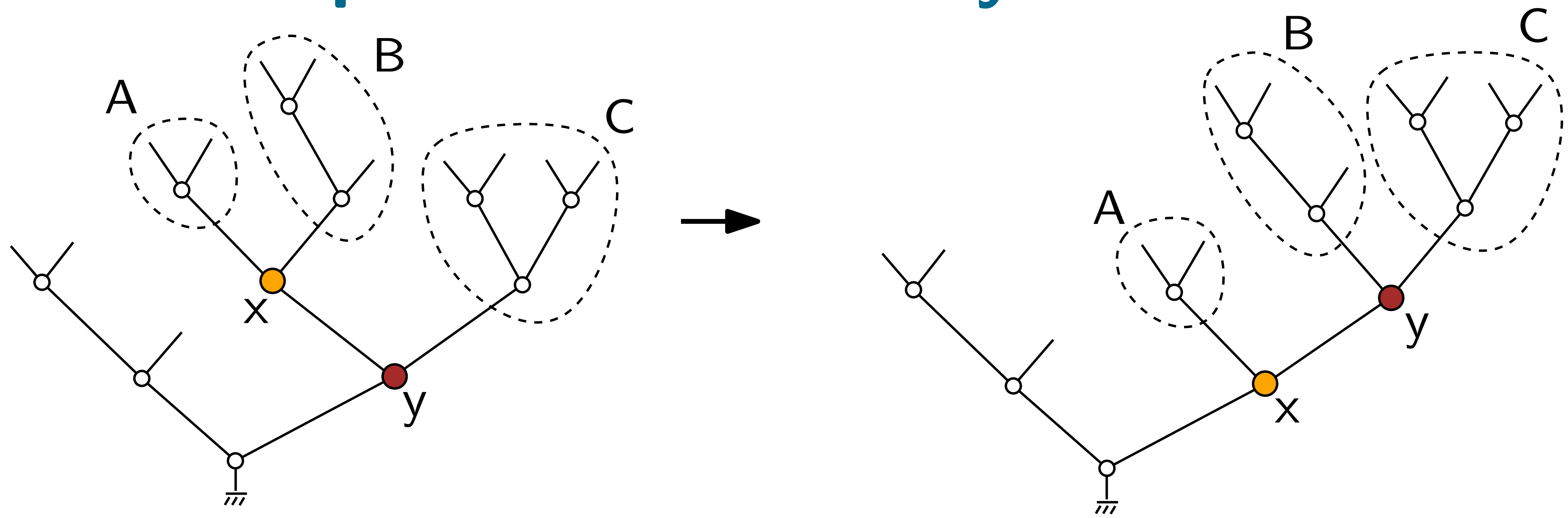
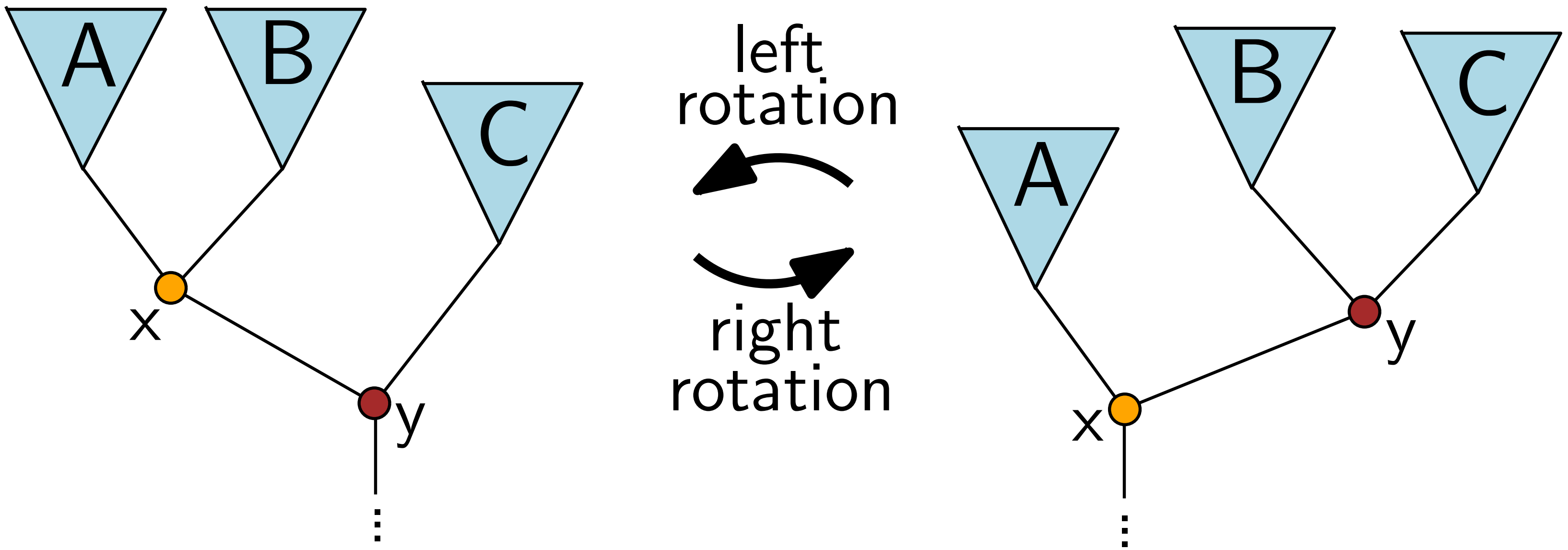
On Tamari intervals and planar maps

Éric Fusy (CNRS/LIX, École Polytechnique)

Berlin, Nov. 6th 2017

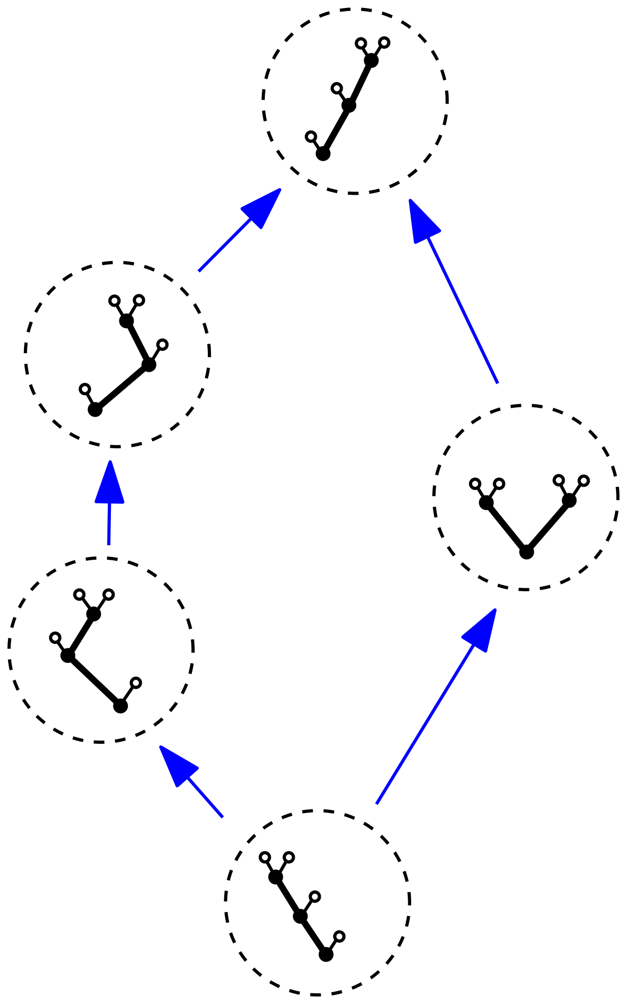
Rotation operations on binary trees



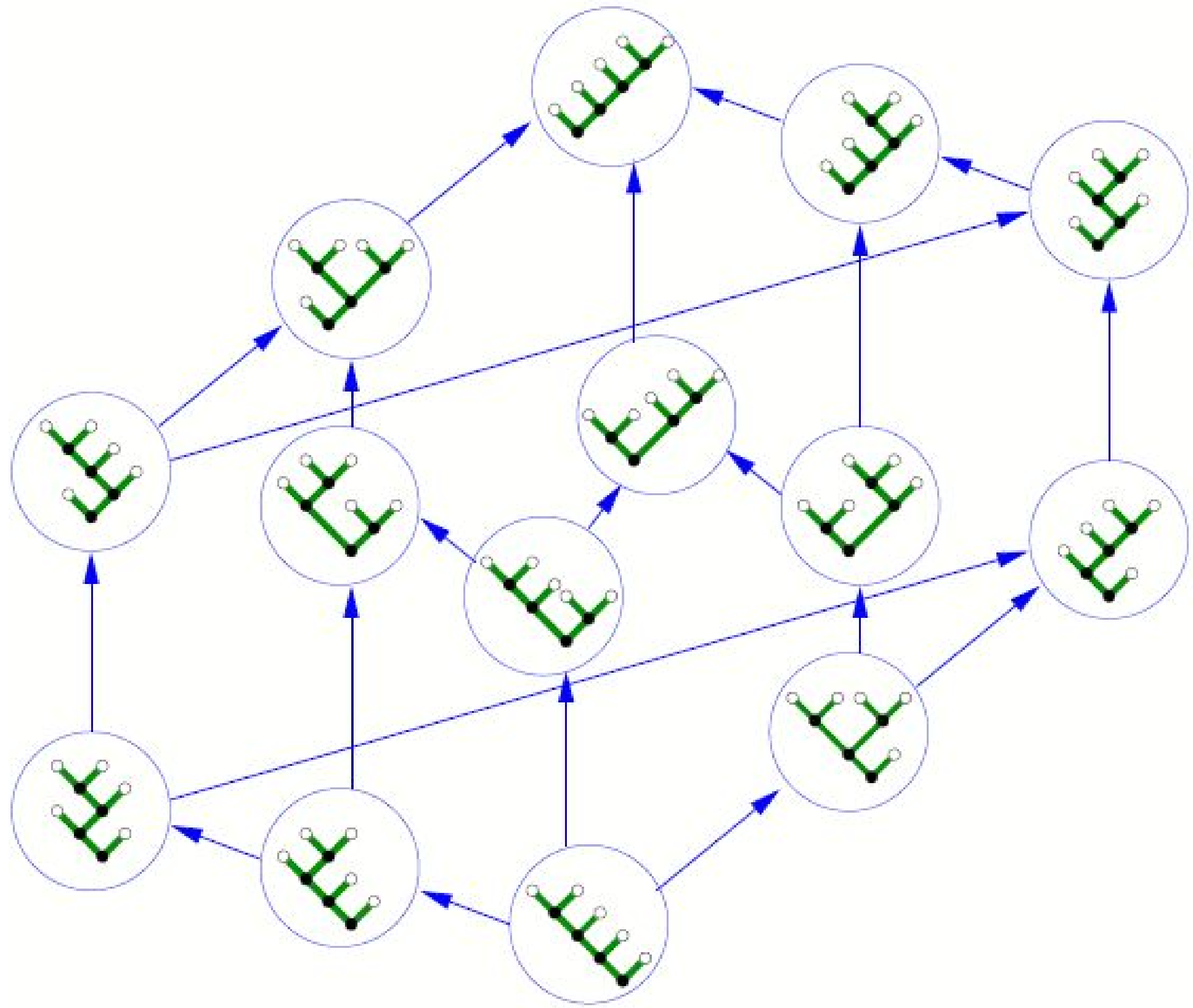


The Tamari lattice

The Tamari lattice $\mathcal{L}_n$ is the partial order on binary trees with n nodes where the covering reation corresponds to right rotation

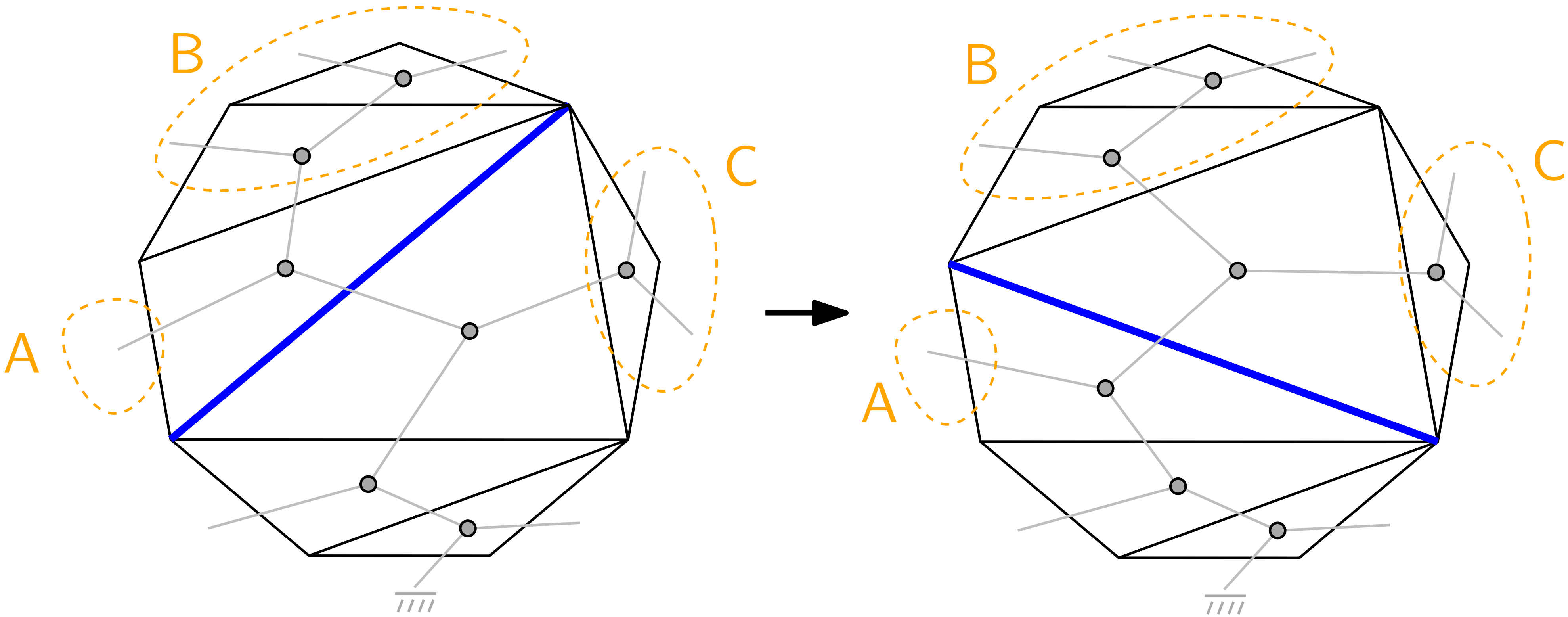


n=3

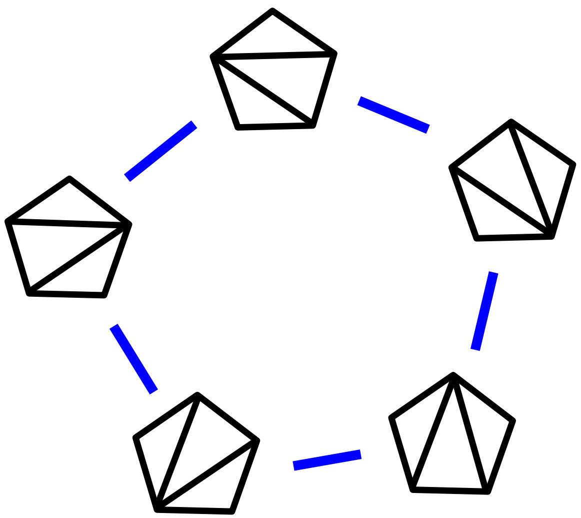


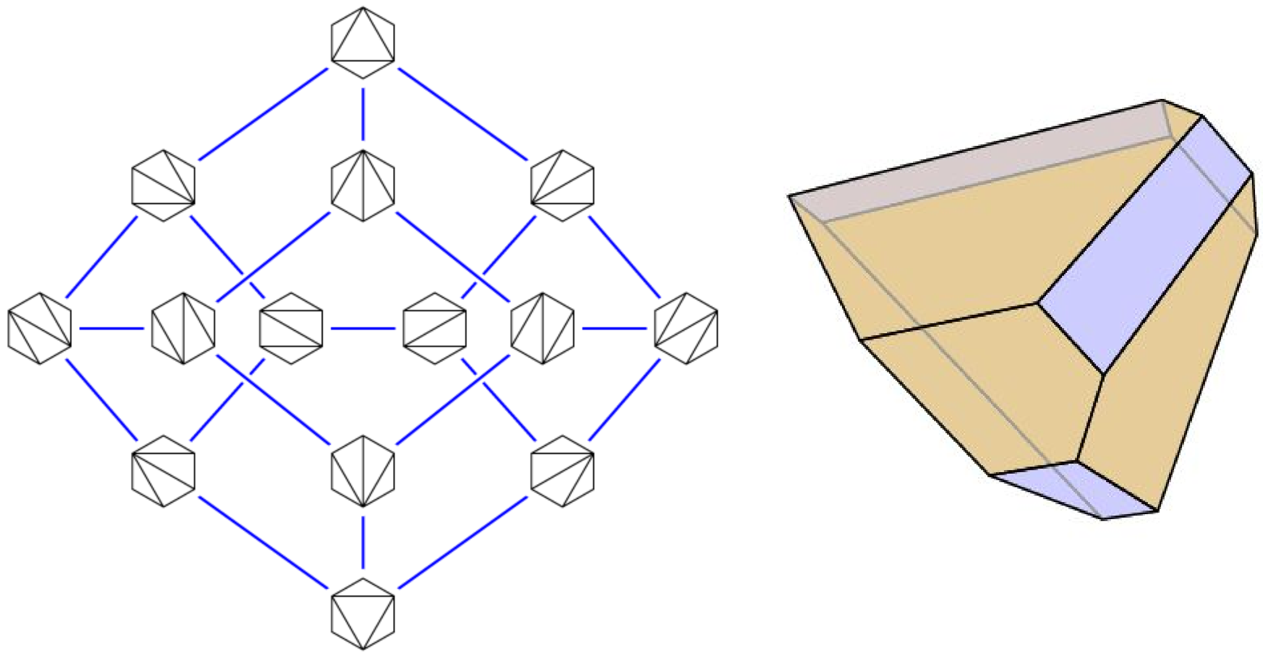
n=4

Rotation ⇔ flip on triangulated dissections



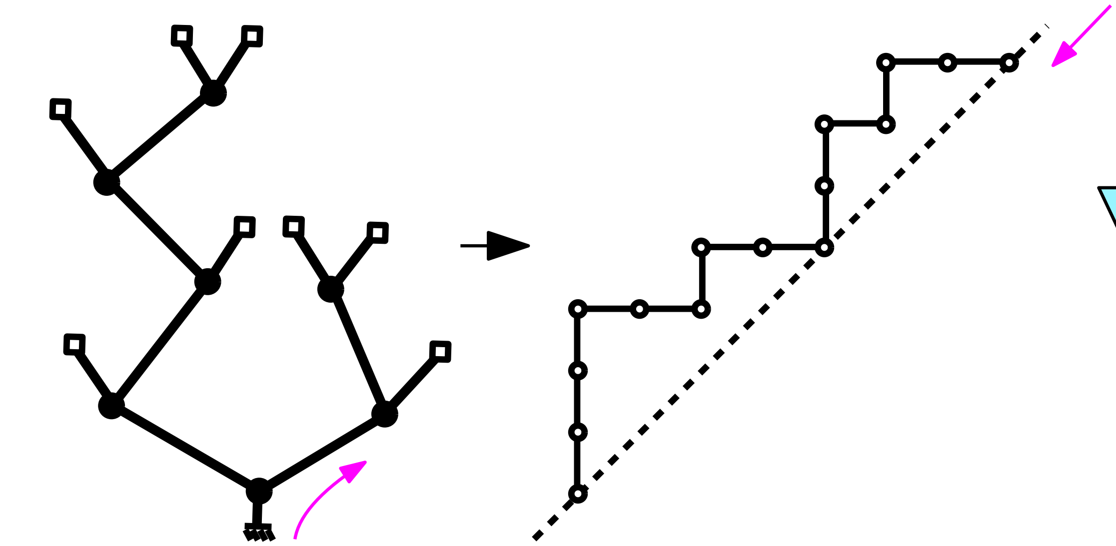
cf the associahedron

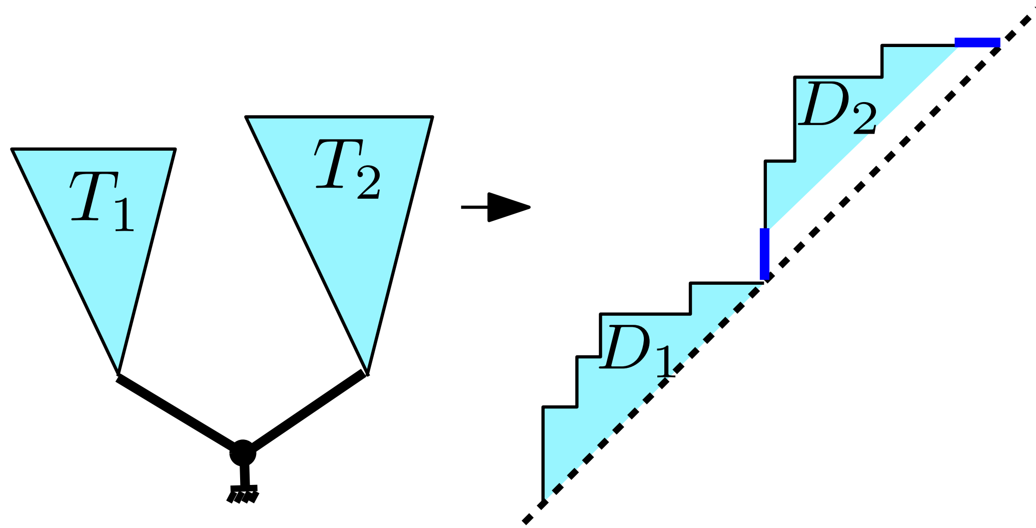




The covering relation for Dyck paths

- Encoding by left-to-right postfix order (⇔ right-to-left prefix order)





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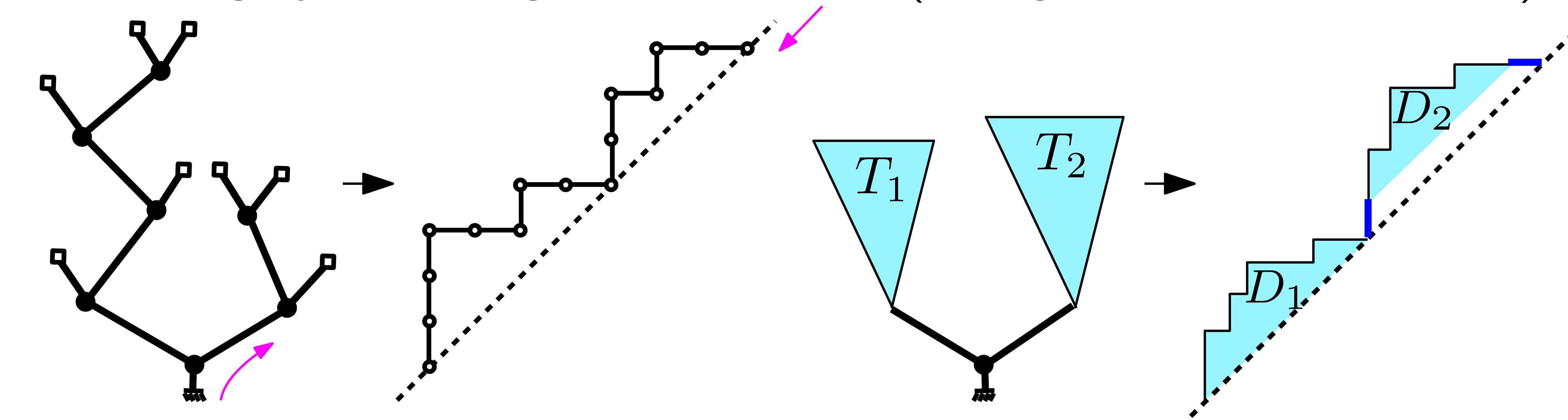
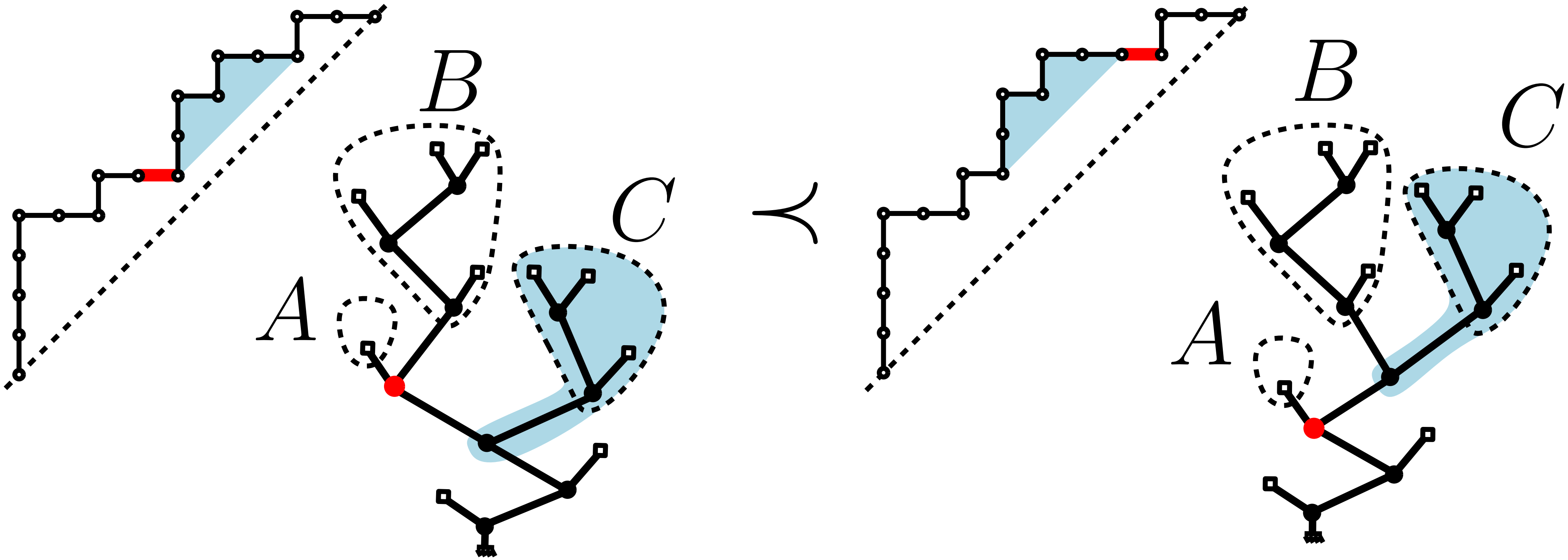
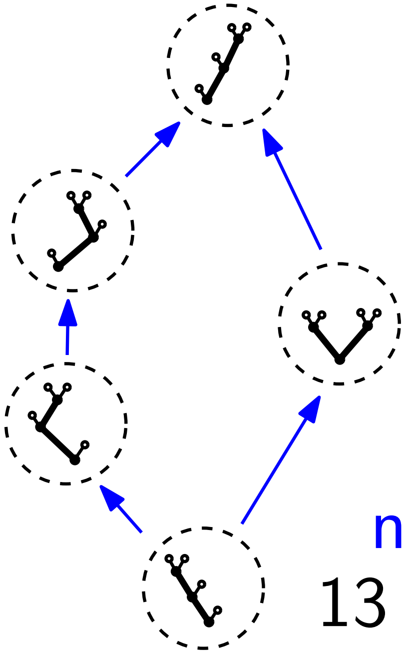


- Effect of a rotation on the associated Dyck path:



Tamari intervals

An interval in $\mathcal{L}_n$ is a pair (t, t') such that $t \leq t'$



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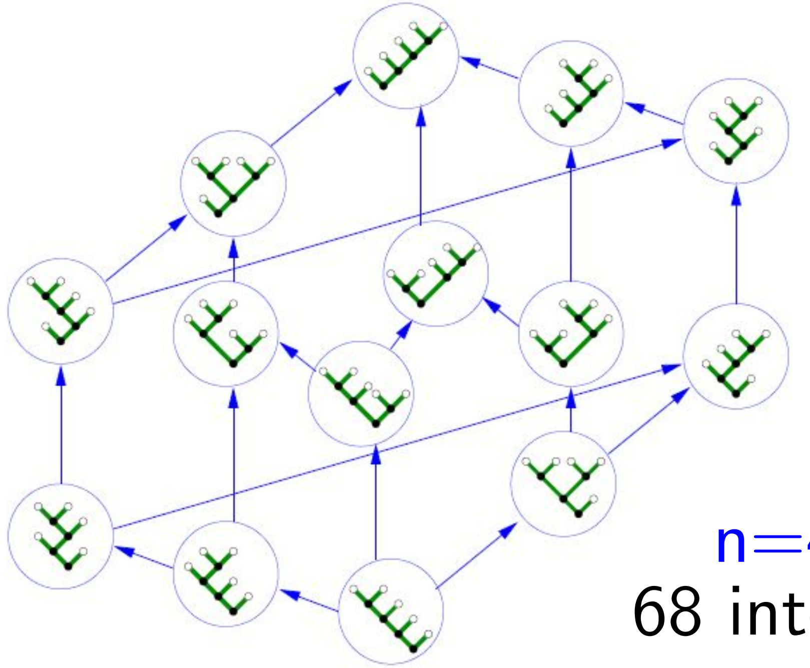
13 intervals



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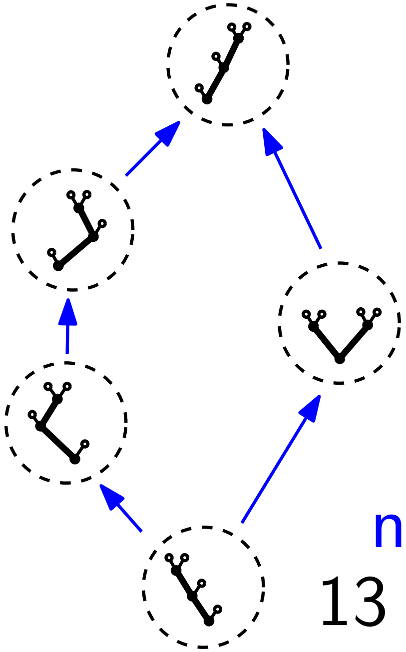

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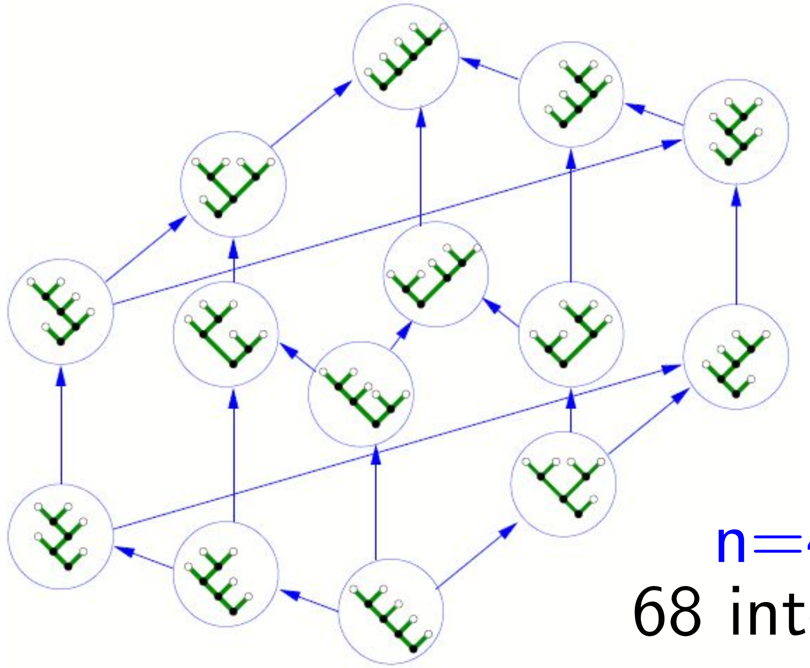

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Theorem [Chapoton'06]: there are $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ intervals in $\mathcal{L}_n$

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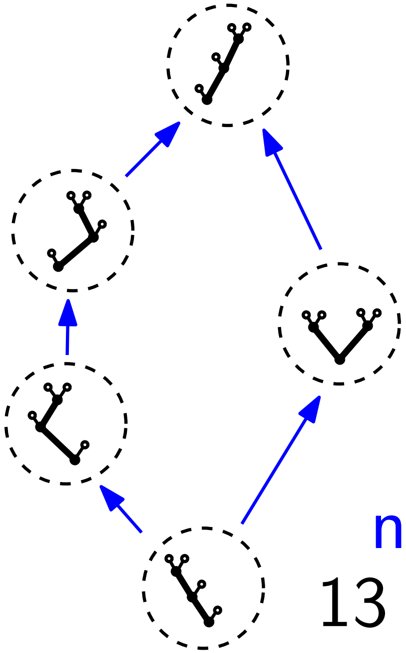

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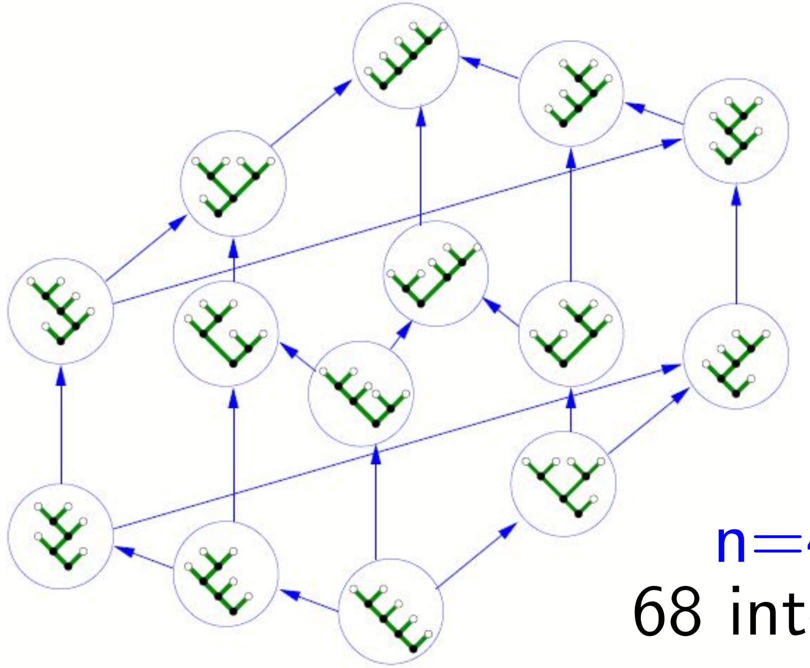

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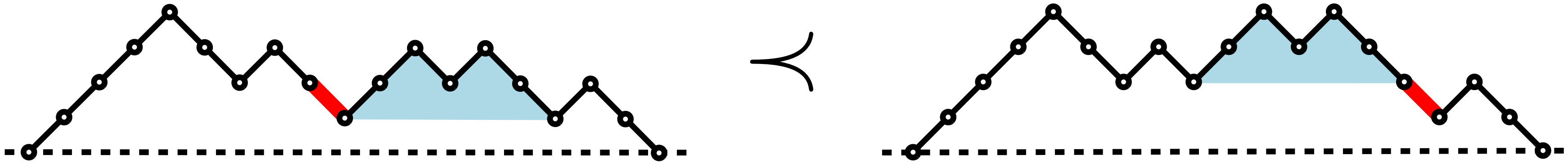
Very active research domain over last 10 years:

- various extensions with nice counting formulas
  - m-Tamari [Bousquet-Mélou,F,Préville-Ratelle'11]
  - labelled m-Tamari [Bousquet-Mélou,Chapuy,Préville-Ratelle'12]
  - v-Tamari [Préville-Ratelle, Viennot'14]
- connections to algebra [Bergeron, Préville-Ratelle'11]
- connections to geometry (associahedron and extensions)
- bijective links: planar maps [Bernardi,Bonichon'07] [Fang, Préville-Ratelle'16]
  interval posets [Chatel,Pons'13]

Characterization of intervals by length-vectors



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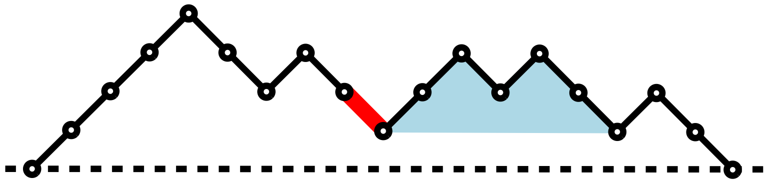

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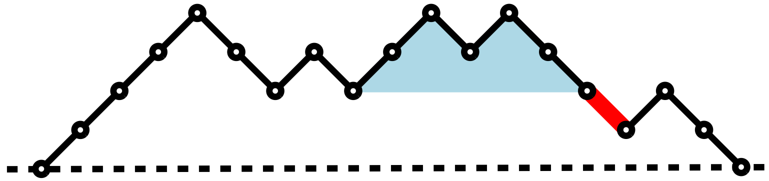
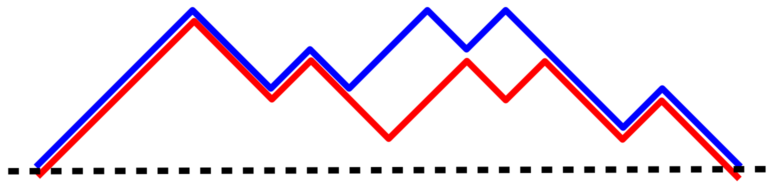


Rk: if $t \leq t'$ in $\mathcal{T}_n$, then t is below t'



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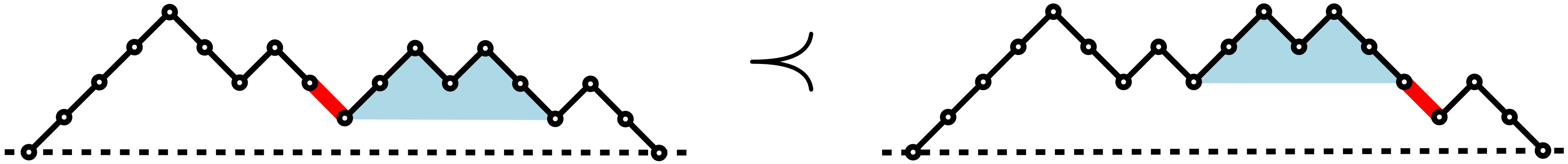

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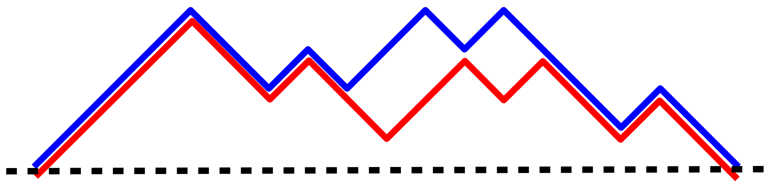

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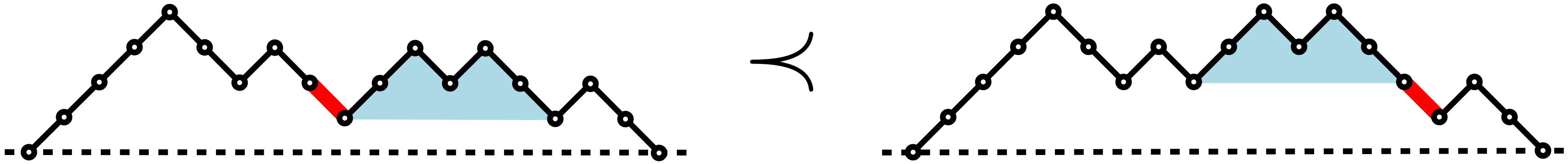

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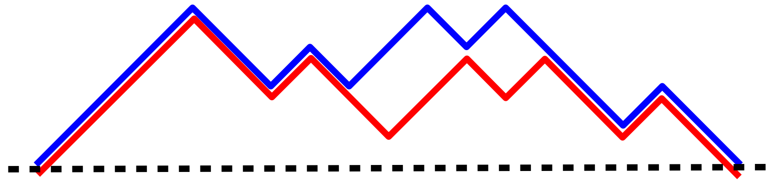
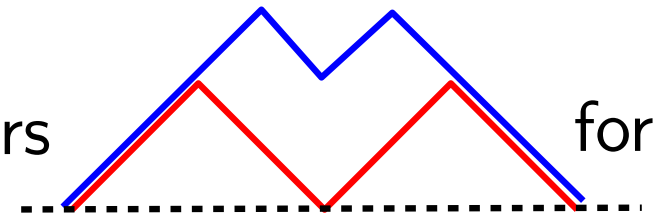


Q: How to characterize pairs  forming an interval in $\mathcal{L}_n$?

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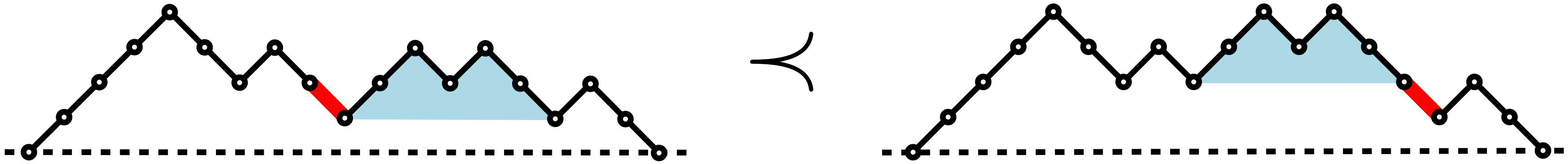

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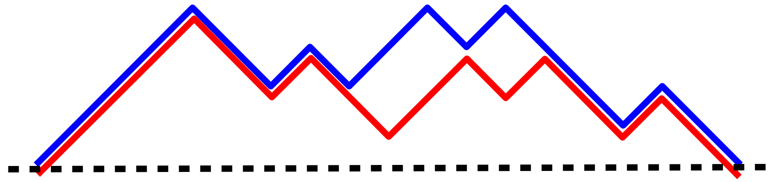
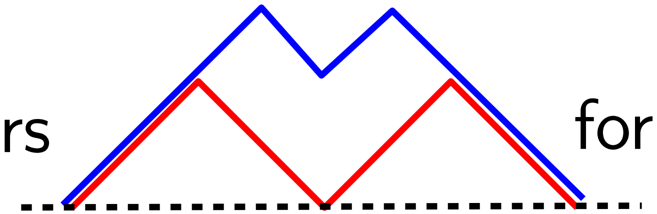
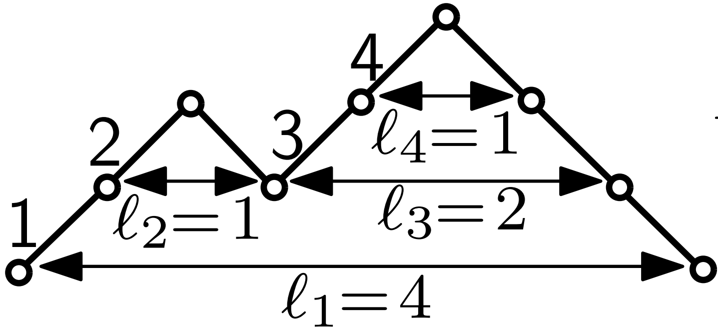


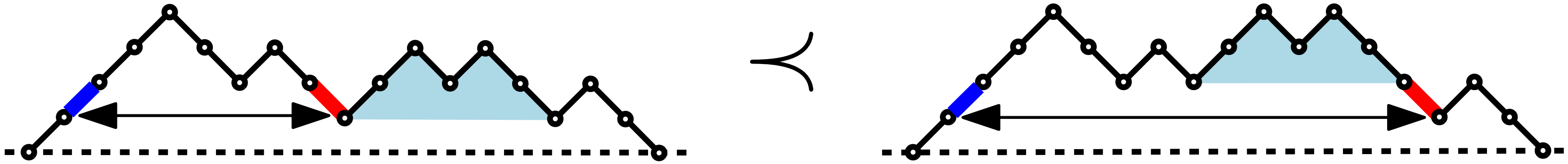
Q: How to characterize pairs  forming an interval in $\mathcal{L}_n$?

Length-vector L_D of D:



$$L_D = (4, 1, 2, 1)$$

Characterization of intervals by length-vectors



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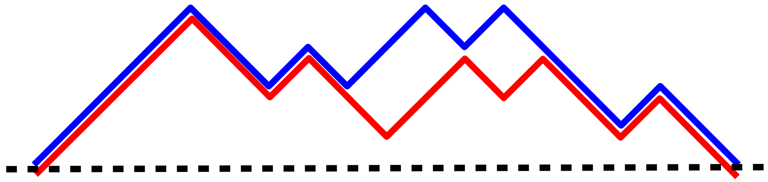

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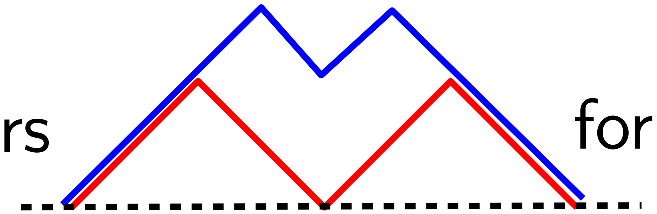

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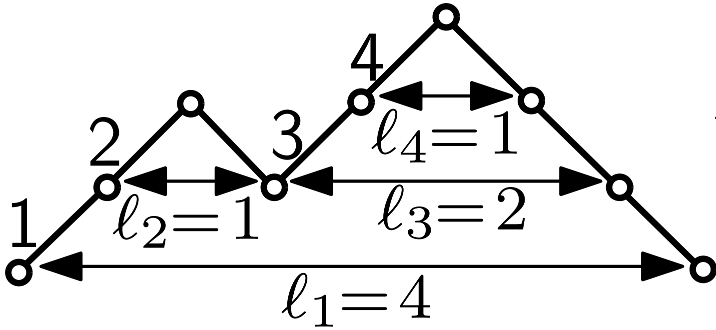
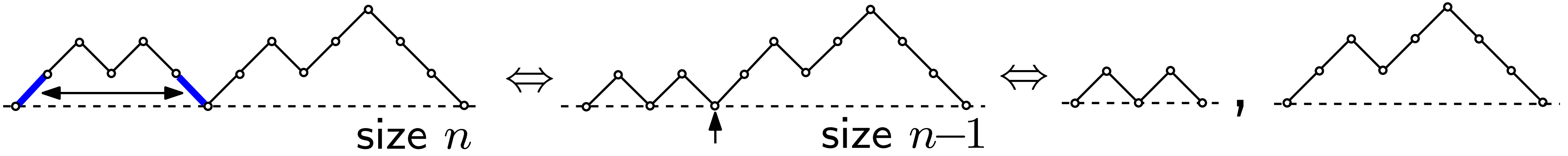


$L_D = (4, 1, 2, 1)$

Lem: $D \leq D'$ in $\mathcal{L}_n$ iff $L_D \leq L_{D'}$

Recursive decomposition of Dyck paths

- Reduction of a Dyck path:



(removes 1st component in length-vector)

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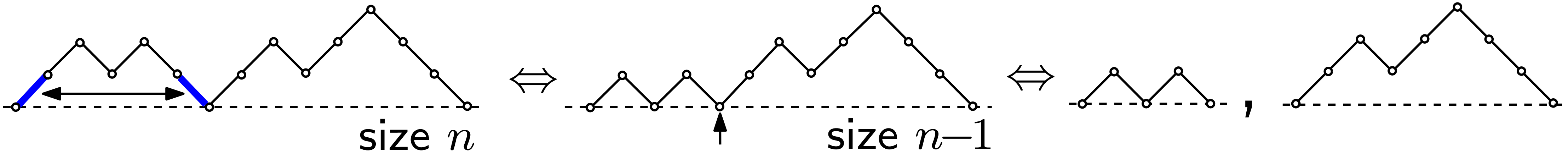

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  Let a_n be the number of Dyck paths of length $2n$.
  Then $a_0 = 1$ and

$$a_n = \sum_{i+j=n-1} a_i \cdot a_j$$

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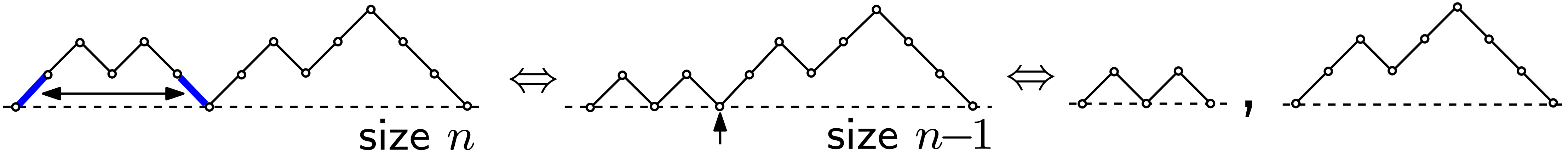

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Functional equation: $A(t) = 1 + tA(t)^2$

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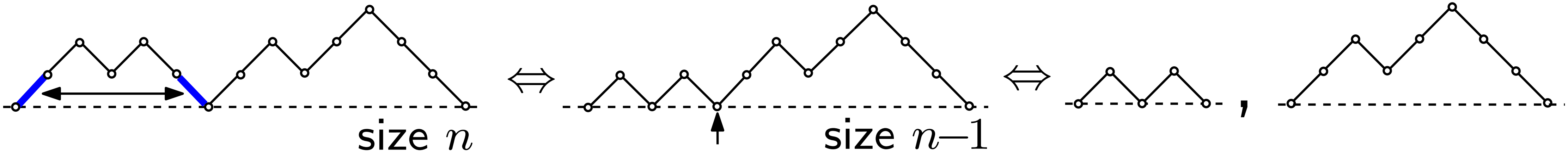

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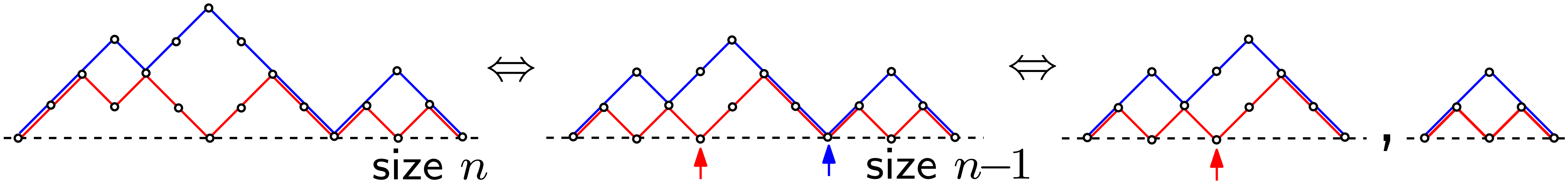
Solution: $A(t) = \frac{1-\sqrt{1-4t}}{2t} \quad \Rightarrow \quad a_n = \frac{(2n)!}{n!(n+1)!}$ Catalan numbers

Recursive decomposition of Tamari intervals

- Reduction of an interval in $\mathcal{L}_n$:



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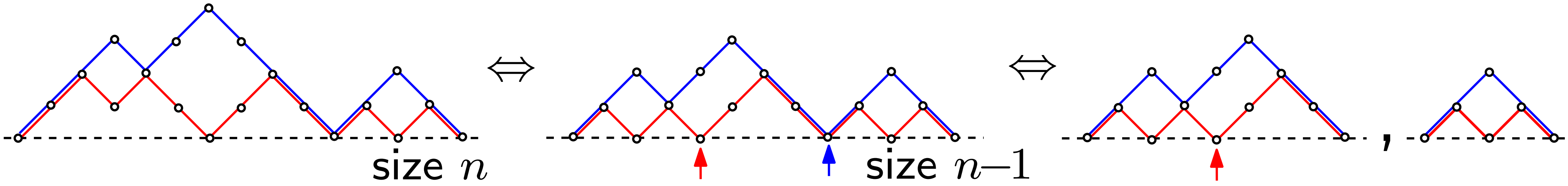
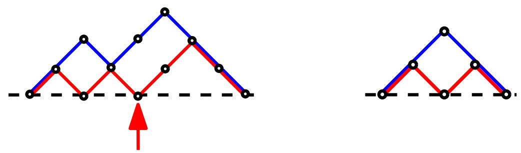


Let $a_{n,i} = \#($ intervals in $\mathcal{L}_n$ with i bottom contacts$)$

Let $F(t,u) := \sum_{n,i} a_{n,i} t^n u^i$

Then: $$F(t,u) = u + t \cdot G(t,u) \cdot F(t,u)$$



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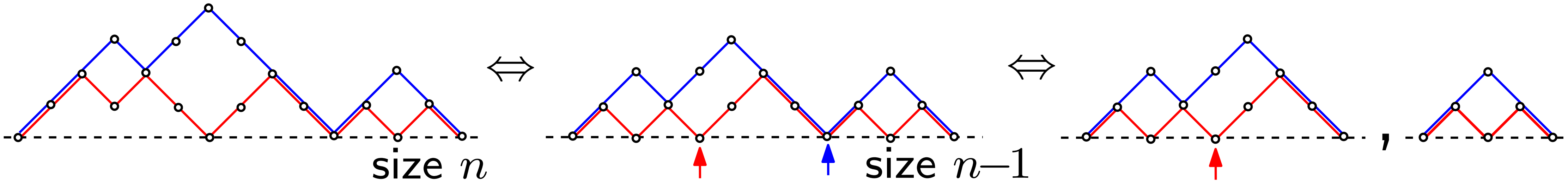

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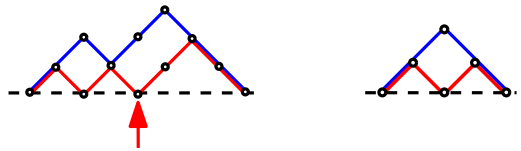

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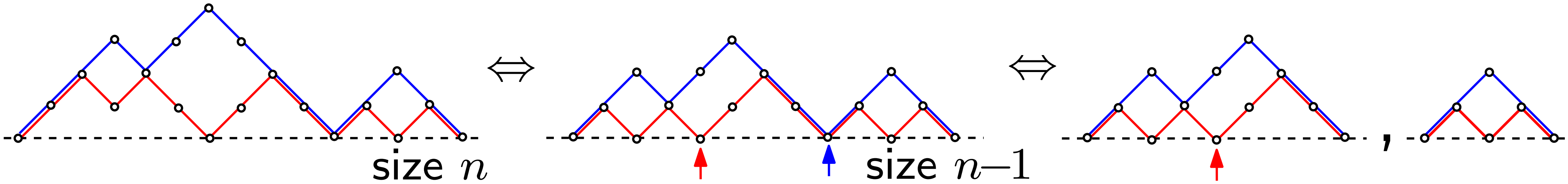

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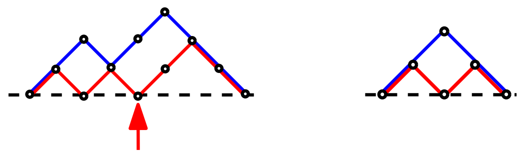

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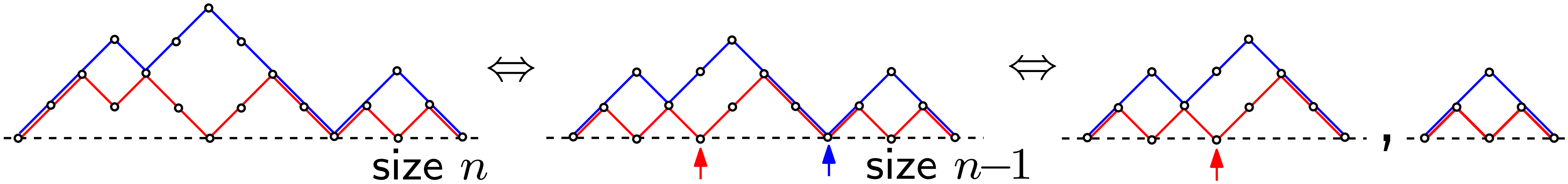

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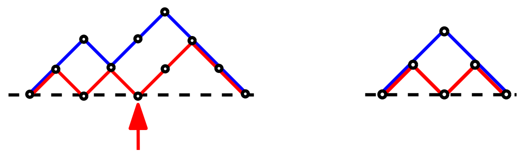

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  [Brown, Tutte, Bousquet-Mélou Jehanne'06]

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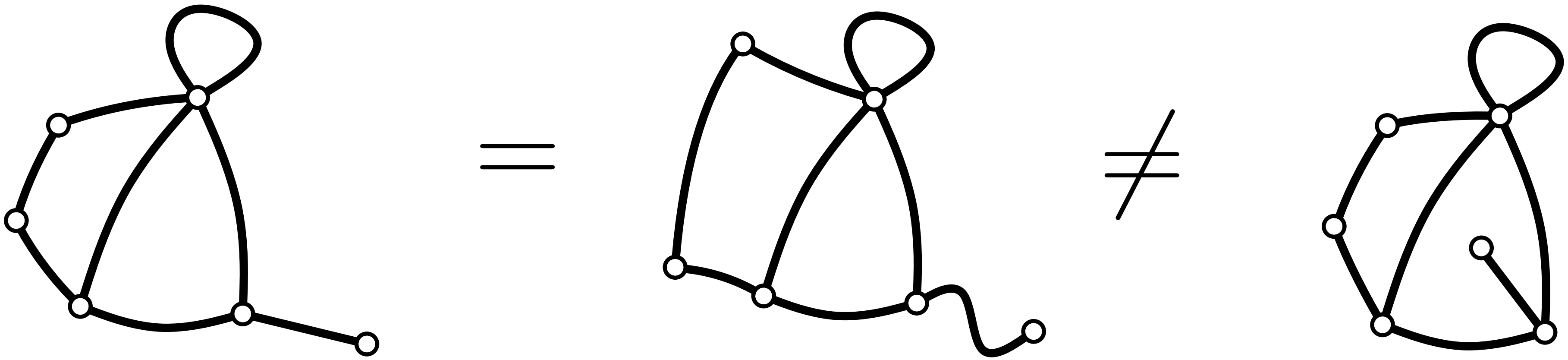
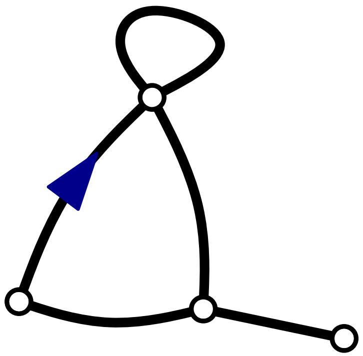
- We explain here how the equation can be "solved" bijectively using triangulations and Schnyder woods (cf L.-F. Préville-Ratelle)

Planar maps, triangulations

Def. Planar map = connected graph embedded in the plane up to isotopy



rooted map = map + marked directed edge
with the outer face on its left



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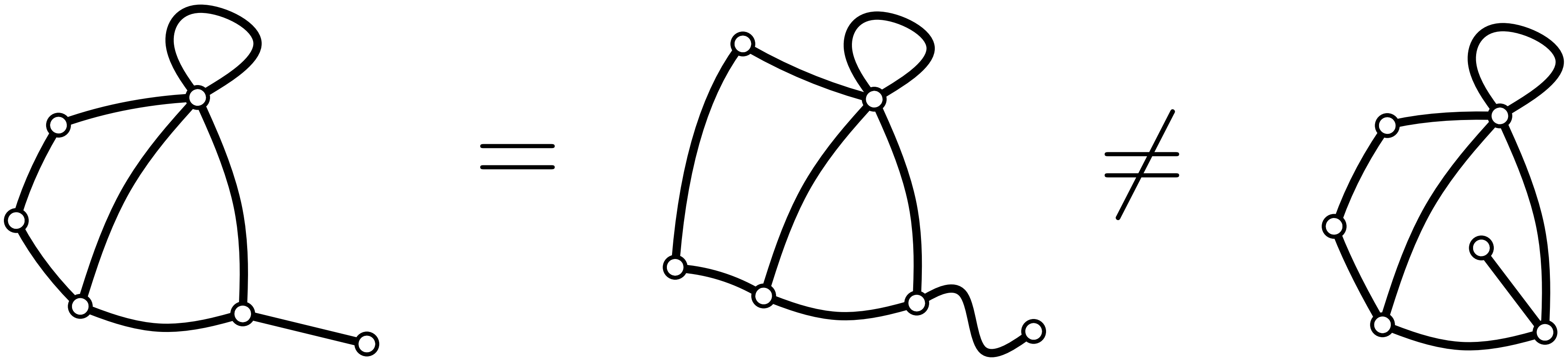

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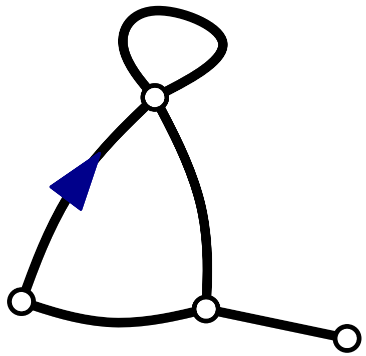
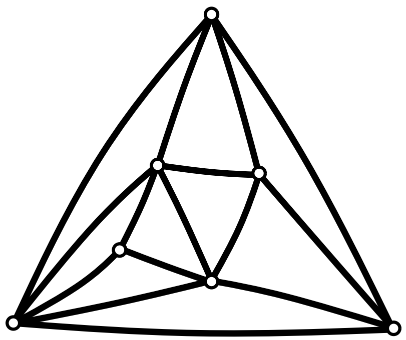


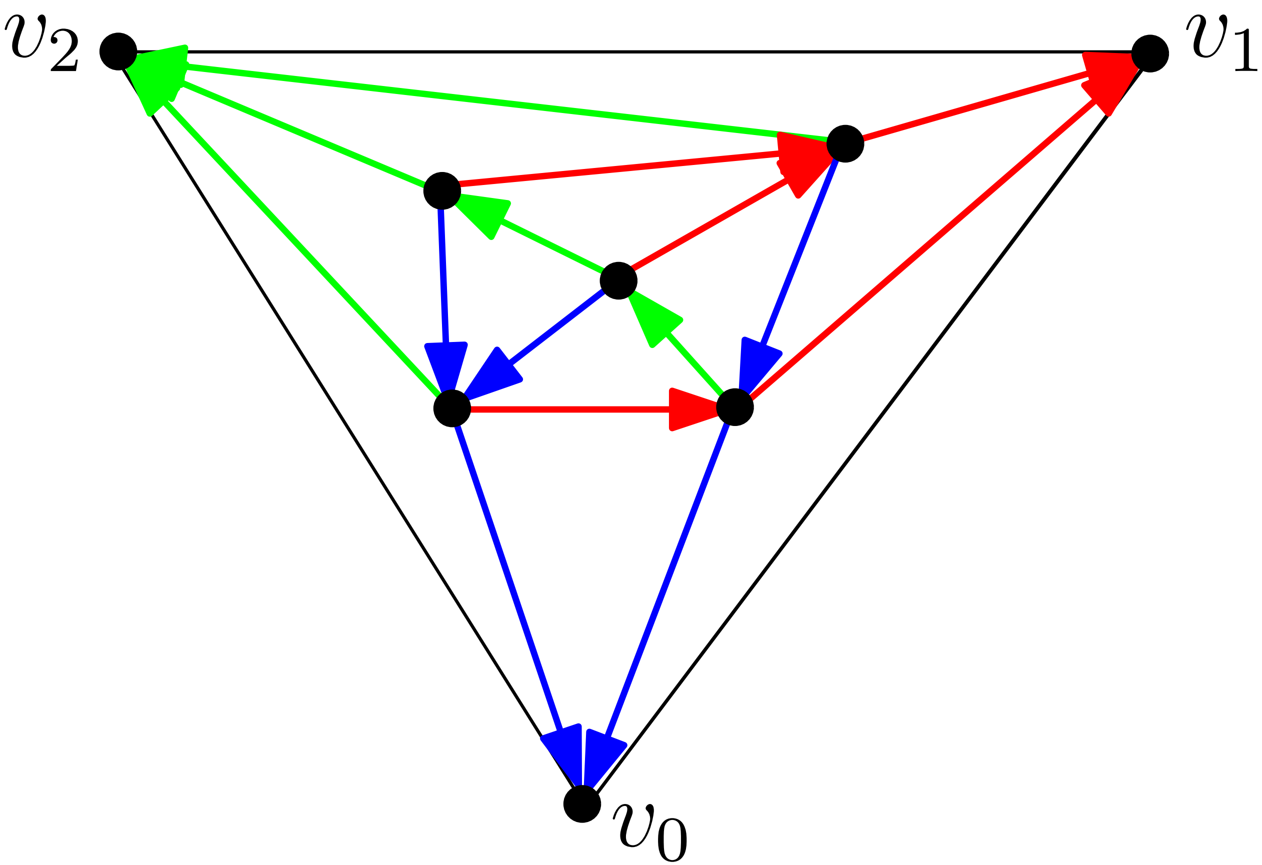
- Triangulation = simple planar map with all faces of degree 3

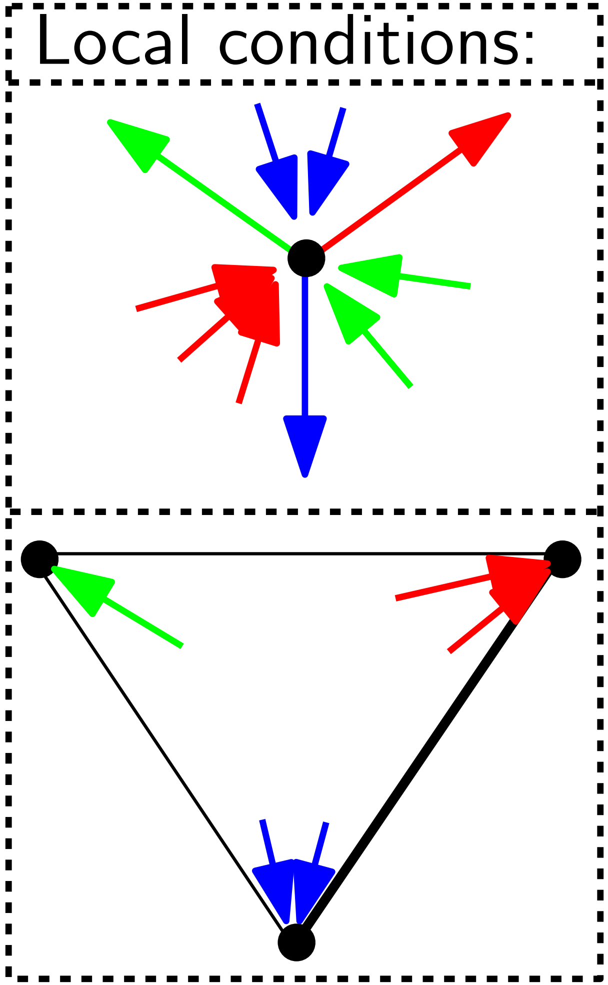


$n = 4$ internal vertices

[Tutte'62] #(triangulations on n internal vertices)= $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$

Schnyder woods





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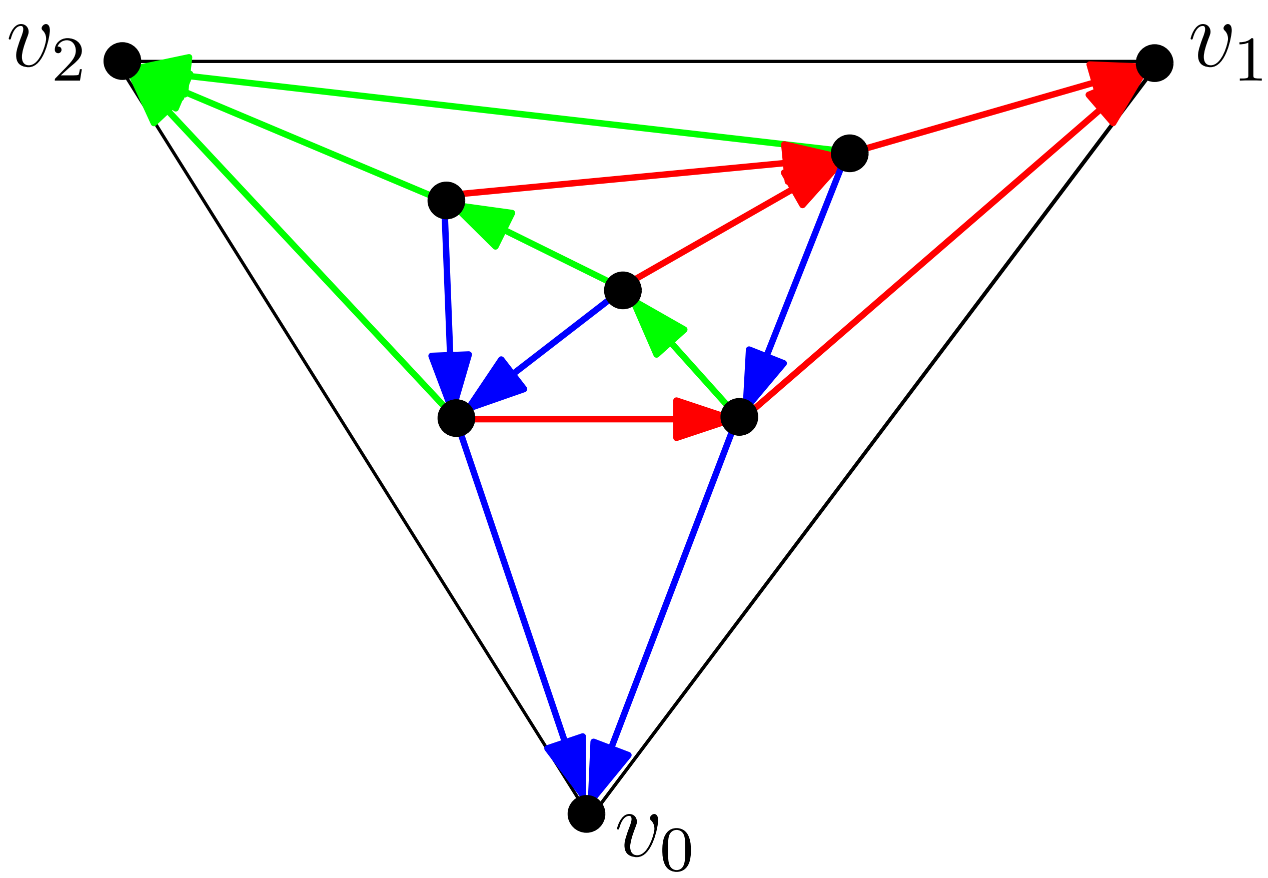

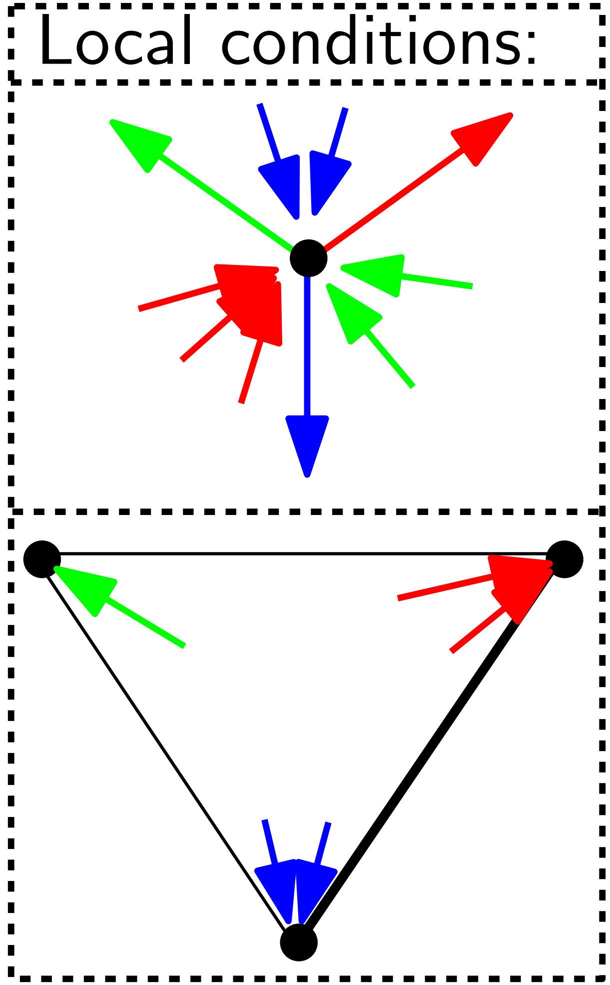

Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

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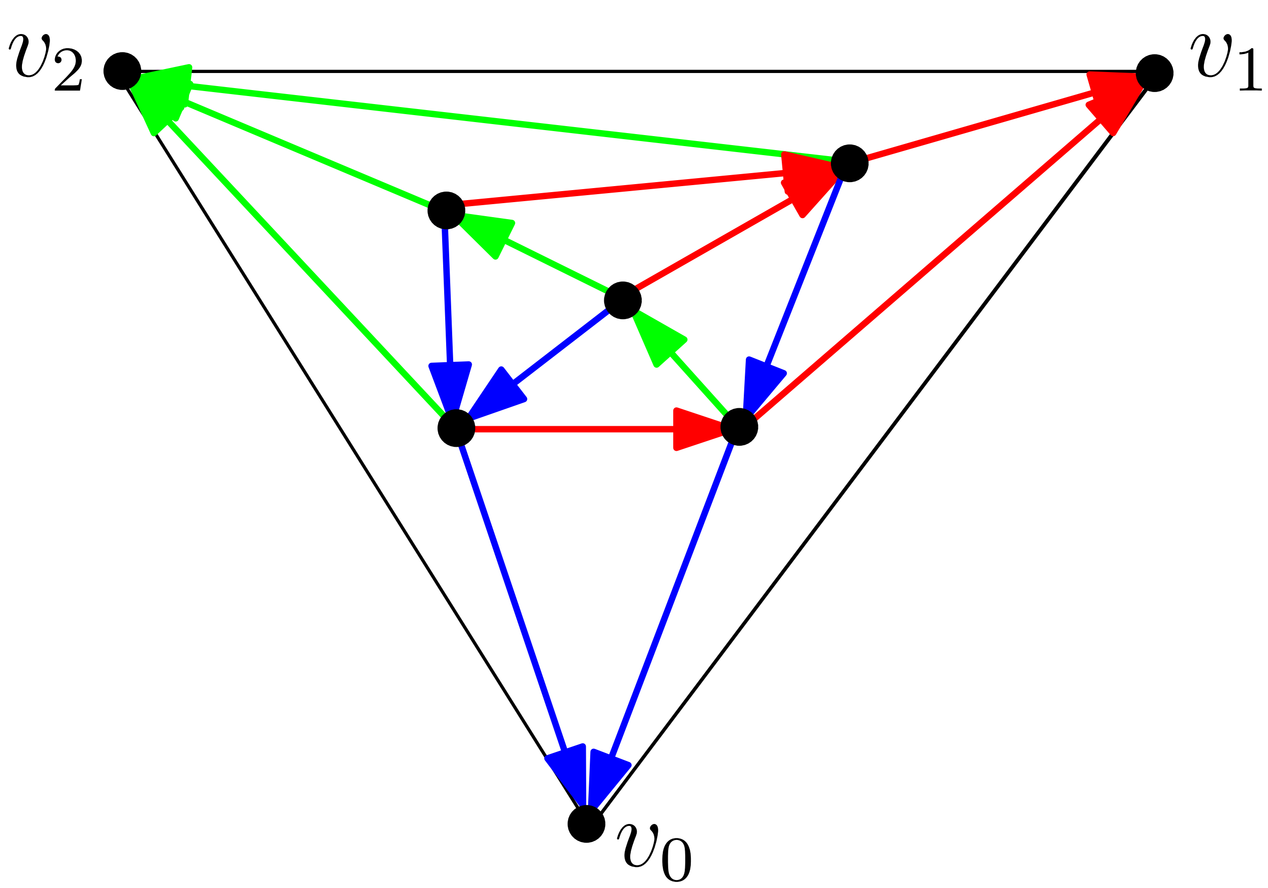

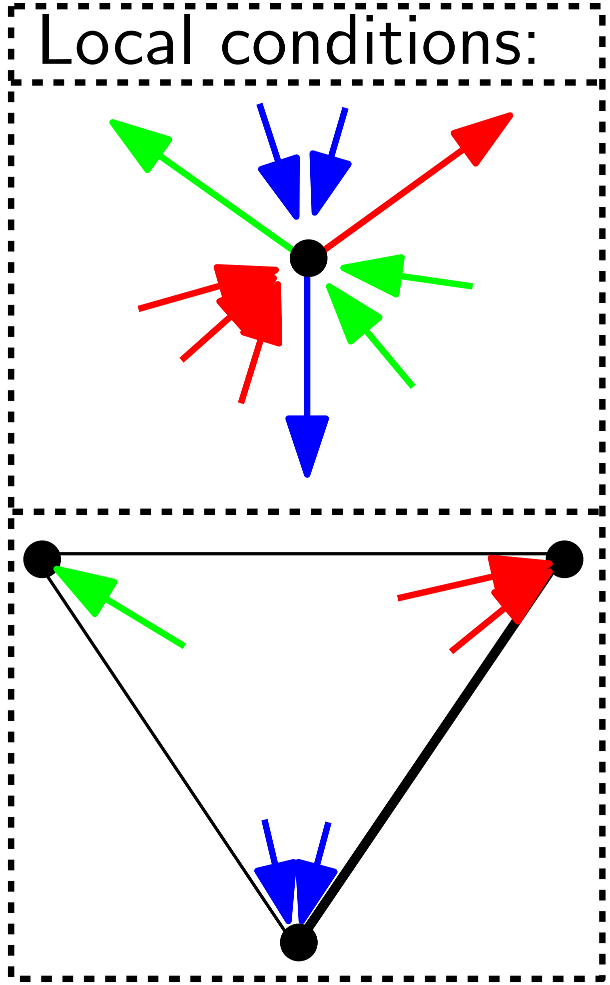

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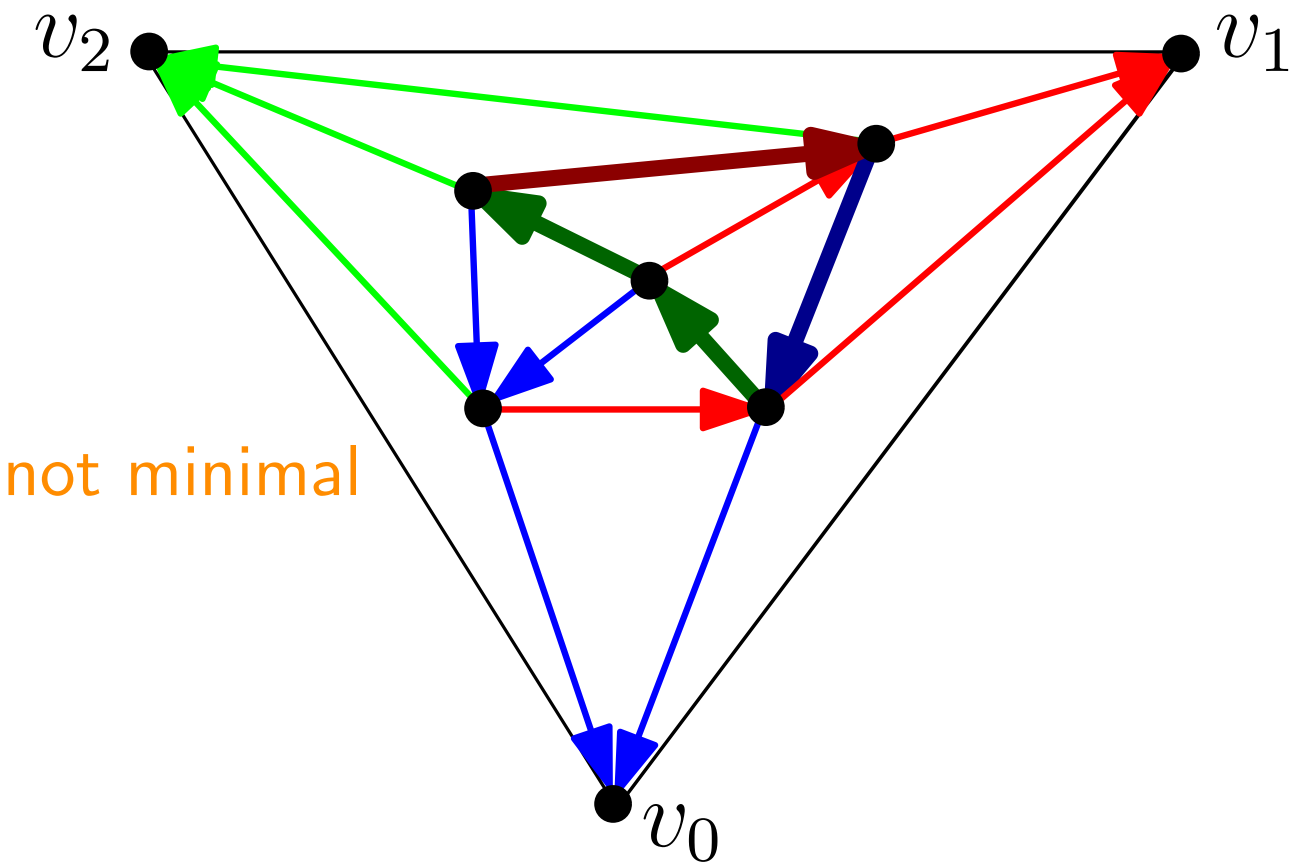

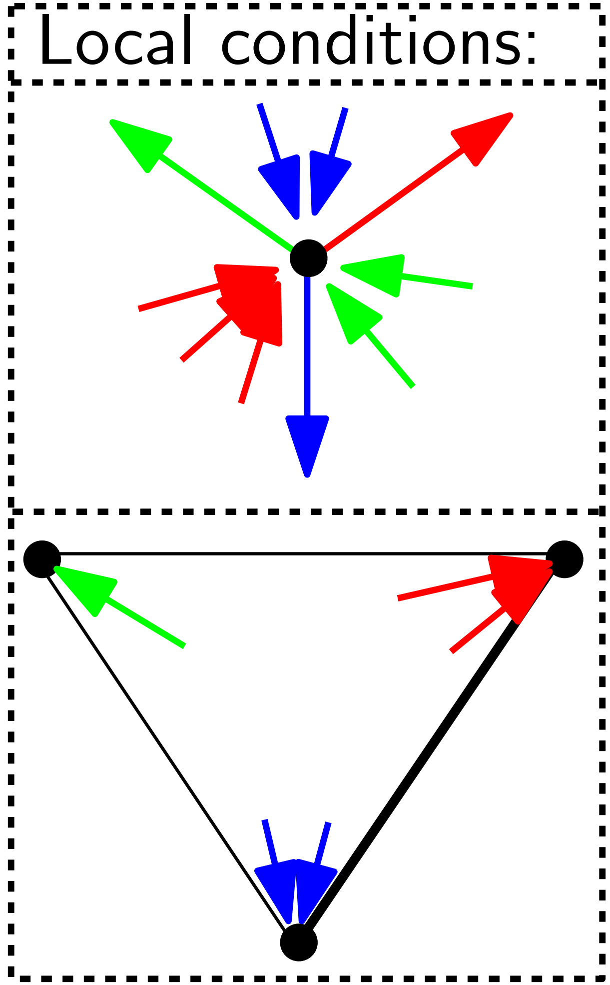

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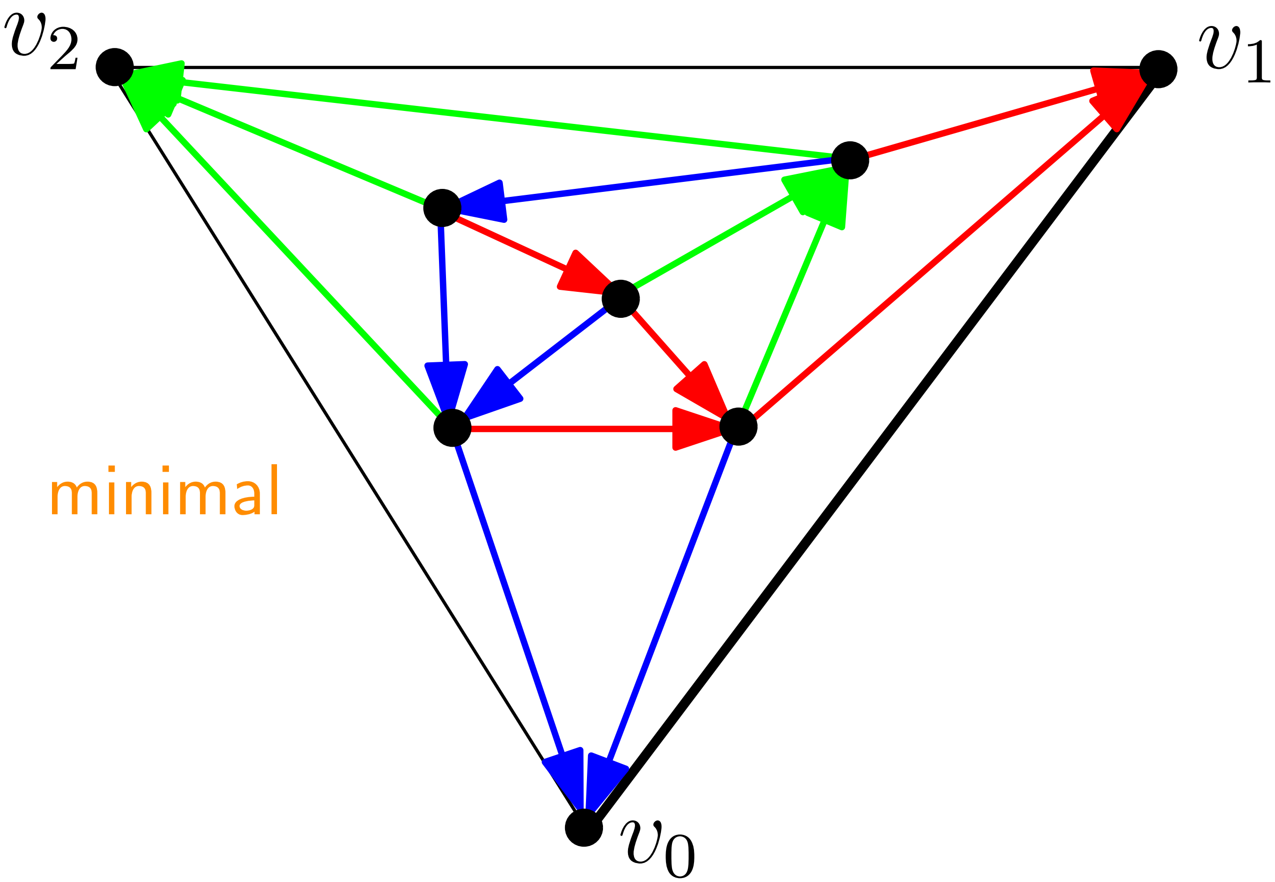

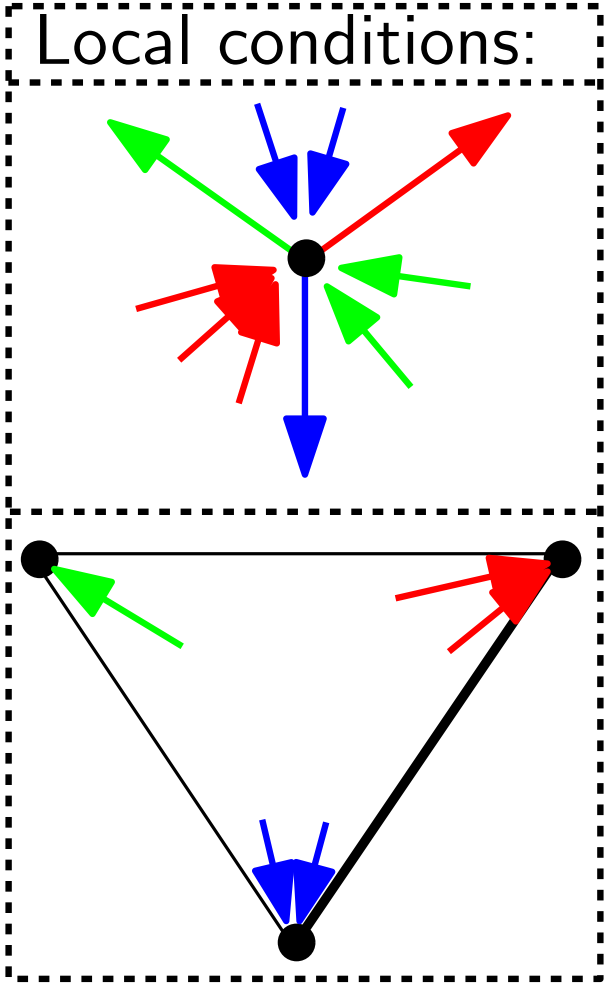

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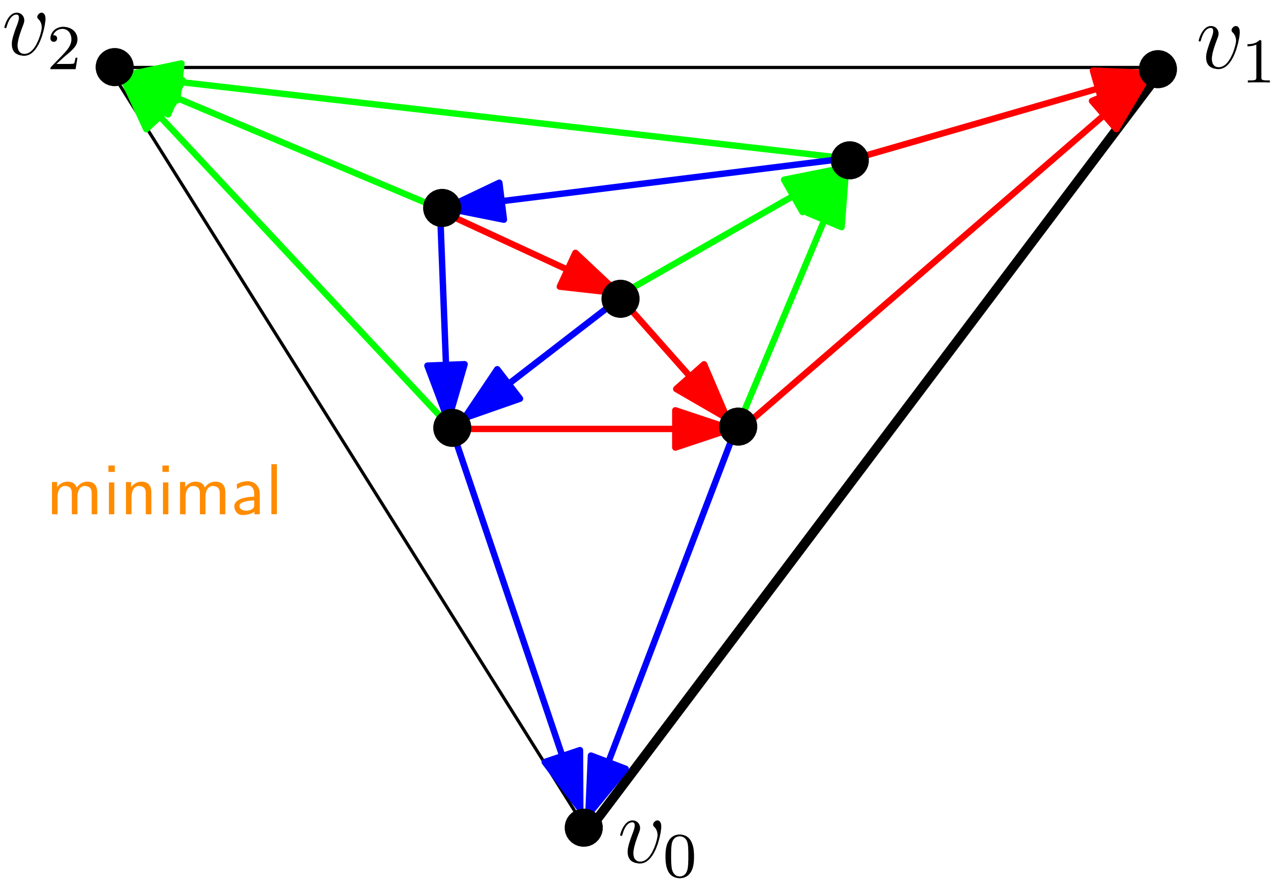

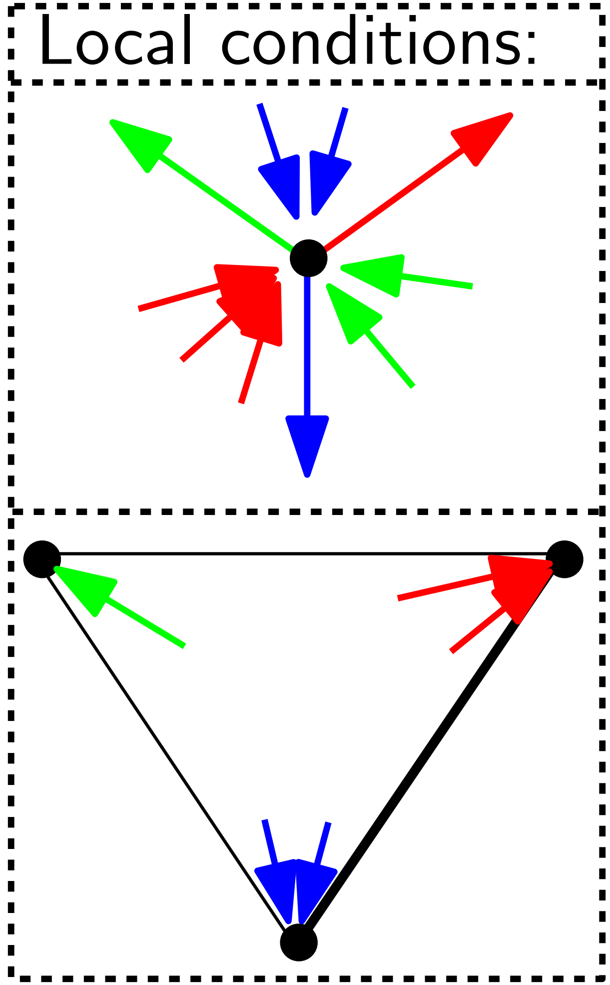

Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

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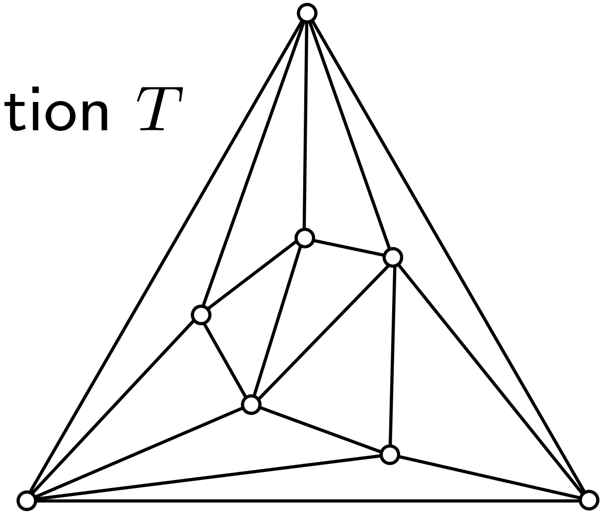
Theo: Any triangulation has a unique minimal Schnyder wood
(cf set of Schnyder woods on fixed triangulation is a distributive lattice)
[Ossona de Mendez'94, Brehm'03, Felsner'03]

Bijective counting of triangulations

triangulation T



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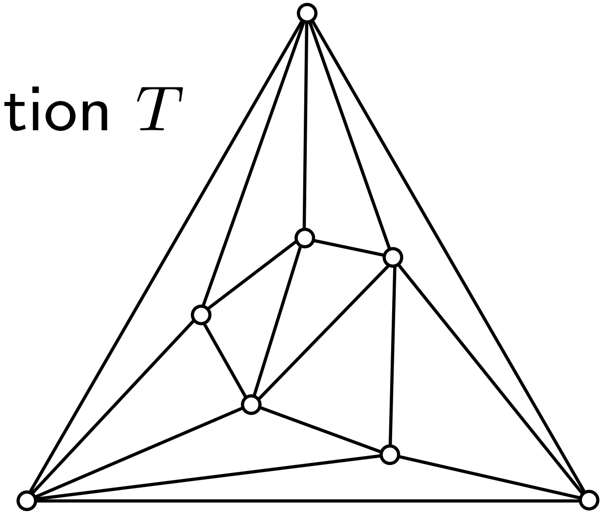

minimal Schnyder wood
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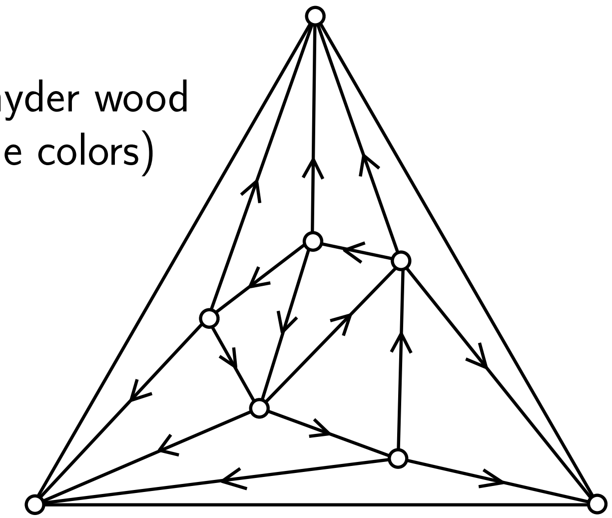

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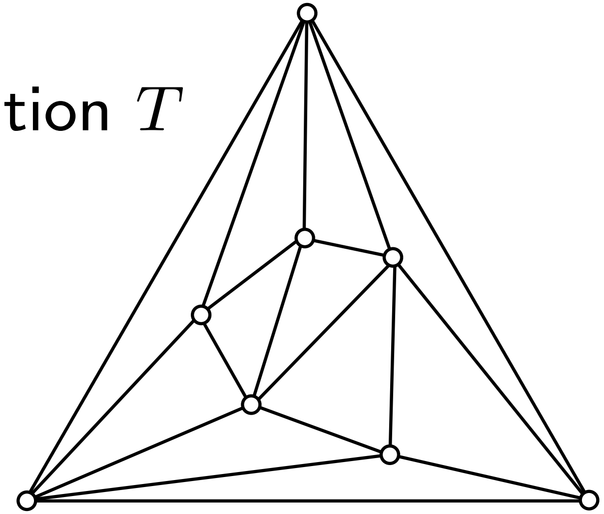

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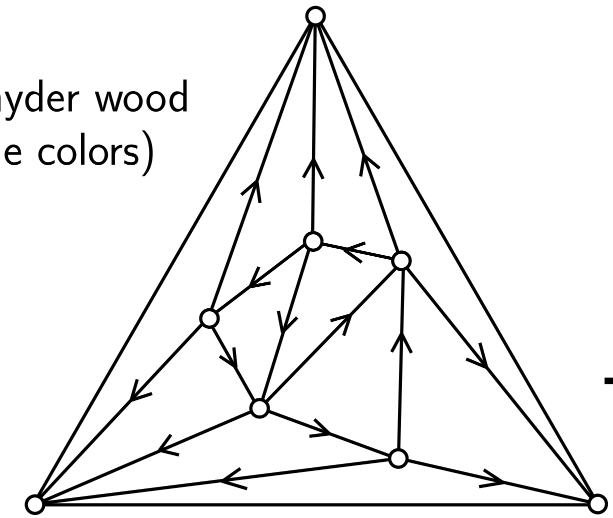
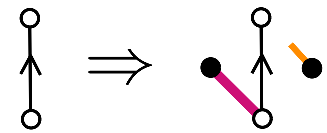
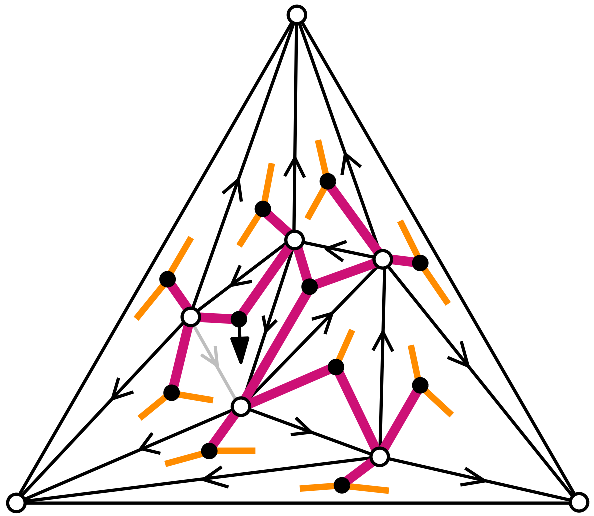


Local rule





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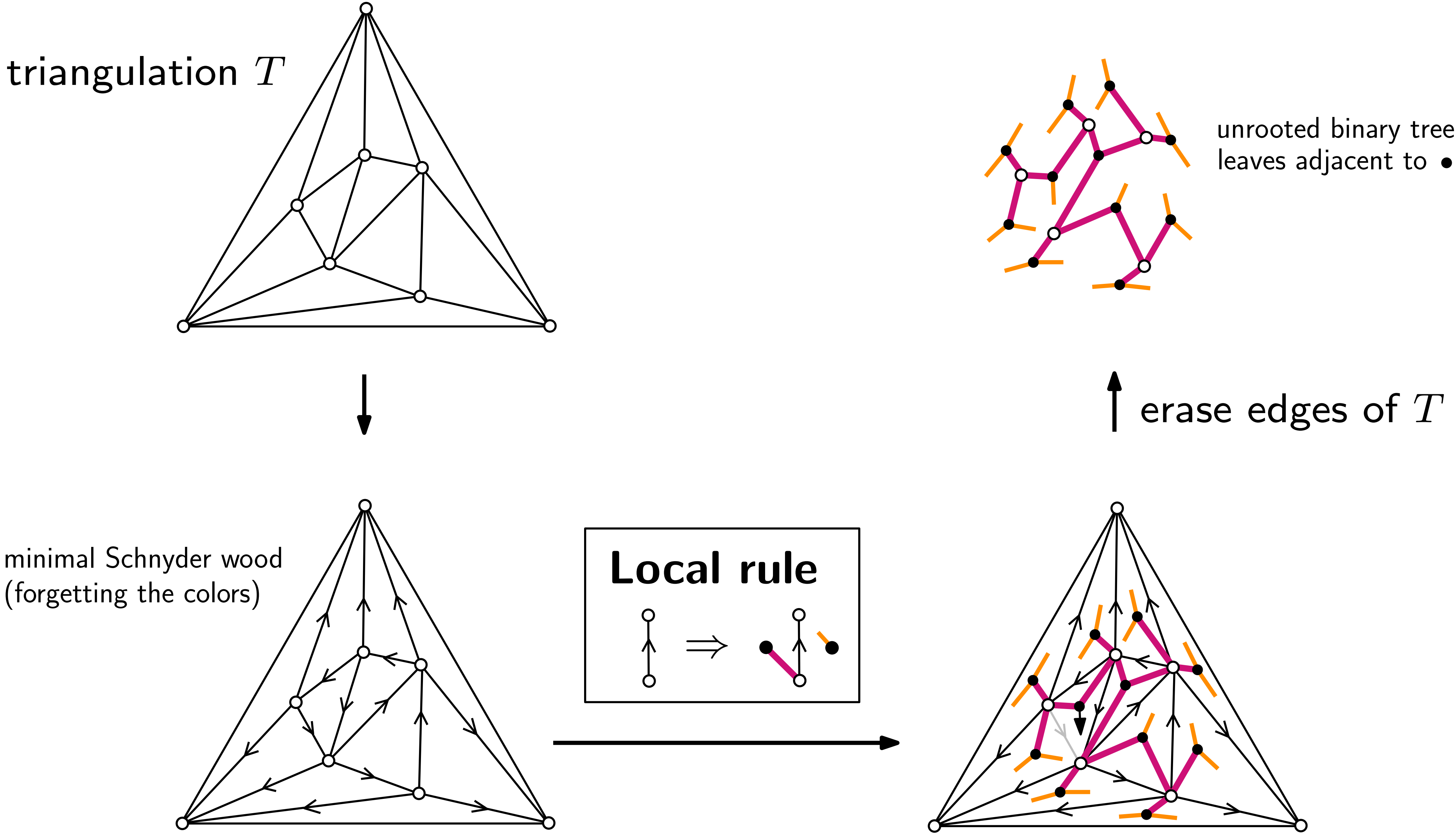

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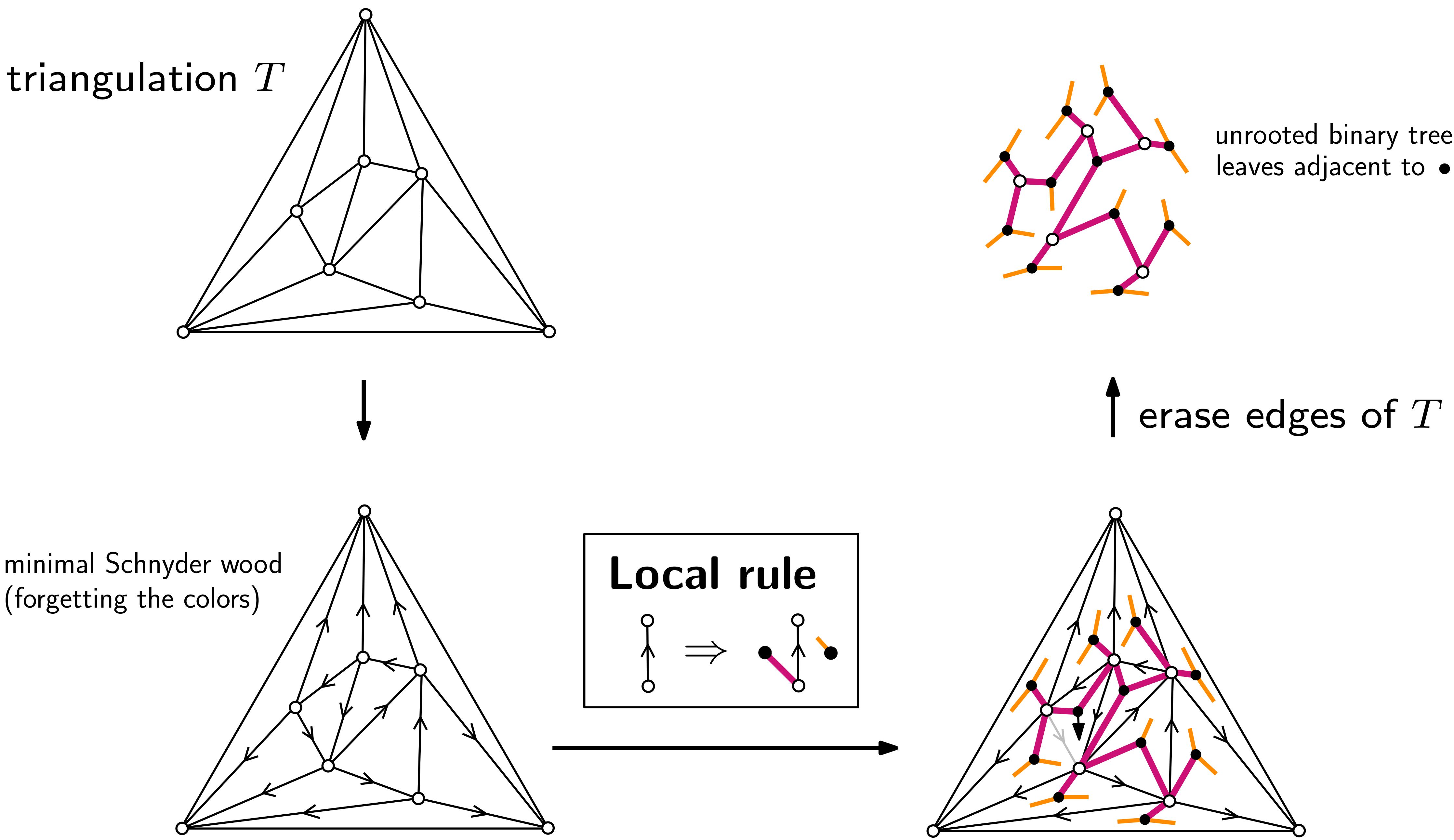

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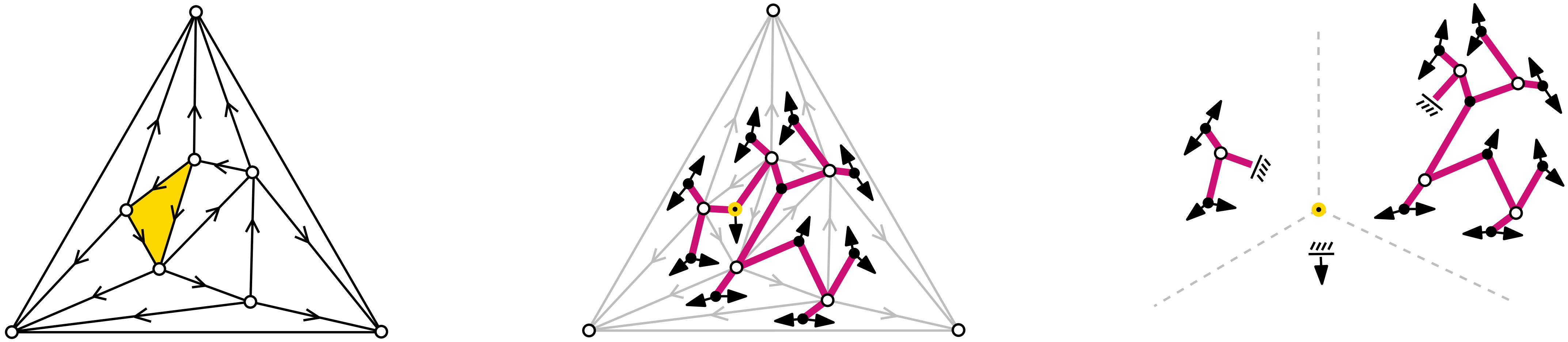
Counting formula

The **bijection** when there is a **marked inner face**:



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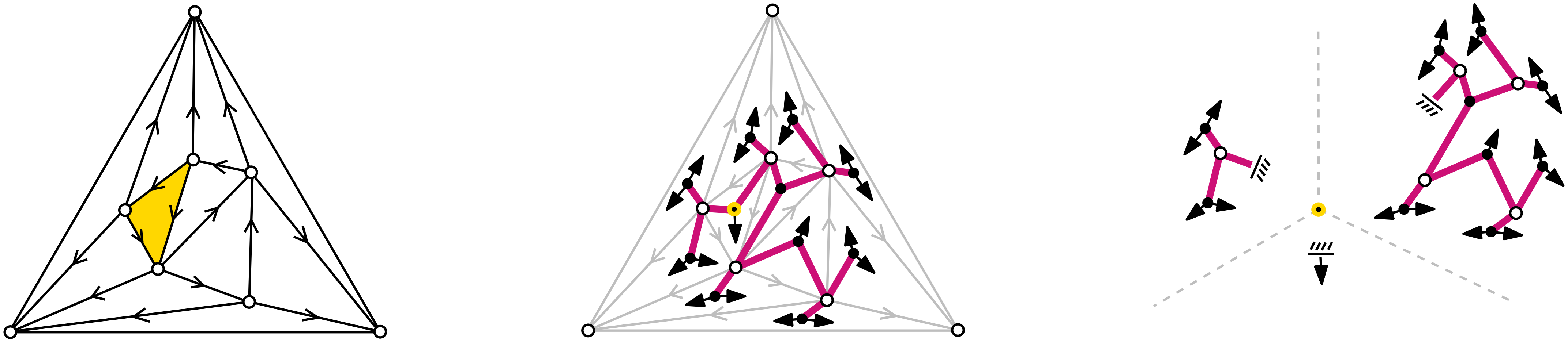
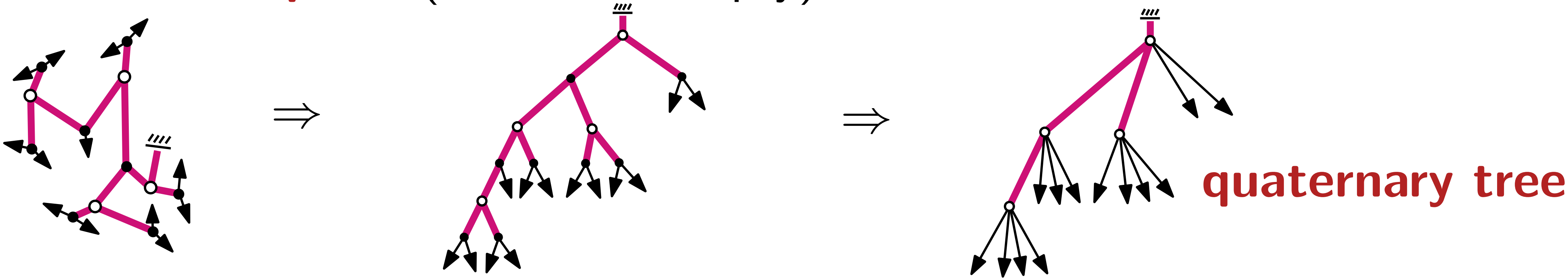


Each of the **3 parts** (when non empty) is **of the form**



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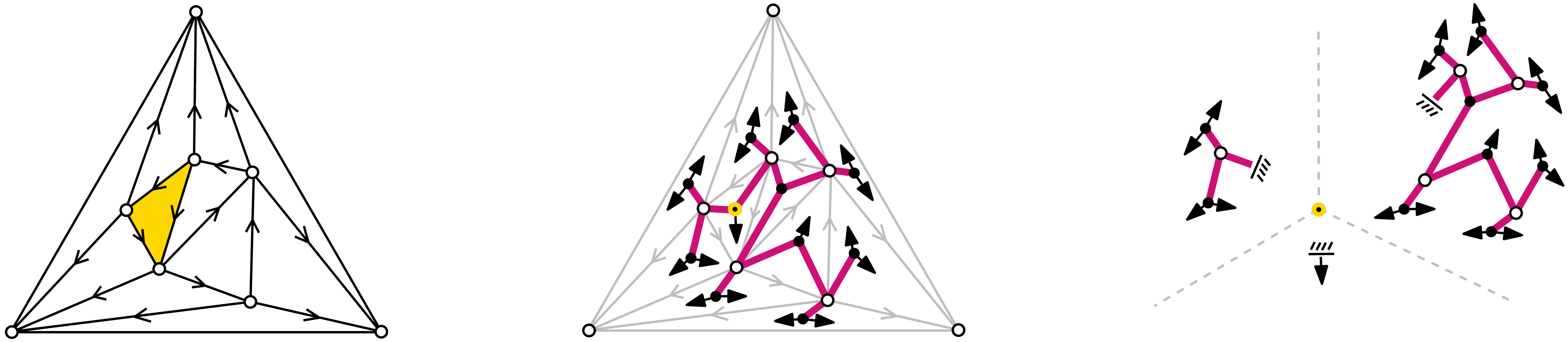

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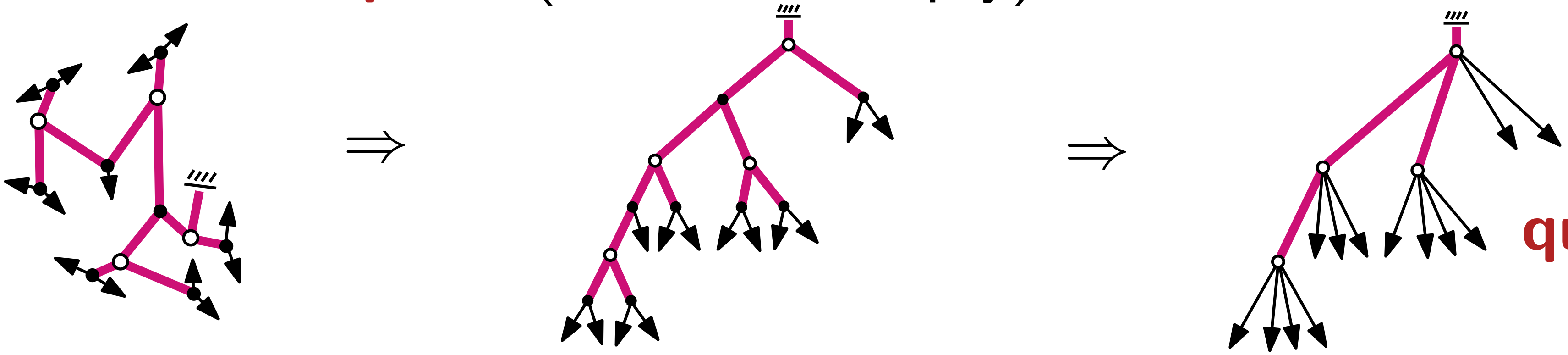

quaternary tree

Let $t_n = \#\{\text{(rooted) triang. with } n+3 \text{ vertices}\}$, $F(x) = \sum_n t_n x^{2n+1}$

Then $\boxed{F'(x) = (1+u)^3}$ where $u = u(x)$ is specified by $\underbrace{u = x^2(1+u)^4}_{\text{quat. trees}}$

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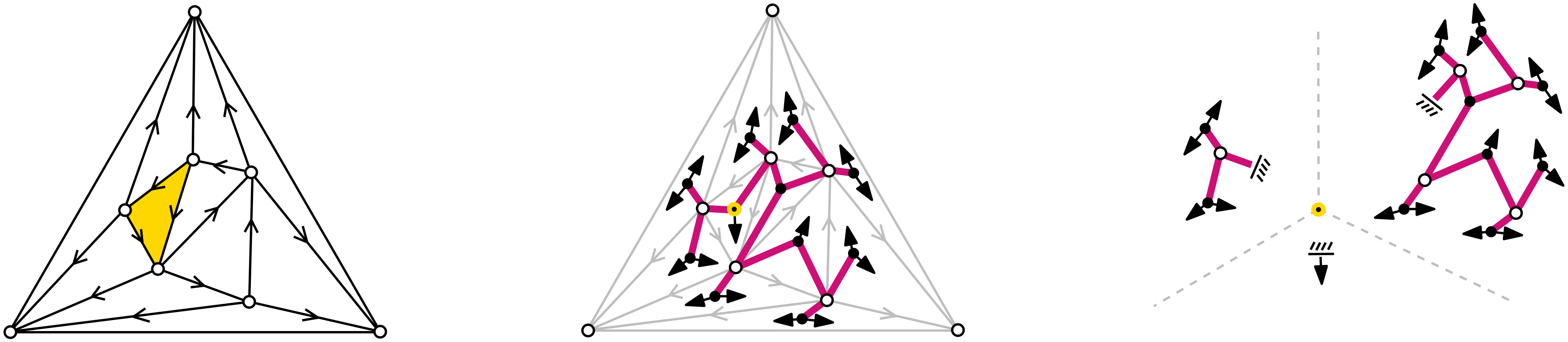

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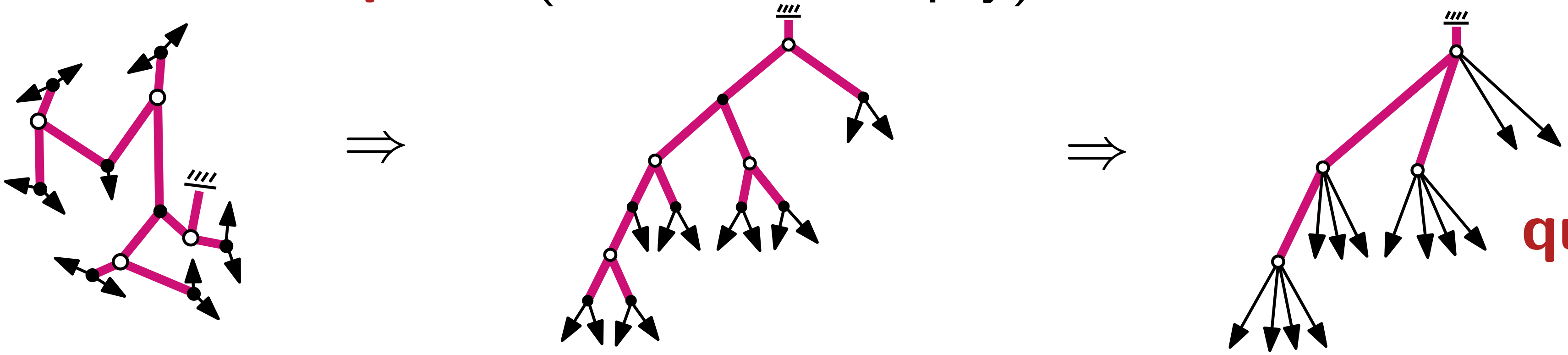

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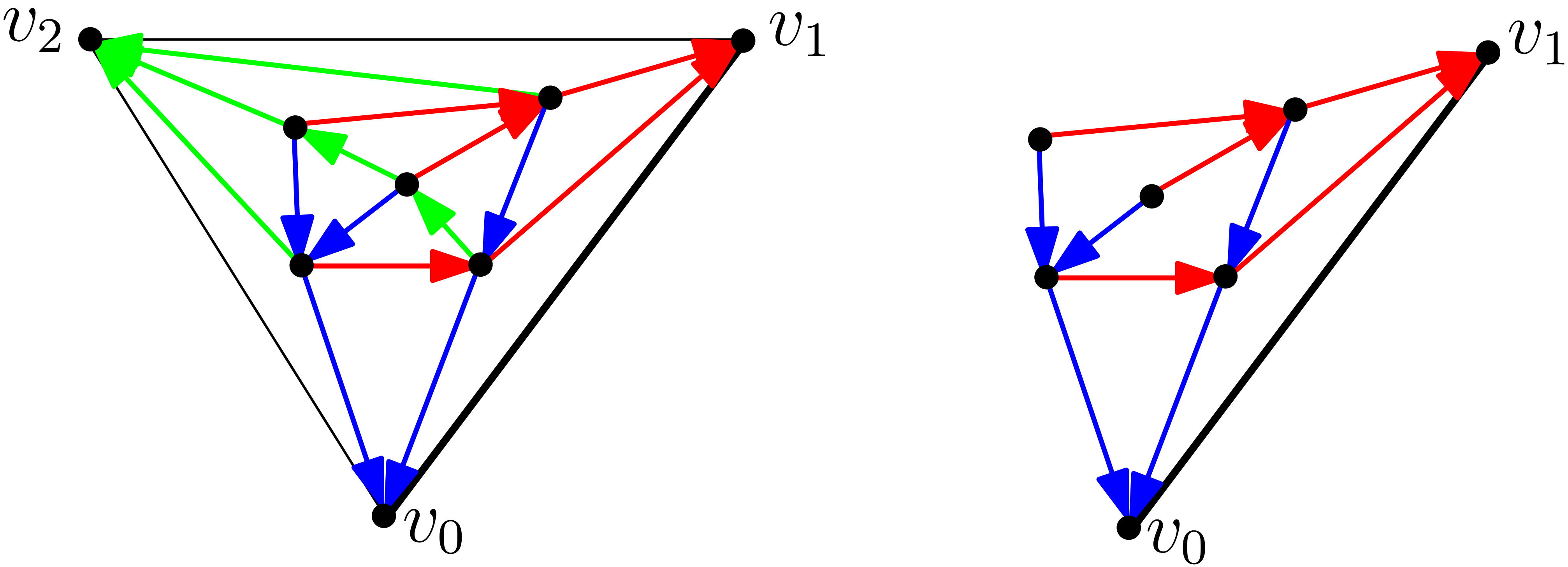
$\Rightarrow$ (Lagrange)

$$t_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

[Tutte'62]

The red-blue induced structure



There is no loss of information in deleting the green edges

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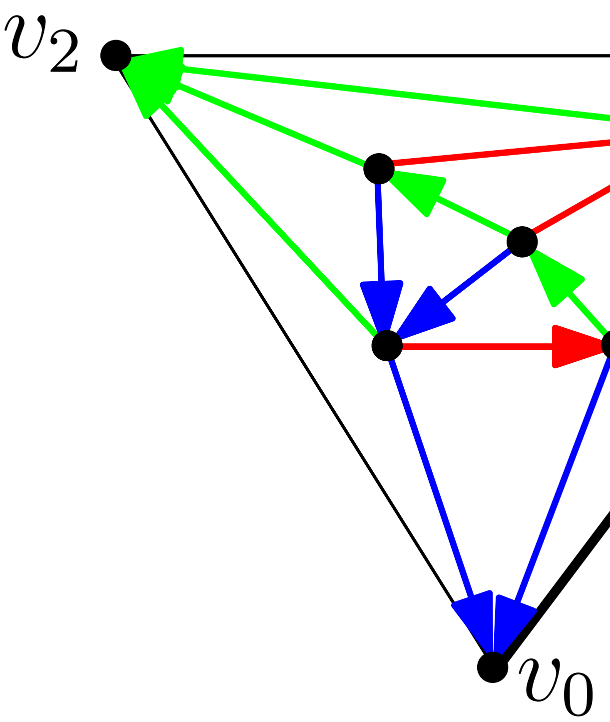

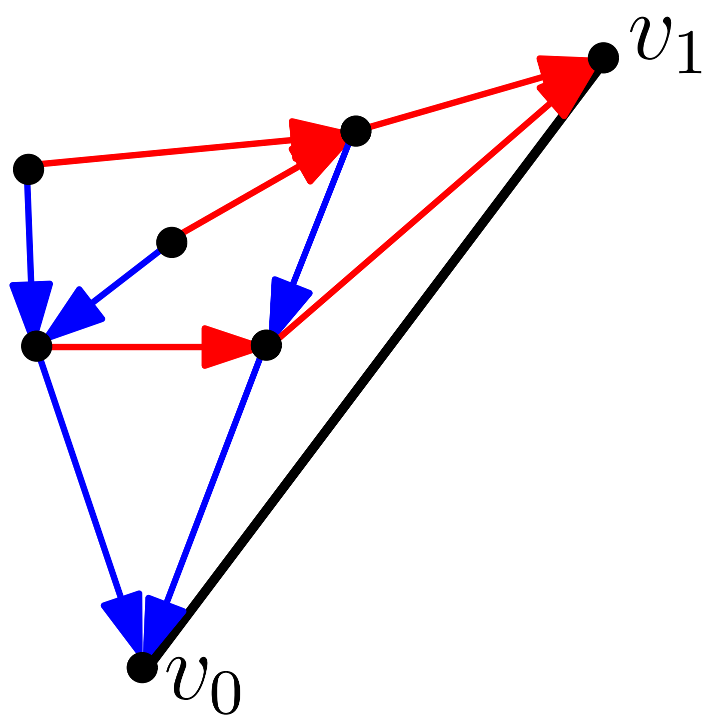

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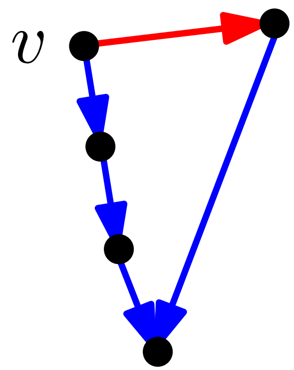
no clockwise circuit
(i.e., minimal) $\Leftrightarrow$



$\forall v$ interval vertex the red parent of the blue parent is a blue ancester

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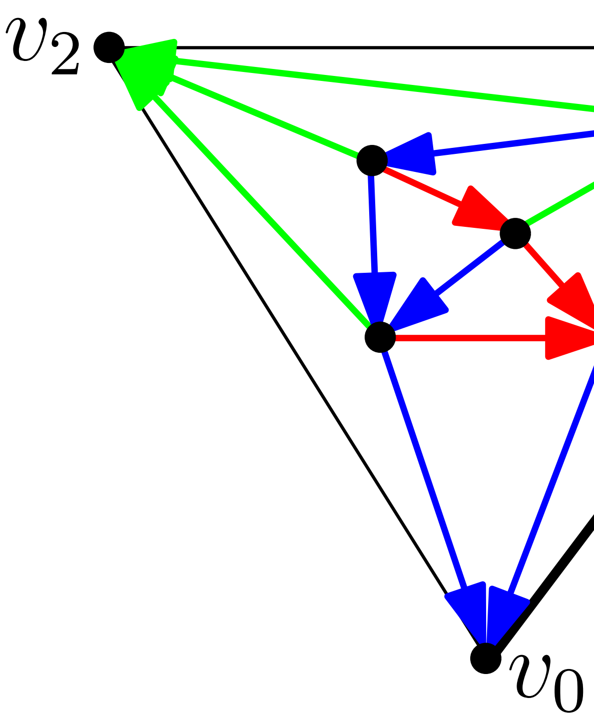

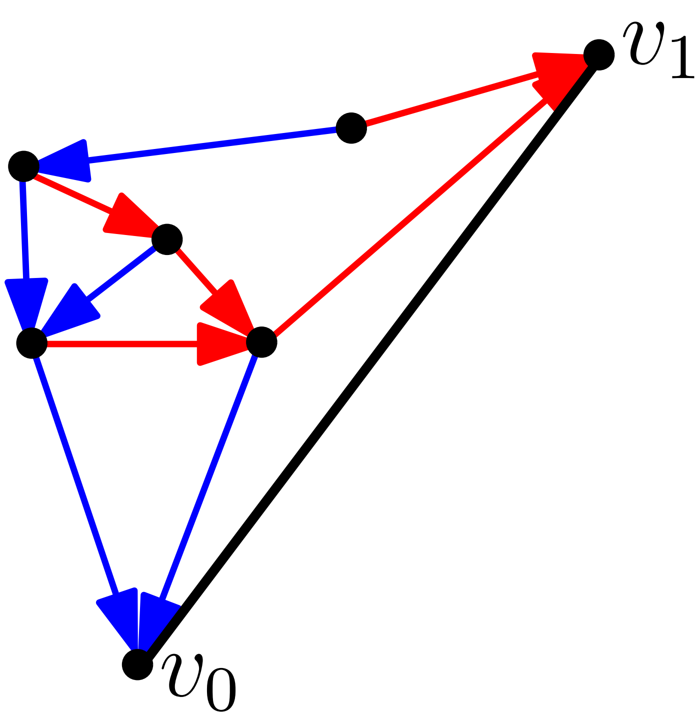

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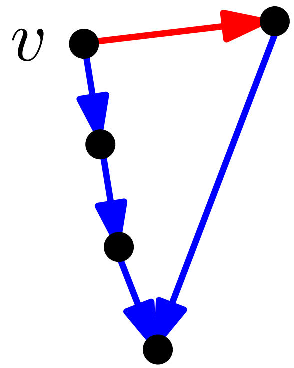

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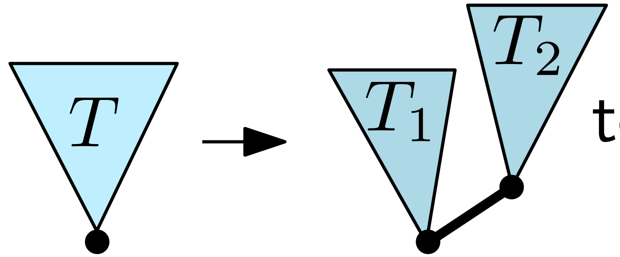
Decomposing a minimal red-blue structure

cf L.-F. Préville-Ratelle. Idea: apply 

to the blue tree

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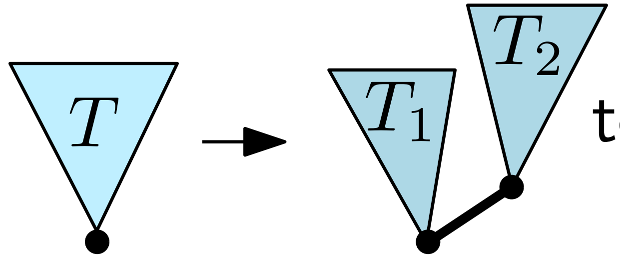
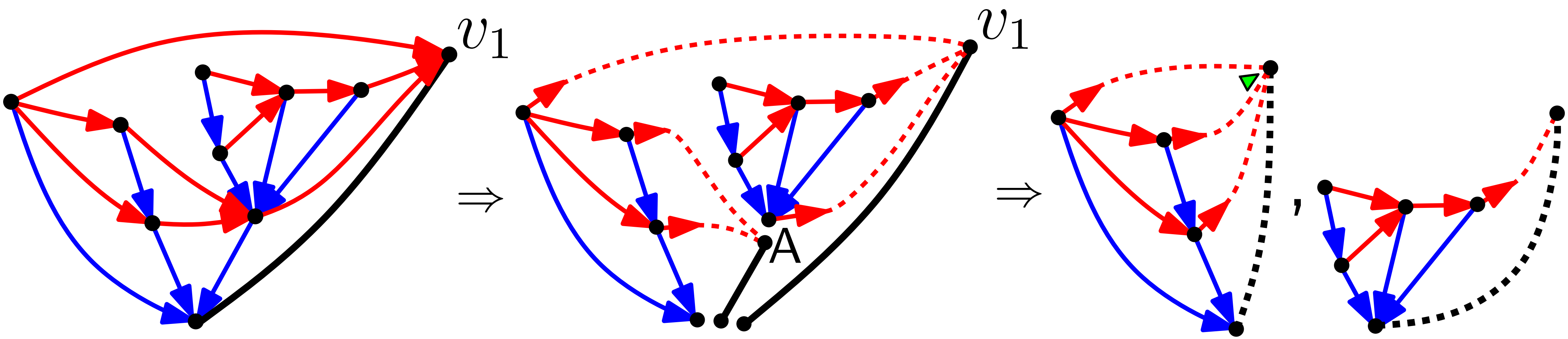
cf L.-F. Préville-Ratelle. Idea: apply  to the blue tree



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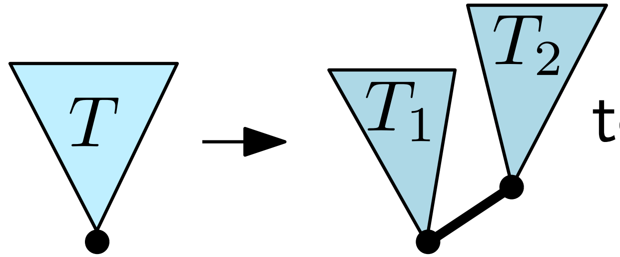
cf L.-F. Préville-Ratelle. Idea: apply 

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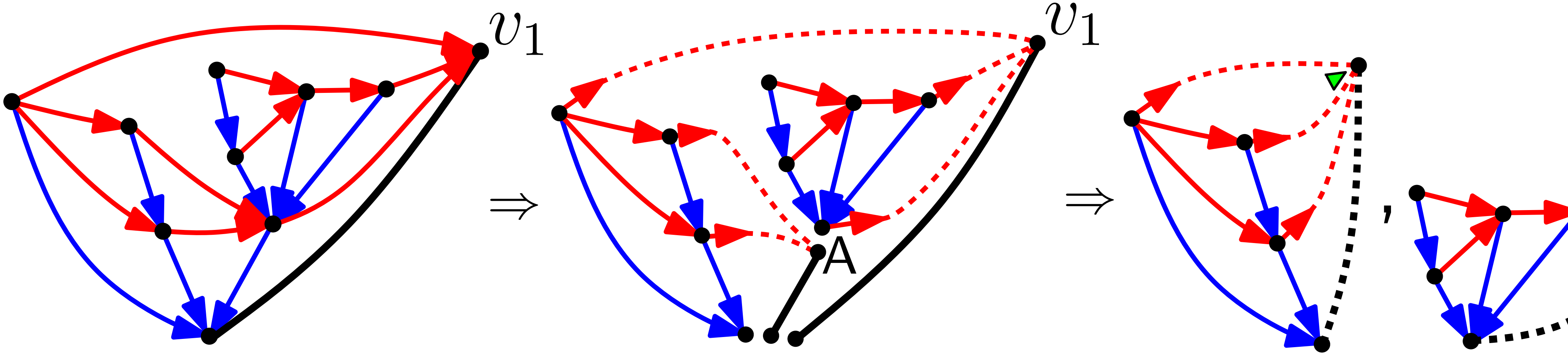

Let $a_{n,i} = \#($ triangulations with $n+3$ vertices, $\deg(v_1) = i+1$ $)$

Let $F(t,u) := \sum_{n,i} a_{n,i} t^n u^i$.

Then
$$F(t,u) = u + tuF(t,u)\frac{F(t,u) - F(t,1)}{u-1}$$

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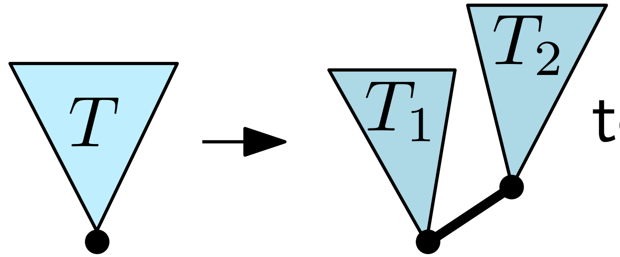
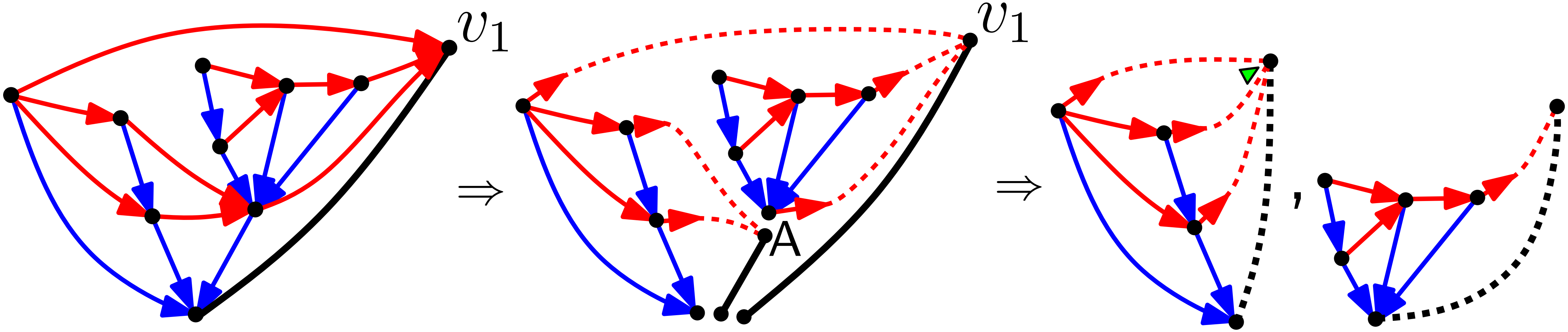
cf L.-F. Préville-Ratelle. Idea: apply  to the blue tree



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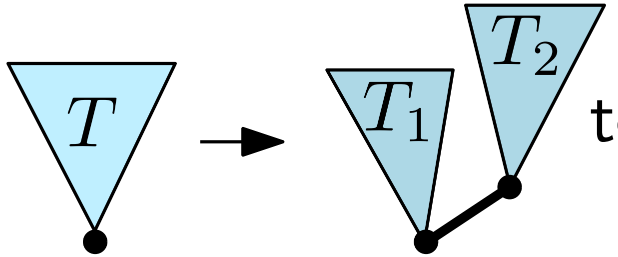
cf L.-F. Préville-Ratelle. Idea: apply 

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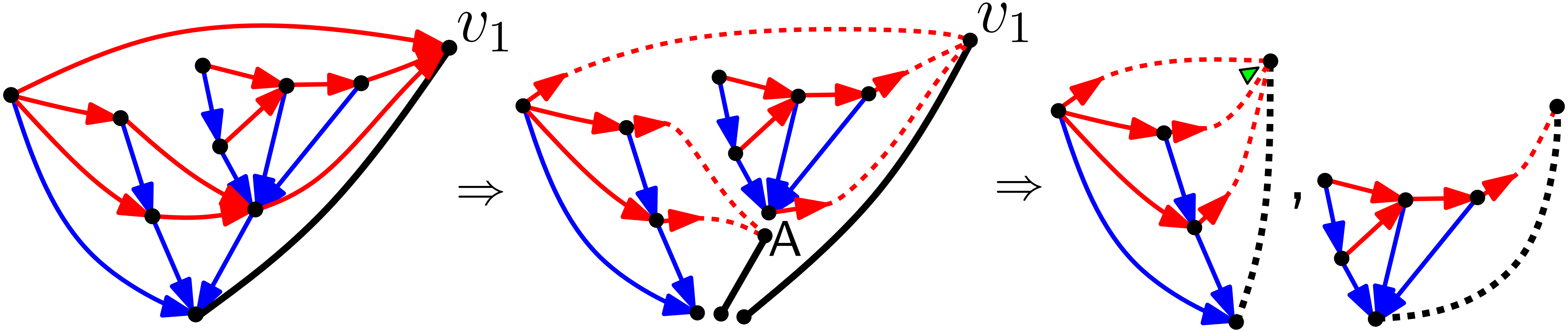

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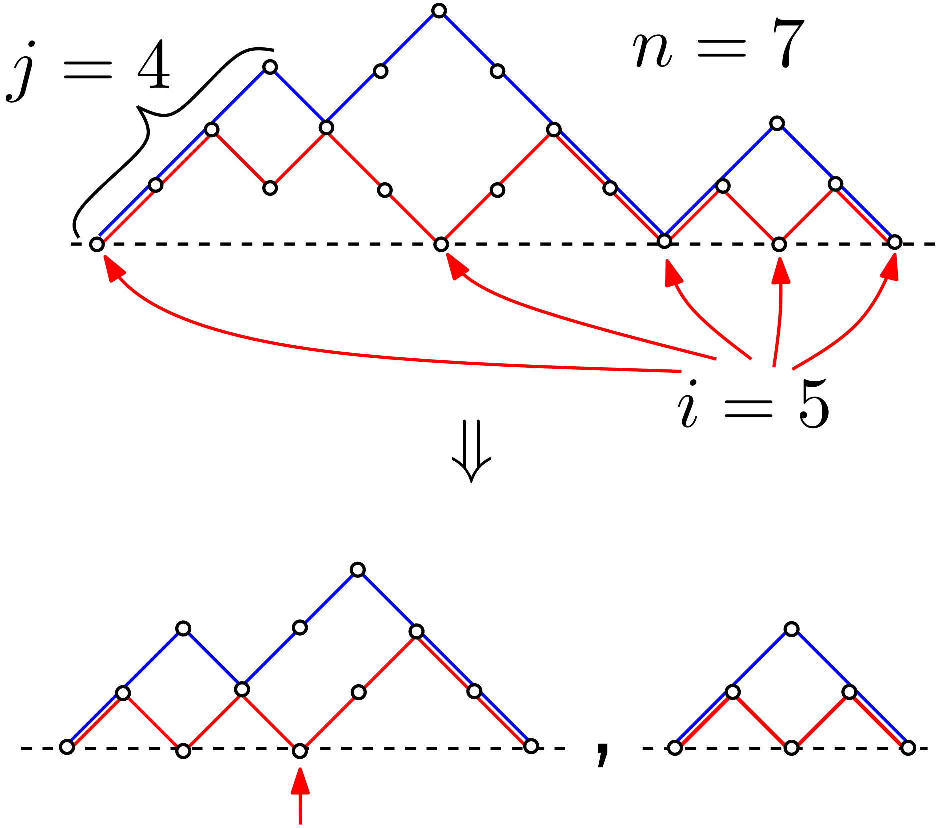
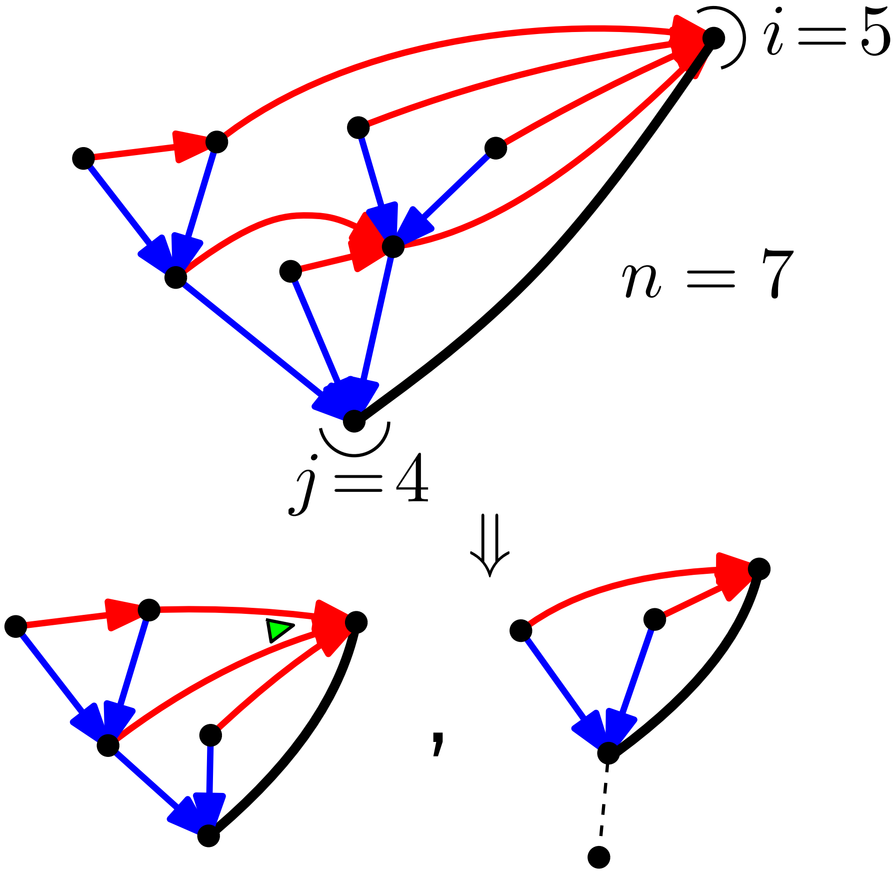
same equation as for Tamari intervals (& recursive bijection)

$\Rightarrow$ coefficient $[t^n]F(t,1)$ is the same in both cases

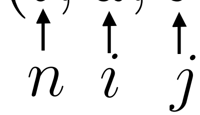
$$\boxed{I_n = T_n = \frac{2}{n(n+1)}\binom{4n+1}{n-1}}$$

A symmetry consequence





In both cases, the trivariate series $F(t; u, v)$ satisfies the equation



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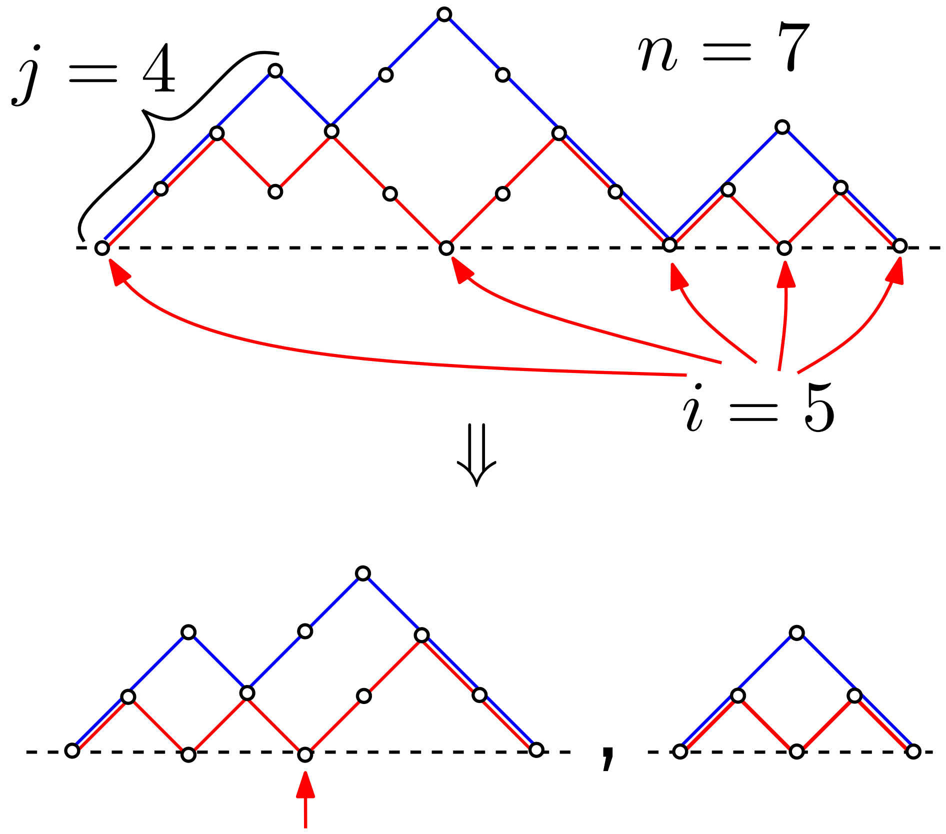
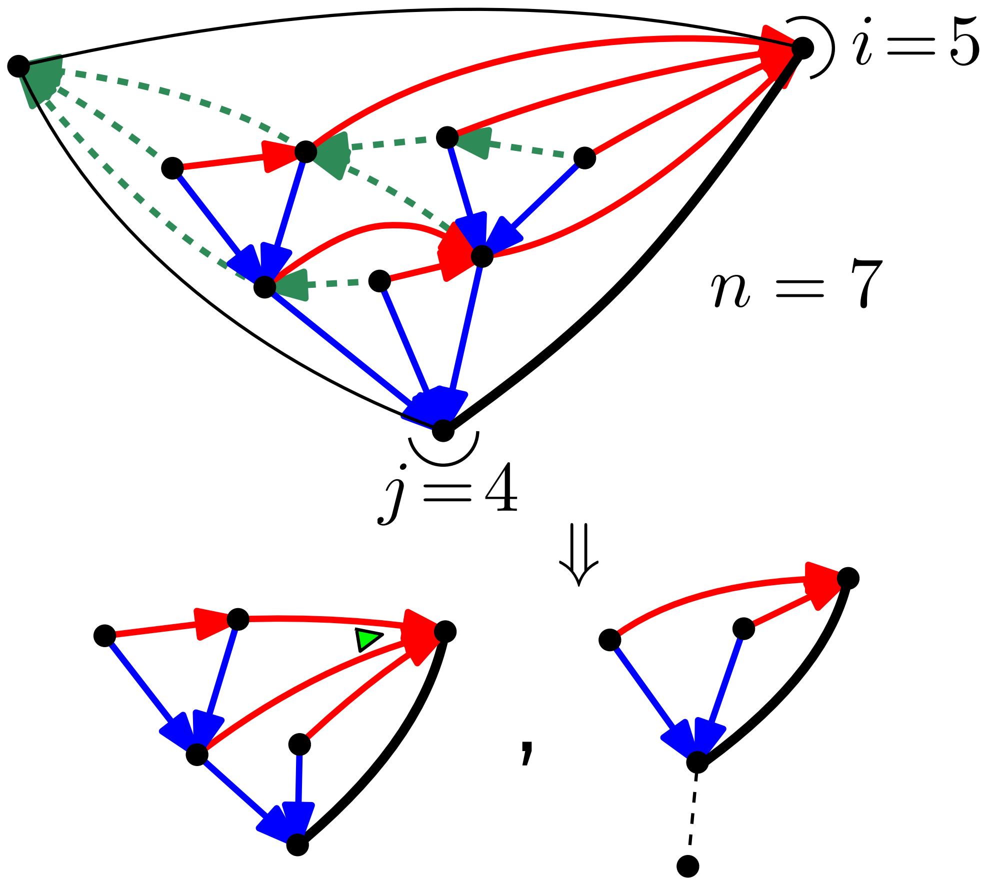




In both cases, the trivariate series $F(t;u,v)$ satisfies the equation ($t \leftarrow n$, $u \leftarrow i$, $v \leftarrow j$)

$$F(t;u,v) = uv + tuvF(t;u,1)\frac{F(t;u,v)-F(t;u,1)}{u-1}$$

$\Rightarrow$ the variables (i,j) are symmetrically distributed over $\mathcal{I}_n$

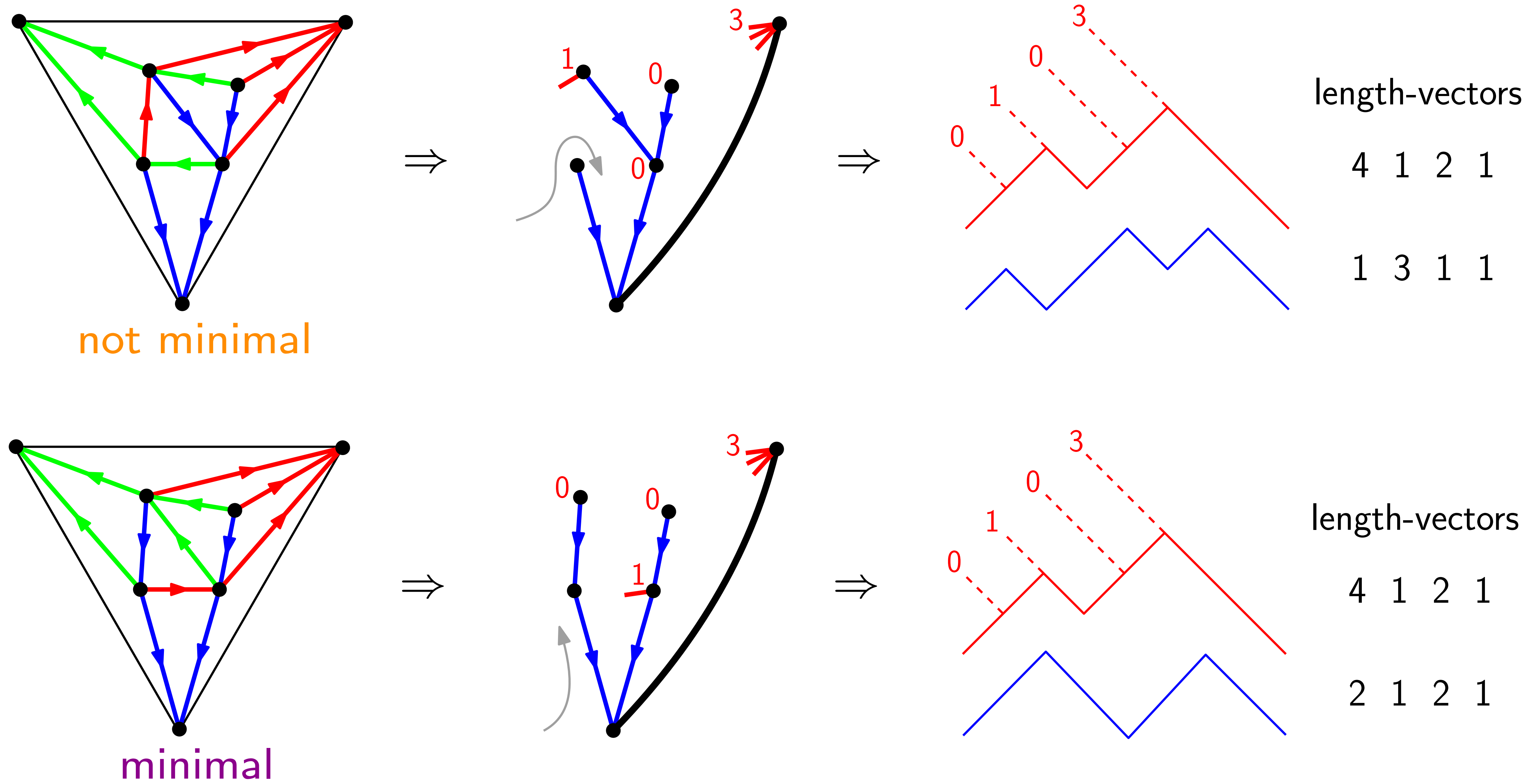
(other combinatorial proof in [Chapoton,Chatel,Pons'15] using interval posets)

The Bernardi-Bonichon bijection [Bernardi, Bonichon'07]

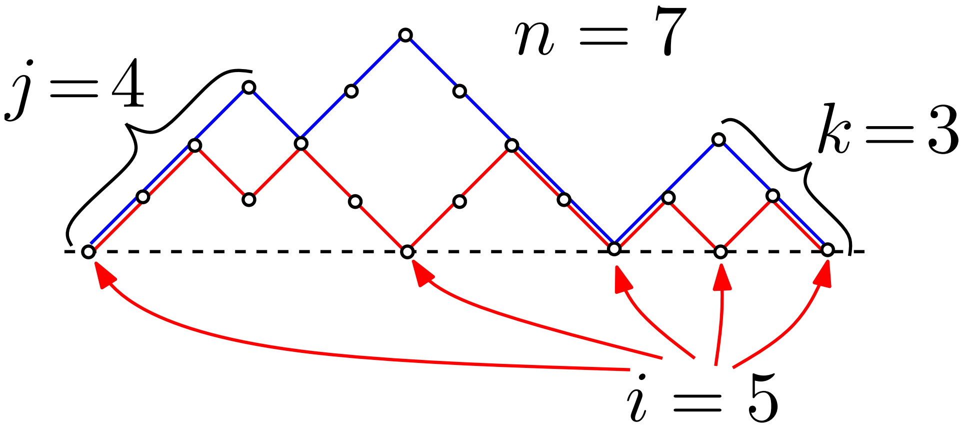
A direct (non-recursive) bijection between $\mathcal{T}_n$ and $\mathcal{I}_n$

Schnyder woods on $n+3$ vertices ↕ non-intersecting pairs of Dyck paths of lengths $2n$

minimal ↕ in $\mathcal{I}_n$



Remarks on symmetric distributions



(i, j, k) not symmetrically distributed over $\mathcal{I}_n$

($j = n$ implies $k = n$ but not $i = n$)

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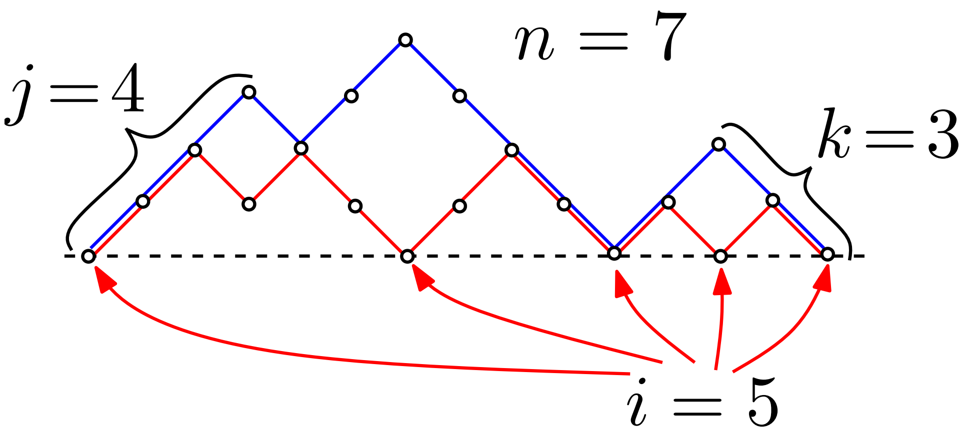

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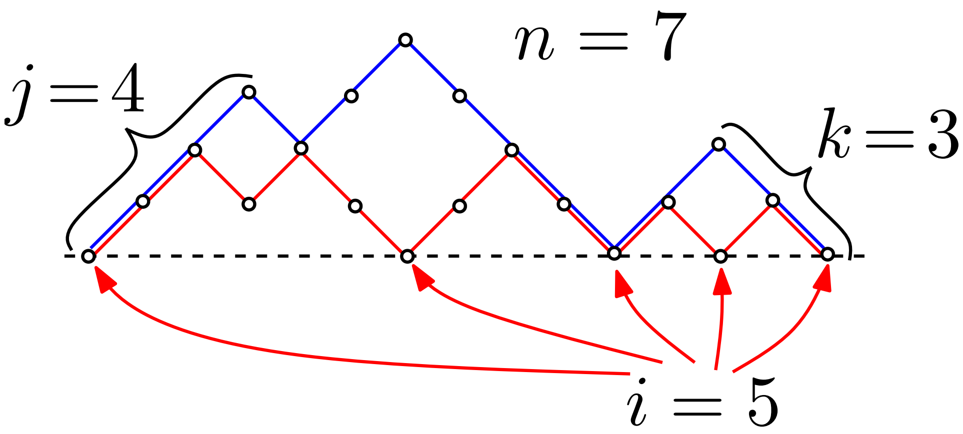

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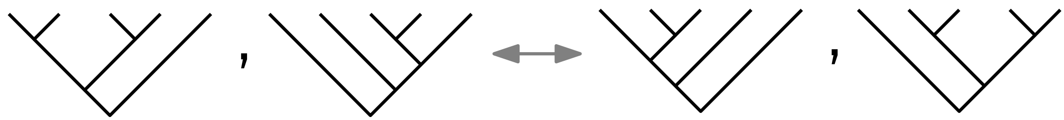
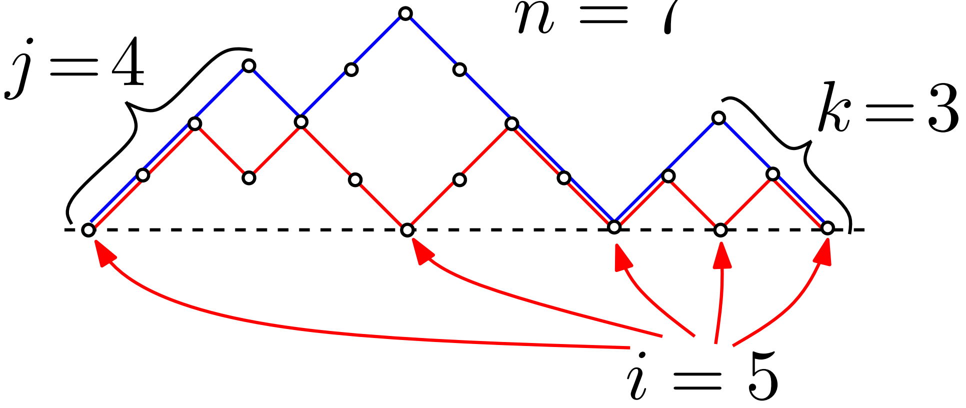
also follows from mirror-involution



Remarks on symmetric distributions



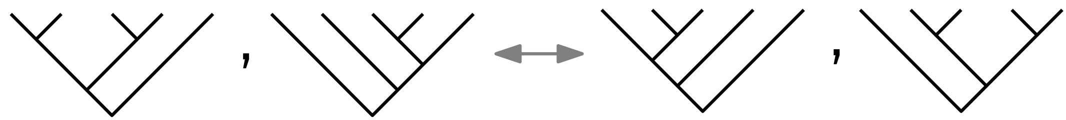
(i,j,k) not symmetrically distributed over $\mathcal{I}_n$

($j=n$ implies $k=n$ but not $i=n$)

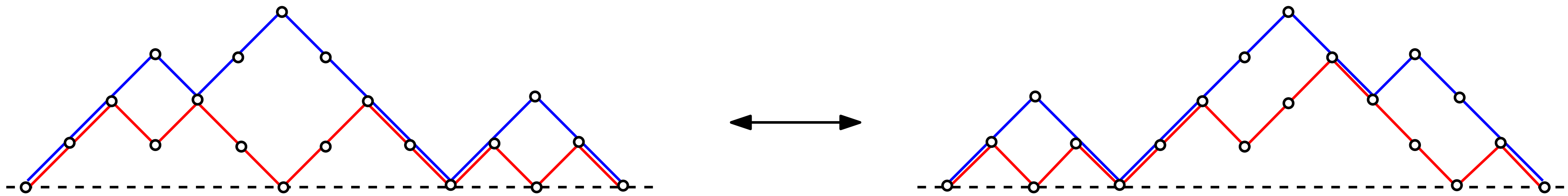
But:

- (i,j) is symmetric (cf bijection of Préville-Ratelle)
- (i,k) is symmetric (cf bijection of Bernardi-Bonichon) & same distribution as (i,j)

also follows from mirror-involution



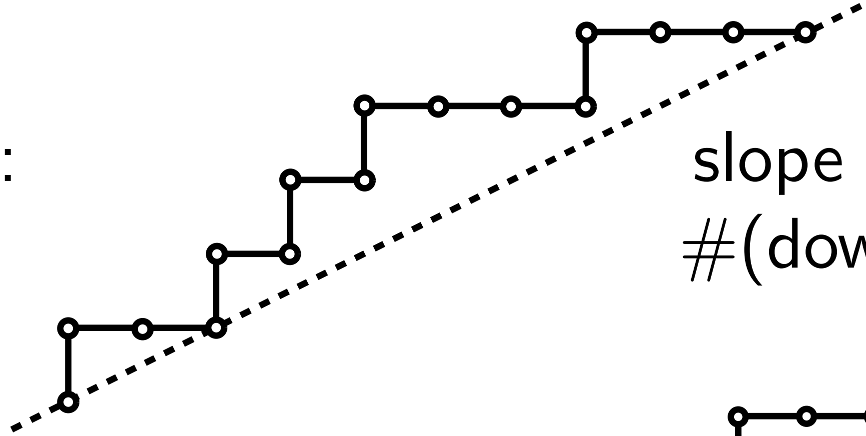
- Strong symmetry in (j,k): easy (inductive) bijection doing mirror of upper path & preserving i



Extension to m-Tamari lattices

[Bergeron,Préville-Ratelle'11] [Bousquet-Mélou,F,Préville-Ratelle'11] [Bousquet-Mélou,Chapuy,Préville-Ratelle'12]

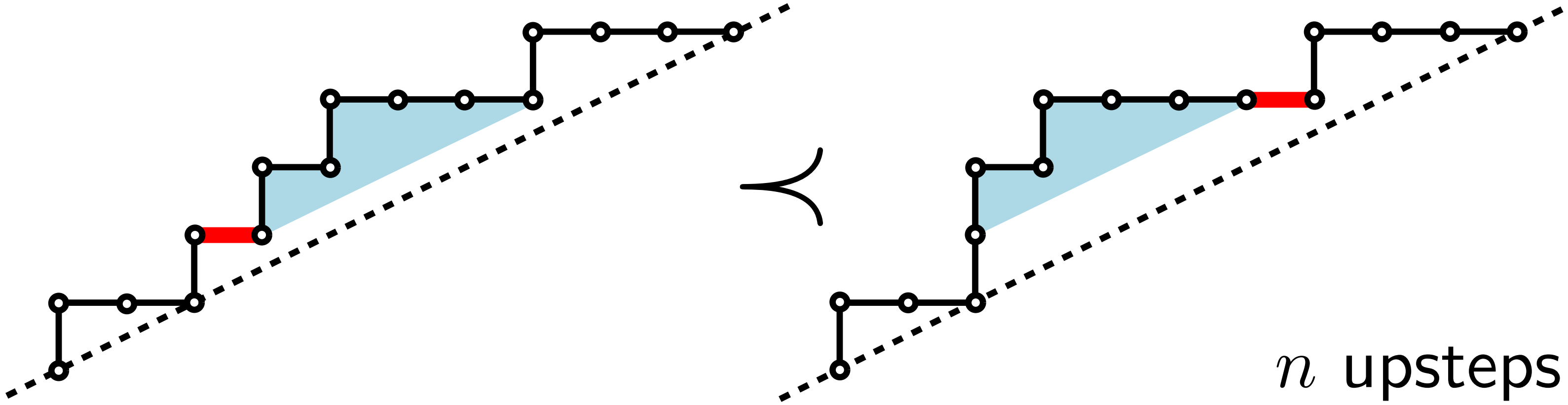
m-Dyck path:



slope $= 1/m$

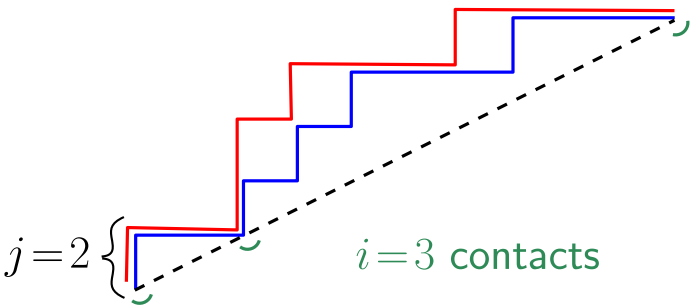
#(downsteps)$= m \cdot$ #(upsteps)

m-Tamari lattice:



Theo: $\#(\text{intervals in size } n) = \dfrac{m+1}{n(mn+1)}\dbinom{(m+1)^2 n + m}{n-1}$ (no bijective proof)

still symmetry in (i, j)



New Tamari intervals and bipartite maps

(discussions with Frédéric Chapoton)

New Tamari intervals

[Chapoton'06]

Obtaining a composed interval out of 2 intervals: operator $\circ_\ell$

Binary trees:

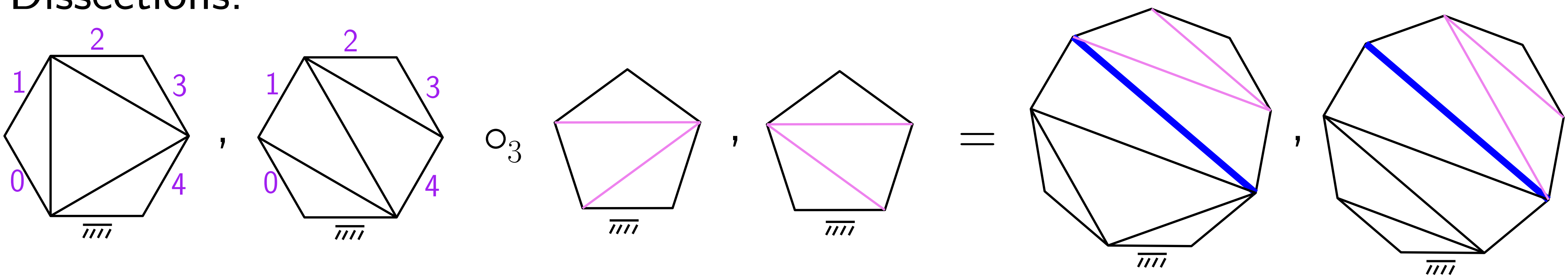
New Tamari intervals

[Chapoton'06]

Obtaining a composed interval out of 2 intervals: operator $\circ_\ell$

Binary trees:

Dissections:

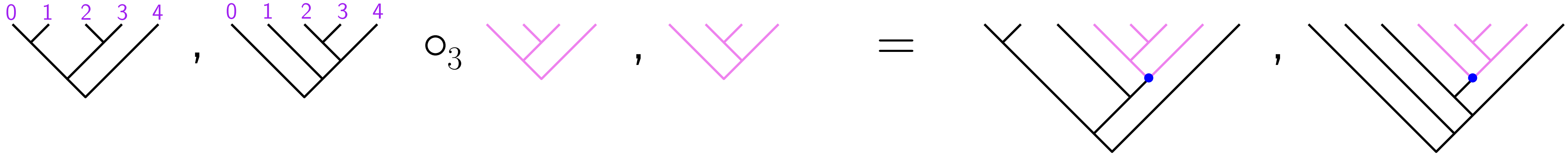


New Tamari intervals

[Chapoton'06]

Obtaining a composed interval out of 2 intervals: **operator $\circ_\ell$**

Binary trees:



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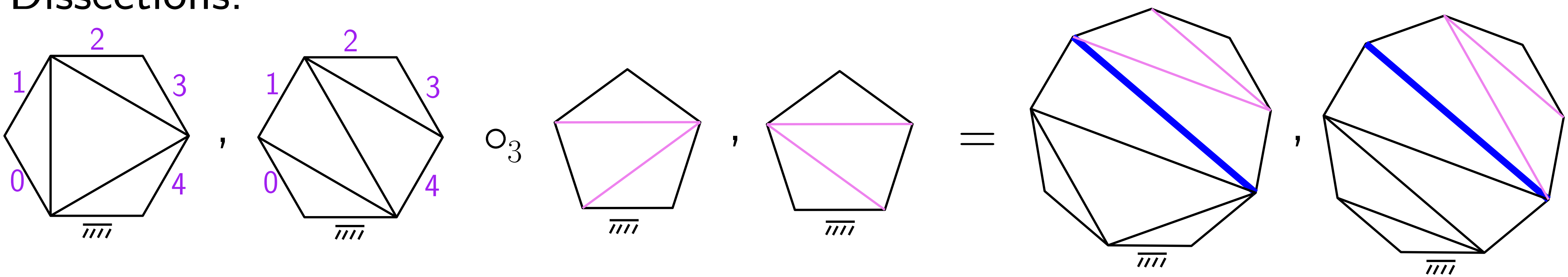
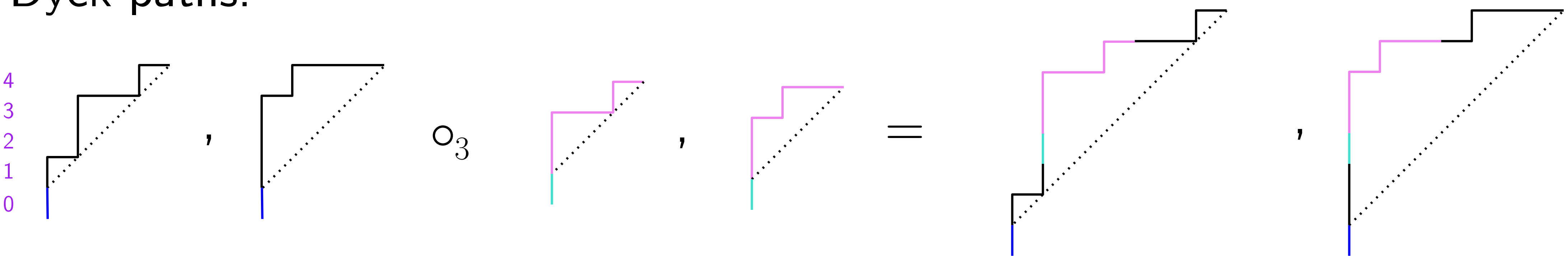


Dyck paths:



New Tamari intervals

[Chapoton'06]

Obtaining a composed interval out of 2 intervals: operator $\circ_\ell$

Binary trees:

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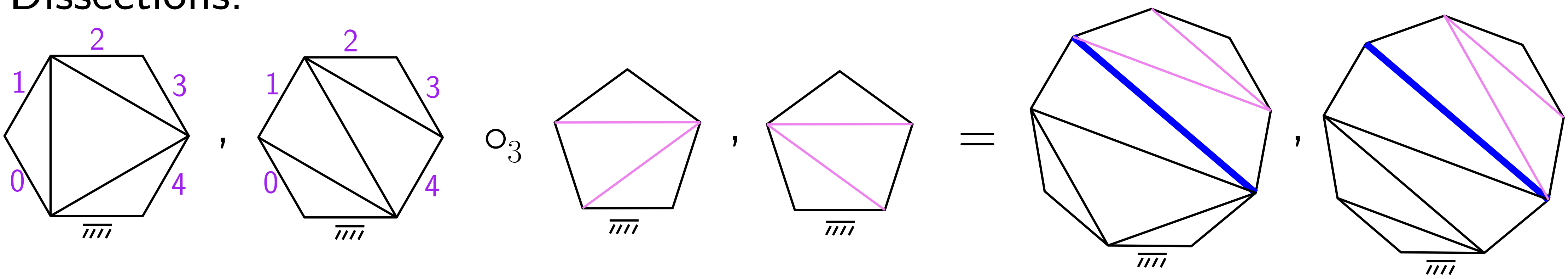

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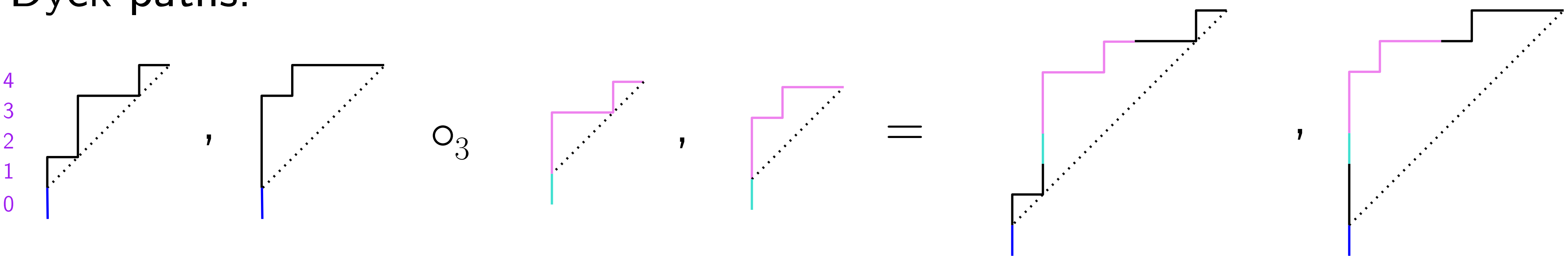


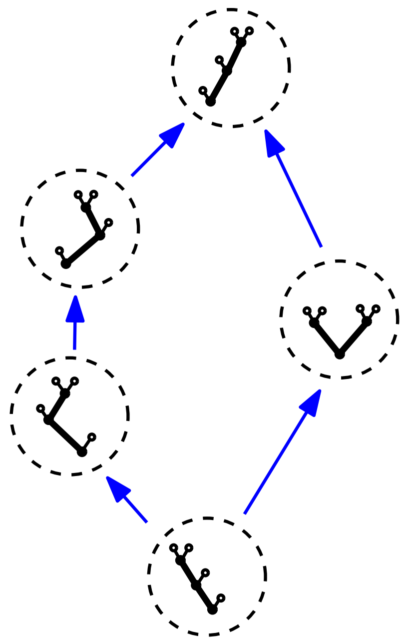
A Tamari interval is called "new" if it can not be obtained this way

Characterization of new intervals

For binary trees: no common node when superimposed

For dissections, no common diagonal when superimposed
(the two dissections do not belong to a same facet in the associahedron)

Ex: among the 13 intervals of size 3,
3 are such that both trees are not adjacent



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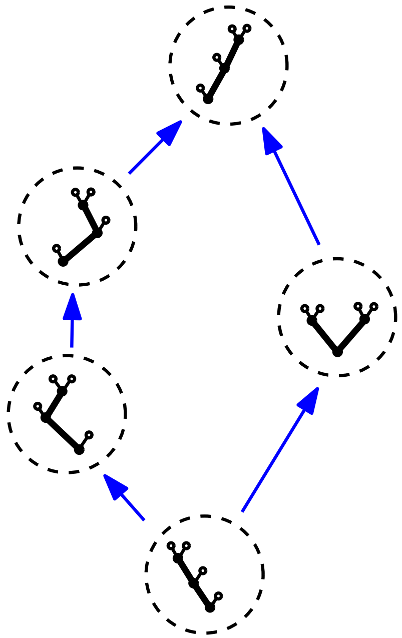
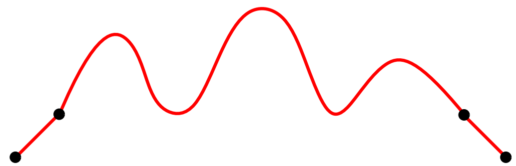
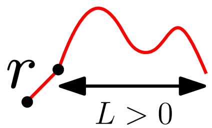
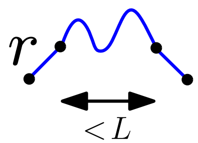


For Dyck paths, 2 conditions:

- upper path has only two contacts with x-axis



- For each $1 \leq r \leq n$, if 

then 

Enumeration of new intervals

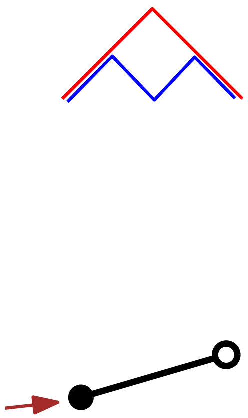
[Chapoton'06]

The number of new intervals of size $n+1$ is $b_n = 3 \cdot 2^{n-1} \frac{(2n)!}{(n+2)!n!}$

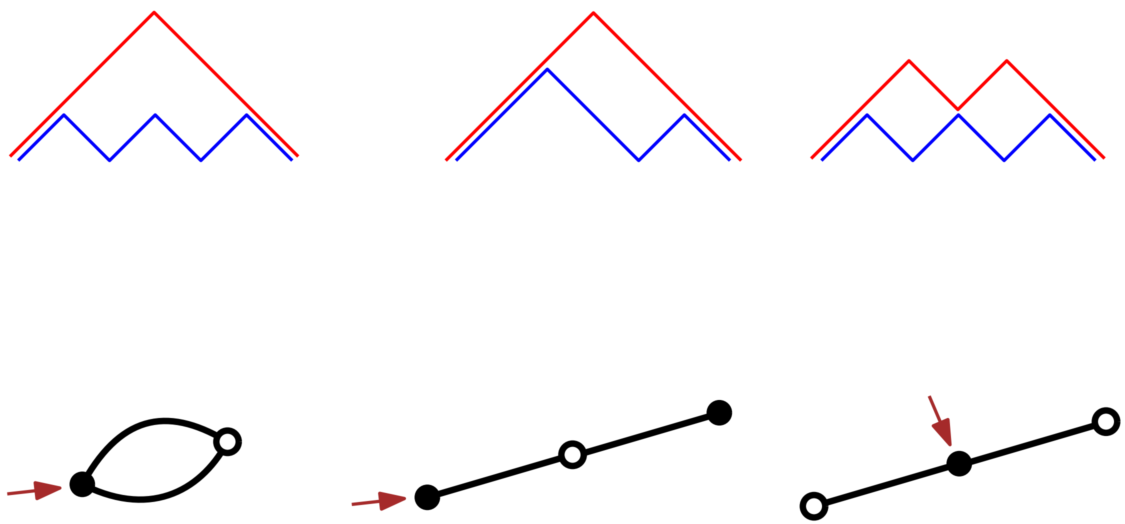
which is also the number of bipartite planar maps with n edges

$$b_n = 1, 3, 12, 56, 88, \ldots$$

$n = 1$



$n = 2$

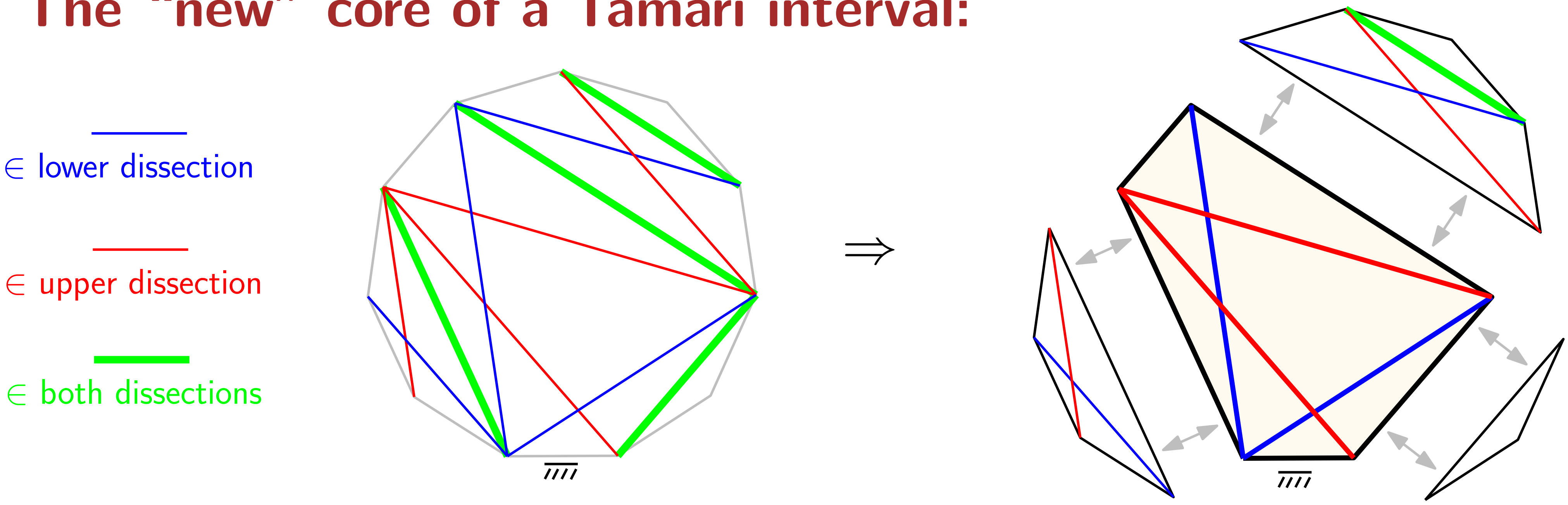


Proof by parallel core-decompositions

Let $I(t)$ be the series of Tamari intervals

$N(t)$ the series of new Tamari intervals

The "new" core of a Tamari interval:



Proof by parallel core-decompositions

Let $I(t)$ be the series of Tamari intervals $\qquad \widetilde{I}(t) := t \cdot I(t)$

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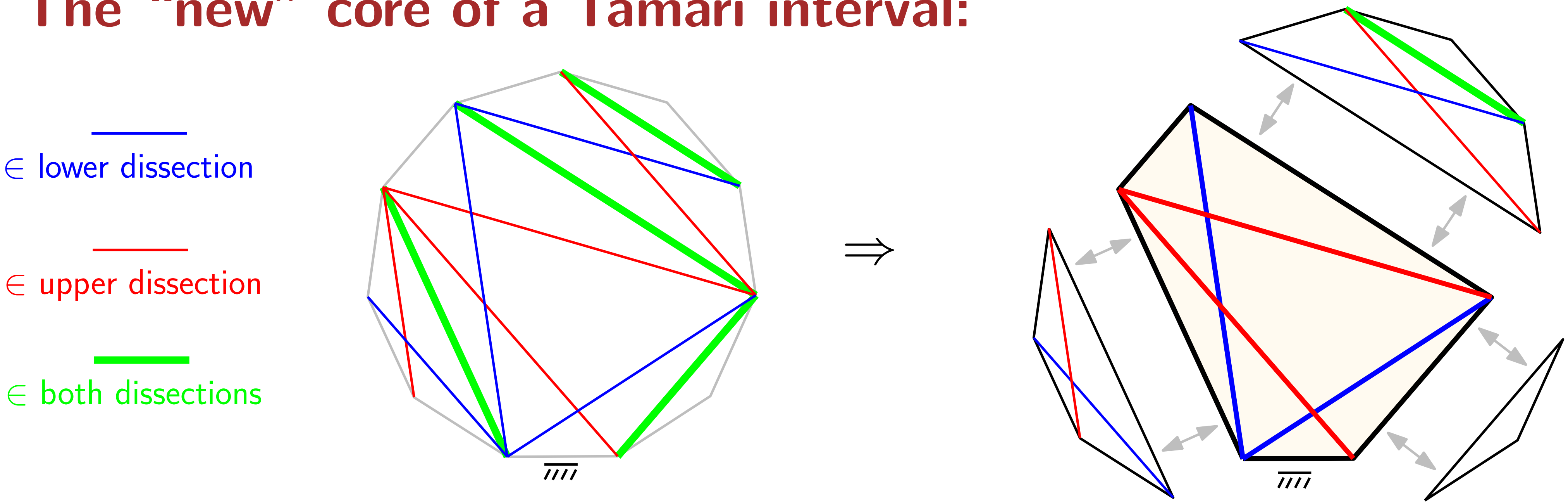

Decomposition implies:

$$\widetilde{I}(t) = \widetilde{N}(s)\Big|_{s=t+\widetilde{I}(t)}$$

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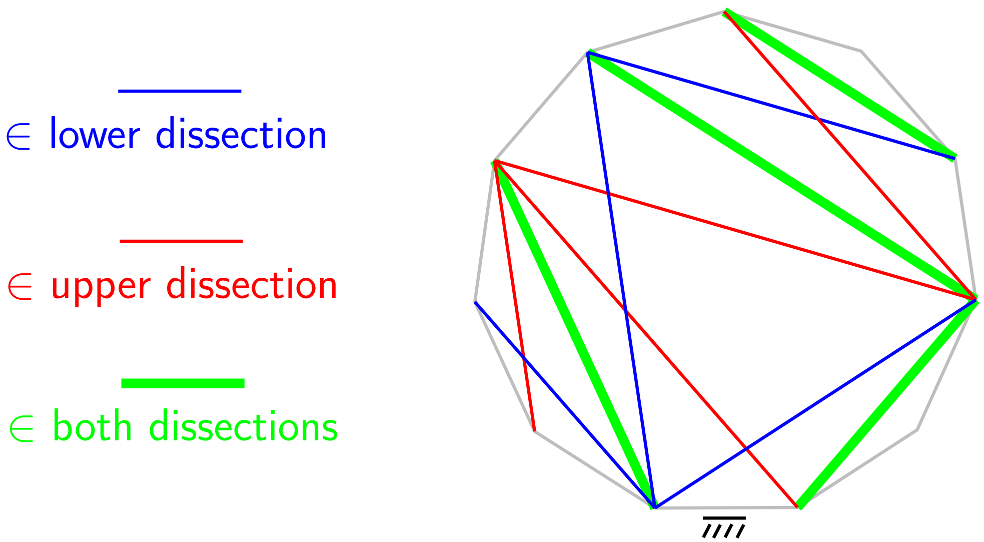

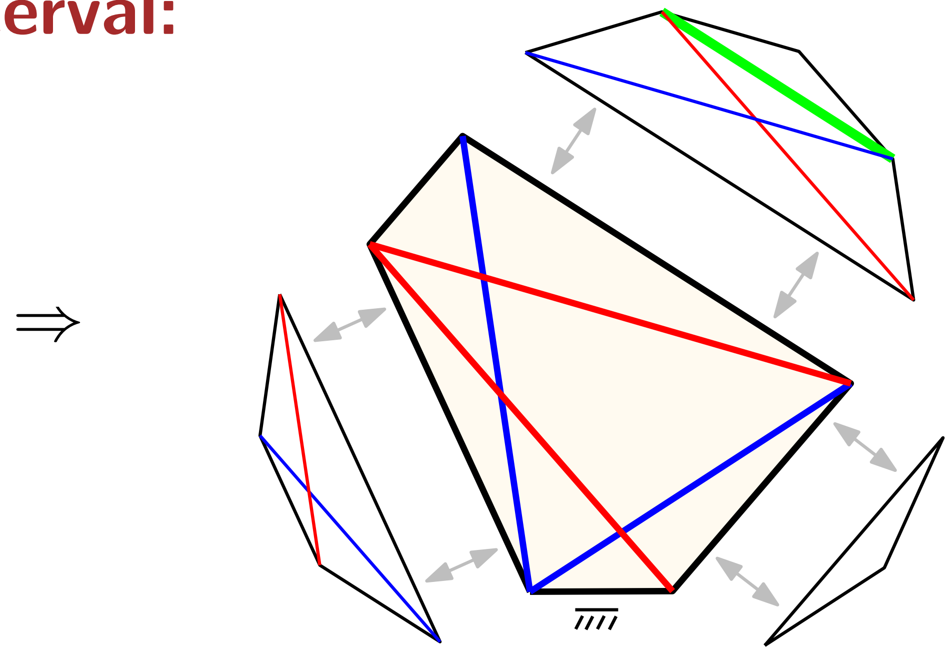

Decomposition implies:

$$\widetilde{I}(t) = \widetilde{N}(s)\Big|_{s=t+\widetilde{I}(t)}$$

$$\Downarrow$$

$$I(t) = \frac{N(s)}{1-N(s)}\Bigg|_{s=t+tI(t)}$$

Proof by parallel core-decompositions

Let $L(t)$ be the series of loopless maps (by edges)
$S(t)$ the series of simple maps (by edges)

The simple core of a loopless map:

Decomposition implies:

map equation

$$L(t) = S(s)\Big|_{s=t+tL(t)}$$

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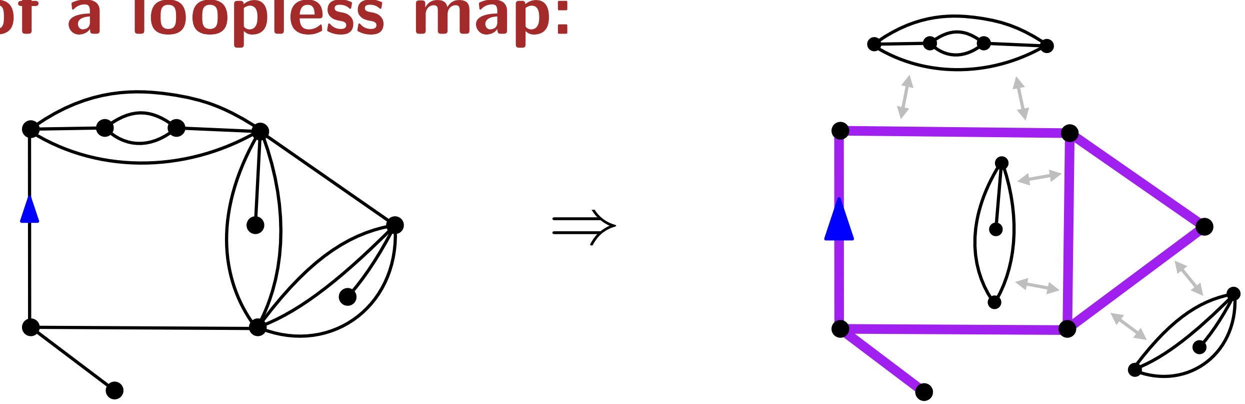

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+ known bijections:

$L(t) =$ series counting triangulations $= I(t)$ [Wormald'80]

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$S(s) = \frac{sB(s)}{1-sB(s)}$ with $B(s)$ the series for bipartite maps [Noy'13] [Bernardi,Collet,F'14]

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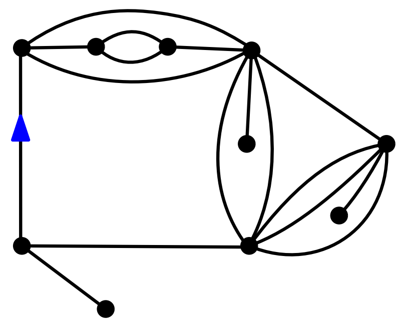
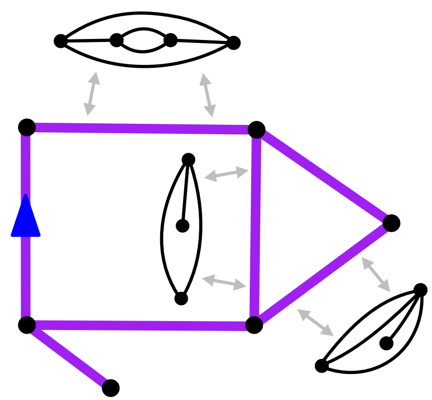


$\Rightarrow$



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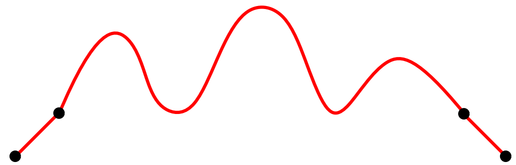
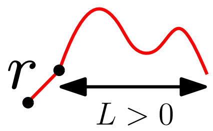
$$\Rightarrow \quad N(s) = sB(s)$$

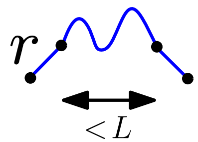
Proof by parallel reduction-decompositions

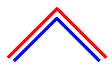
Let $b_{n,i} = \#($ new intervals of size $n+1$ with $i+2$ bottom contacts$)$

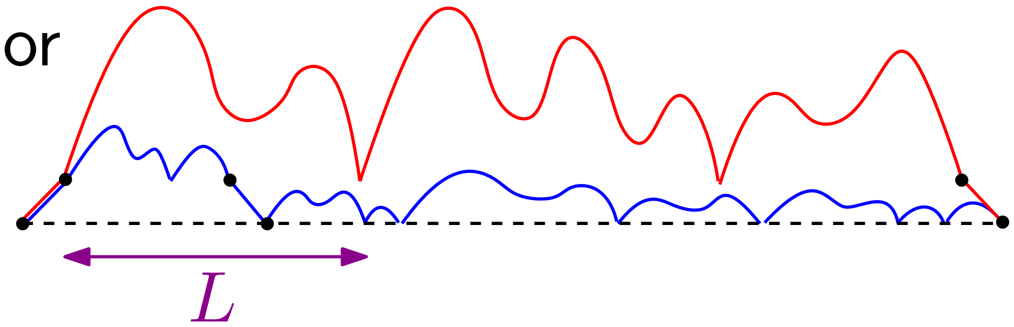
Let $G(t,u) := \sum_{n,i} b_{n,i} t^n u^i$

Interval is new $\Leftrightarrow$ for Dyck paths, 2 additional conditions:

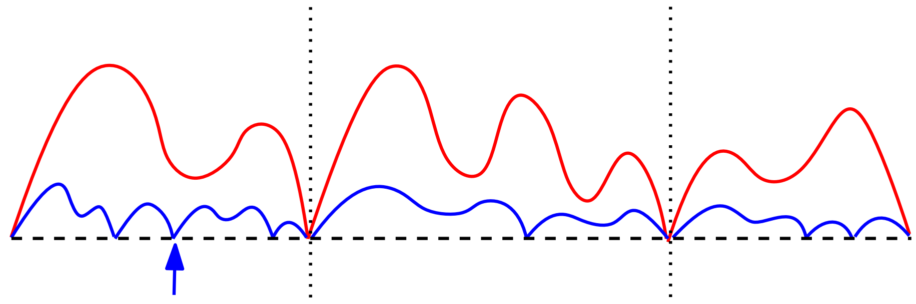
- upper path has only two contacts with x-axis 
- For each $1 \le r \le n$, if 

then 

Interval is either 

or 

$\Rightarrow$ reduction



$$G(t,u) = 1 + t \cdot \mathrm{subs}(u^i = u + \cdots + u^{i+1}, G(t,u)) \cdot \frac{1}{1 - tuG(t,u)}$$

$$= 1 + tu \frac{uG(t,u) - G(t,1)}{u-1} \frac{1}{1 - tuG(t,u)}$$

Proof by parallel reduction-decompositions

Let $b_{n,i} = \#($ new intervals of size $n+1$ with $i+2$ bottom contacts$)$

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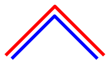
Interval is either 

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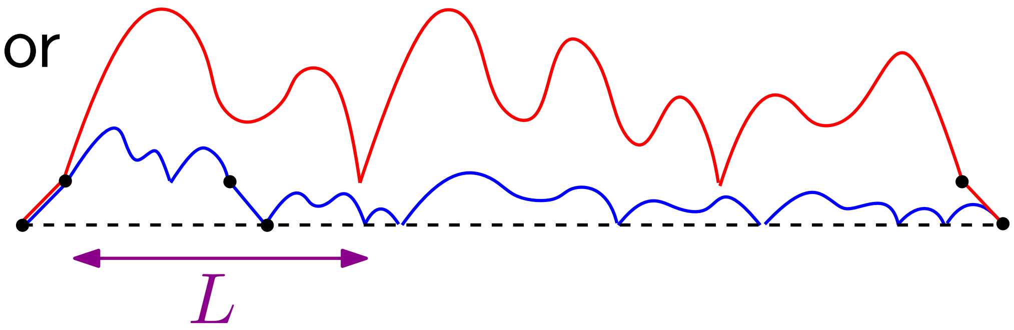

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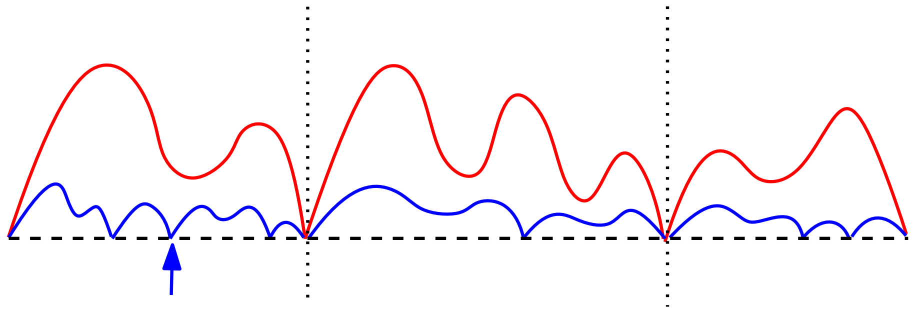

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$$\Updownarrow$$

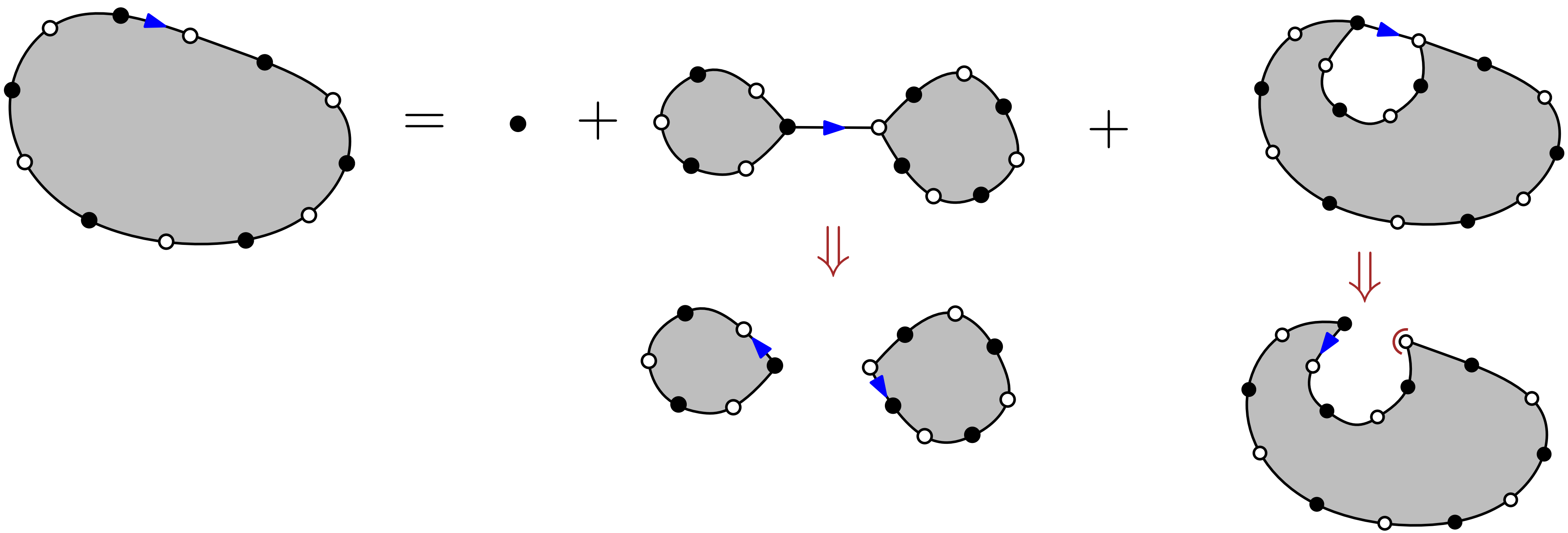
$$\boxed{G(t,u) = 1 + tuG(t,u)^2 + tu\frac{G(t,u) - G(t,1)}{u-1}}$$

Proof by parallel reduction-decompositions

Same equation as for bipartite maps!

Let $b_{n,i} = \#($ bipartite maps with n edges and outer degree $2i)$

Let $G(t,u) := \sum_{n,i} b_{n,i} t^n u^i$

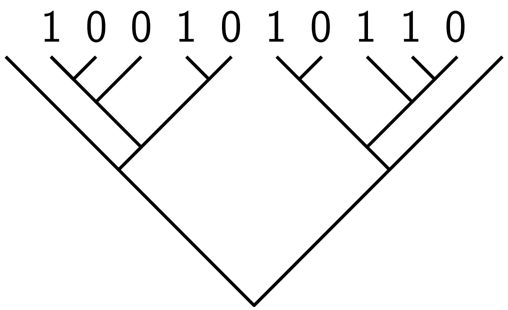


$$G(t,u) \quad = \quad 1 \; + \; tuG(t,u)^2 \; + t\,\mathrm{subs}(u^i\!=\!u\!+\!\cdots+u^i, G(t,u))$$

$$\boxed{G(t,u) = 1 + tuG(t,u)^2 + tu\frac{G(t,u) - G(t,1)}{u-1}}$$

Canopy triple of parameters for intervals

- Canopy of a binary tree: word giving the types of the leaves read left to right

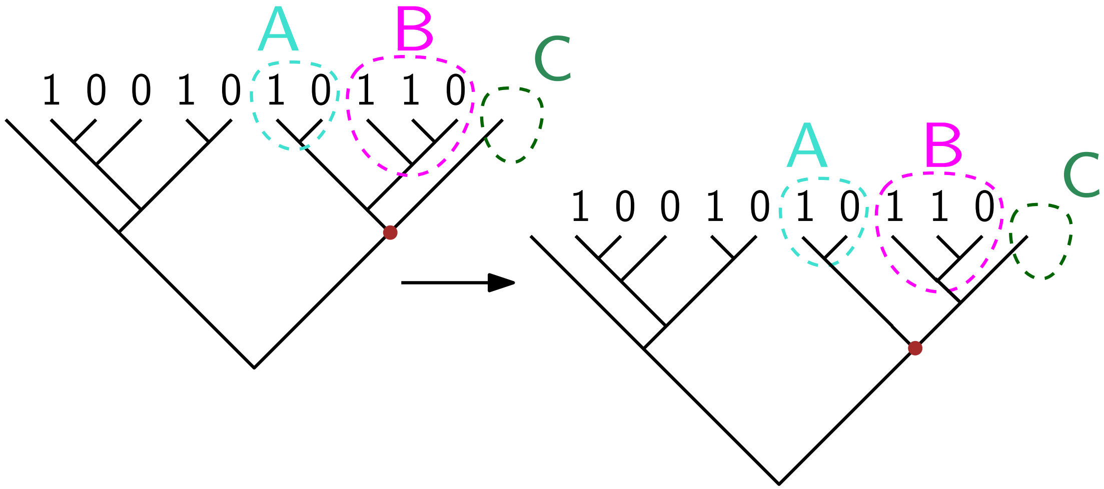


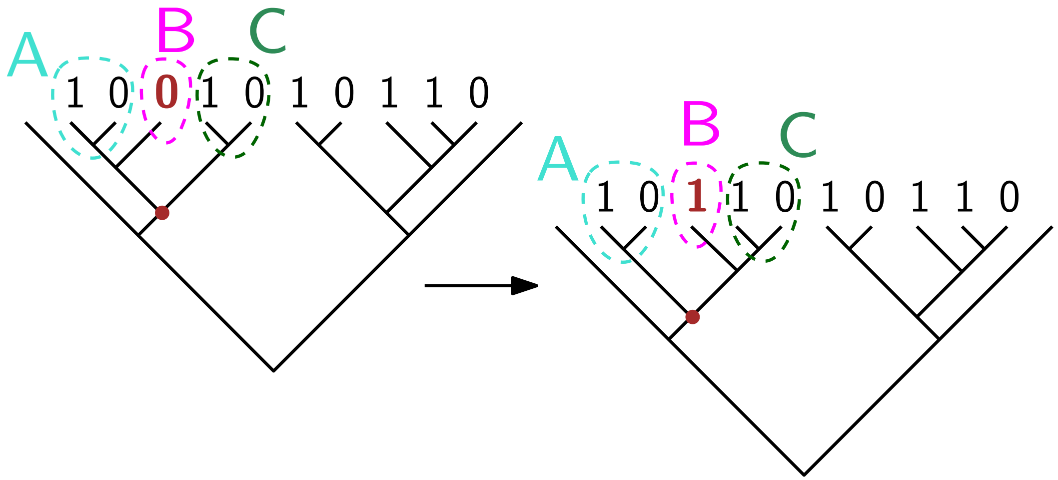
Canopy triple of parameters for intervals

- Canopy of a binary tree: word giving the types of the leaves read left to right

effect of a right rotation on the canopy: nothing or switches a 0 to a 1

$B \neq$ leaf $\quad$ $B =$ leaf





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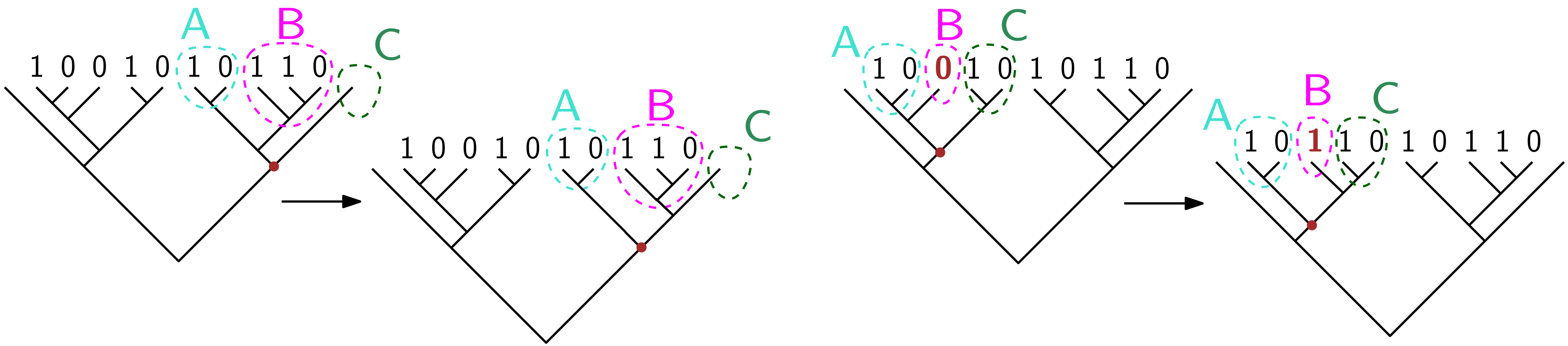
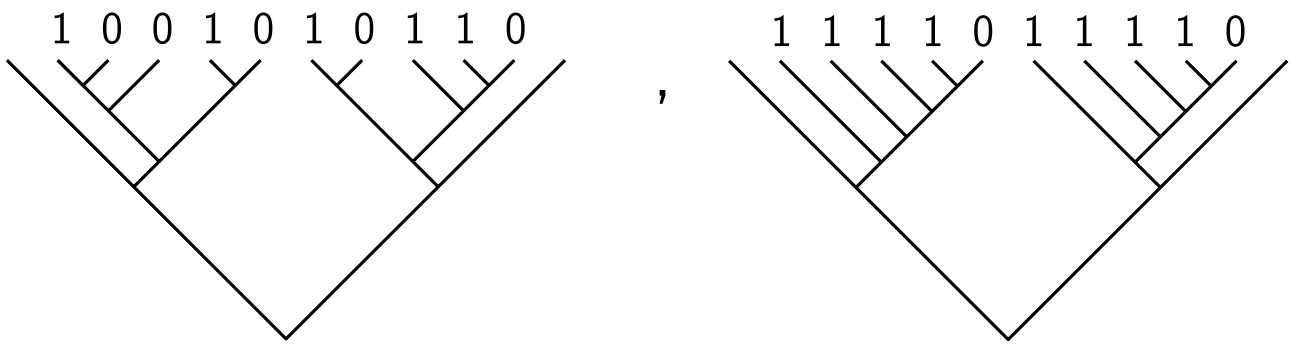


Hence, for $t \leq t'$, we have $\mathrm{Canopy}(t) \leq \mathrm{Canopy}(t')$



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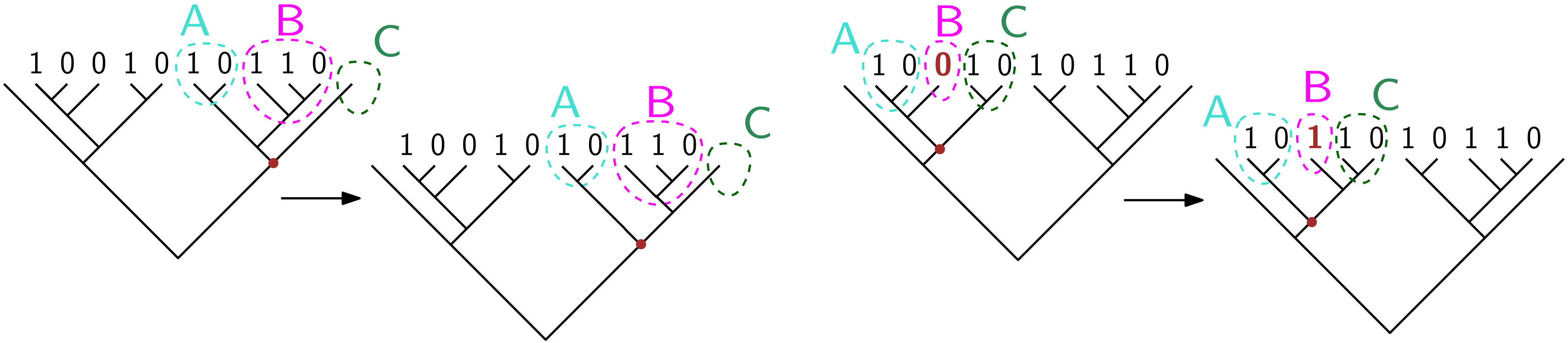

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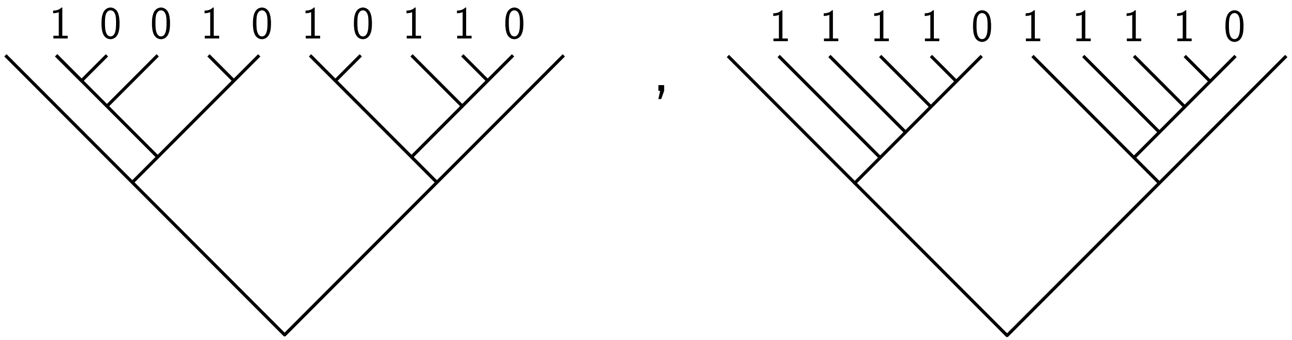

canopy-word $= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ $\quad$ weight $= x^{\#\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} y^{\#\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} z^{\#\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}$

weight $= x^2 y^3 z^5$

Generating functions by canopy triples

- Let $F(x,y,z) :=$ series of Tamari intervals, with $x^{\#\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} y^{\#\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} z^{\#\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}$

$$F(x,y,z) = 1 + (x+y+z) + (x^2+y^2+z^2+3xy+3yz+4xz)$$
$$x^3+y^3+z^3+6x^2y+6xy^2+10x^2z+10xz^2+6y^2z+6yz^2+21xyz$$

only symmetry $x \leftrightarrow z$ (cf mirror-symmetry)

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only symmetry $x \leftrightarrow z$ (cf mirror-symmetry)

Rk: $F(x,0,z)$ is the series for intervals where both canopies are equal, counts non-separable maps by vertices and faces [Fang,Préville-Ratelle'16]

Generating functions by canopy triples

- Let $F(x,y,z) :=$ series of Tamari intervals, with $x^{\#\left[\substack{0\\0}\right]} y^{\#\left[\substack{1\\0}\right]} z^{\#\left[\substack{1\\1}\right]}$

$$F(x,y,z) = 1 + (x+y+z) + (x^2+y^2+z^2+3xy+3yz+4xz)$$
$$x^3+y^3+z^3+6x^2y+6xy^2+10x^2z+10xz^2+6y^2z+6yz^2+21xyz$$

only symmetry $x \leftrightarrow z$ (cf mirror-symmetry)

Rk: $F(x,0,z)$ is the series for intervals where both canopies are equal, counts non-separable maps by vertices and faces [Fang,Préville-Ratelle'16]

- Let $G(x,y,z) :=$ series restricted to **new** Tamari intervals

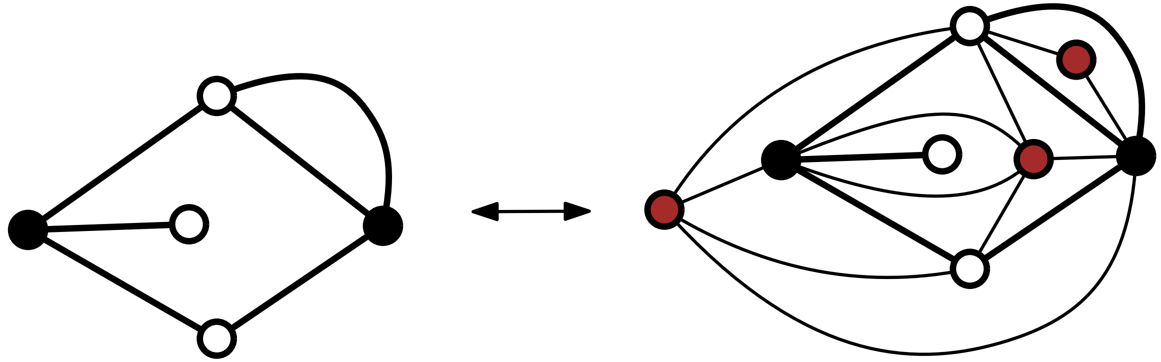
$$\frac{1}{y}G(x,y,z) = 1 + (x+y+z) + (x^2+y^2+z^2+3xy+3xz+3yz)$$
$$+(x^3+y^3+z^3+6x^2y+6xy^2+6x^2z+6xz^2+6y^2z+6yz^2+17xyz) + \cdots$$

symmetry in the 3 variables!

Equidistributed triple for bipartite maps

Let $M(x, y, z)$ be the series of bipartite maps $x^{\#\bullet}\, y^{\#\circ}\, z^{\#\,\text{faces}}$

symmetric in x, y, z, cf



3-colored triangulation

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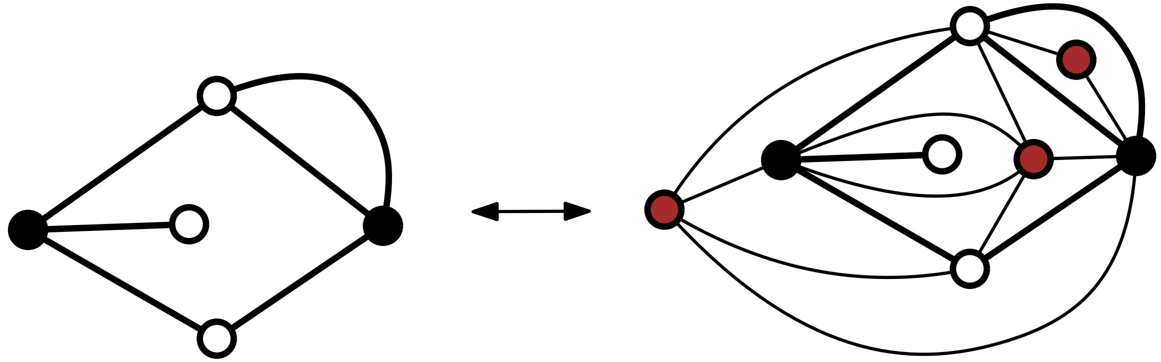

3-colored triangulation

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coincides with the series for new intervals!

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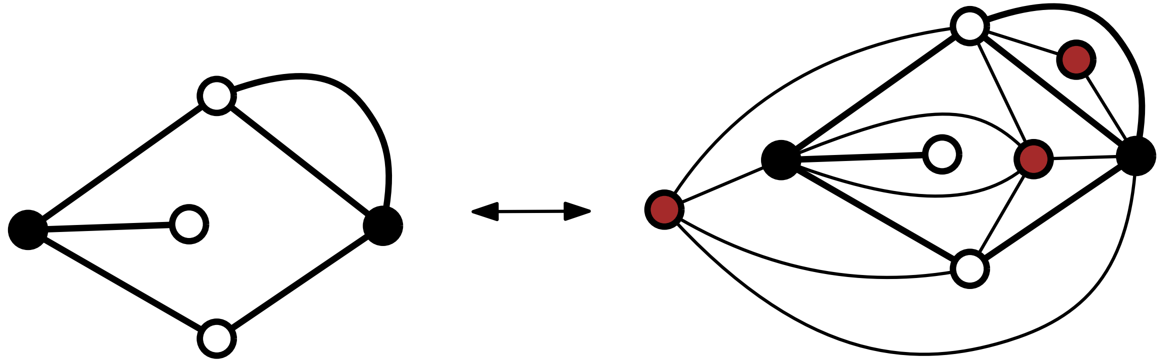

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- Symmetric parametrized expression (from bijection with some trees)

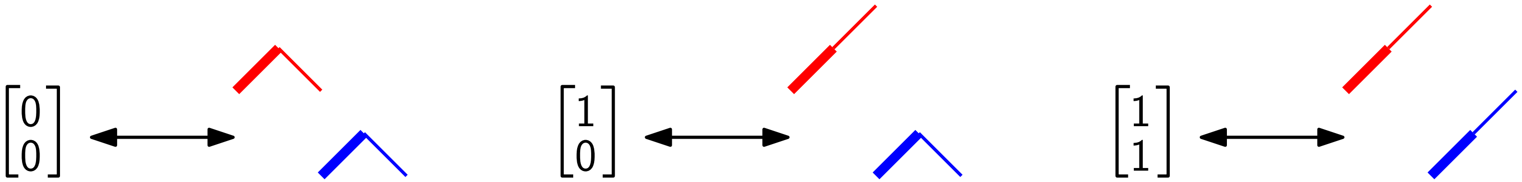
[Bousquet-Mélou, Schaeffer'02]
[Bouttier, Di Francesco, Guitter'02]

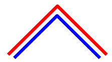
$$M = U_1U_2U_3(1-U_1-U_2-U_3)$$

with
$$\begin{cases} U_1 = x + U_1U_3 + U_2U_1 \\ U_2 = y + U_1U_2 + U_2U_3 \\ U_3 = z + U_2U_3 + U_1U_3 \end{cases}$$

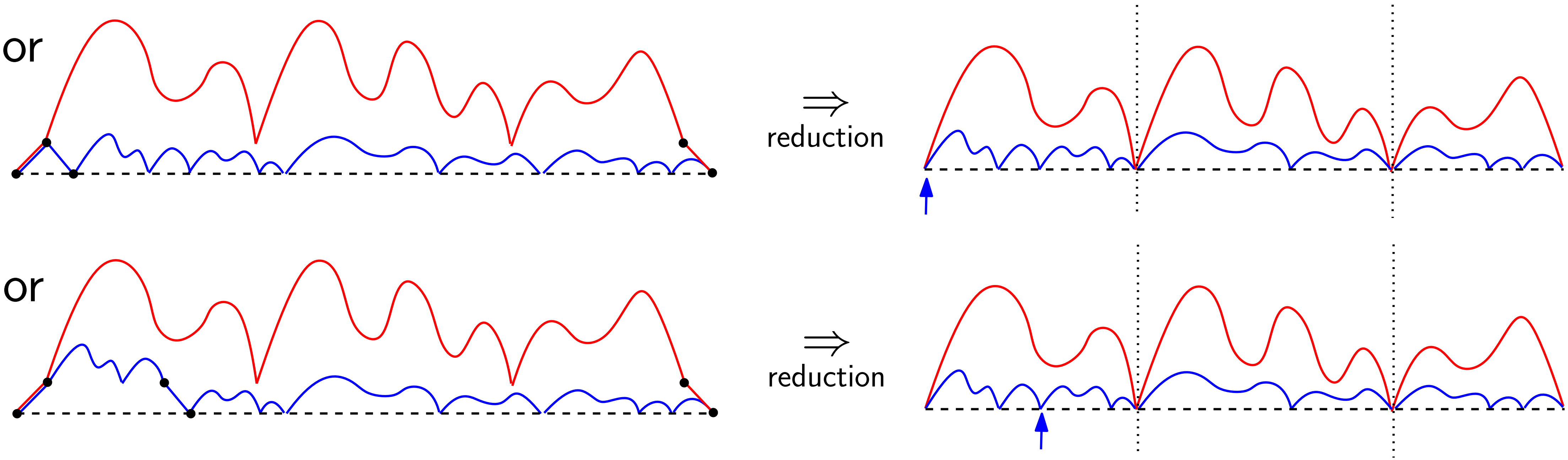
Proof by parallel reduction-decompositions

On Dyck paths,

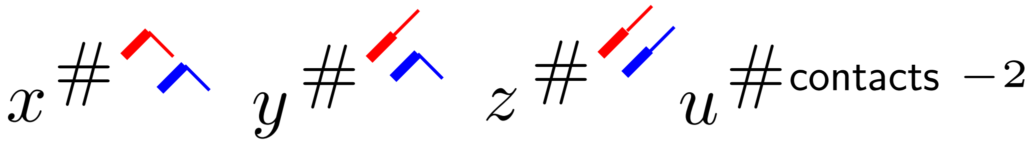


Interval is either 

or



Let $L \equiv L(x, y, z; u)$ the series, with



$L_1 := L$

$$L = x + yu\frac{L}{1 - uL} + zu\frac{L - L_1}{u - 1}\frac{1}{1 - uL}$$

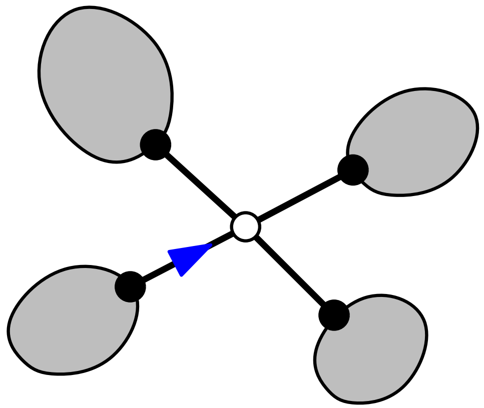
Proof by parallel reduction-decompositions

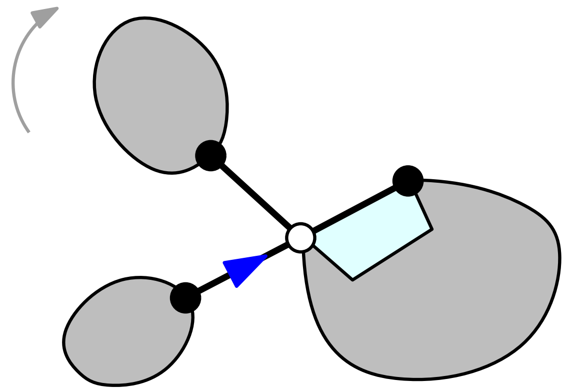
Let $M \equiv M(x, y, z; u)$ be the series of bipartite maps,

with $\quad x^{\#\bullet}\, y^{\#\circ}\, z^{\#\text{ faces -1}}\, u^{\frac{1}{2}\text{outer}-\text{degree}}$

$$M_1 := M\Big|_{u=1}$$

$$M \;=\; x \;+\; y\frac{uM}{1-uM} \;+\; zu\frac{M-M_1}{u-1}\frac{1}{1-uM}$$





Same equation as for L, hence $L = M$ (recursive bijection)

and $zL_1 = zM_1 = zx + U_1U_2U_3(1 - U_1 - U_2 - U_3)$

with $\begin{cases} U_1 = x + U_1U_3 + U_2U_1 \\ U_2 = y + U_1U_2 + U_2U_3 \\ U_3 = z + U_2U_3 + U_1U_3 \end{cases}$