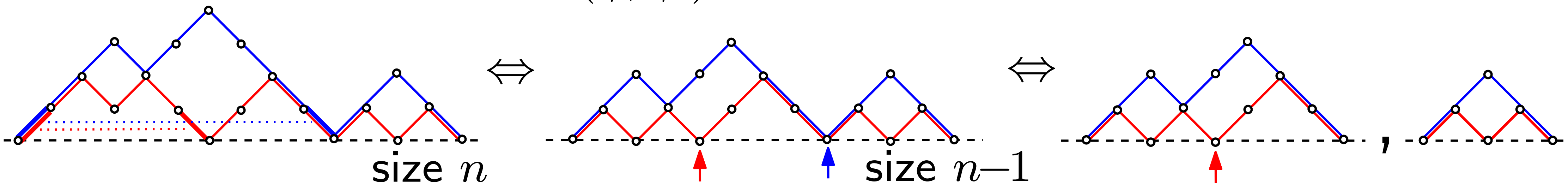
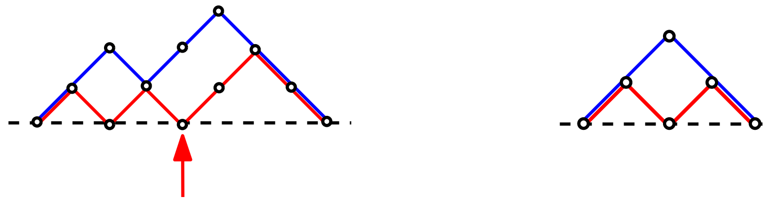


Let $F(t,u)$ the GF, with $t \leftrightarrow$ size and $u \leftrightarrow$ #(bottom-contacts)

(**Rk:** $|\mathcal{I}_n| = [t^n]F(t,1)$)

Then: $$F(t,u) = u + t \cdot u\frac{F(t,u) - F(t,1)}{u-1} \cdot F(t,u)$$



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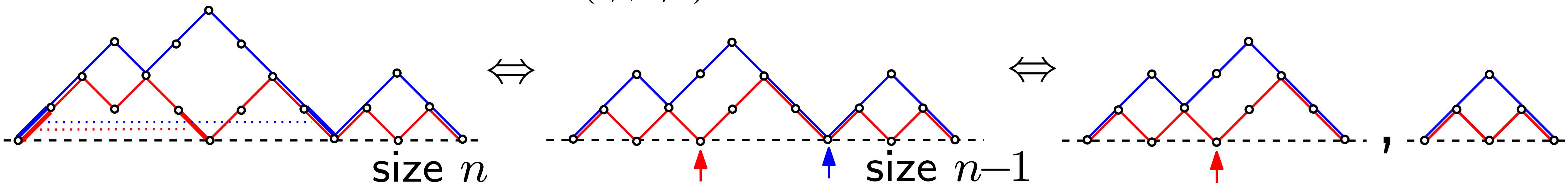

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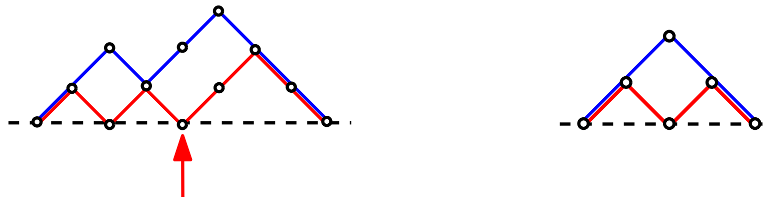
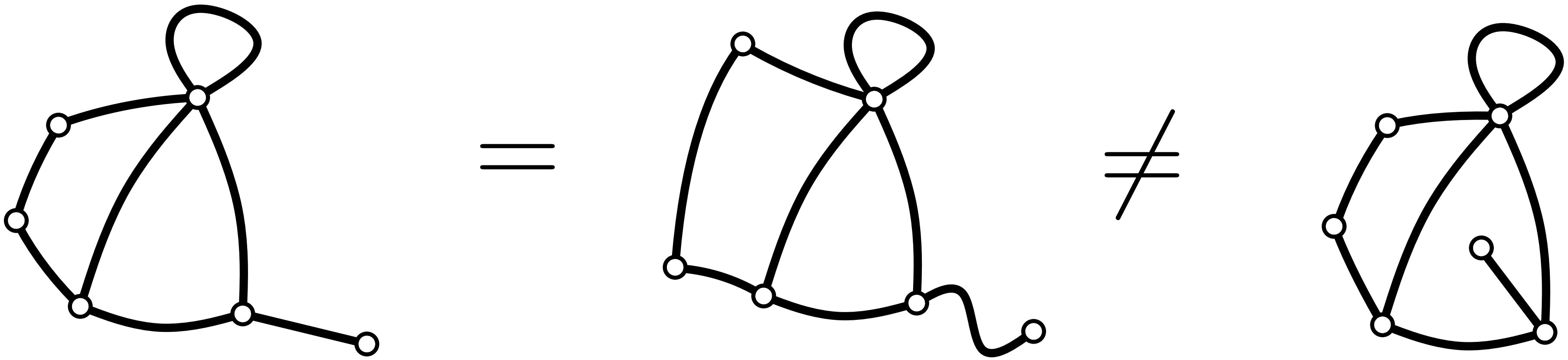


Quadratic method (or guessing-checking) gives $\boxed{[t^n]F(t,1) = \frac{2}{n(n+1)}\binom{4n+1}{n-1}}$

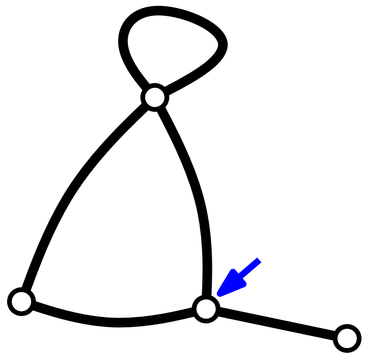
[Brown, Tutte, Bousquet-Mélou Jehanne'06]

Planar maps, triangulations

Def. Planar map = connected graph embedded in the plane up to isotopy



rooted map = map + marked corner
with the outer face on its left



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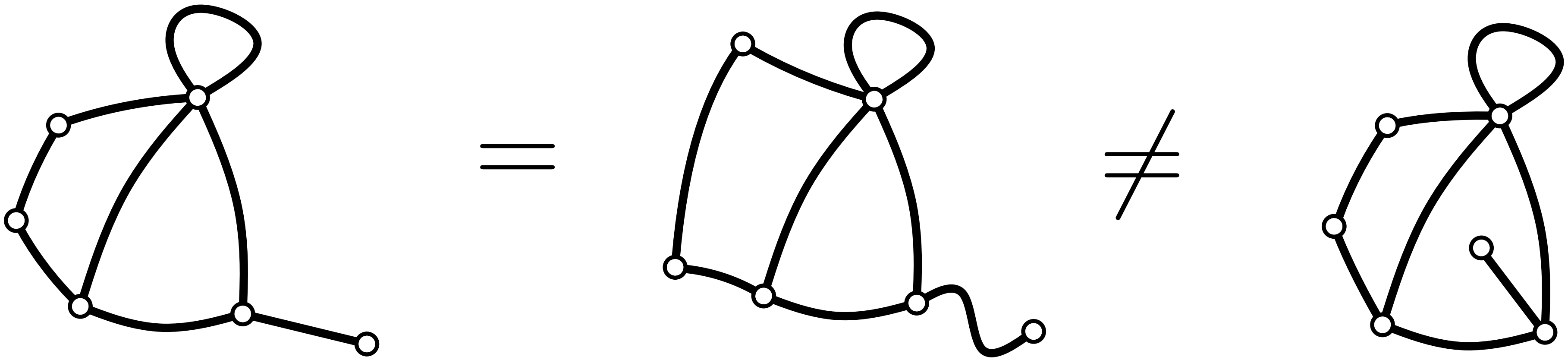

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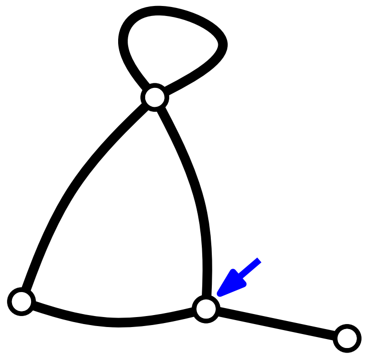
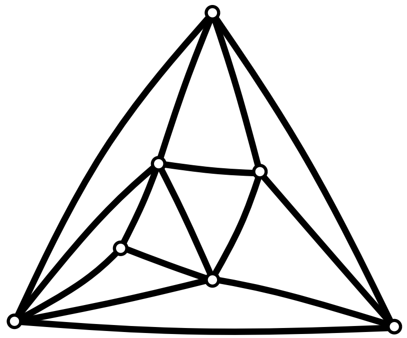


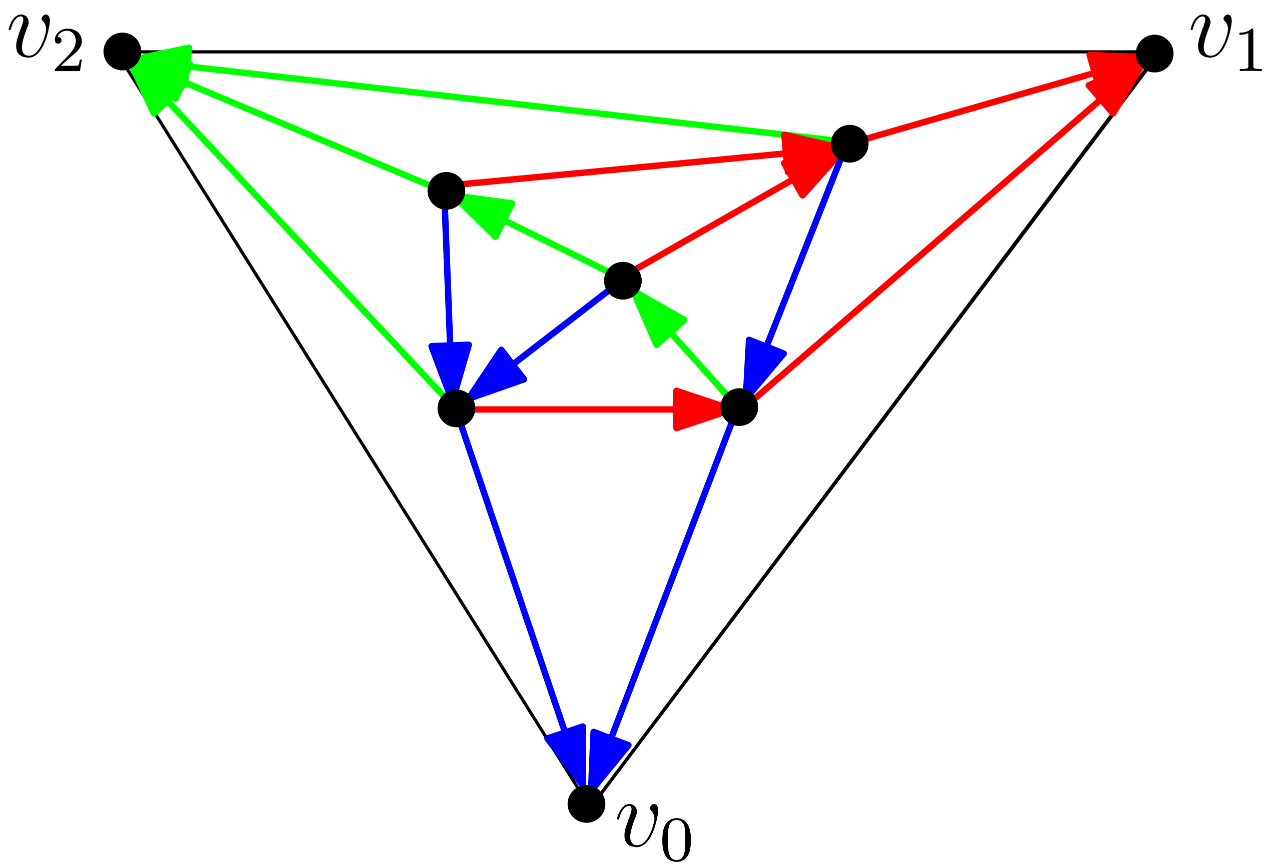
- Triangulation = simple planar map
  with all faces of degree 3

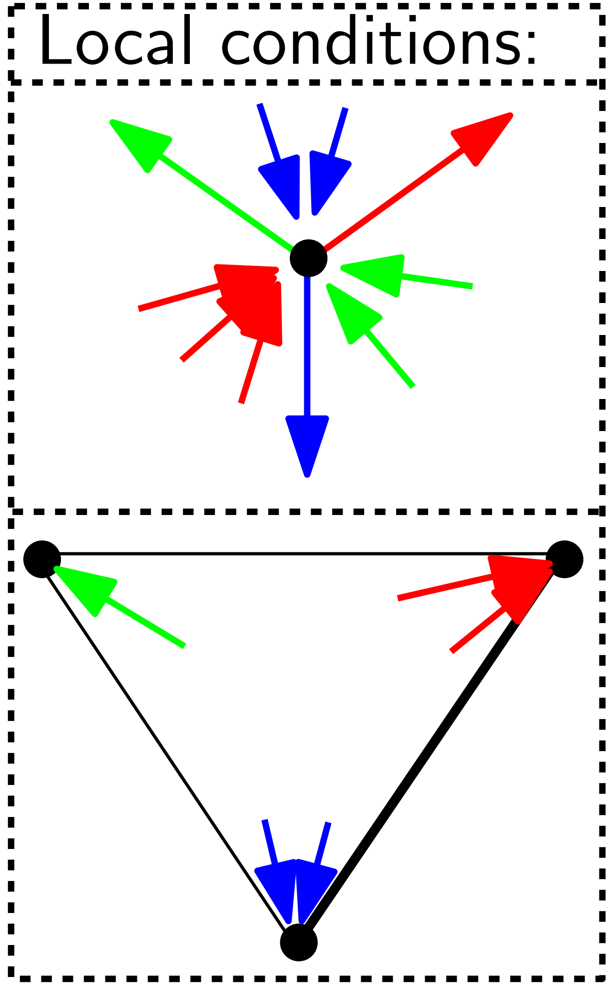


Let $\mathcal{T}_n :=$ set of rooted triangulations on $n+3$ vertices

[Tutte'62]: $|\mathcal{T}_n| = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$ (bijective proof **[Poulalhon,Schaeffer'06]**)

Schnyder woods





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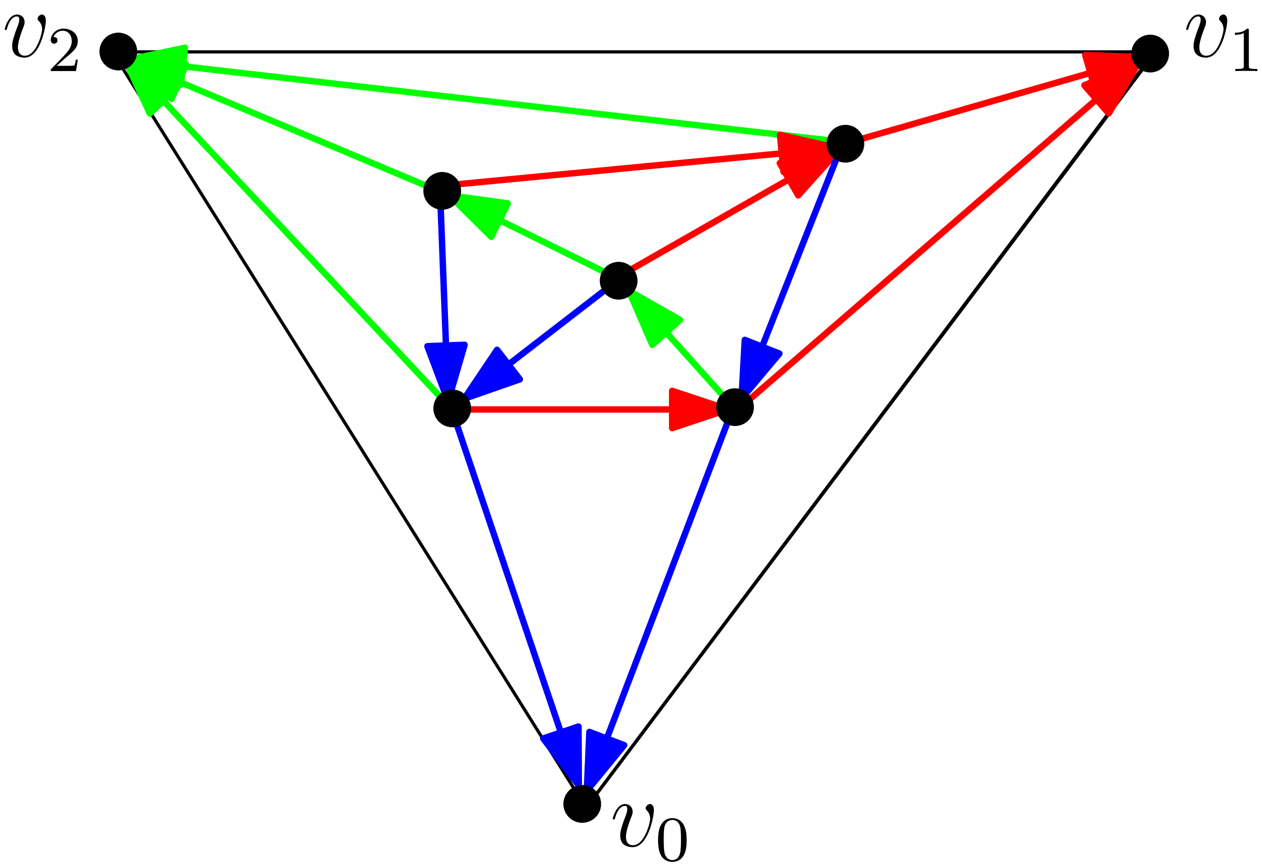
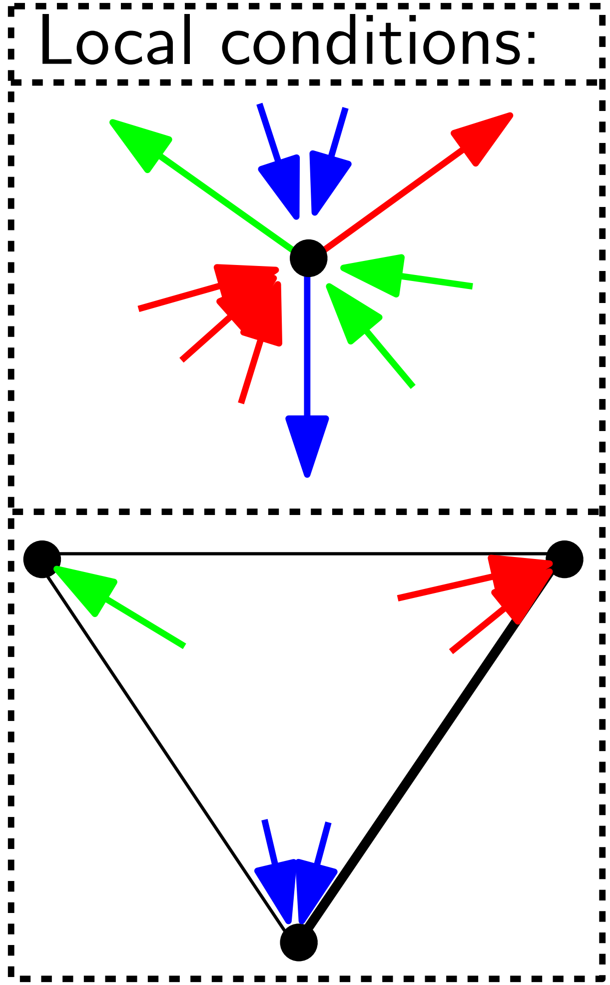
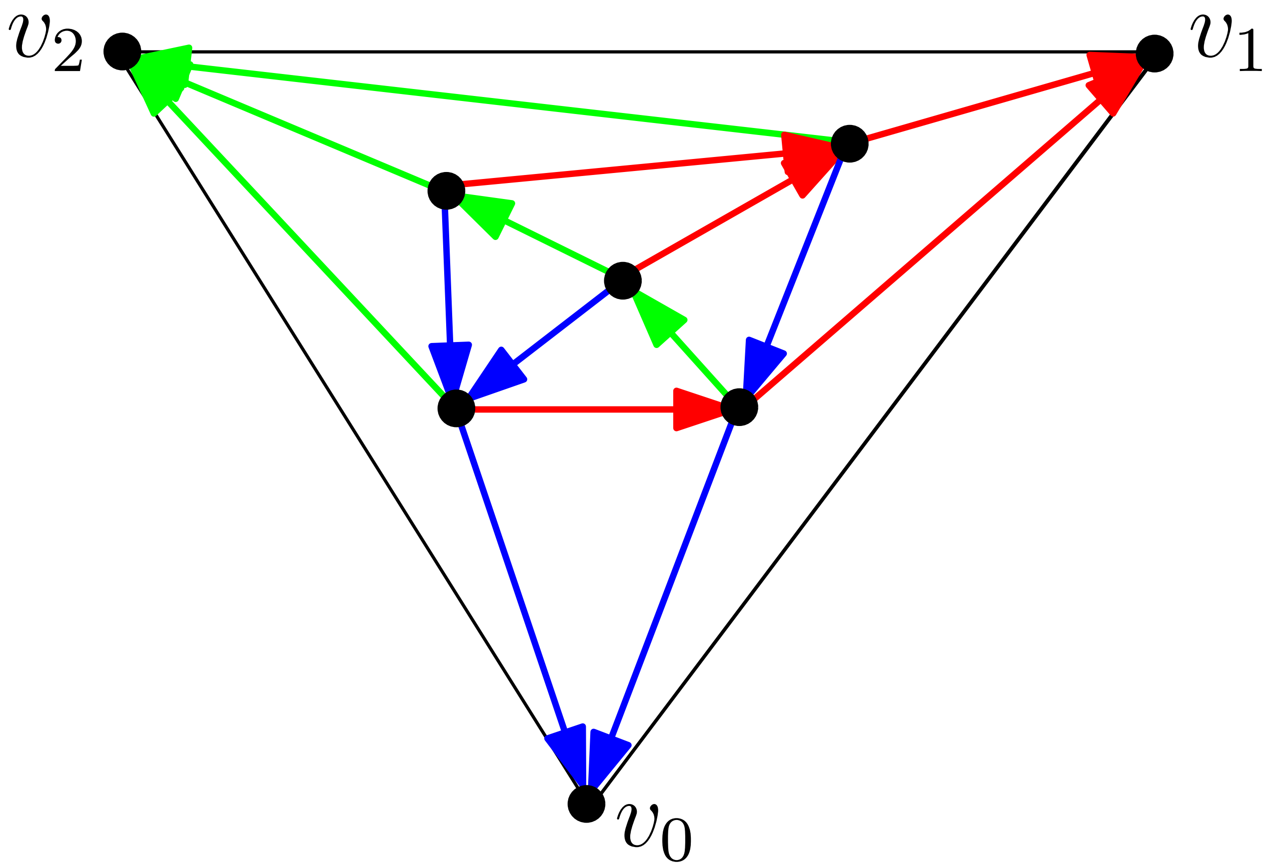


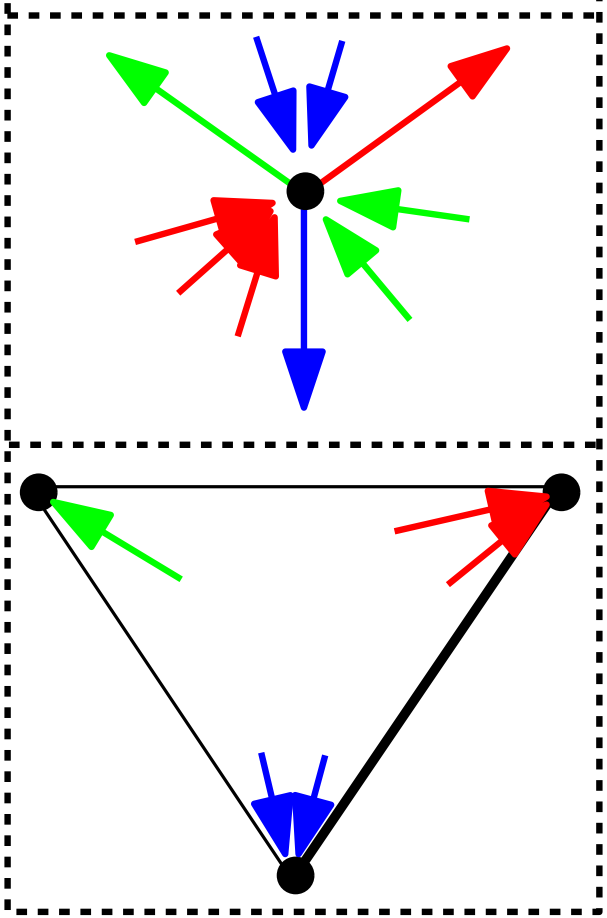


Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**
Property: the edges in each color form a tree

Schnyder woods



Local conditions:



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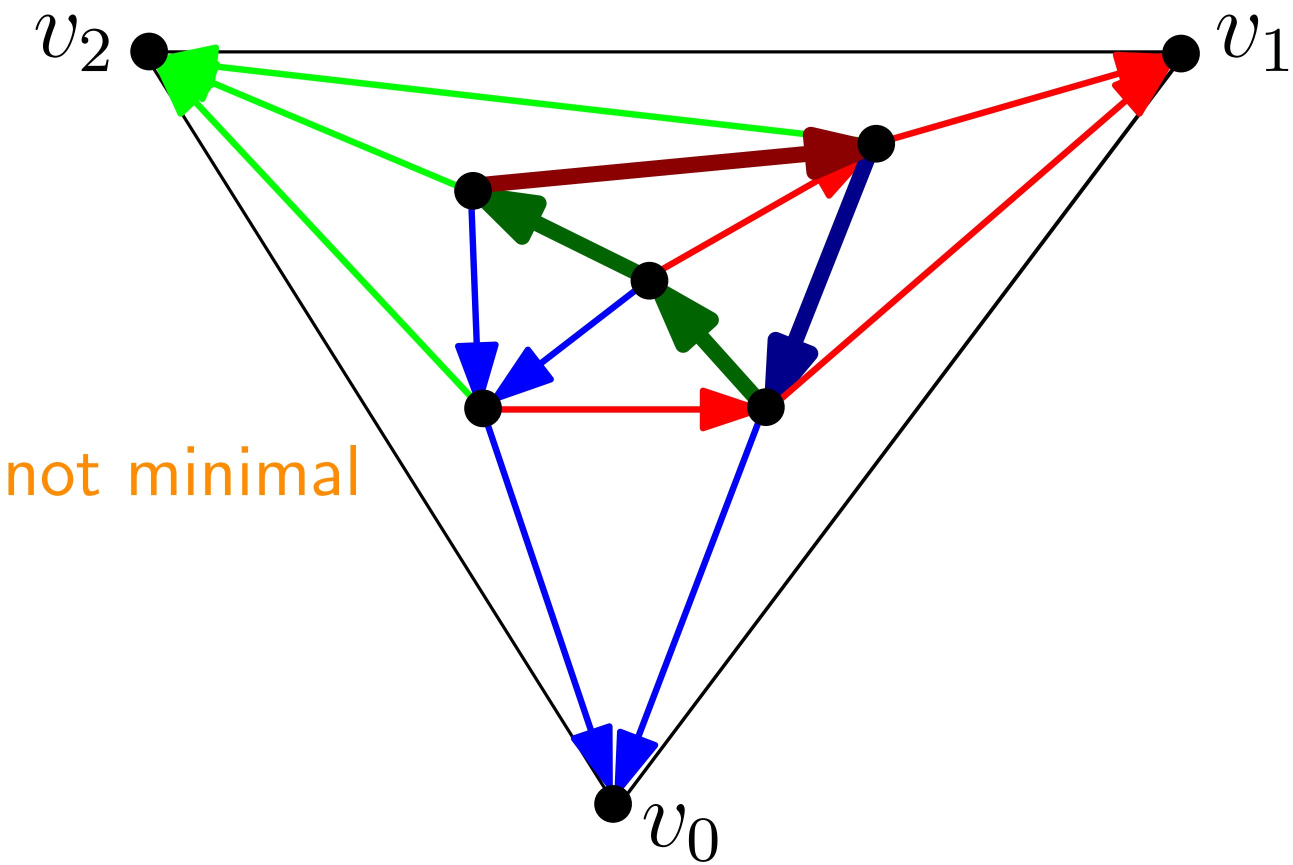
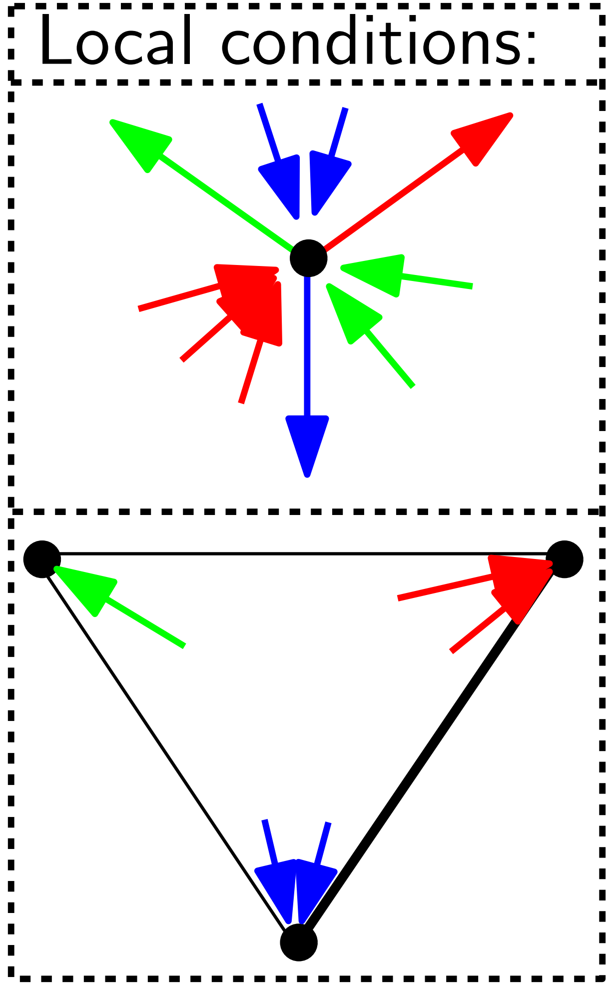
- A Schnyder wood with no cw circuit is called **minimal**

Schnyder woods





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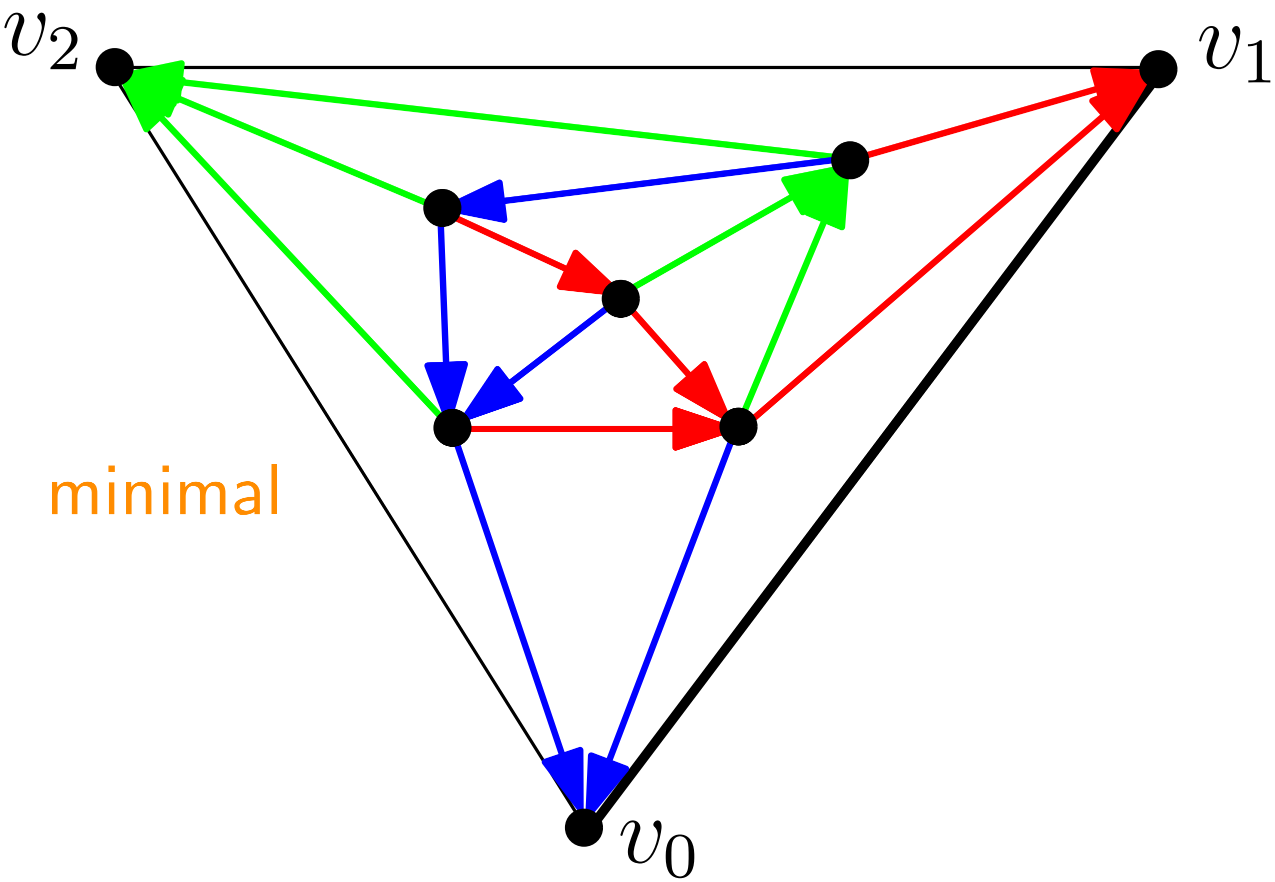

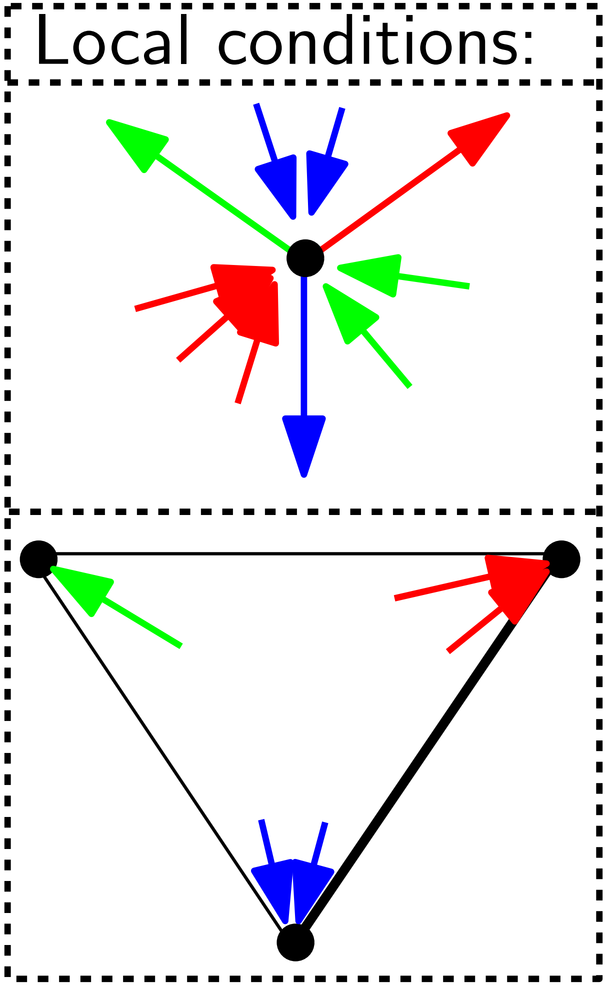

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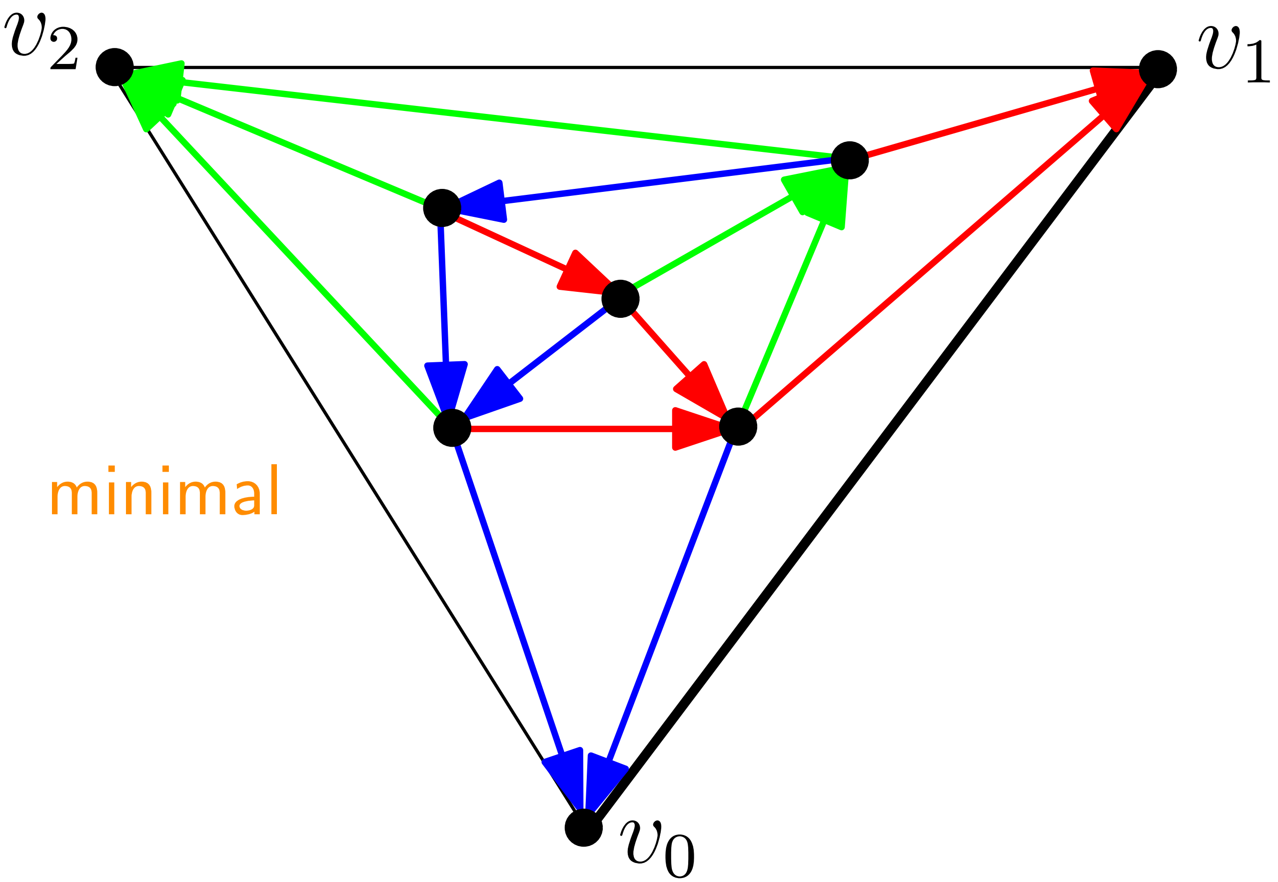

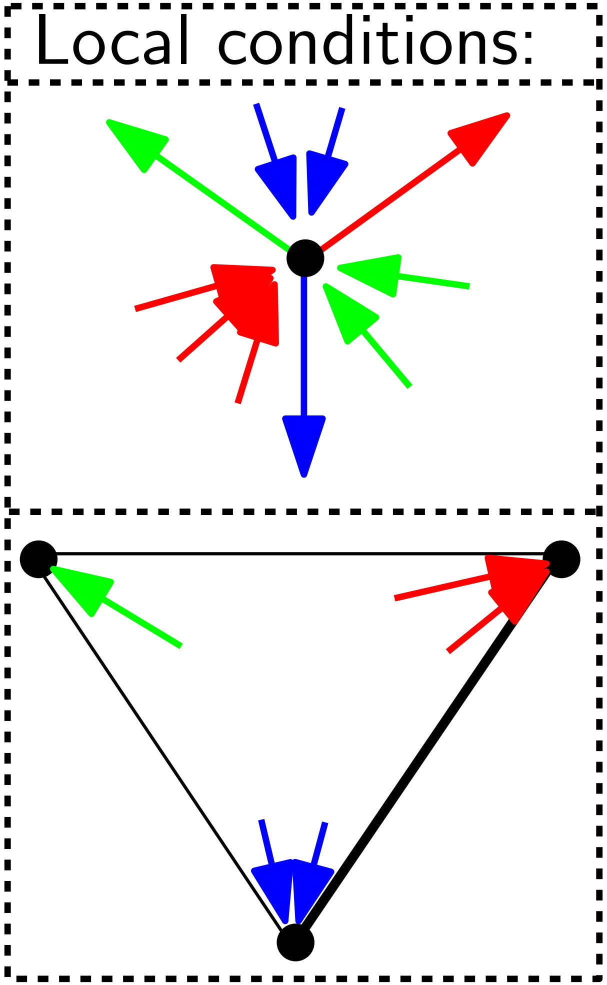

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Theo: Any triangulation has a unique minimal Schnyder wood
(cf set of Schnyder woods on fixed triangulation is a distributive lattice)
[Ossona de Mendez'94, Brehm'03, Felsner'03]

The Bernardi-Bonichon bijection

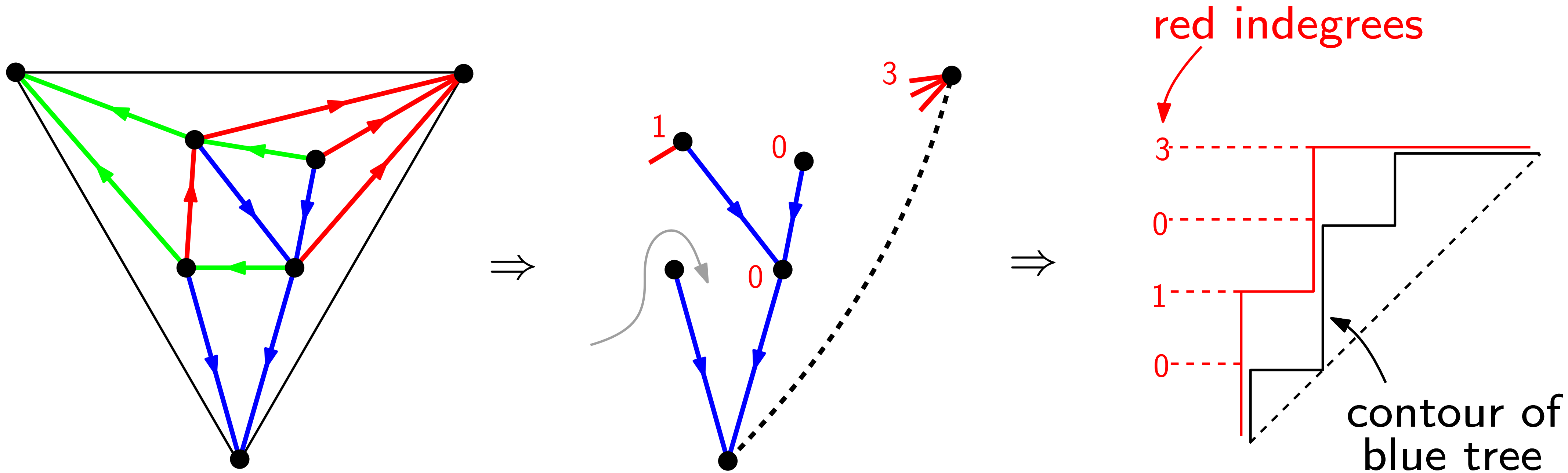
[Bernardi, Bonichon'07]

Bijection between $\mathcal{T}_n$ and $\mathcal{I}_n$ via superfamilies

Schnyder woods on $n+3$ vertices

↕

non-crossing pairs of Dyck paths of lengths $2n$



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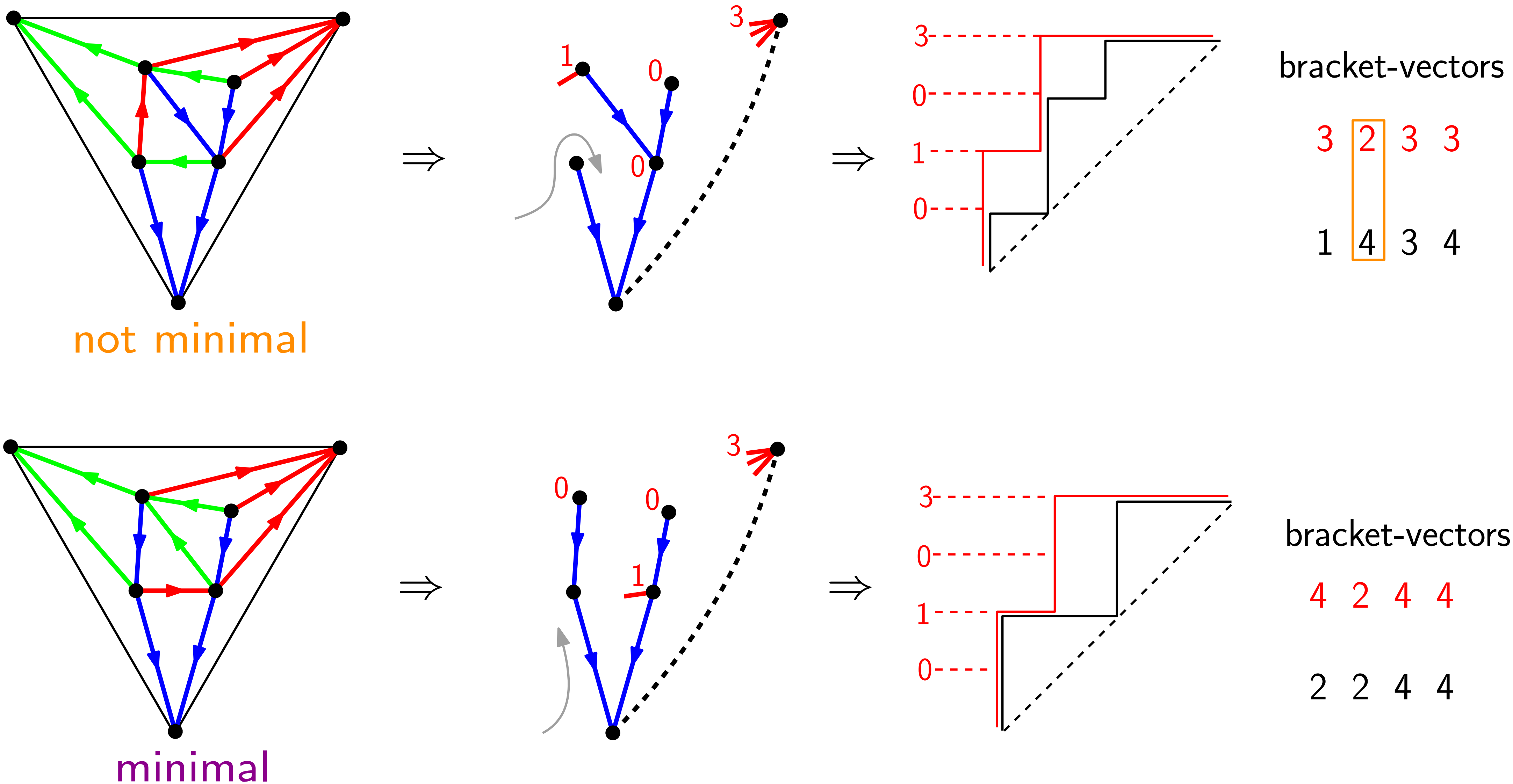
Schnyder woods on $n+3$ vertices

$\updownarrow$

non-crossing pairs of Dyck paths of lengths $2n$

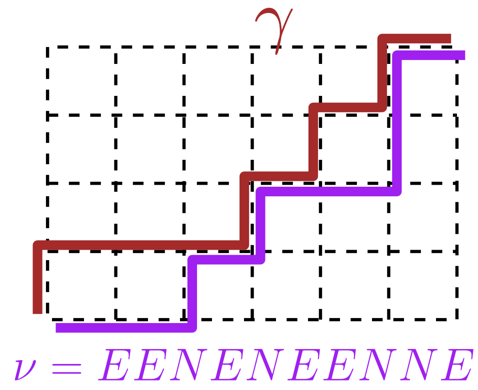
minimal $\updownarrow$ in $\mathcal{I}_n$



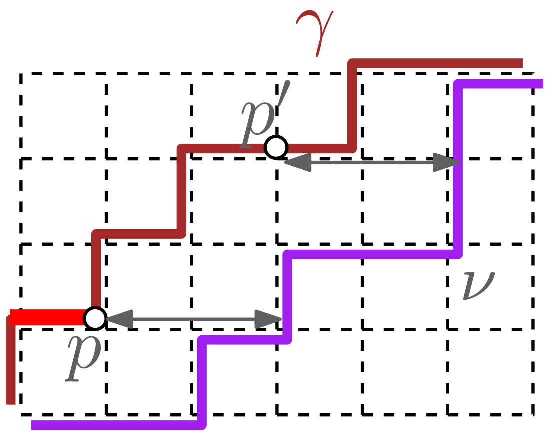
The ν-Tamari lattice

[Préville-Ratelle,Viennot'16]

For ν any walk in $\{E, N\}^n$, let $\mathcal{W}_\nu := \{\text{walks above } \nu\}$.



ν-Tamari lattice: poset Tam_ν on $\mathcal{W}_\nu$ for the covering relation



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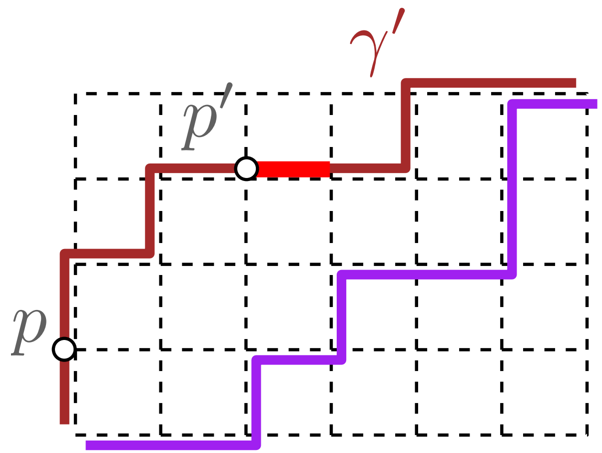
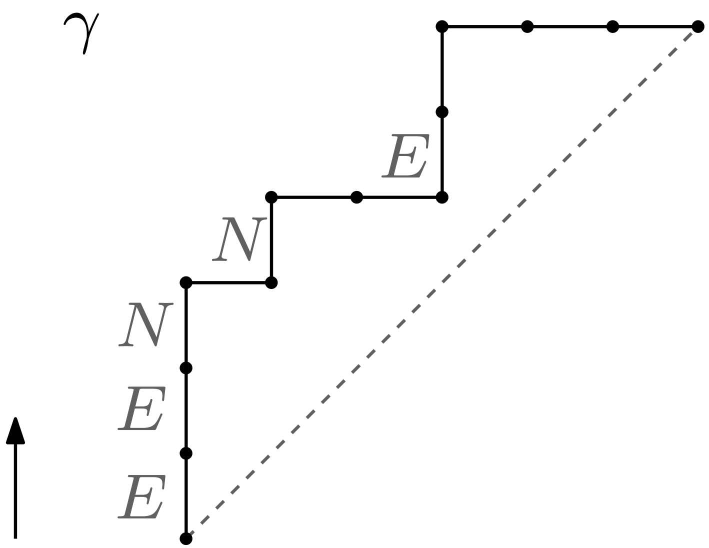


$p'=$ next point after p with same horizontal distance to ν

Other realization from the canopy



$$\mathrm{Can}(\gamma) = (E, E, N, N, E)$$

north step → E if followed by North step

north step → N if followed by East step

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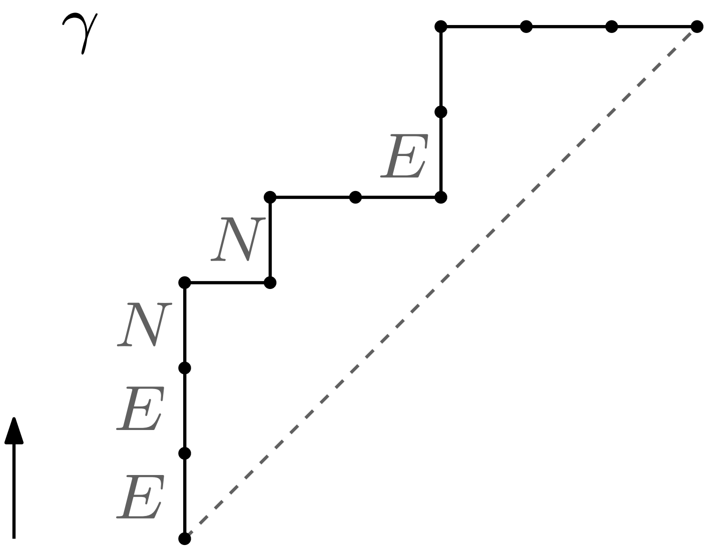

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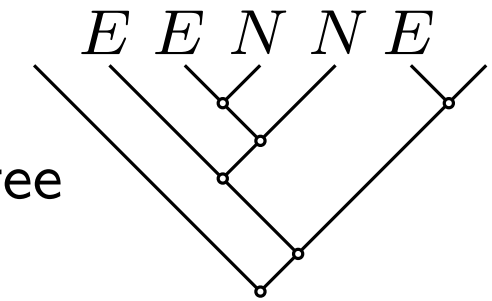
on binary tree



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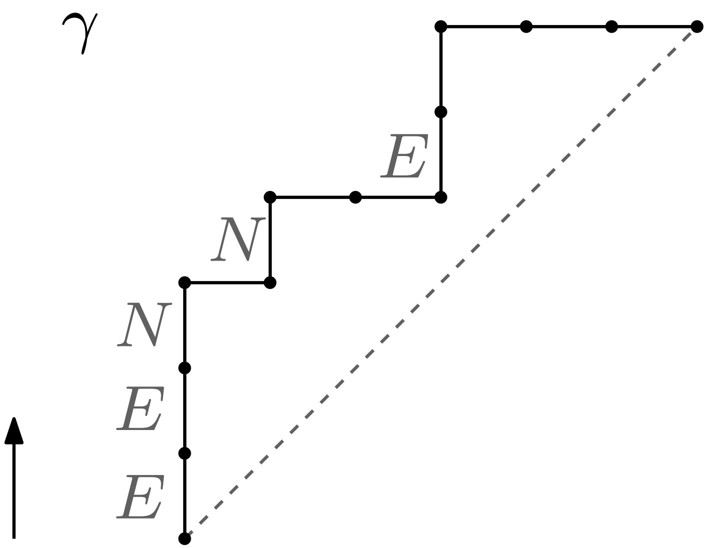
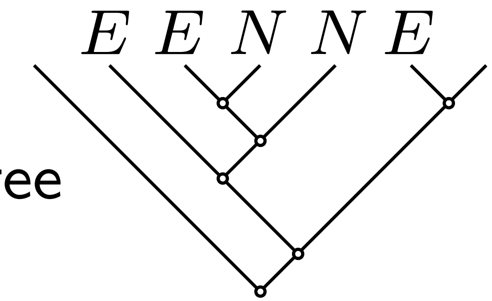


north step → E if followed by North step

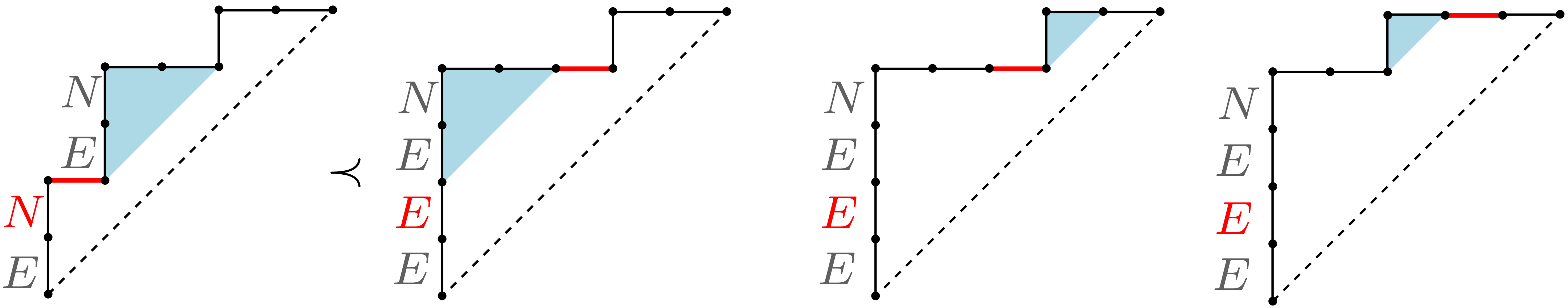
north step → N if followed by East step

on binary tree



$\mathrm{Can}(\gamma) = (E, E, N, N, E)$

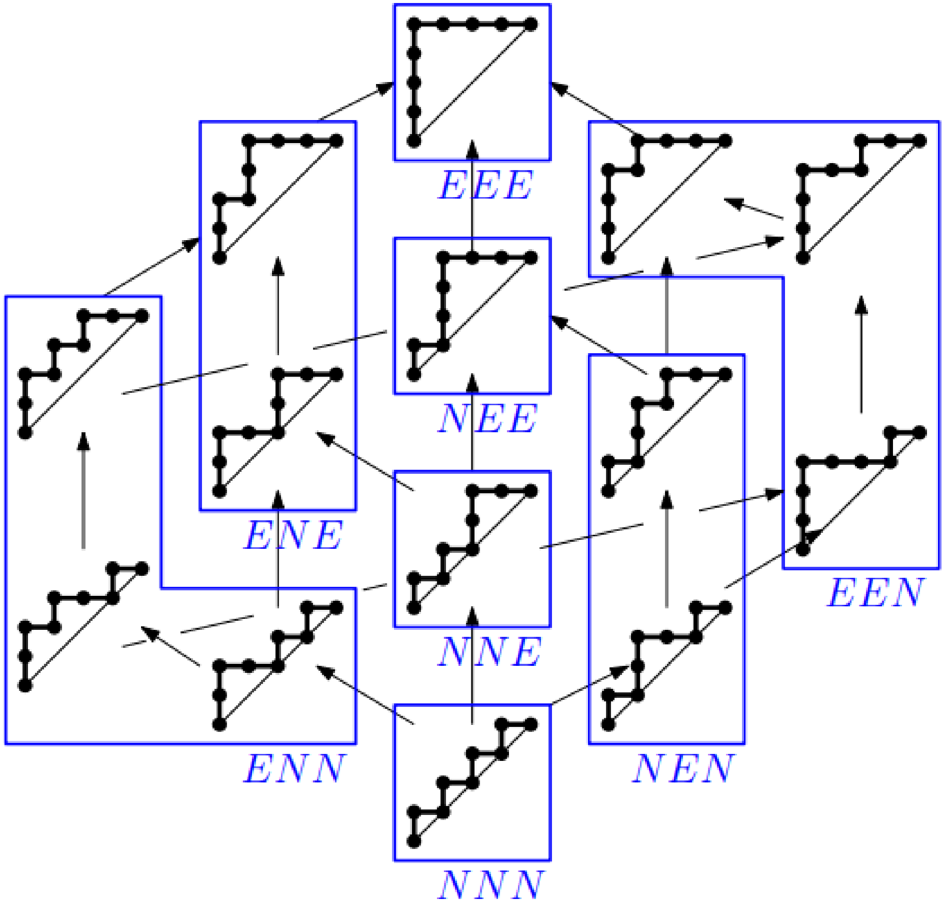
- Two types of covering relation

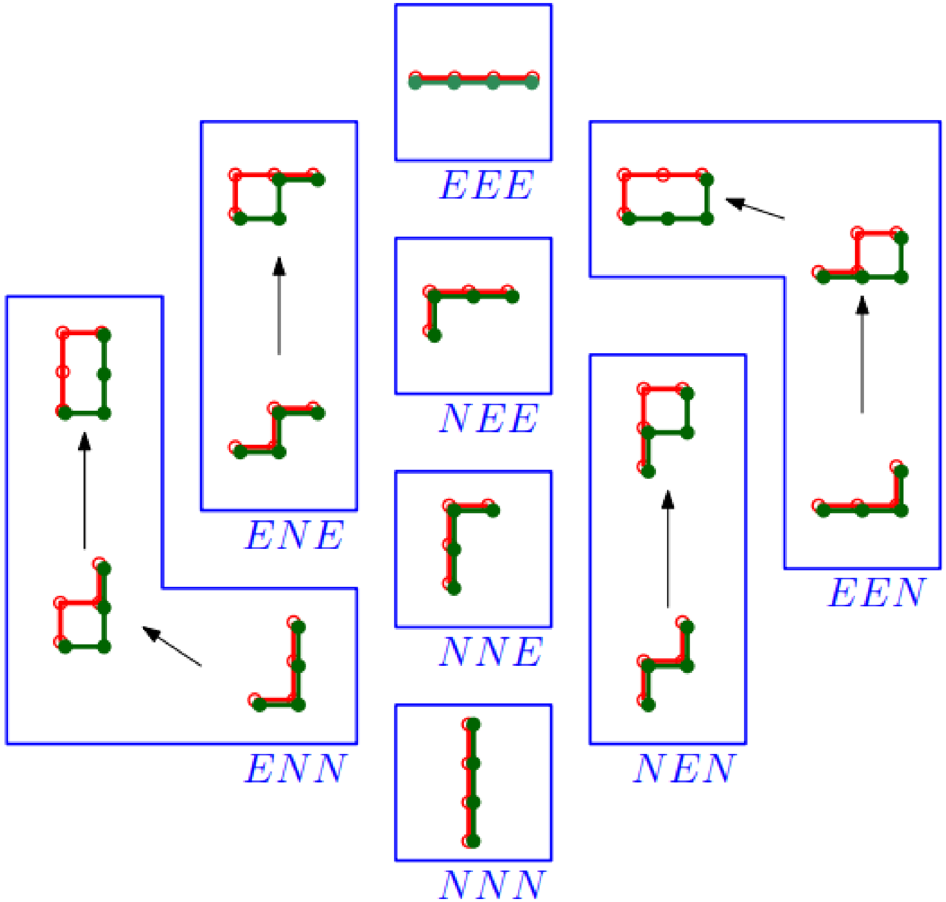


Hence $\gamma \leq \gamma'$ in $\mathrm{Tam}_n \Rightarrow \mathrm{Canopy}(\gamma) \leq \mathrm{Canopy}(\gamma')$ (with $N < E$)

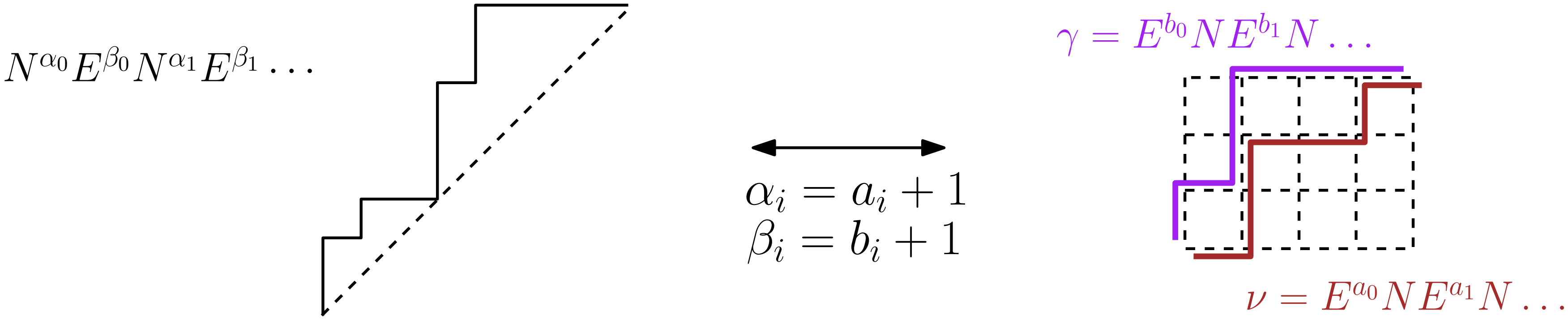
Other realization from the canopy

[Préville-Ratelle, Viennot'16], [Fang, Préville-Ratelle'17]





covering relations commute under the bijection

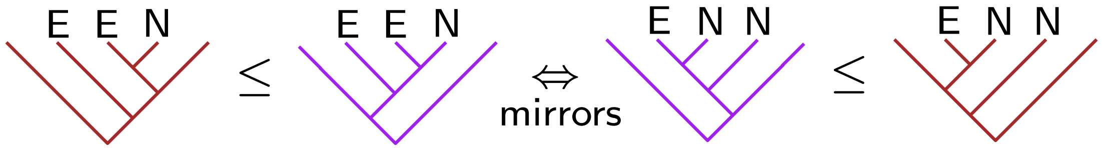


Generalized Tamari intervals

$\mathcal{I}_\nu := \{\gamma, \gamma' \mid \gamma \leq \gamma' \text{ in } \mathrm{Tam}_\nu\}$

$\mathcal{G}_n := \cup_{\nu \in \{E,N\}^n} \mathcal{I}_\nu$ $\qquad$ $\mathcal{G}_{i,j} := \cup_{\nu \in \mathfrak{S}(E^i N^j)} \mathcal{I}_\nu$

Rk: $|\mathcal{G}_{i,j}| = |\mathcal{G}_{j,i}|$ from involution



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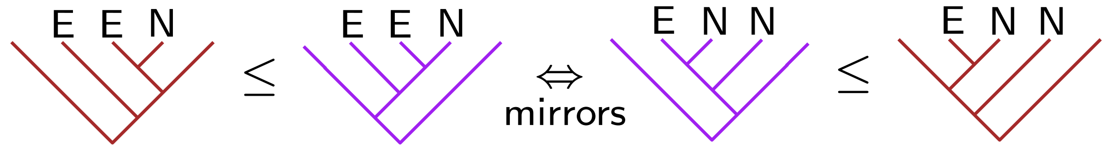
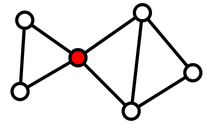


[Fang, Préville-Ratelle'17] (non-separable map = no cut-vertex such as )

Let $\mathcal{N}_n := \{\text{rooted non}-\text{sep. maps with } n{+}2 \text{ edges}\}$

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Then $\boxed{\mathcal{G}_n \longleftrightarrow \mathcal{N}_n}$ and more precisely $\boxed{\mathcal{G}_{i,j} \longleftrightarrow \mathcal{N}_{i,j}}$

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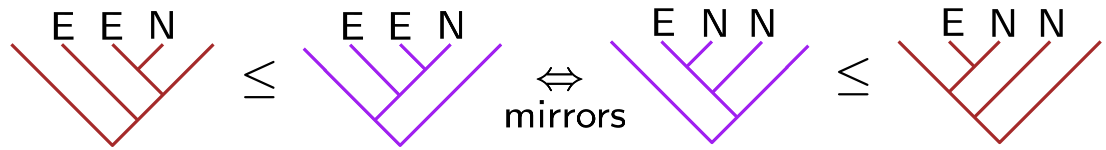

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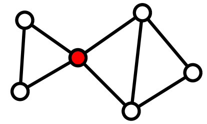
(non-separable map = no cut-vertex such as )

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[Tutte'63] $\qquad$ **[Brown-Tutte'64]**

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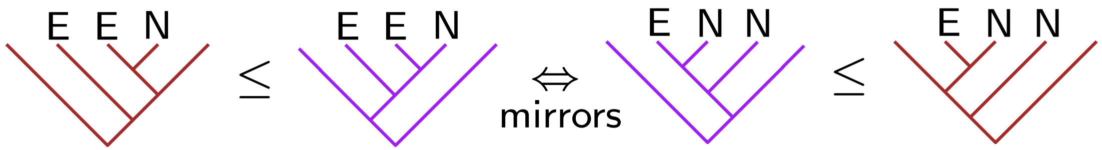
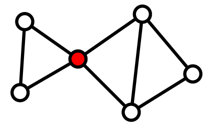


[Fang, Préville-Ratelle'17] $\qquad$ (non-separable map = no cut-vertex such as )

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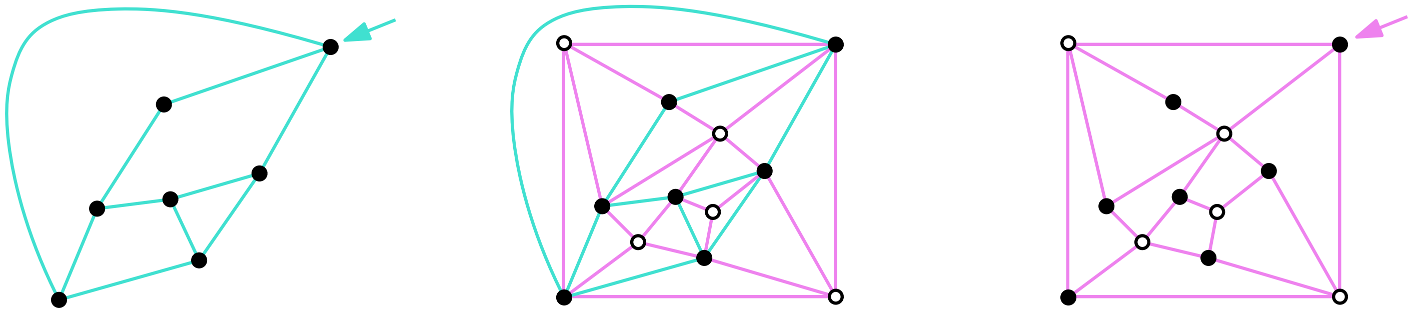
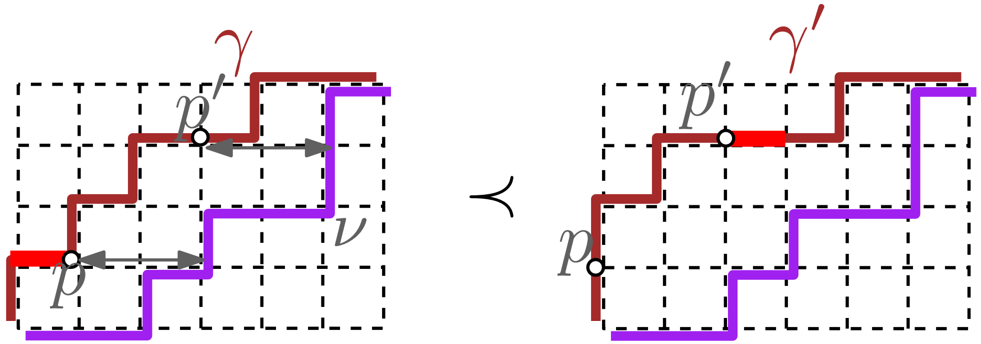
Rk: $\mathcal{N}_{i,j} \longleftrightarrow \mathcal{Q}_{i,j} := \{\text{rooted simple quadrang. } i+2 \text{ vertices } j+2 \text{ faces}\}$

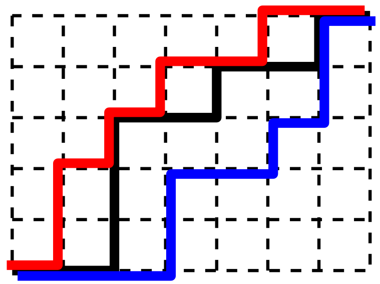


2 Superfamilies for generalized Tamari intervals

- If $\gamma \leq \gamma'$ in Tam_ν then γ is below γ' (and above ν)



$\Rightarrow$ $\boxed{\mathcal{G}_{i,j} \subseteq \mathcal{R}_{i,j}}$ with $\mathcal{R}_{i,j} :=$ {non-crossing triples from $(0,0)$ to (i,j)}



a triple in $\mathcal{R}_{7,5}$

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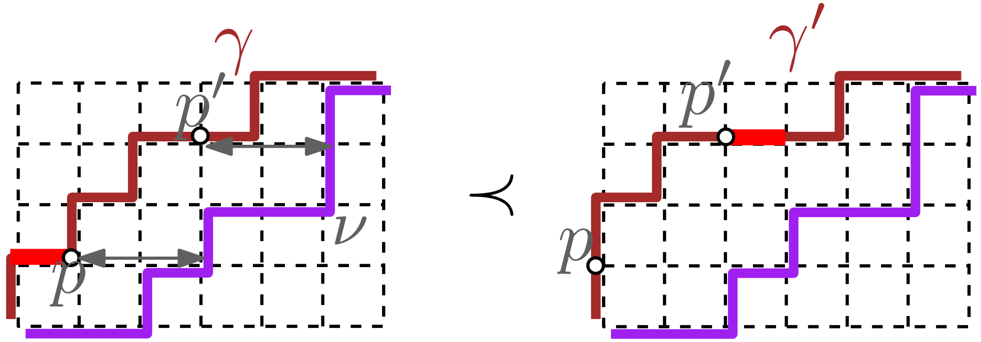

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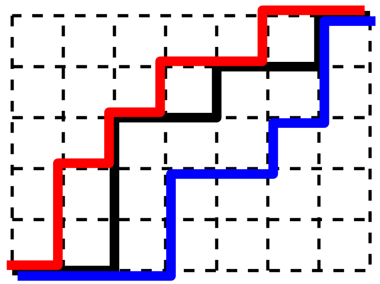

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- An interval $(\gamma, \gamma') \in \mathcal{I}_n$ is called **synchronized** if $\mathrm{Can}(\gamma) = \mathrm{Can}(\gamma')$
Let $\mathcal{S}_n :=$ subfamily of **synchronized** intervals from $\mathcal{I}_n$.

Then $\boxed{\mathcal{G}_n \simeq \mathcal{S}_n \subseteq \mathcal{I}_n}$ **(Rk: on the other hand $\mathcal{I}_n \subset \mathcal{G}_{2n}$)**

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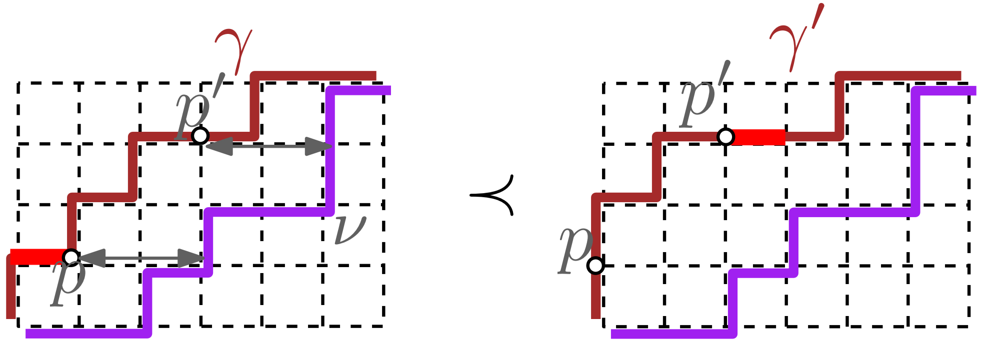

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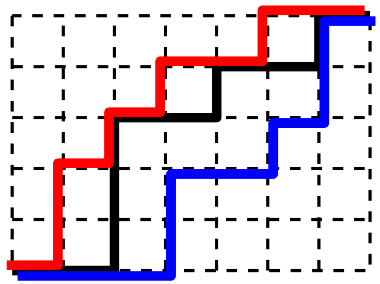

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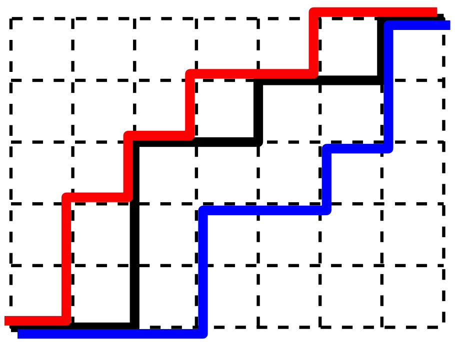
  where common canopy word is in $\mathfrak{S}(E^i N^j)$

  Then $\boxed{\mathcal{G}_{i,j} \simeq \mathcal{S}_{i,j} \subseteq \mathcal{I}_{i+j-1}}$

Non-intersecting triples and Baxter families

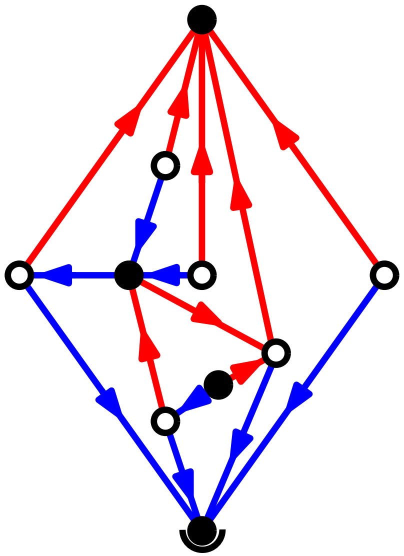
A **Baxter family** is a family $\mathcal{B}_{i,j}$ indexed by two parameters i, j such that

$$|\mathcal{B}_{i,j}| = 2\frac{(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}$$



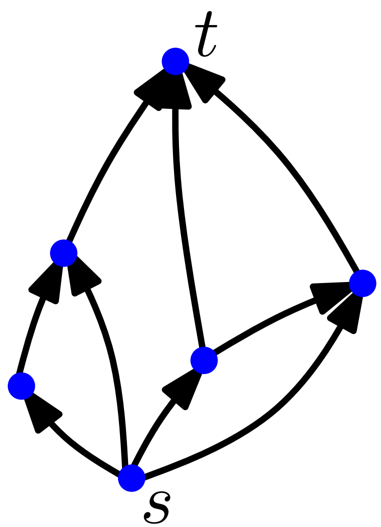
non-intersecting triples of walks

Lindström-Gessel-Viennot lemma



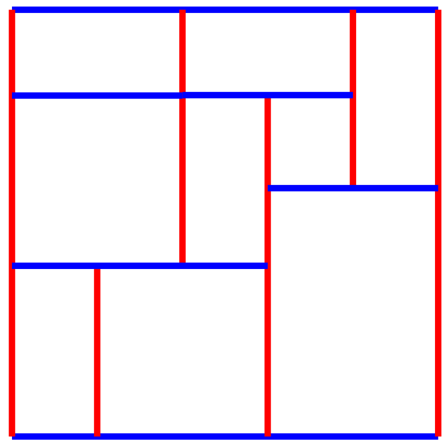
separating decompositions

[F, Poulalhon, Schaeffer'09]
[Felsner,F,Noy,Orden'11]



plane bipolar orientations

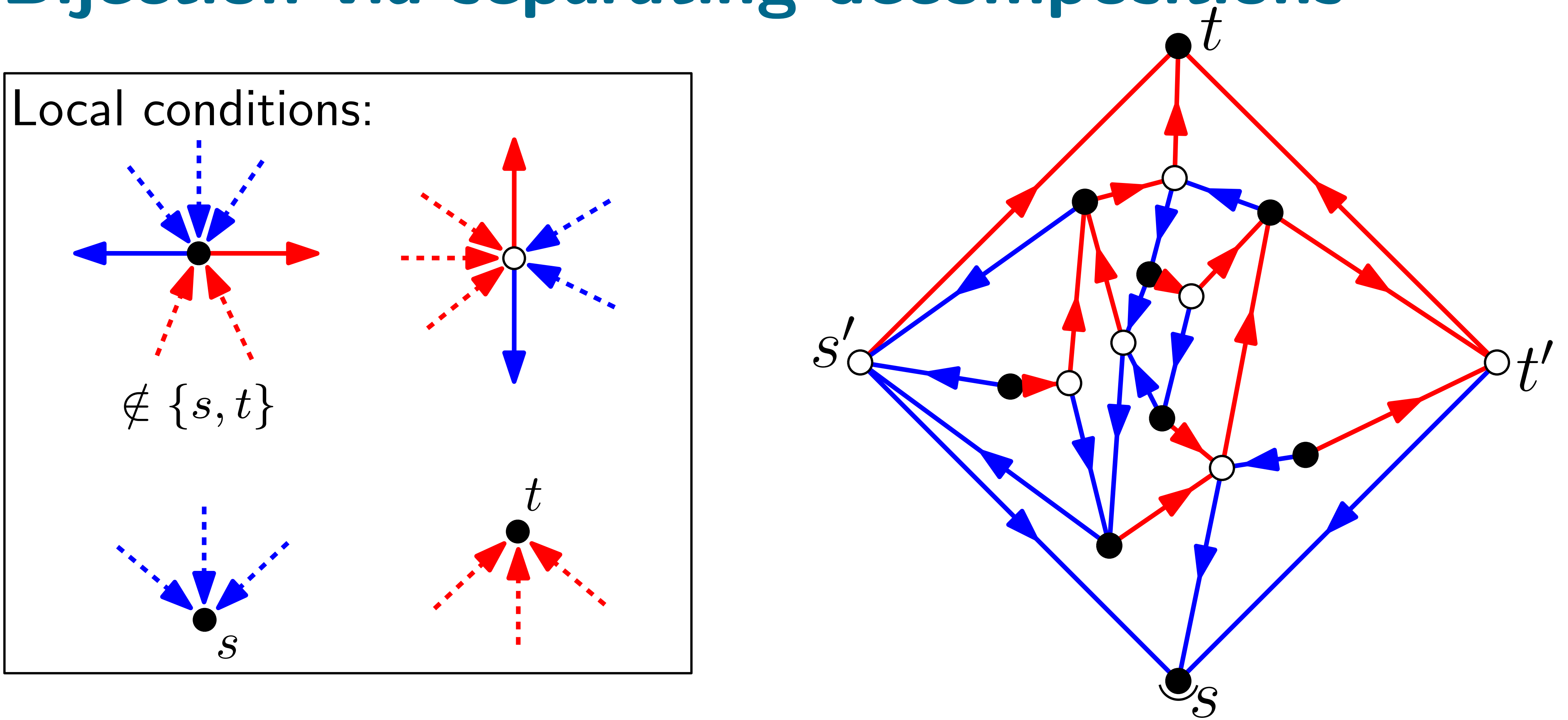
[Albenque,Poulalhon'15]
[Kenyon et al.'19]



rectangular floorplans

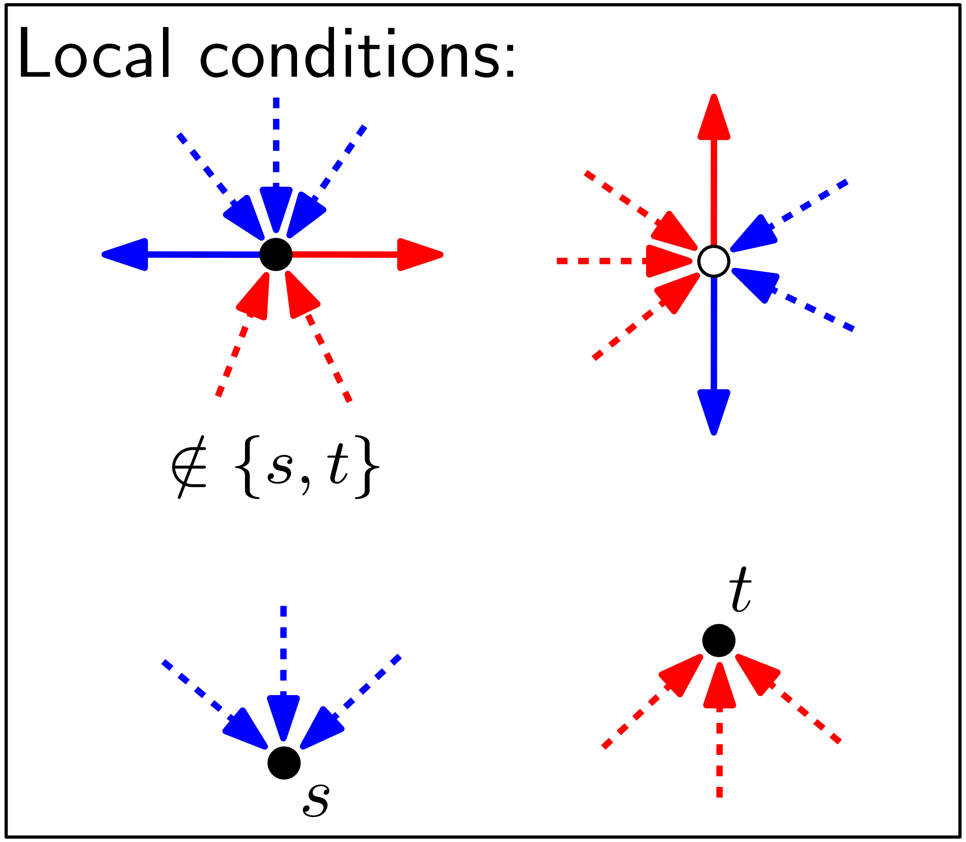
[Dulucq,Guibert'96]
[Ackerman et al.'06]

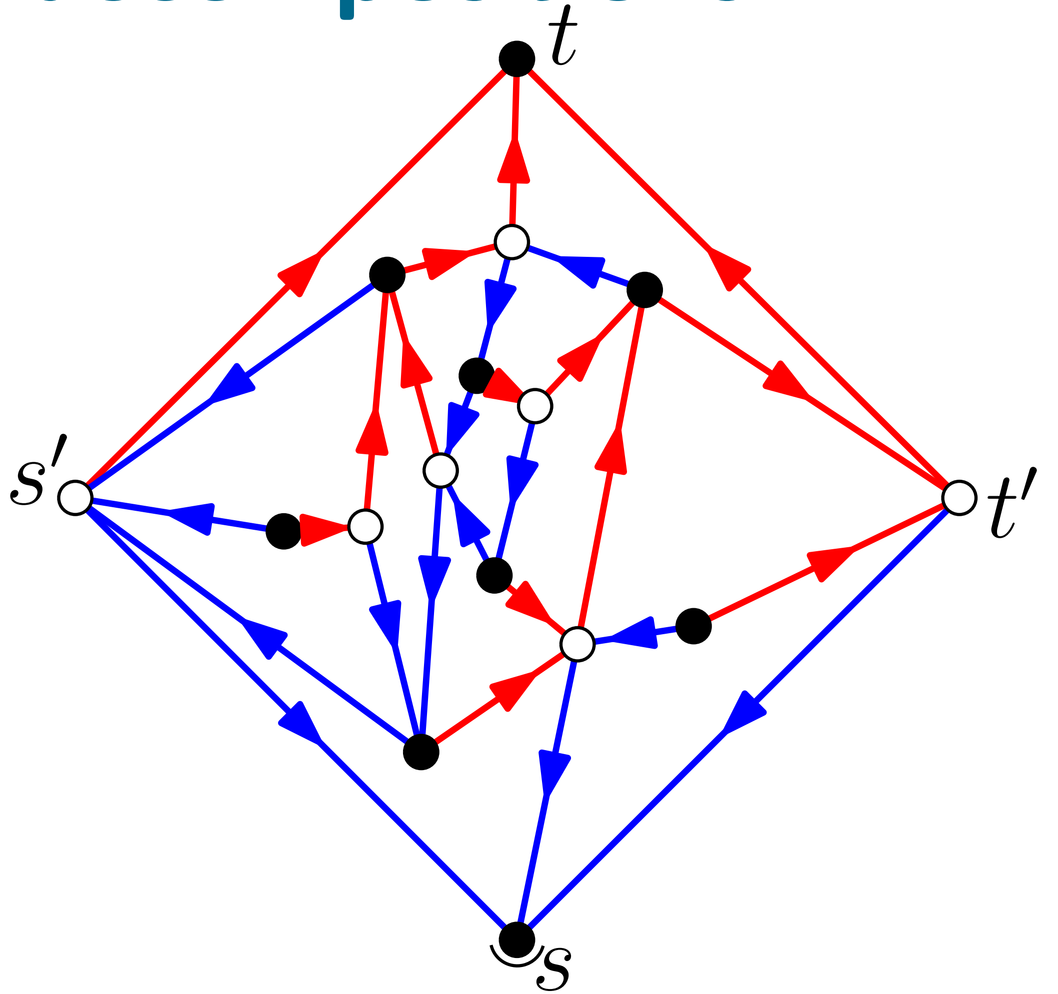
Bijection via separating decompositions



$\mathrm{Sep}_{i,j}$:= set of separating decompositions with $i+2$ vertices, $j+2$ faces

Bijection via separating decompositions





$\mathrm{Sep}_{i,j}$:= set of separating decompositions with $i+2$ vertices, $j+2$ faces

Theorem: [de Fraysseix et al.95]

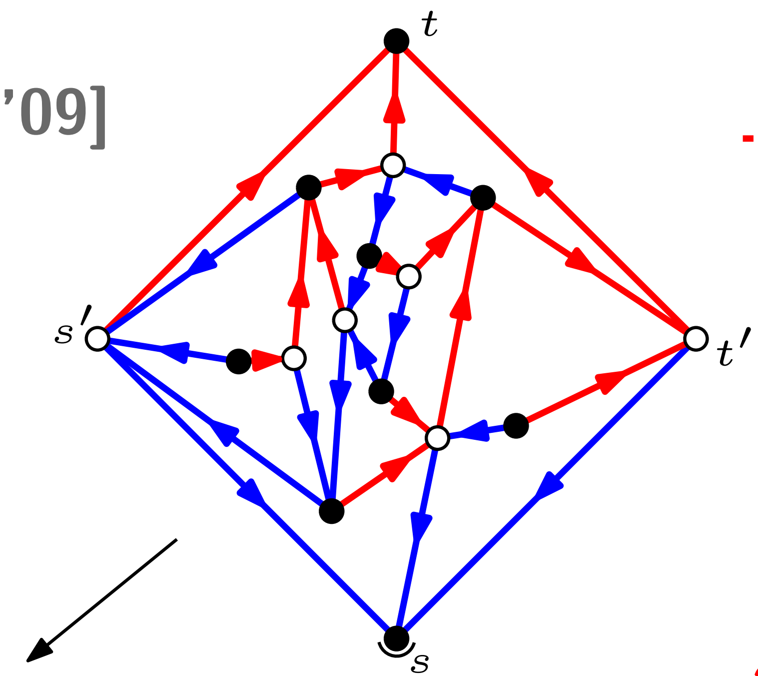
Any simple quadrangulation admits a separating decomposition

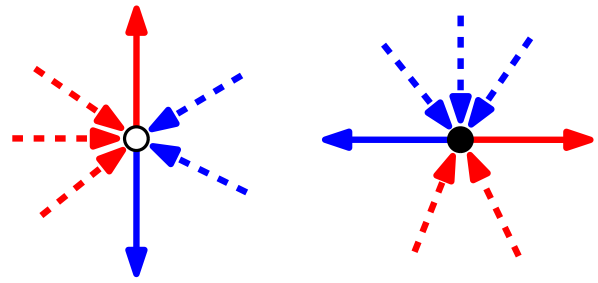
It has a unique one that is **minimal** (no cw cycle)

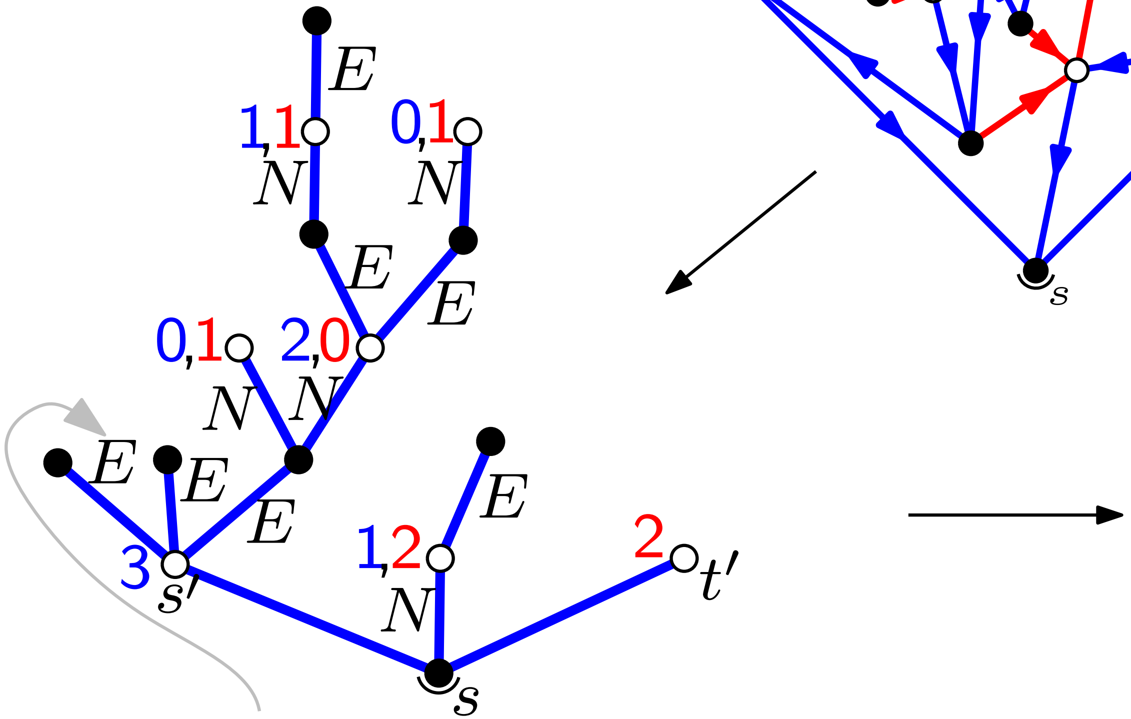
Property: edges in each color form a tree

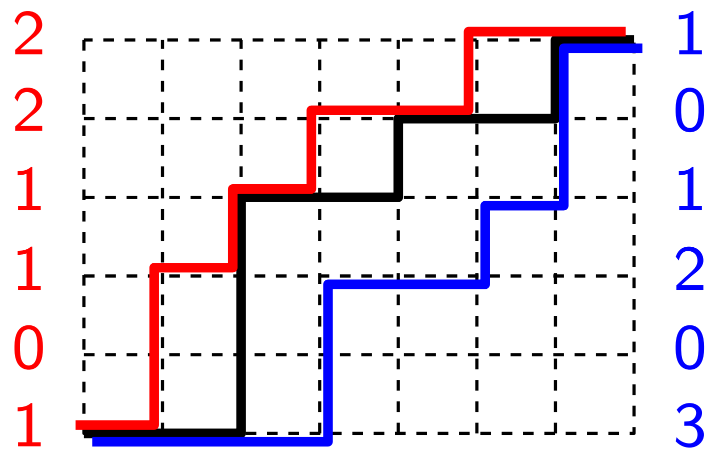
Bijection via separating decompositions

close to bijection in
[F, Poulalhon, Schaeffer'09]









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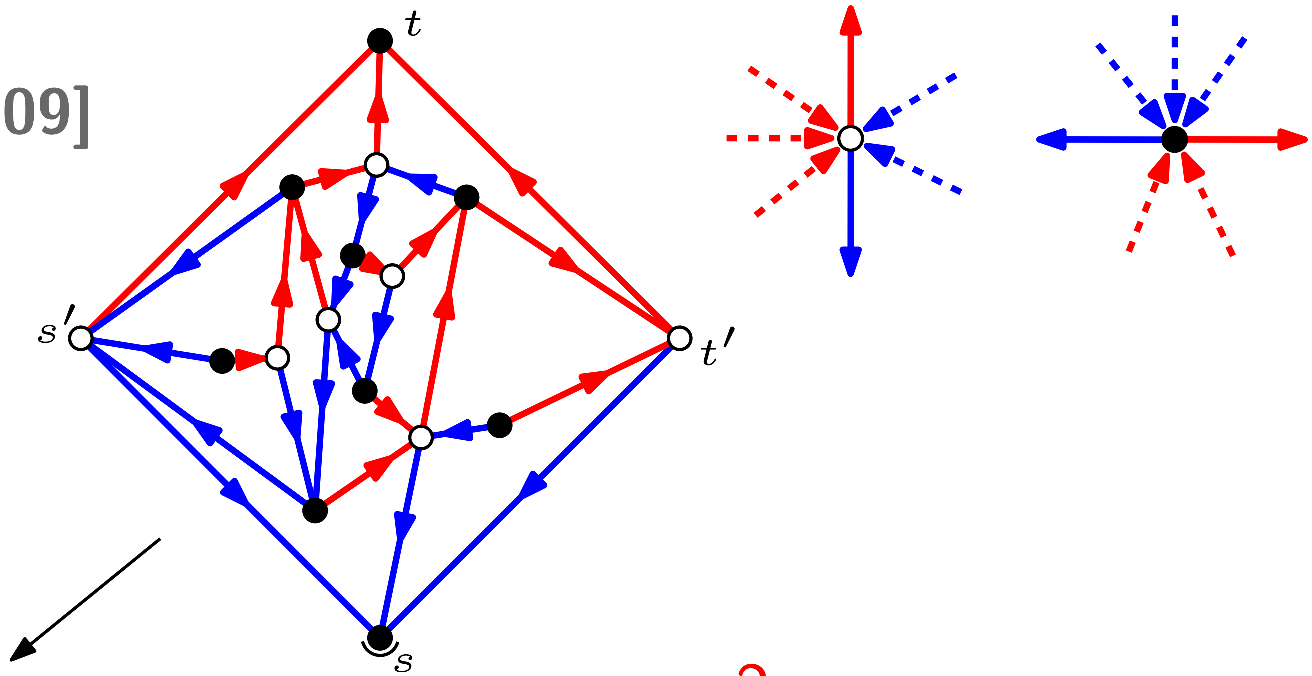
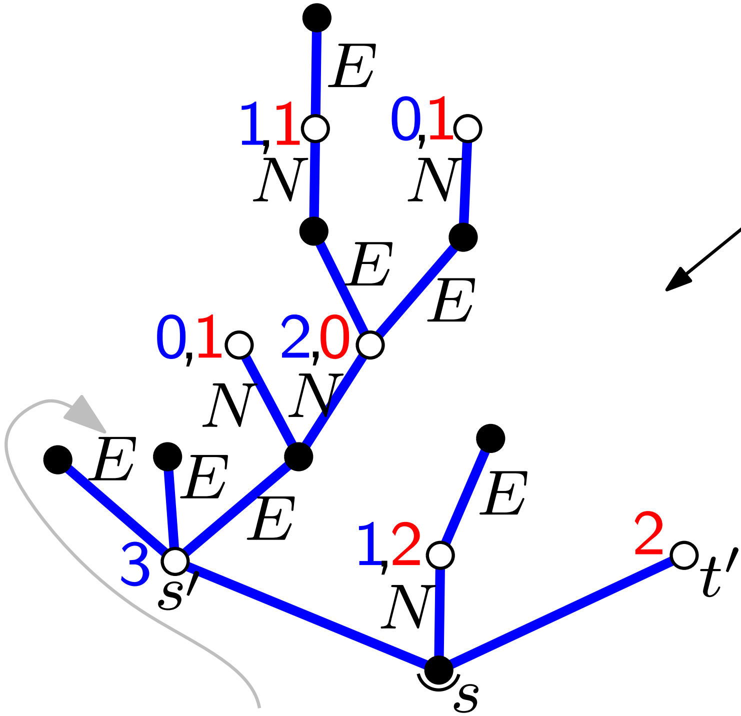
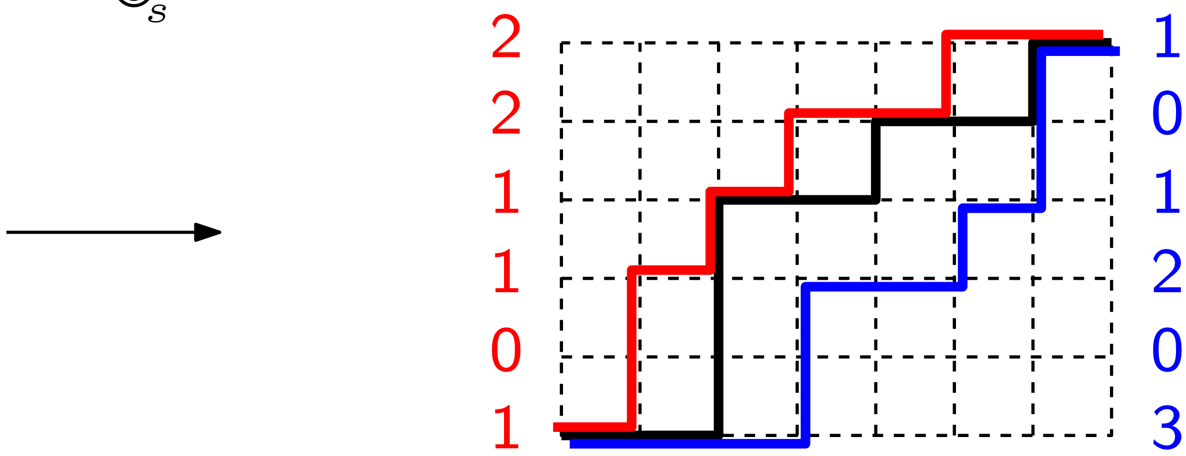


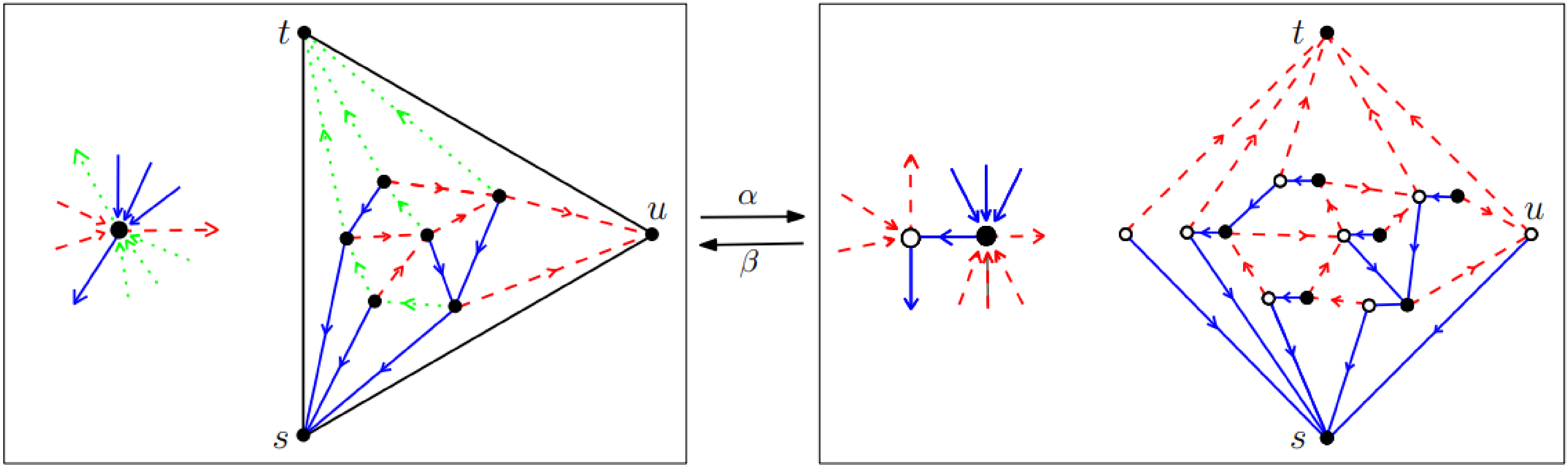




[F, Humbert'19]

The mapping is a bijection between $\mathrm{Sep}_{i,j}$ and $\mathcal{R}_{i,j}$
A separating decomposition is minimal iff its image is in $\mathcal{G}_{i,j}$
$\Rightarrow$ specialization into a bijection from $\mathcal{Q}_{i,j}$ to $\mathcal{G}_{i,j}$

Link to the Bernardi-Bonichon bijection

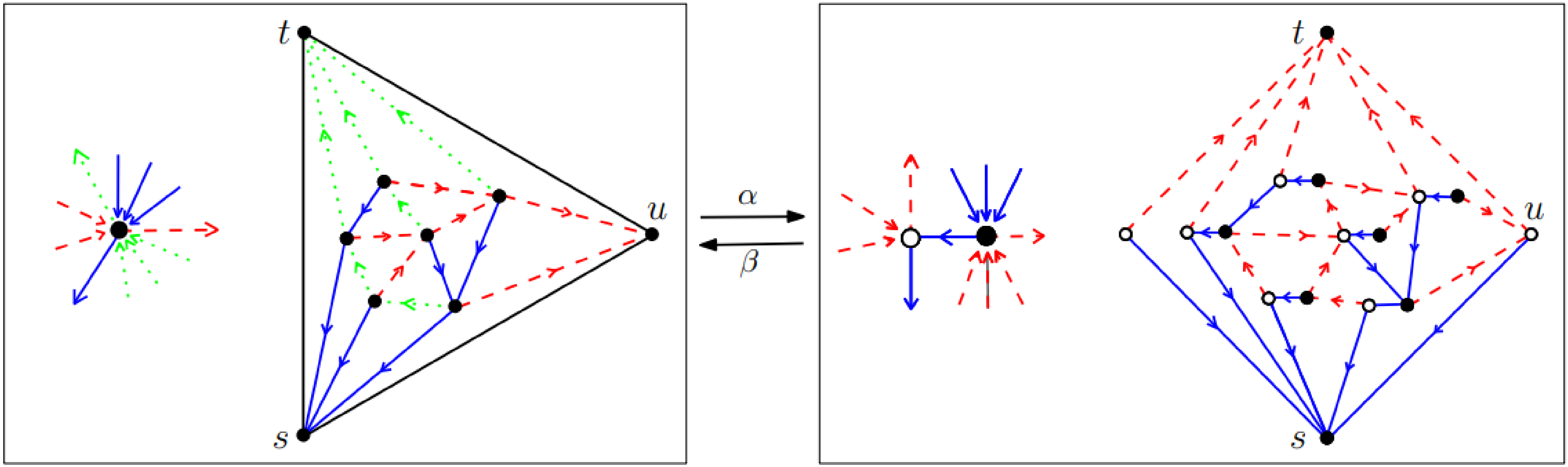


mapping preserves minimality

⇒ Bernardi-Bonichon bijection $\simeq$ case where white vertices have blue indegree 1

(bottom-walk$= (NE)^n$)

Link to the Bernardi-Bonichon bijection



mapping preserves minimality

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(bottom-walk$=(NE)^n$)

More generally,

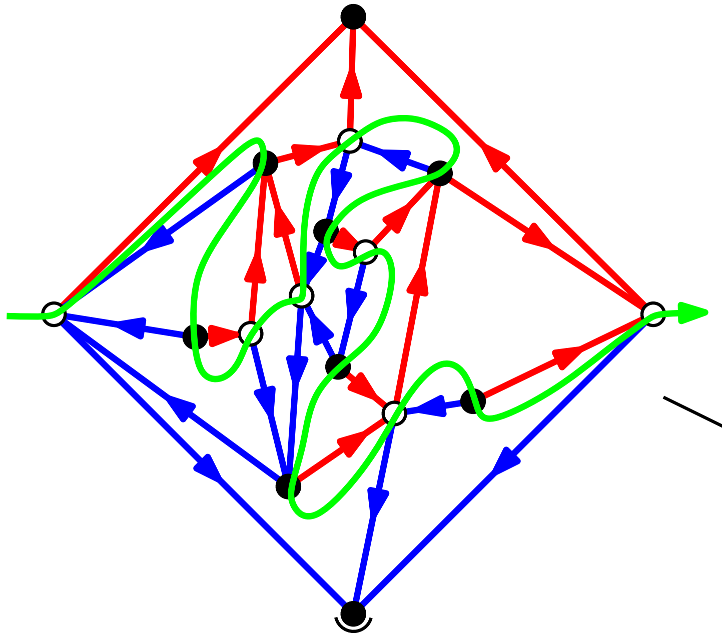
m-Tamari intervals $\longleftrightarrow$ minimal separating decompositions
$\nu = (NE^m)^n$ where white vertices have blue indegree m

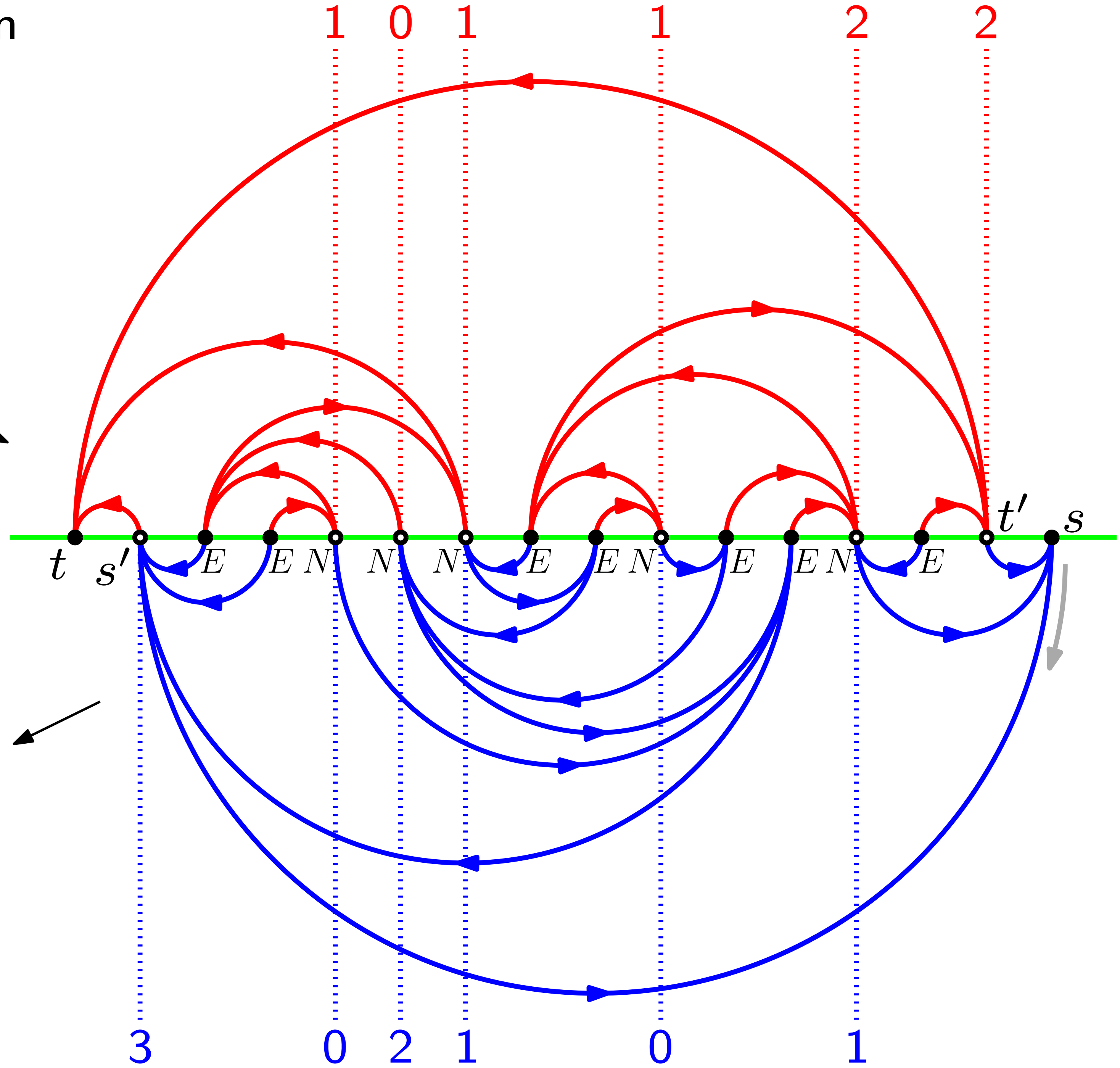
Not yet a bijective interpretation of the formula

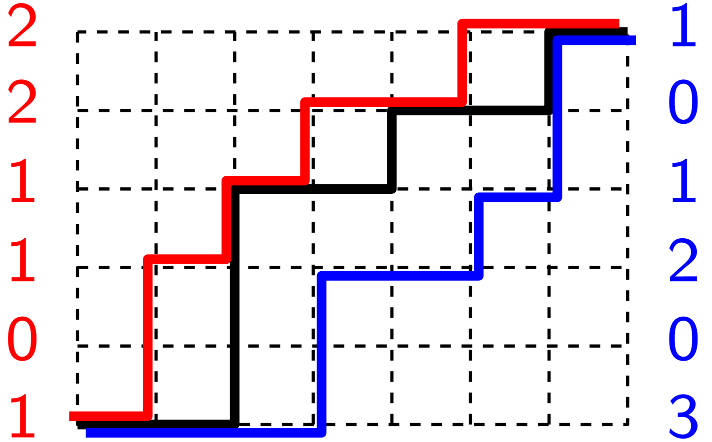
$$I_n^{(m)} = \frac{m+1}{m(nm+1)}\binom{(m+1)^2 n + m}{n-1}$$

Symmetric reformulation of the bijection

2-book embedding of
a separating decomposition
[Felsner et al.'07]



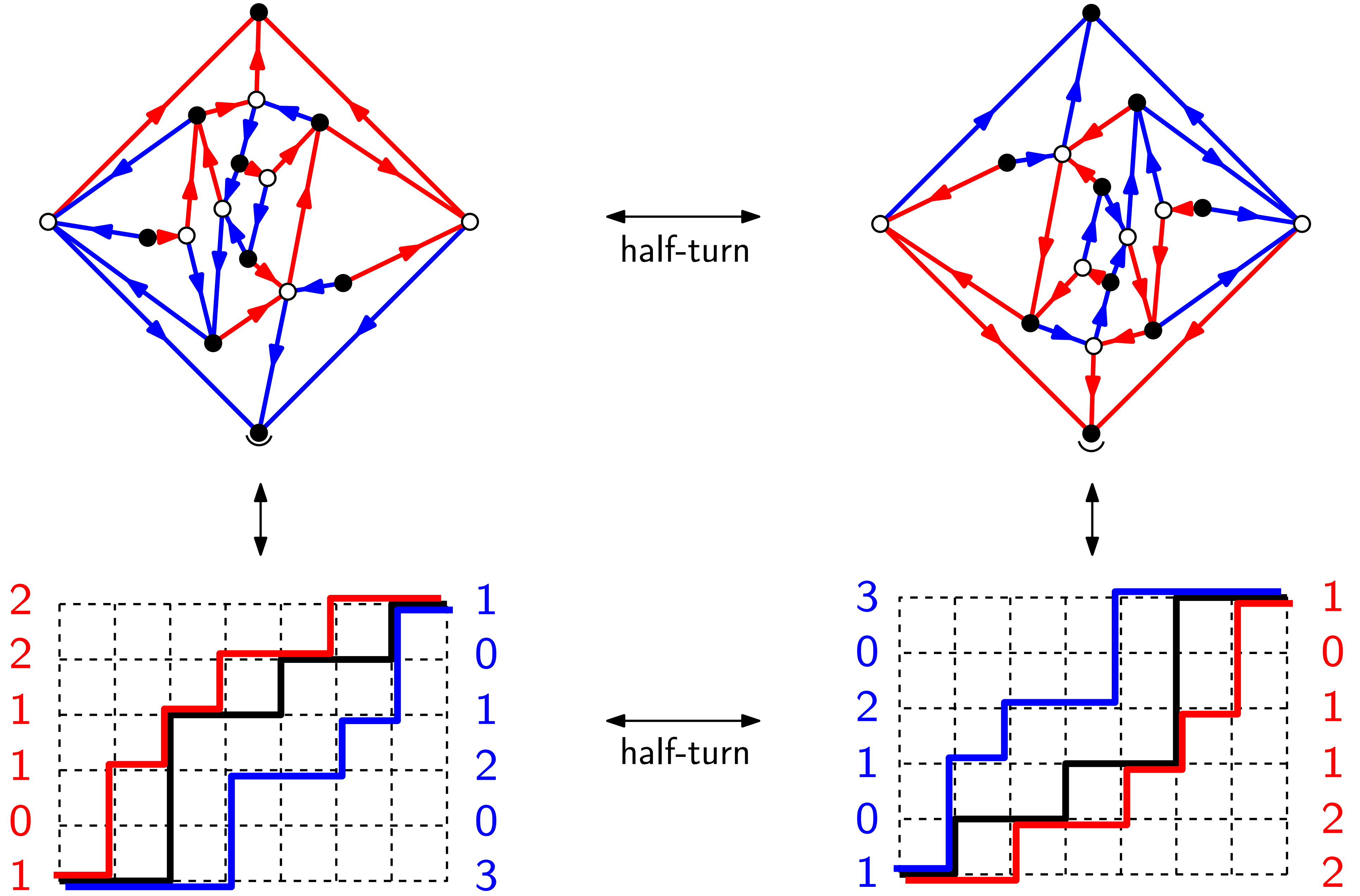




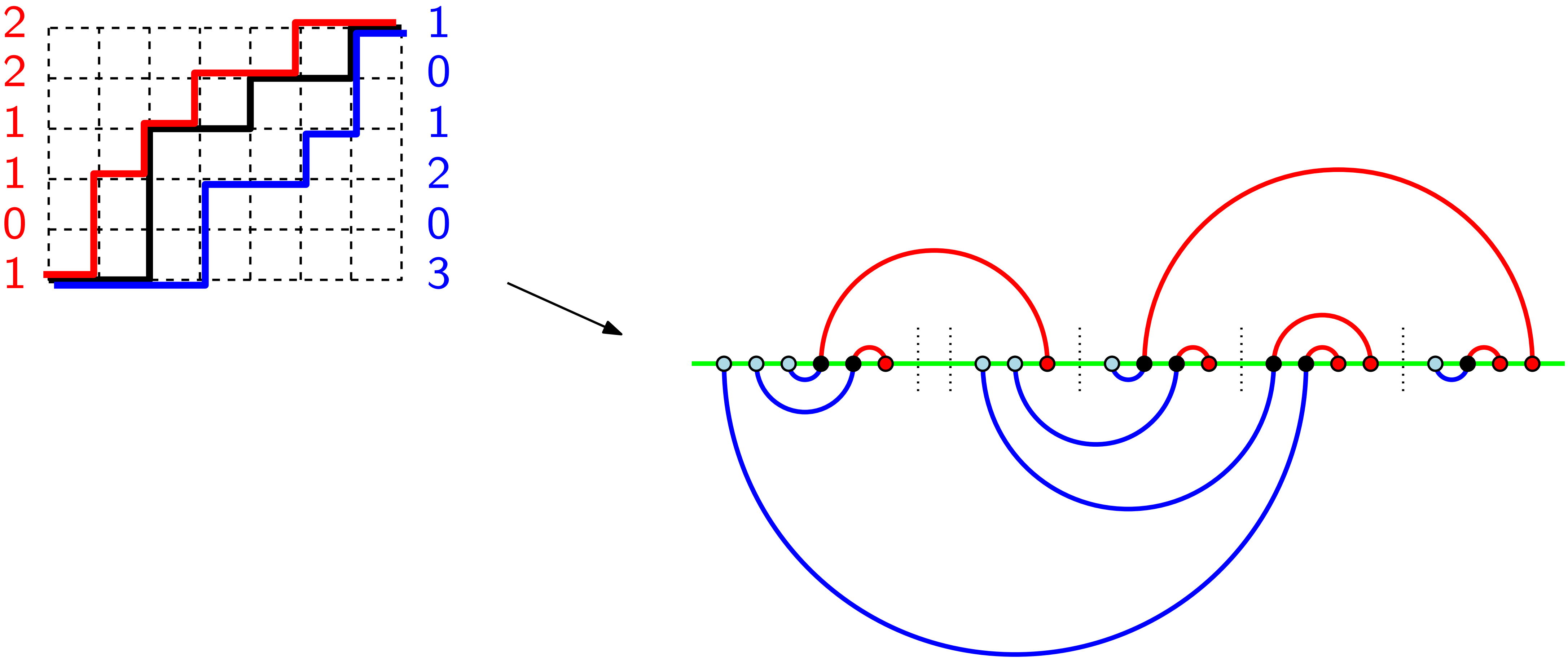
Symmetric reformulation of the bijection

Corollary: bijection commutes with half-turn rotation

$\Rightarrow$ stability of $\mathcal{G}_{i,j} \subset \mathcal{R}_{i,j}$ under half-turn rotation



Proof of the bijection $\mathrm{Sep}_{i,j} \longleftrightarrow \mathcal{R}_{i,j}$



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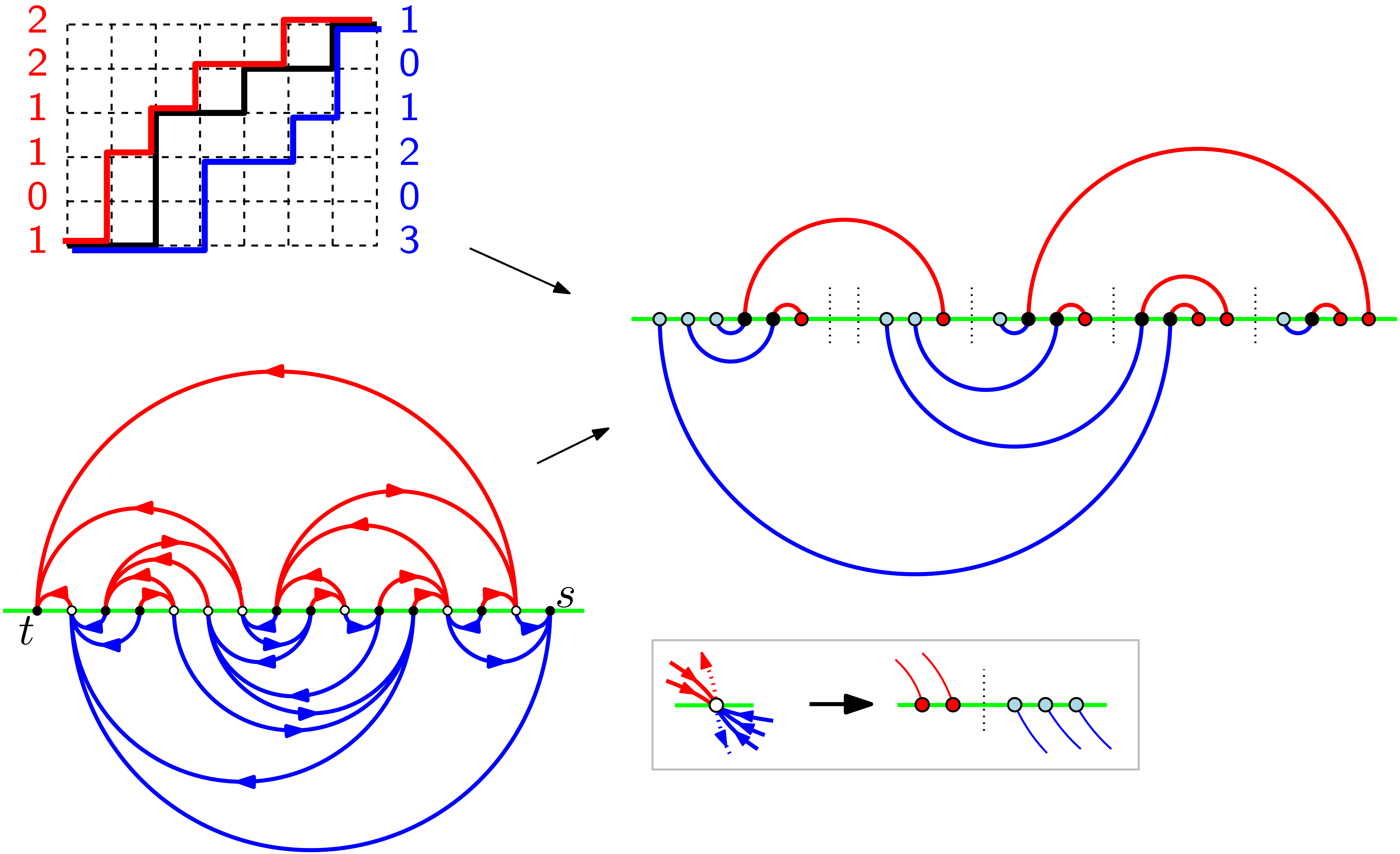
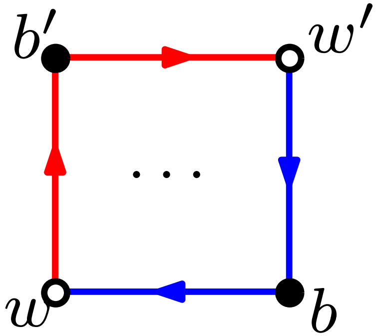


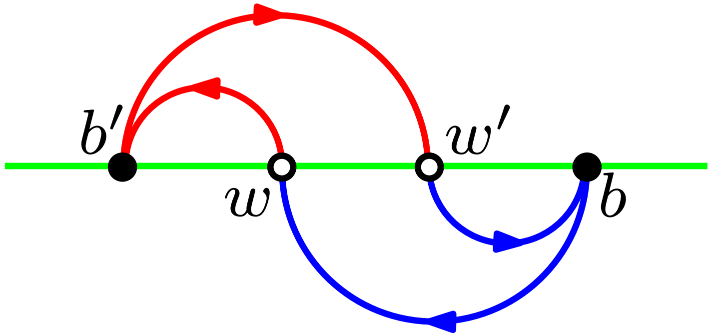
Non-minimality on arc-diagrams

non-minimal (i.e., $\exists$ clockwise cycle)

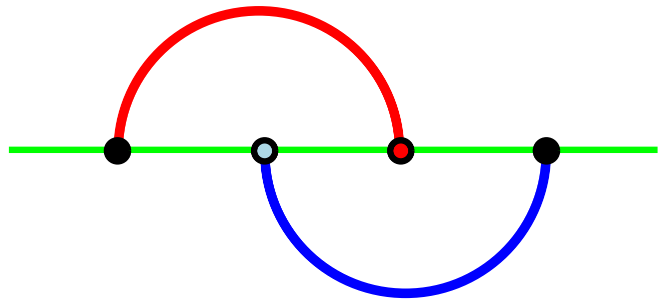
$\Downarrow$

$\exists$ clockwise 4-cycle





2-book embedding



arc-diagram

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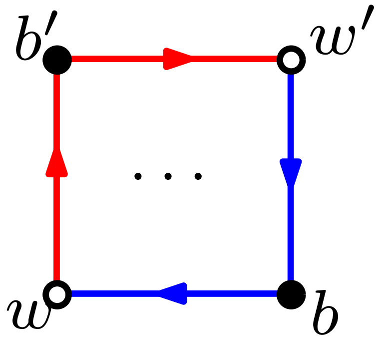

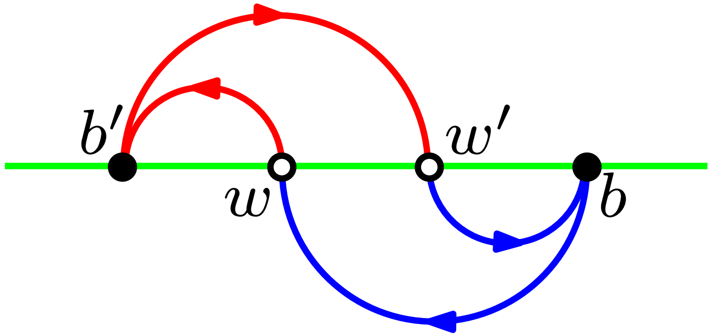

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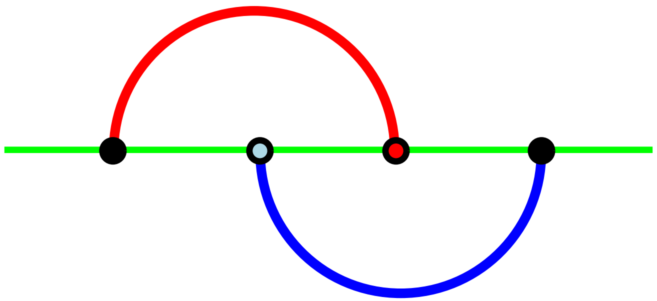
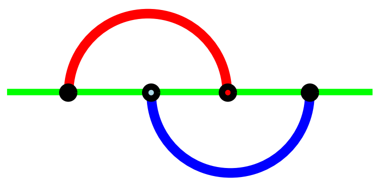


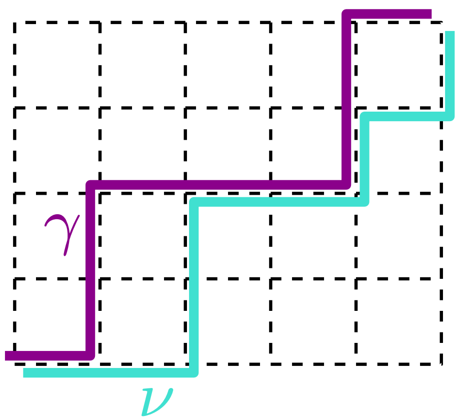
arc-diagram

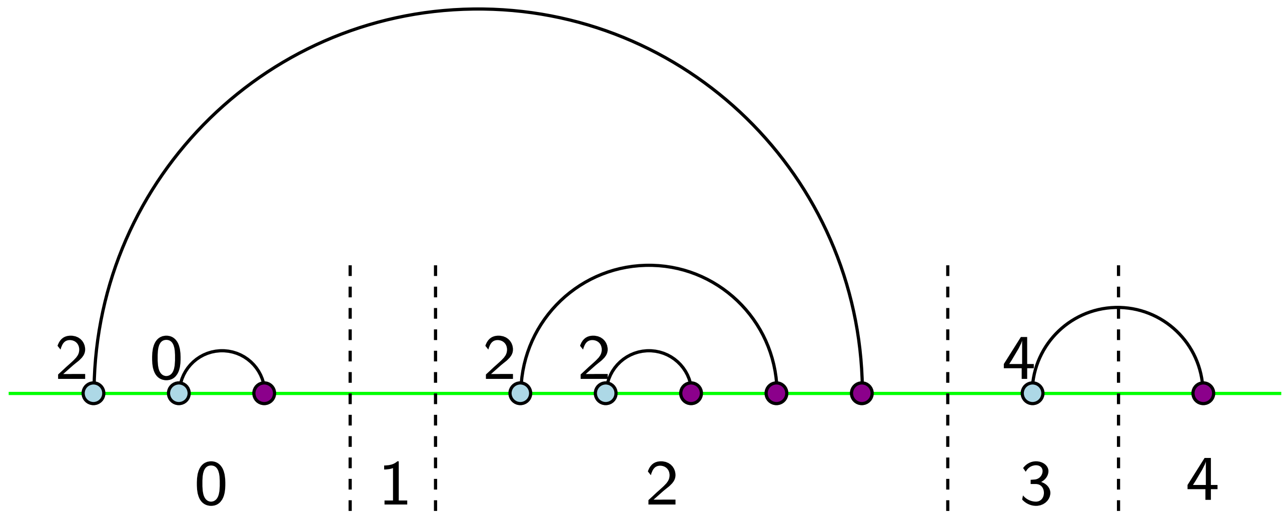
Remains to see that for $R \in \mathcal{R}_{i,j}$

R is in $\mathcal{G}_{i,j}$ iff arc-diagram of R has no 

Bracket-vectors in the ν-Tamari lattice

[Ceballos,Padrol,Sarniento'18]

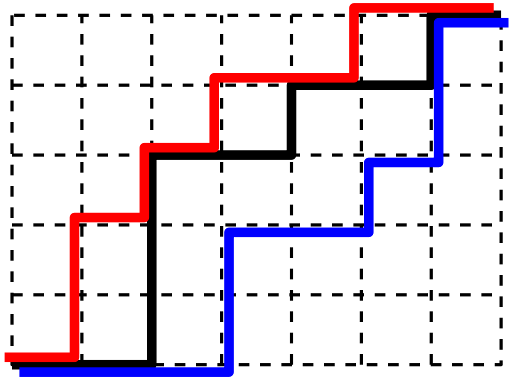




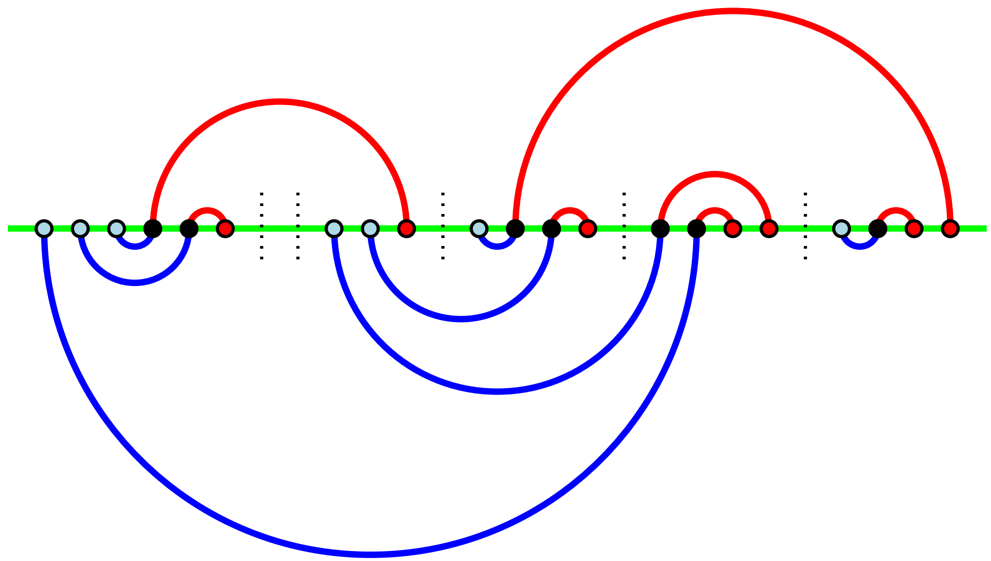
$$V_\nu(\gamma) = (2, 0, 2, 2, 4)$$

Property: $\gamma \leq \gamma'$ in Tam_ν iff $V_\nu(\gamma) \leq V_\nu(\gamma')$

Condition for $R \in \mathcal{R}_{i,j}$ to be in $\mathcal{G}_{i,j}$



$$(\nu, \gamma, \gamma') \in \mathcal{R}_{7,5}$$



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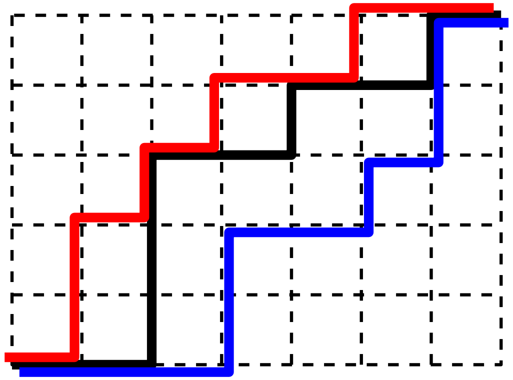

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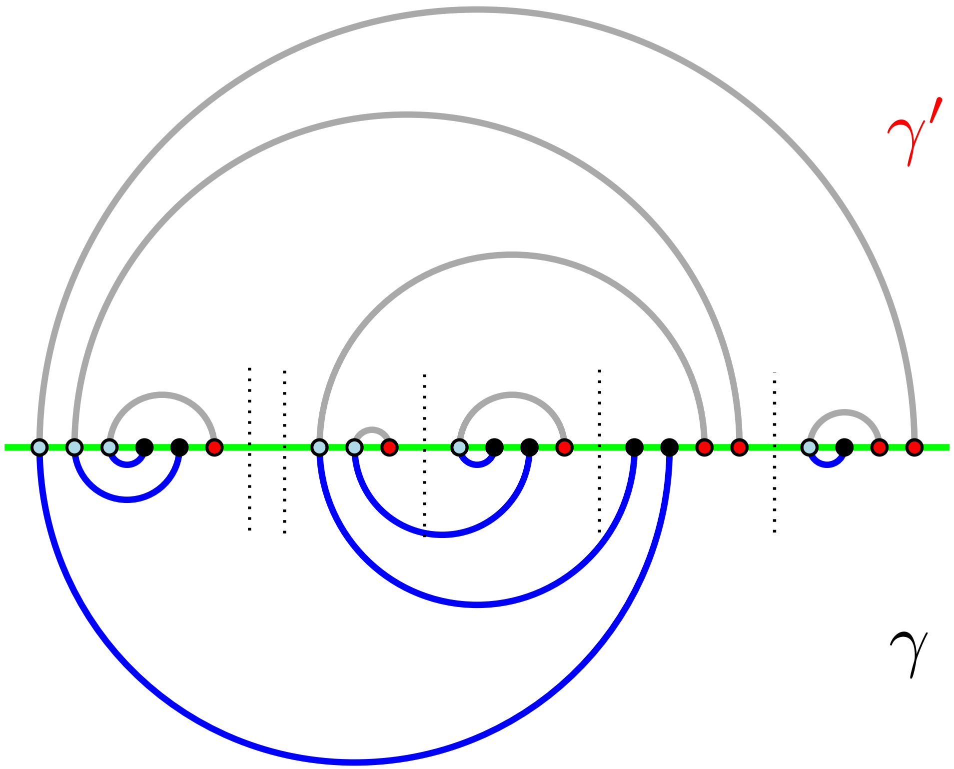

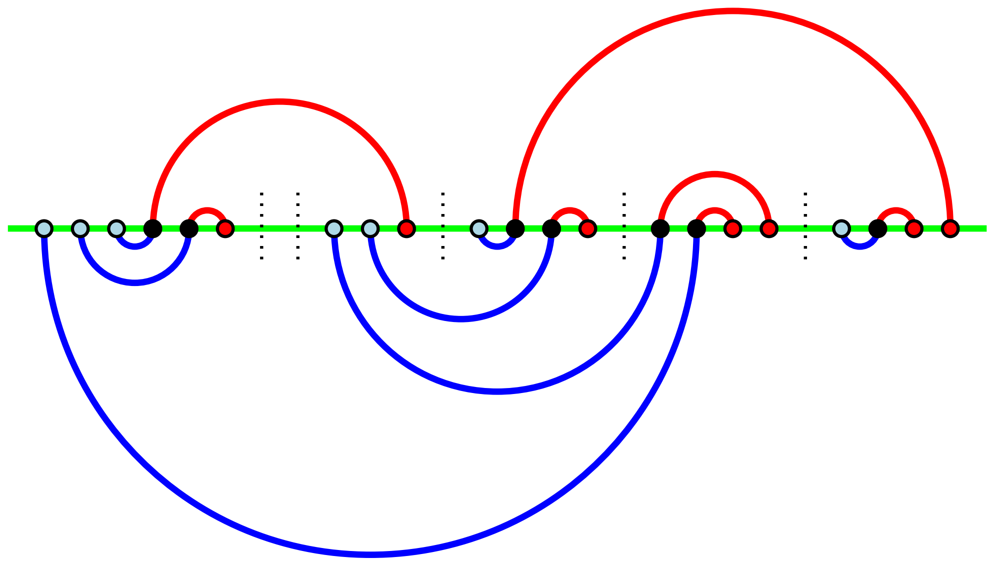

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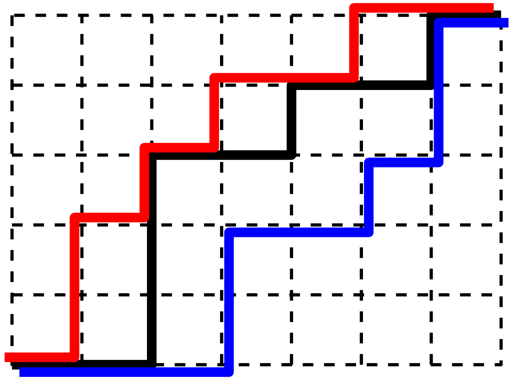

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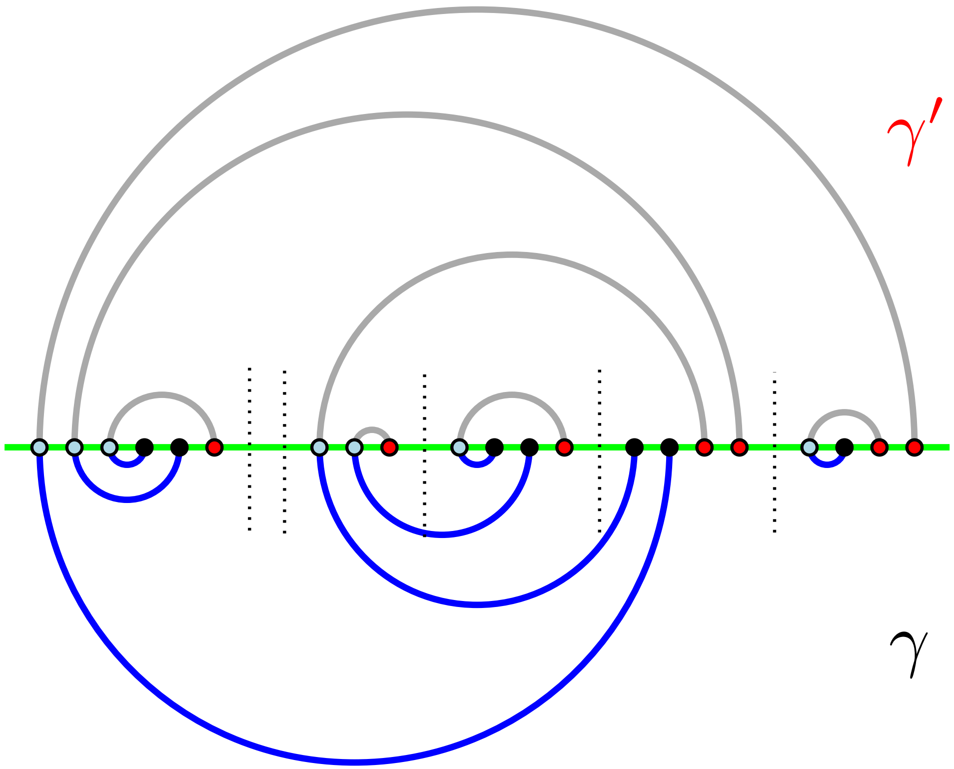
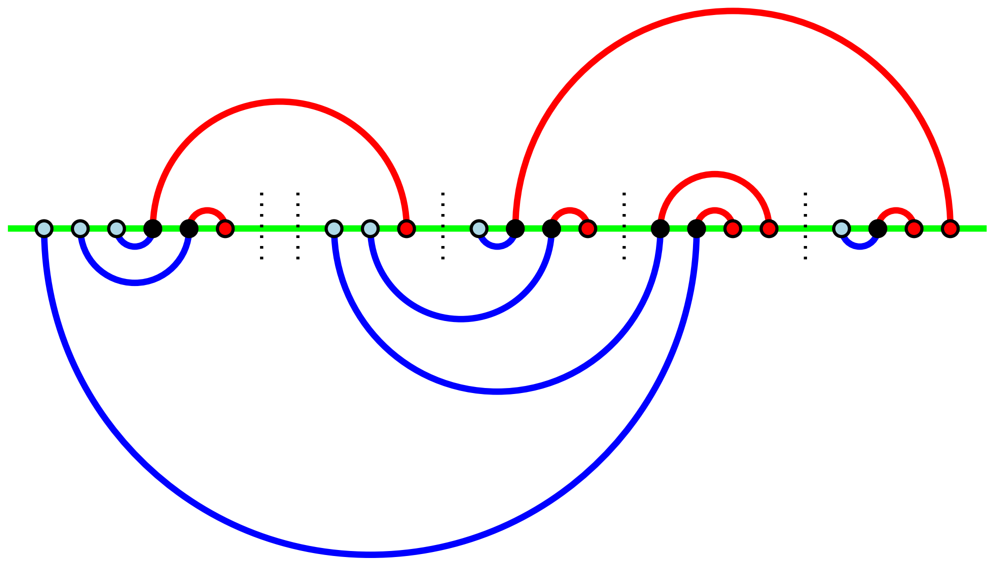
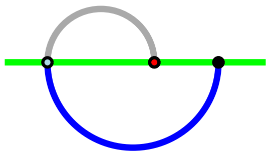
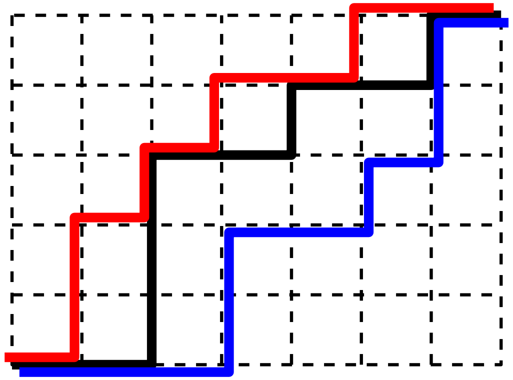




$V_\nu(\gamma) \leq V_\nu(\gamma') \Leftrightarrow$ no



Condition for $R \in \mathcal{R}_{i,j}$ to be in $\mathcal{G}_{i,j}$



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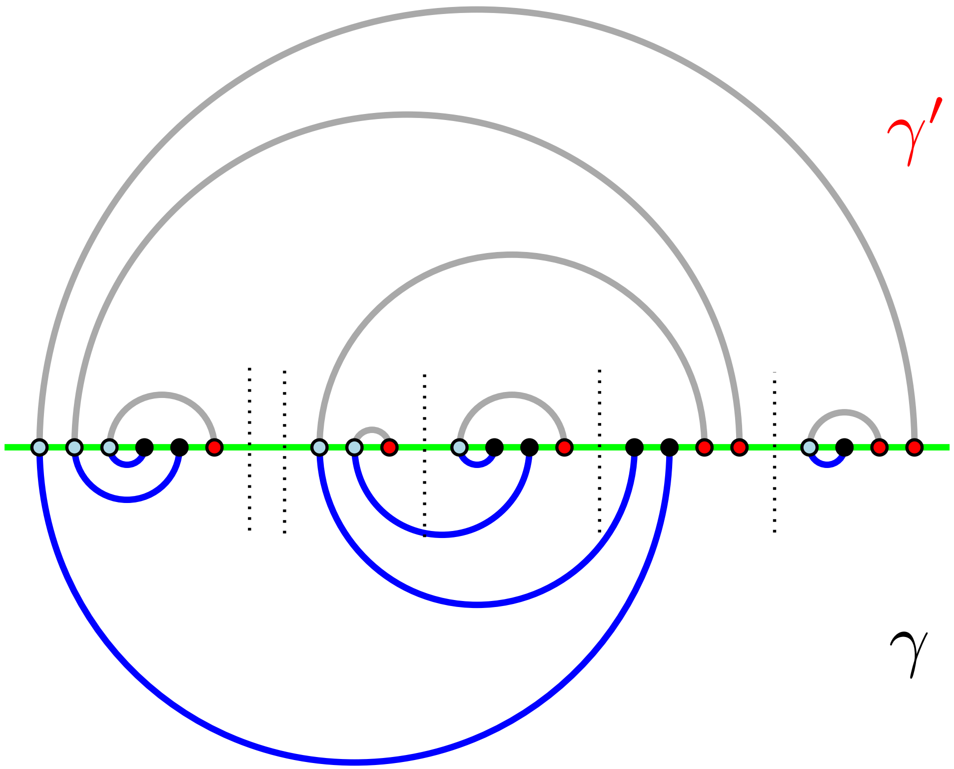
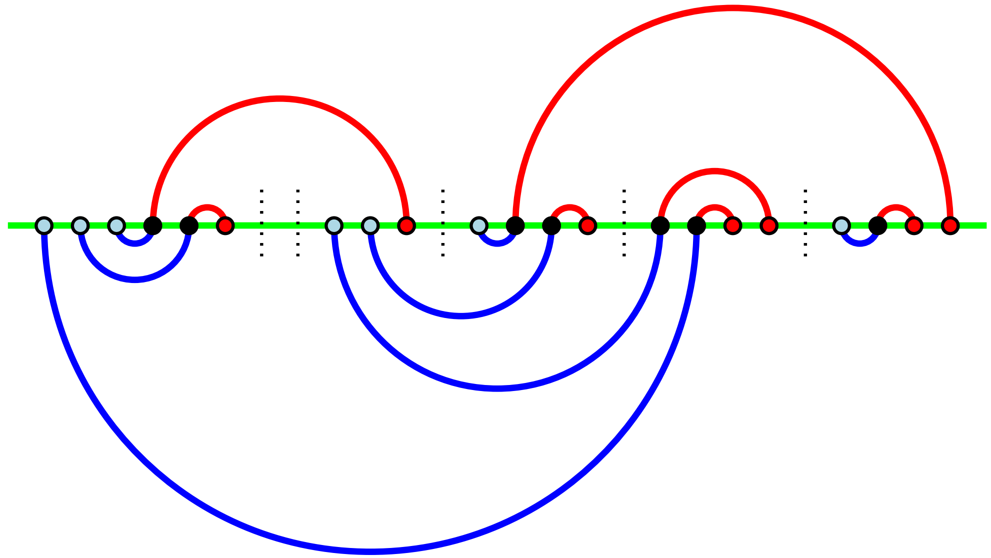
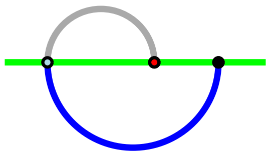
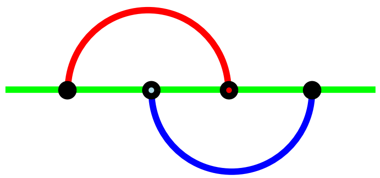




$V_\nu(\gamma) \leq V_\nu(\gamma') \Leftrightarrow$ no  $\Leftrightarrow$ no 

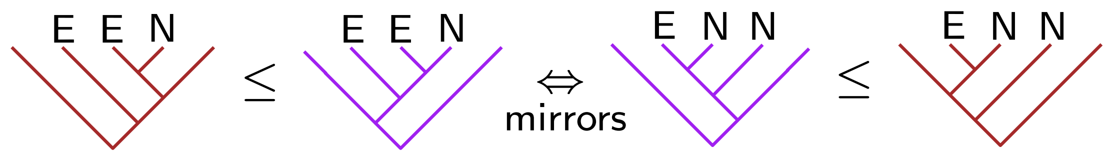
Other approach via the canopy

Recall that if $\gamma \leq \gamma'$ in Tam_n then $\mathrm{Can}(\gamma) \leq \mathrm{Can}(\gamma')$ (with $N < E$)

- Let $F(x, y, z) :=$ series of Tamari intervals, with $x^{\#\left[\begin{smallmatrix}E\\E\end{smallmatrix}\right]} y^{\#\left[\begin{smallmatrix}N\\N\end{smallmatrix}\right]} z^{\#\left[\begin{smallmatrix}E\\N\end{smallmatrix}\right]}$

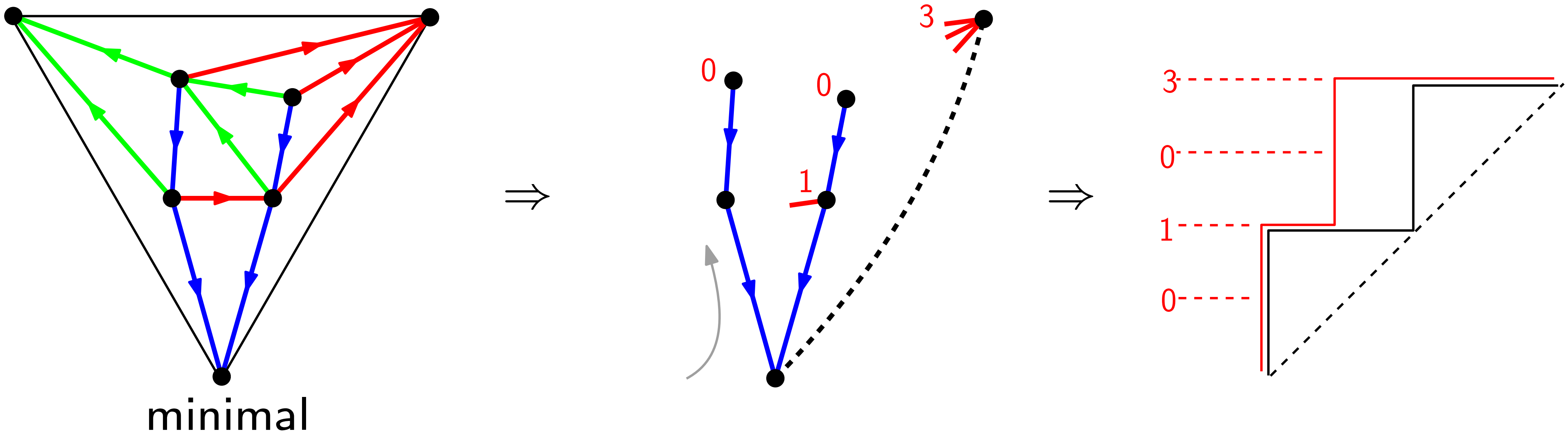
$$F(x, y, z) = 1 + (x + y + z) + (x^2 + y^2 + z^2 + 3xz + 3yz + 4xy)$$
$$x^3 + y^3 + z^3 + 6x^2z + 6xz^2 + 10x^2y + 10xy^2 + 6y^2z + 6yz^2 + 21xyz$$

Rk: symmetry $x \leftrightarrow y$ cf

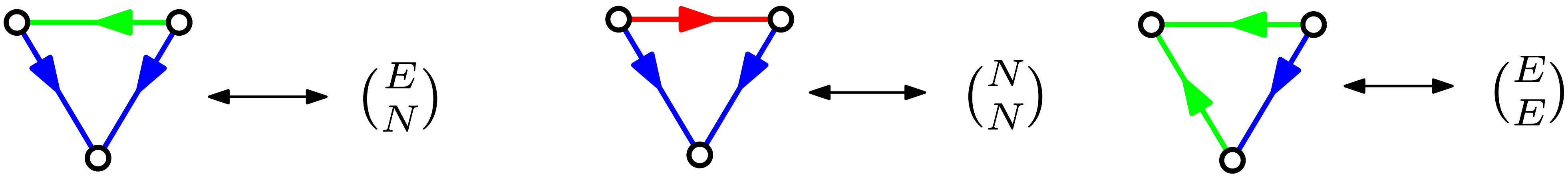


Rk:
$$t\,F(t, t, t) = \sum_{n \geq 1} |\mathcal{I}_n| t^n$$
$$F(x, y, 0) = \sum_{i,j} |\mathcal{G}_{i,j}| x^i y^j$$

3 parameters via Bernardi-Bonichon bijection

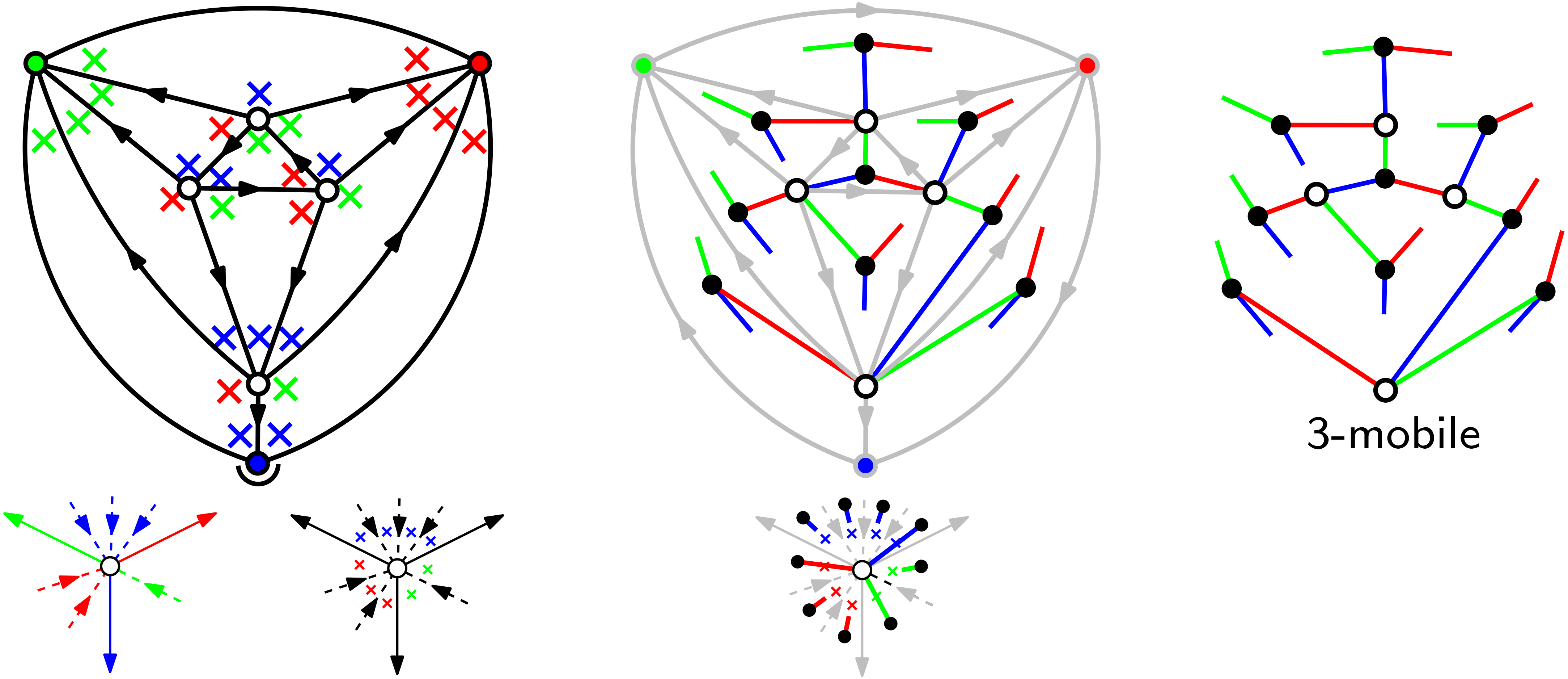


canopy-parameters via the bijection:



Composition to bijection with tree-structures

[F, Poulalhon, Schaeffer'07]



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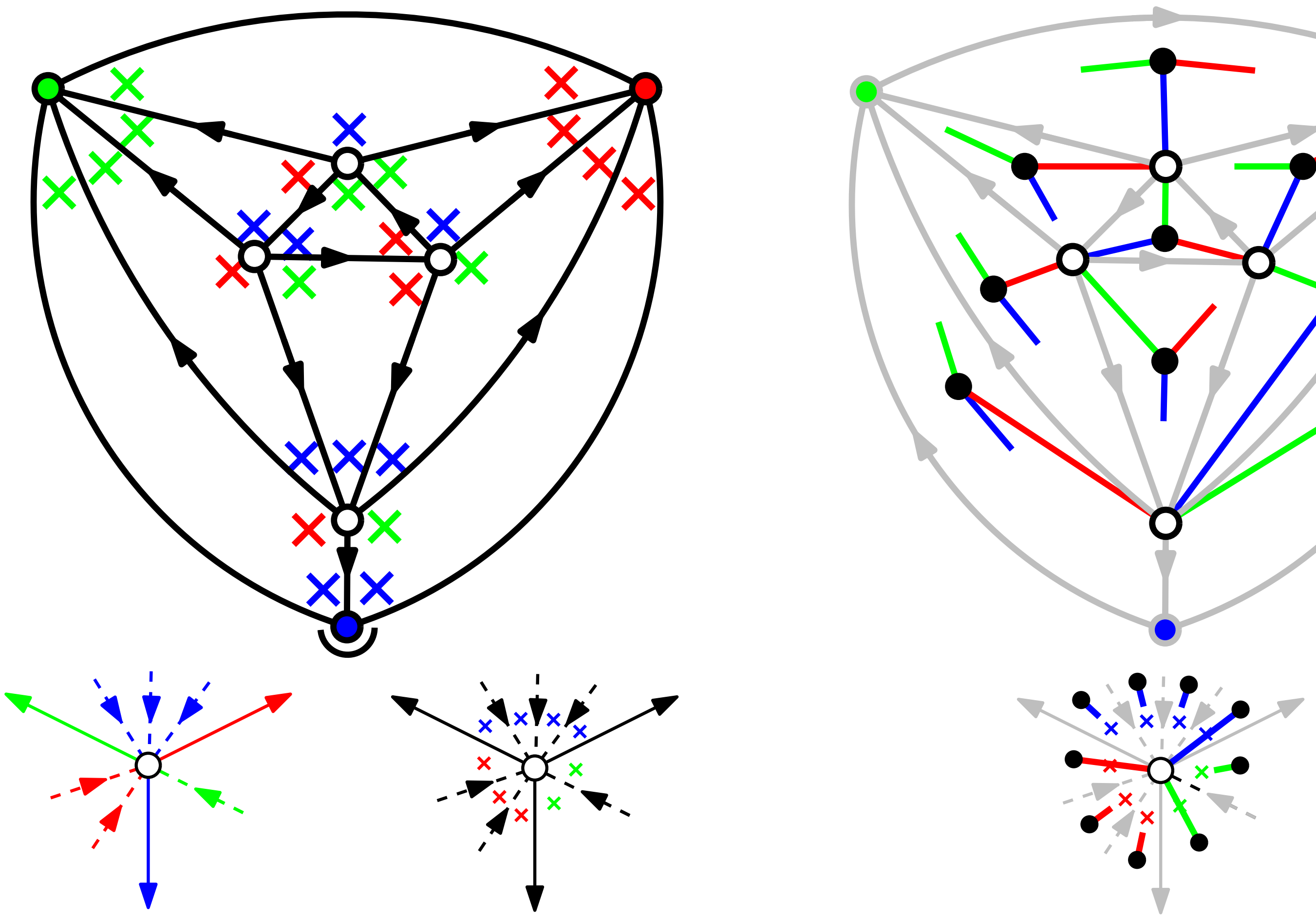

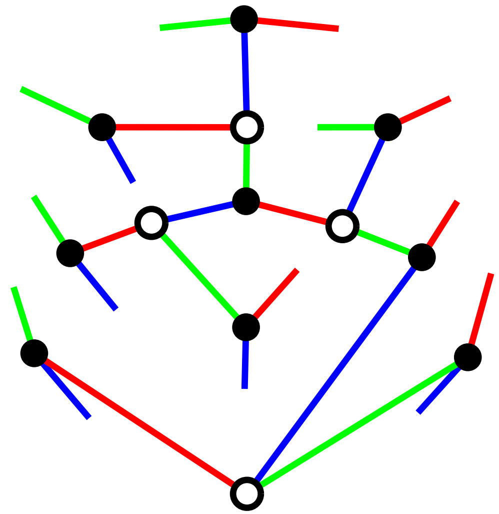

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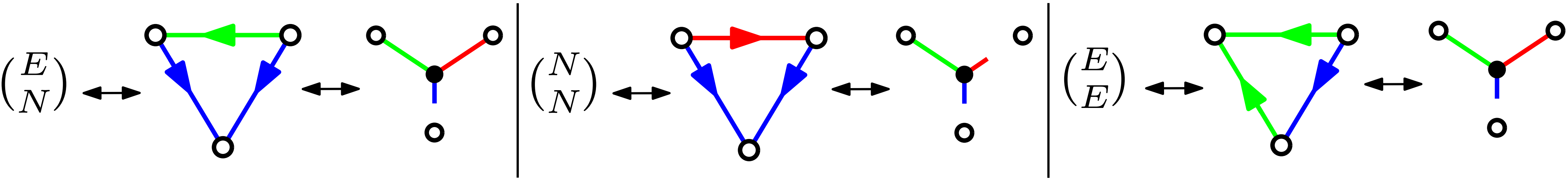


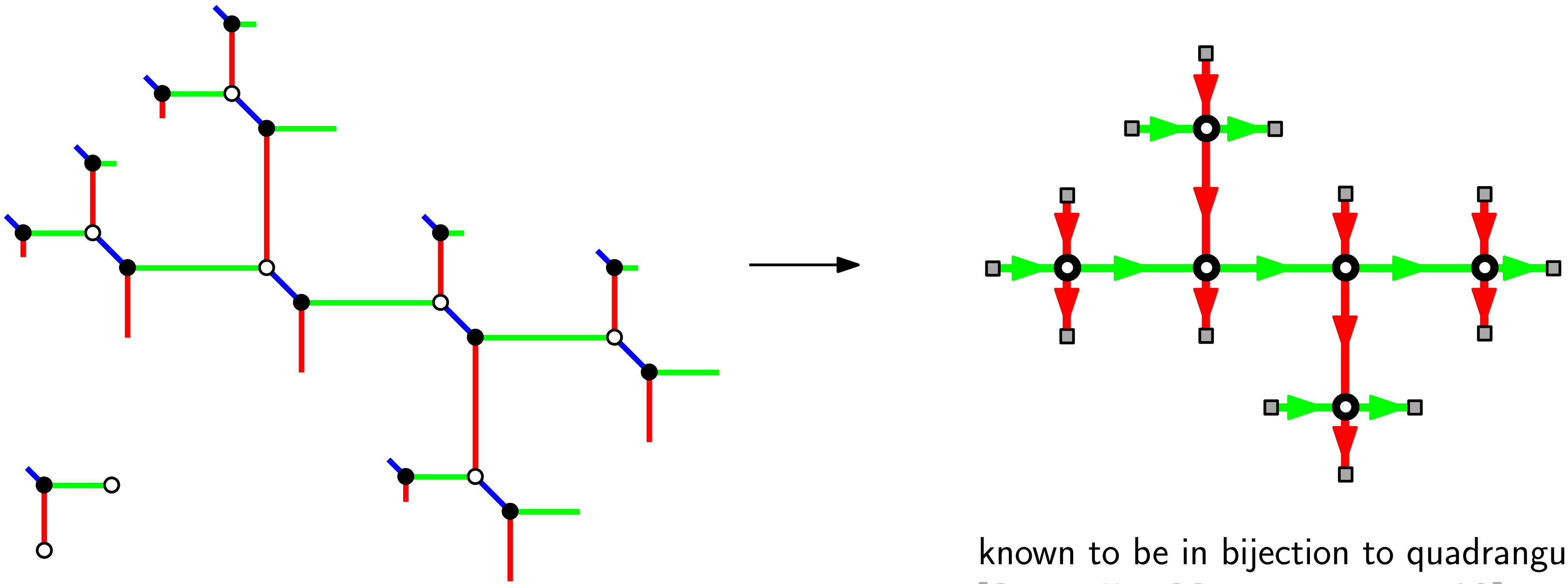
Results

- Trivariate generating function expression:

$$F = xR + yG + zRG - \frac{RG}{(1+R)(1+G)}$$

where $\begin{cases} R = (y+zR)(1+R)(1+G)^2 \\ G = (x+zG)(1+G)(1+R)^2 \end{cases}$

- Simplification of the trees in the synchronized case:

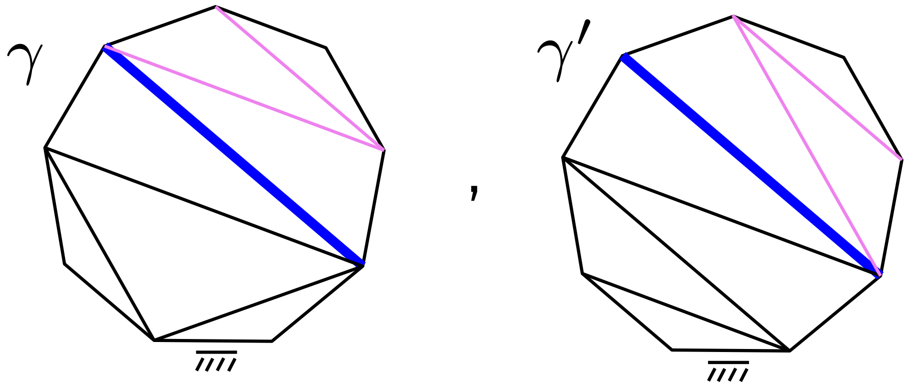


no

known to be in bijection to quadrangulations
[Schaeffer'98, Bernardi, F'10]

New Tamari intervals and canopy symmetry

- An interval $(\gamma, \gamma') \in \mathcal{I}_n$ is called **new** if (with dissection point of view) γ and γ' have **no common chord** [Chapoton'06]



an interval $\gamma \leq \gamma'$ that is not new

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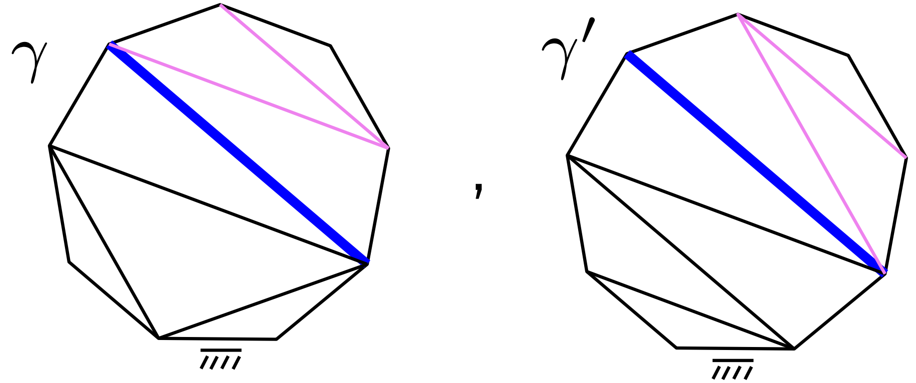

an interval $\gamma \leq \gamma'$ that is not new

- Let $G(x, y, z) :=$ series $F(x, y, z)$ restricted to **new** Tamari intervals

$$\frac{1}{z}G(x,y,z) = 1 + (x+y+z) + (x^2+y^2+z^2+3xy+3xz+3yz)$$
$$+(x^3+y^3+z^3+6x^2y+6xy^2+6x^2z+6xz^2+6y^2z+6yz^2+17xyz)+\cdots$$

symmetry in the 3 variables!

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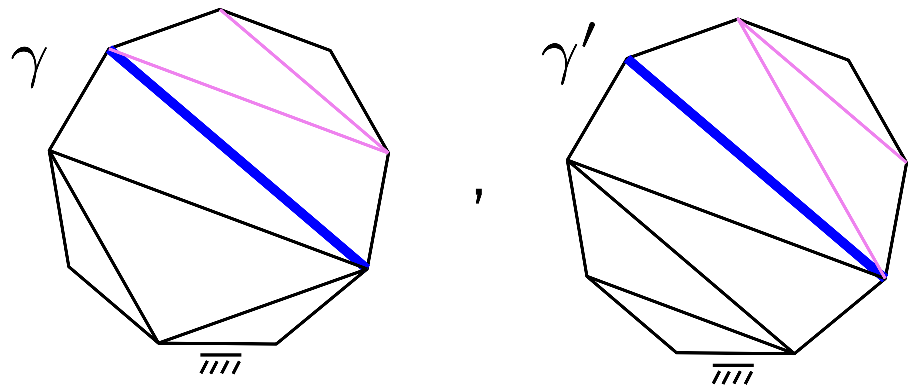

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Bijective explanation via bipartite maps! **[Fang'19+]**