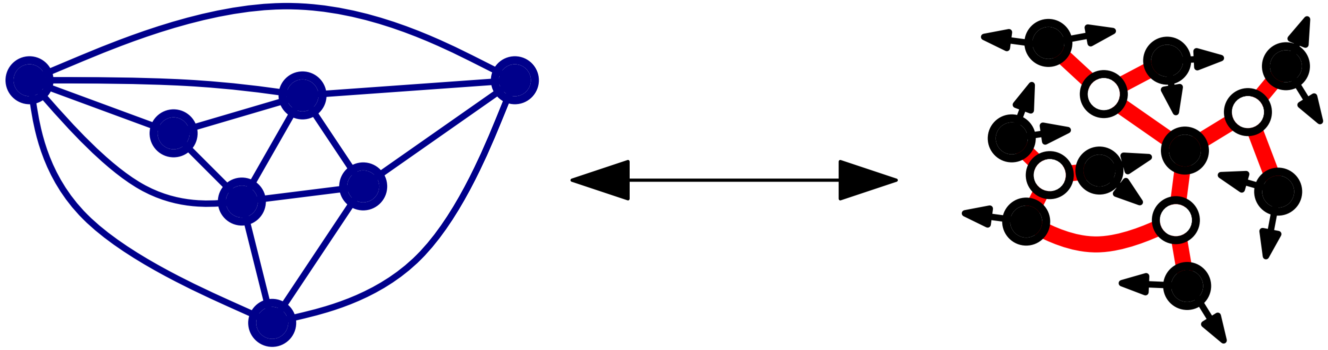
A master bijection for planar maps and its applications

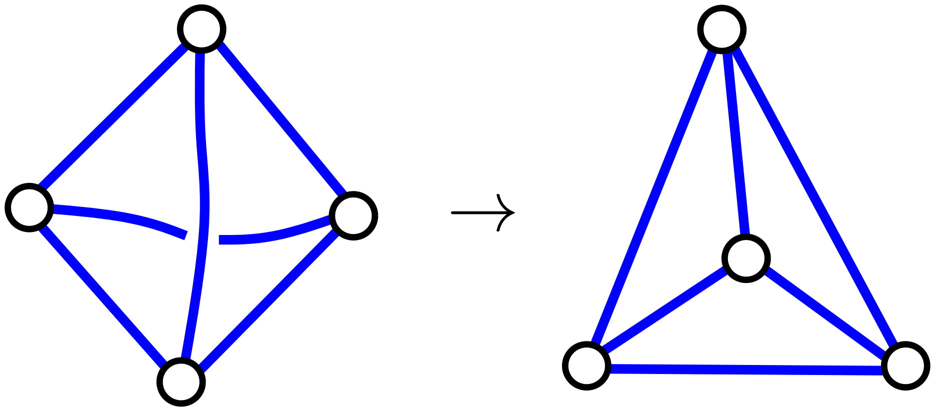
Éric Fusy (CNRS/LIX)
Joint work with Olivier Bernardi (MIT)



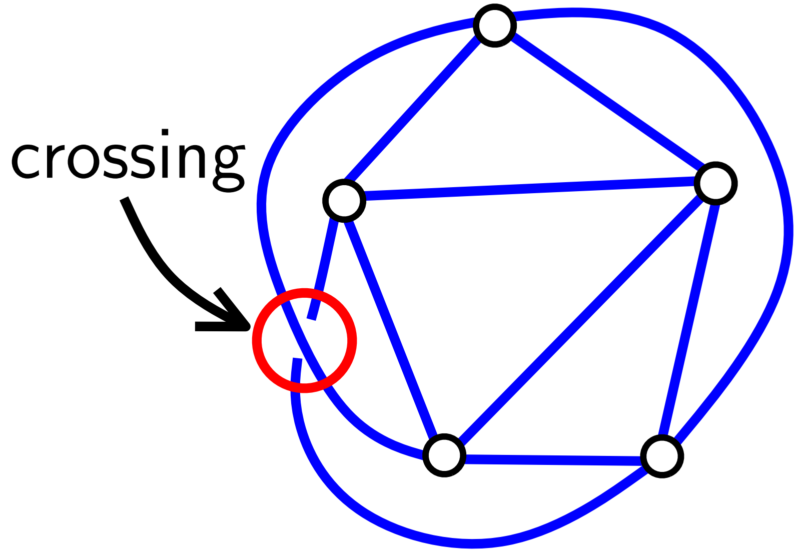
Madrid, June 2011

Planar graphs. Definition

A **planar graph** is a graph that can be drawn in $\mathbb{R}^2$ without edge-crossing



K_4 is planar



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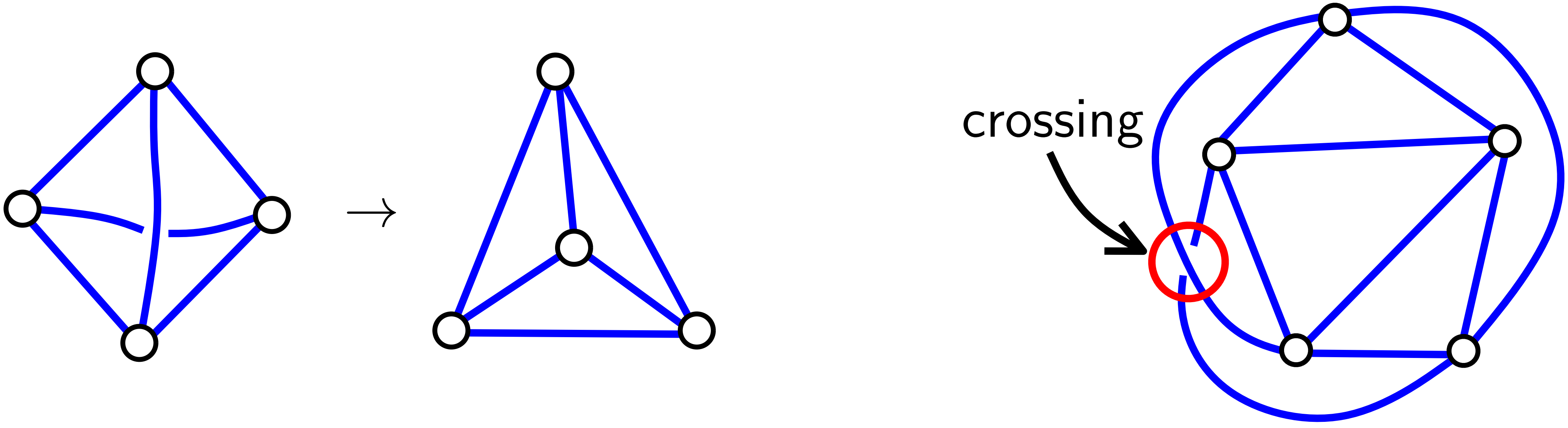

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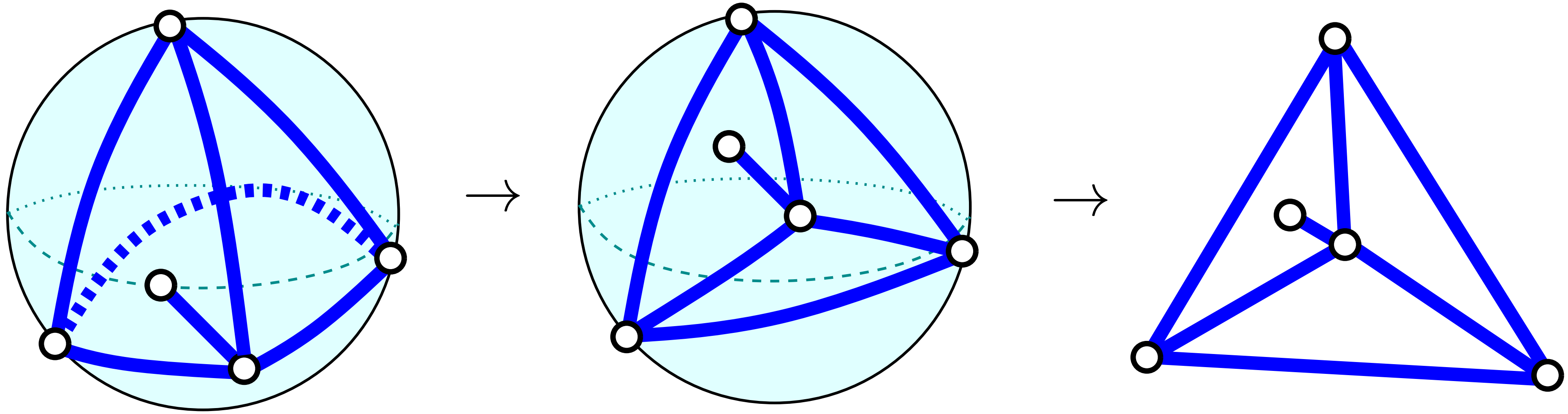

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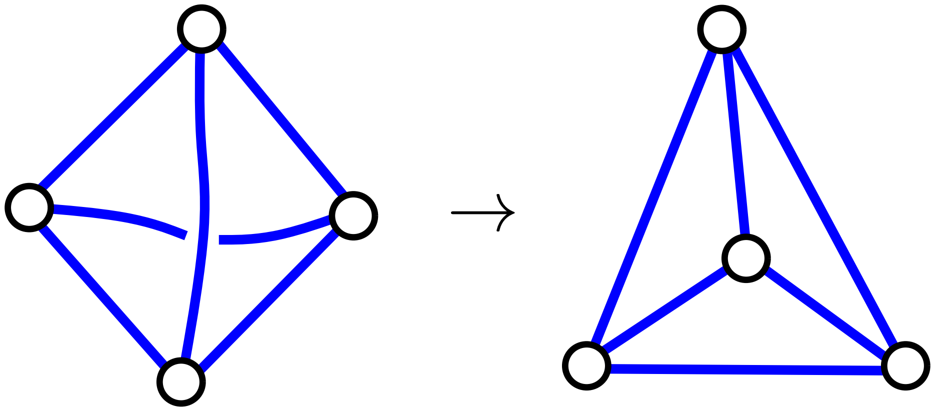

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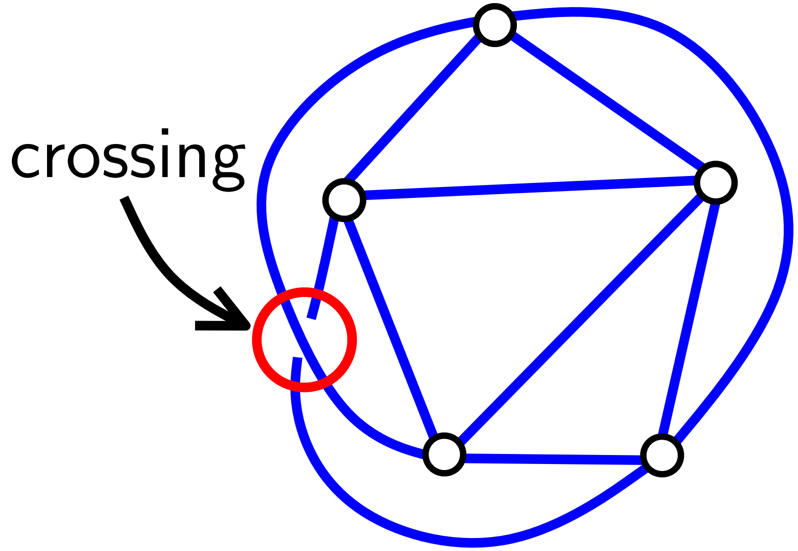

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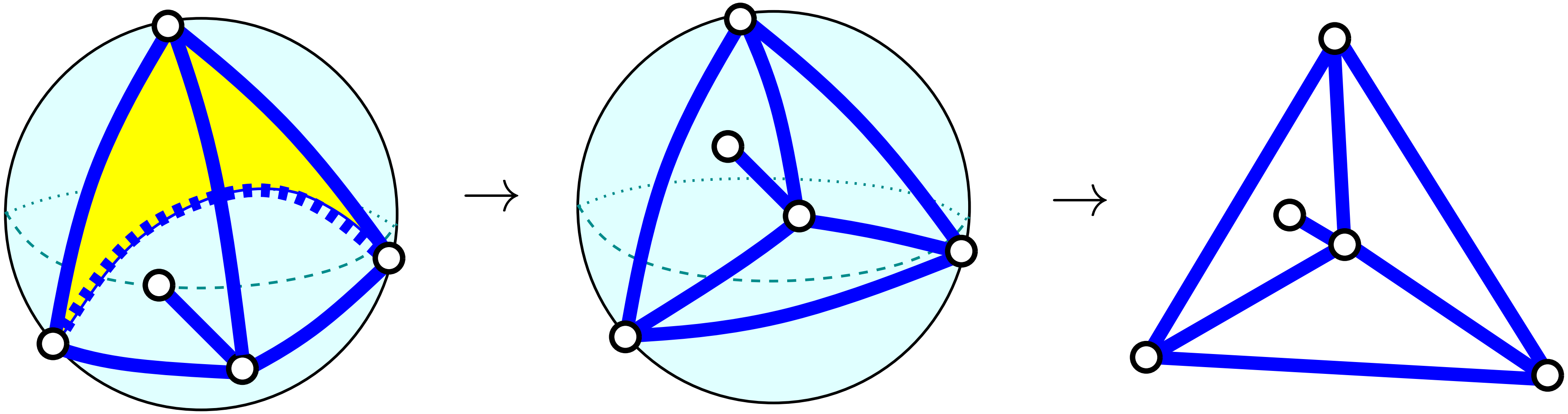

Planar maps and plane maps. Definition

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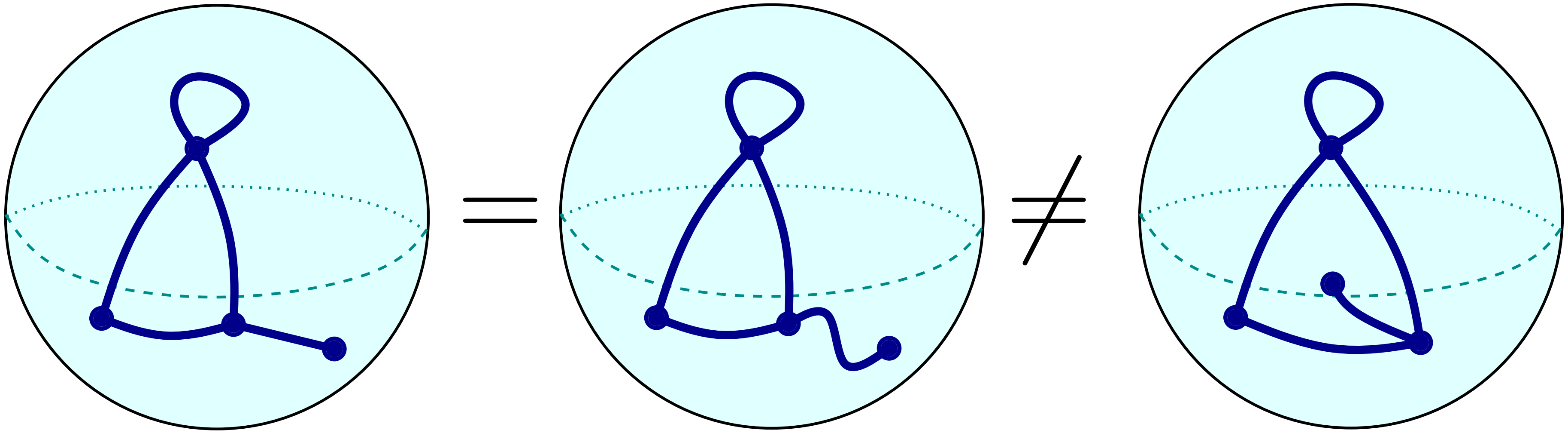

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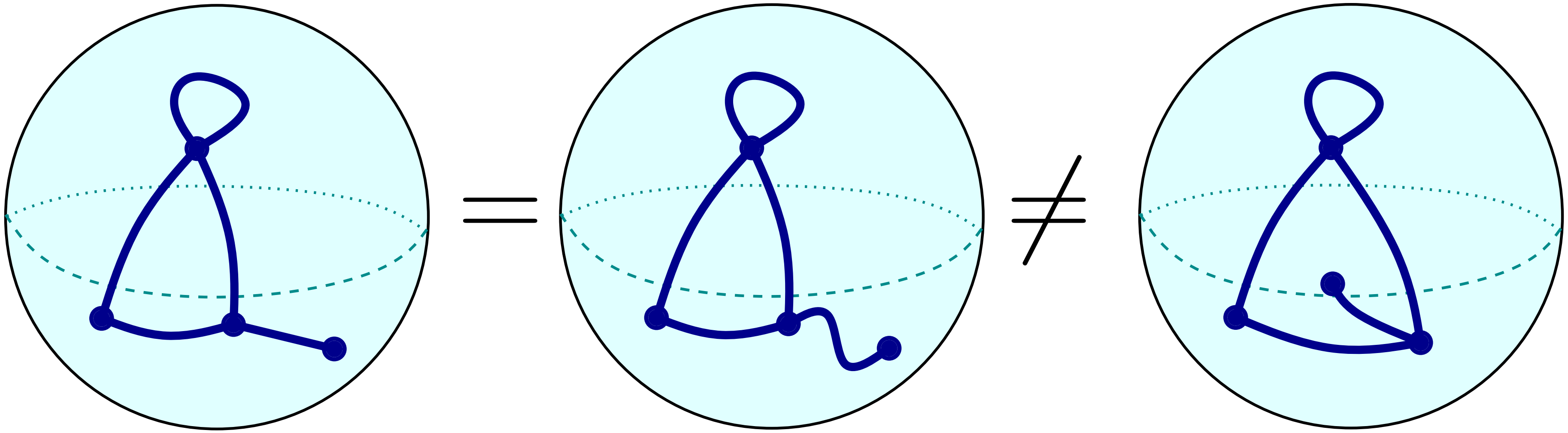

(i) A map has vertices and edges (like a graph), **and also faces**

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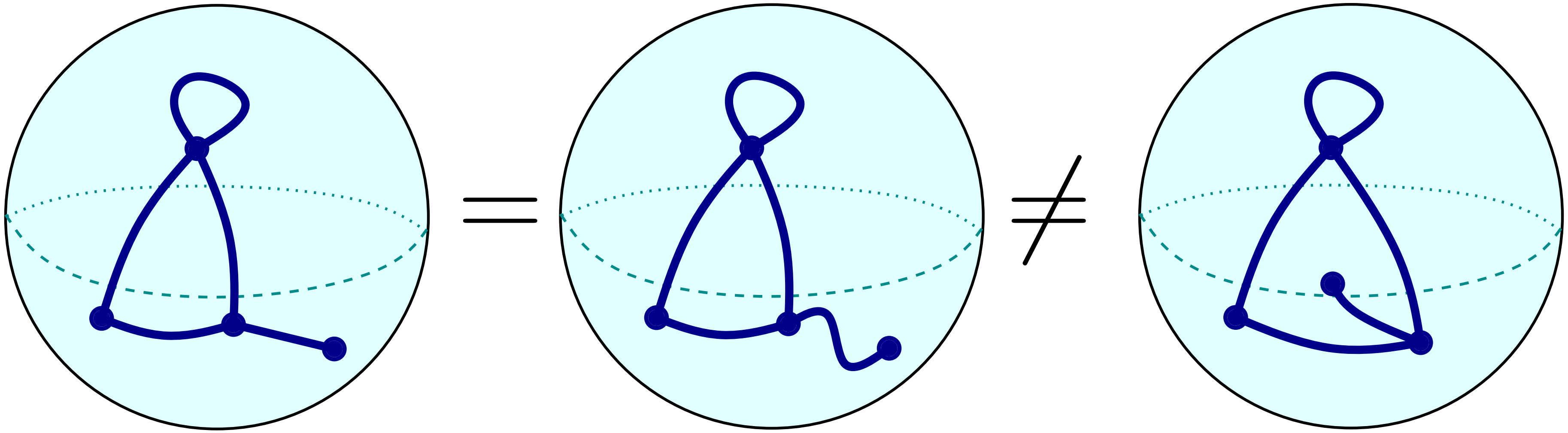

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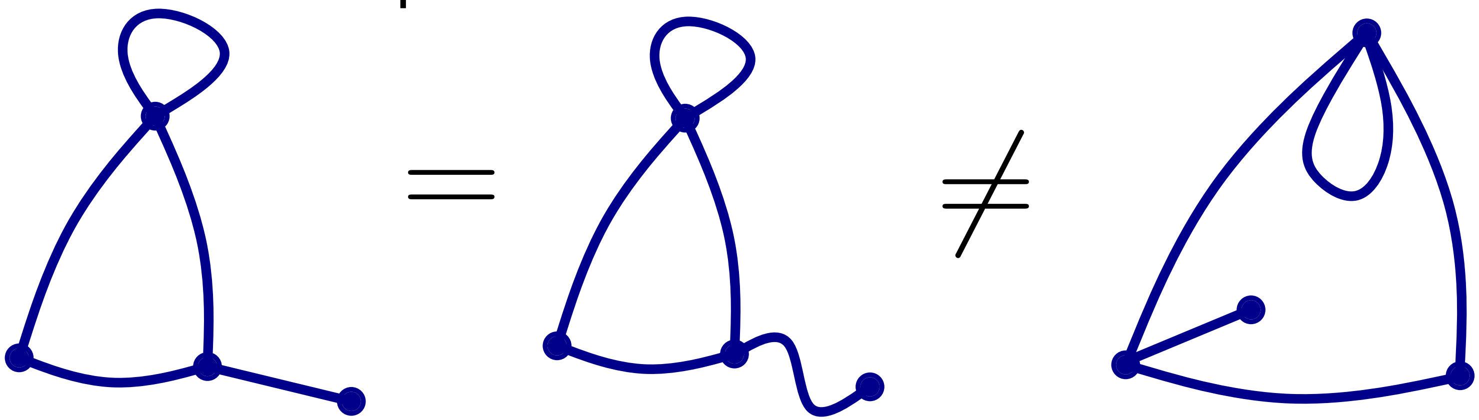

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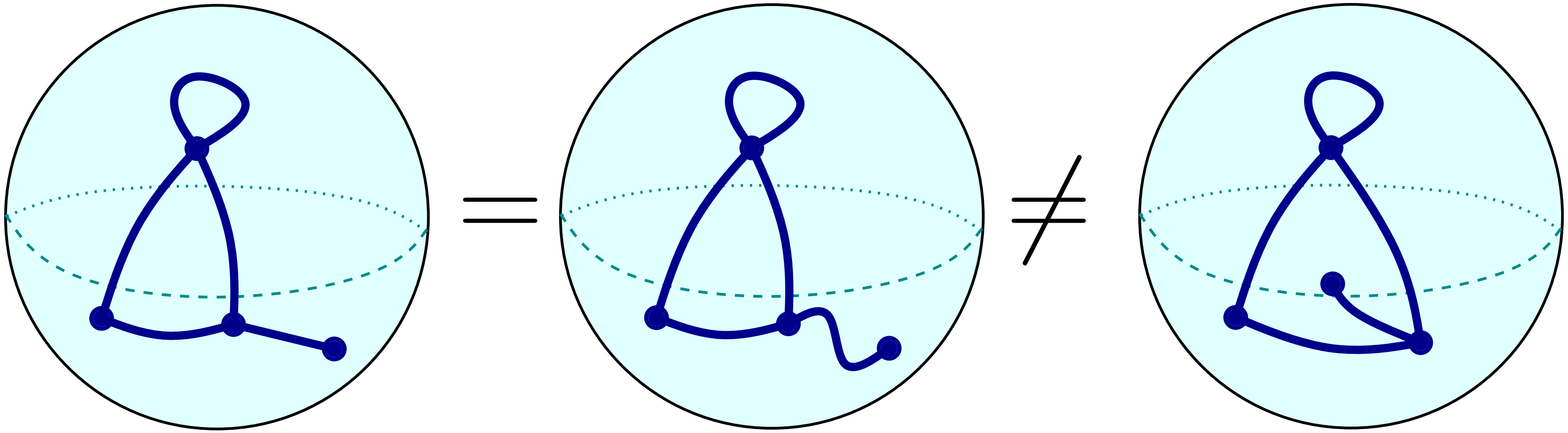

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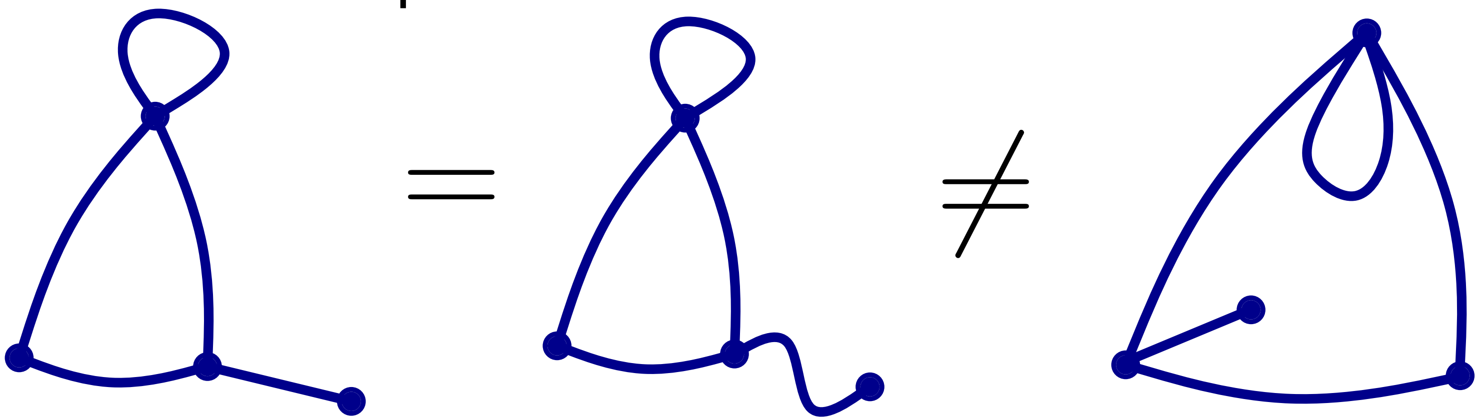
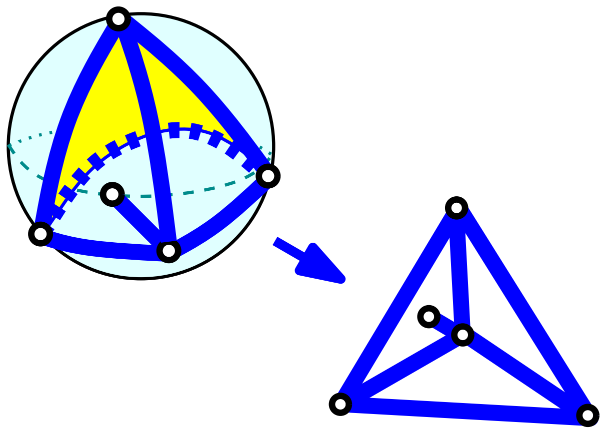


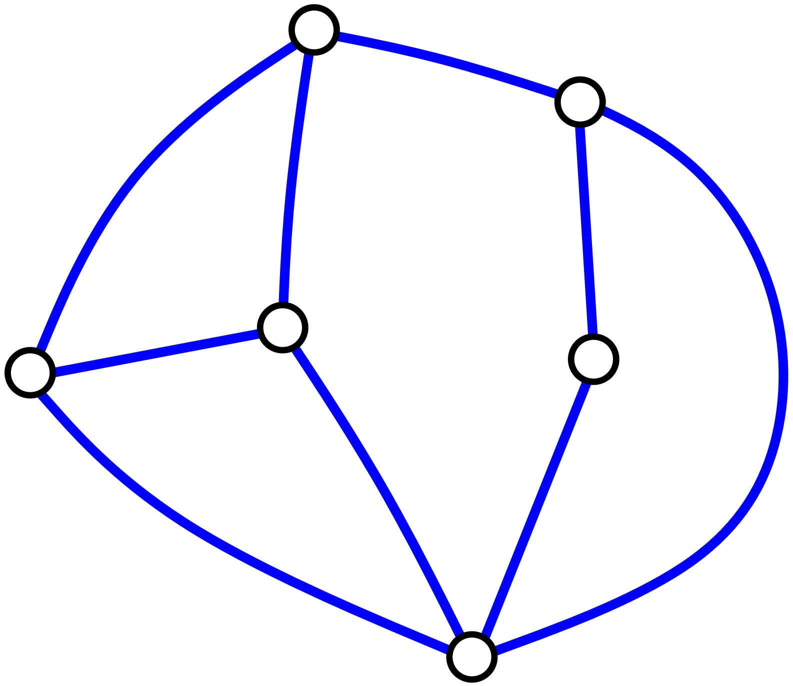


Rk: Plane map = planar map **with a marked face** (the outer face)

The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

$$|V| - |E| + |F| = 2$$



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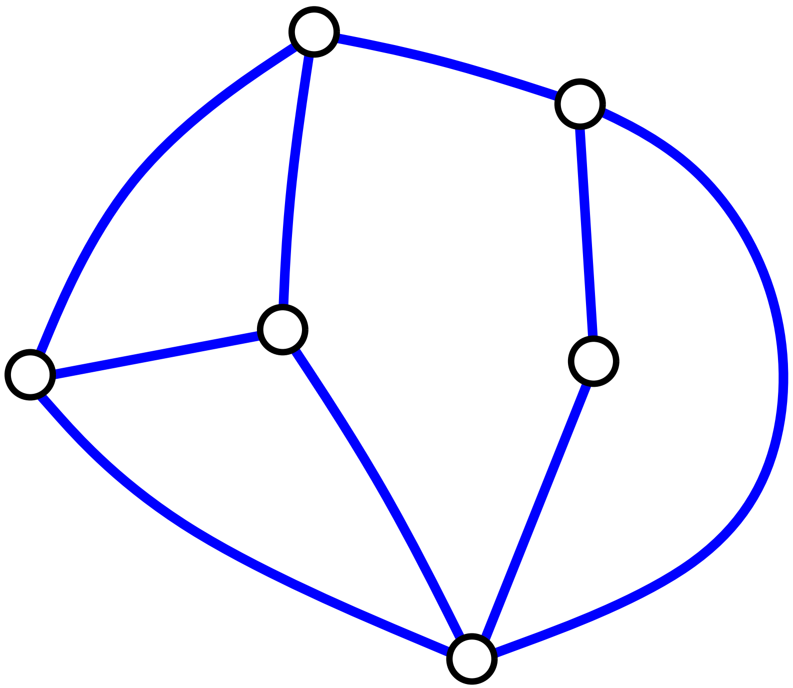

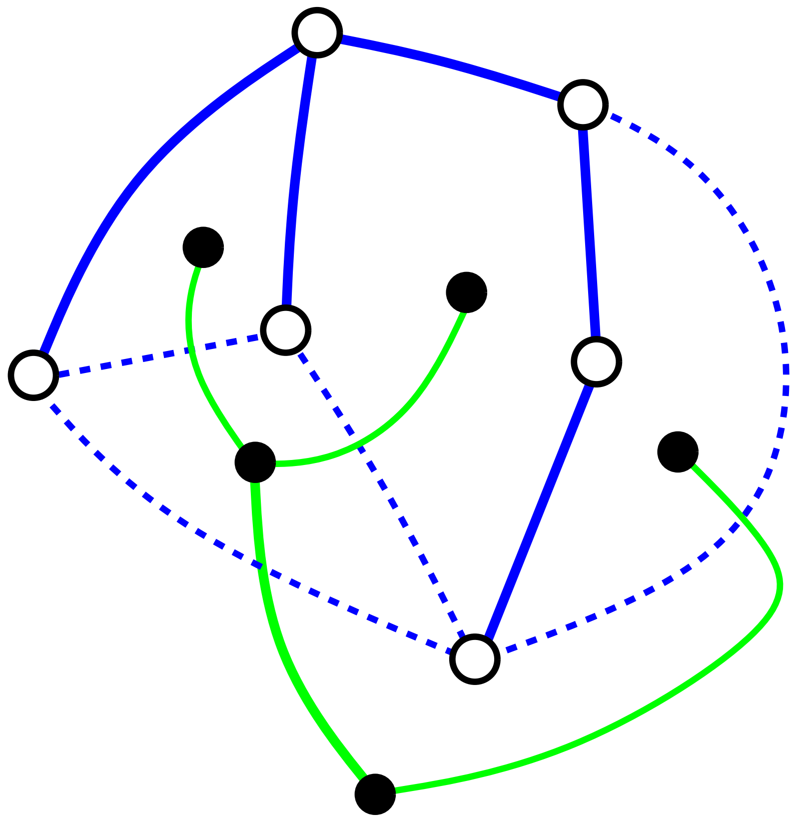

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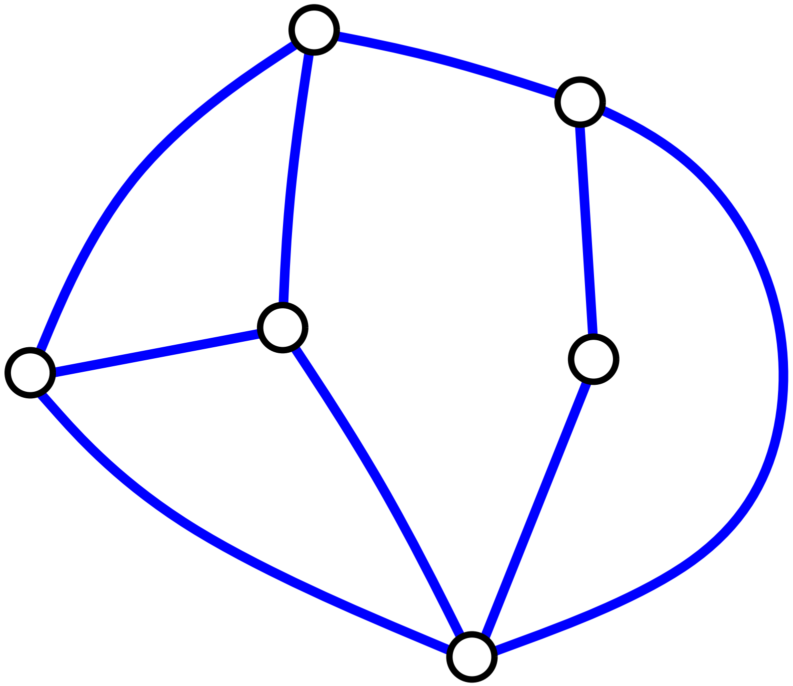
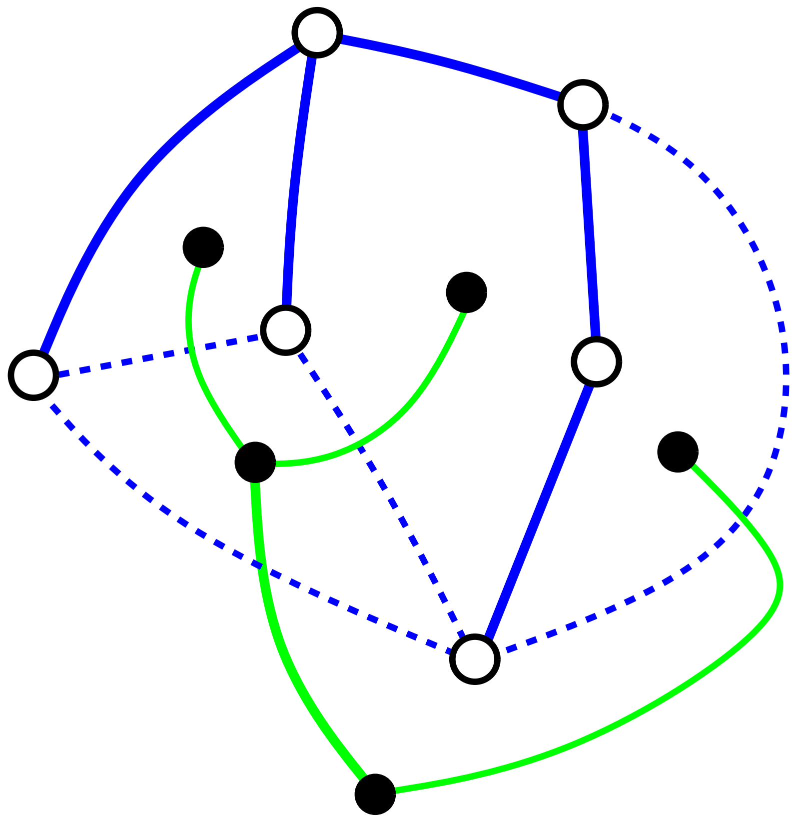
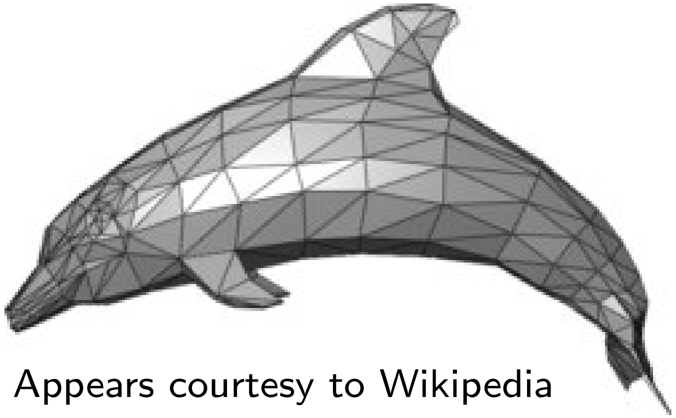




$\Rightarrow$ simple planar graph $G = (V, E)$ satisfies $|E| \leq 3|V| - 6$
(hence K_5 has too many edges to be planar)

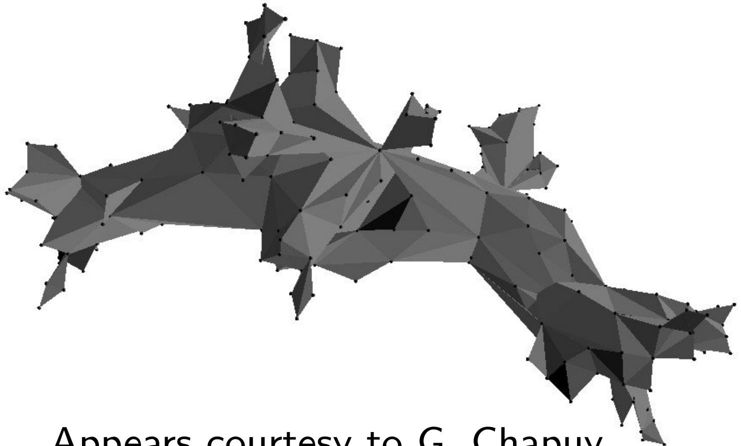
Planar maps. Motivations

- Algorithmic applications: efficient encoding of meshed surfaces.



Appears courtesy to Wikipedia

- Probability and Physics: random lattices, random surfaces.



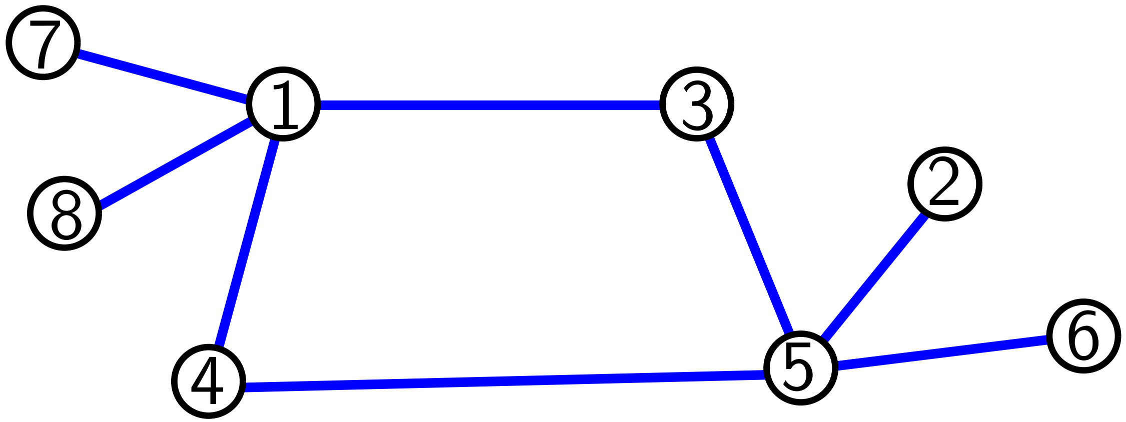
Appears courtesy to G. Chapuy

- Representation Theory: factorization problems.

Symmetry issues.

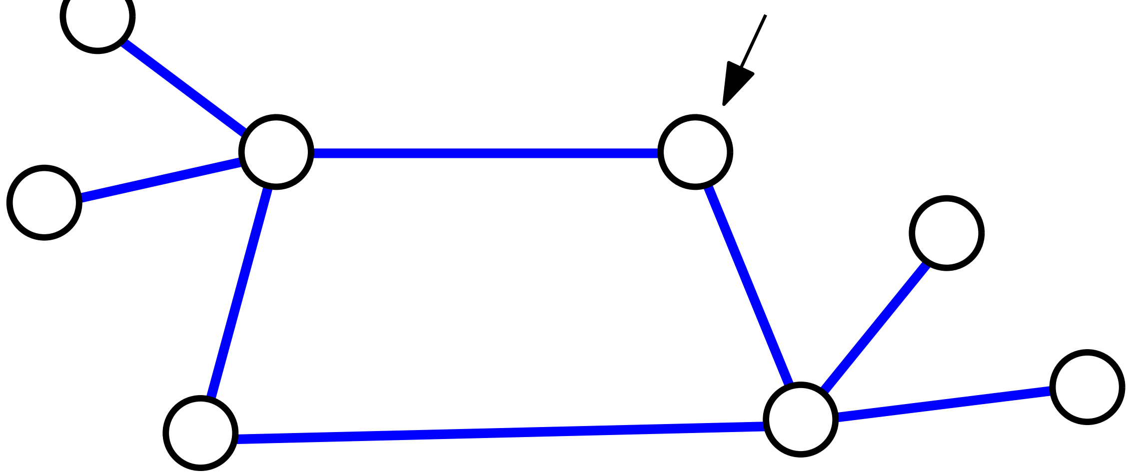
In order to identify vertices unambiguously (to **avoid symmetry issues**):

- Planar graphs: need to **label the vertices**



A labelled planar graph

- Planar maps: only need to **mark a corner**



A rooted planar map

Asymptotic behaviour of planar maps/graphs

• **Asymptotic number**:

Labelled planar graphs n vertices:

$$\boxed{\sim \; n!\, c\, n^{-7/2} \gamma^n}$$ [Giménez,Noy'05]

Rooted planar maps n edges

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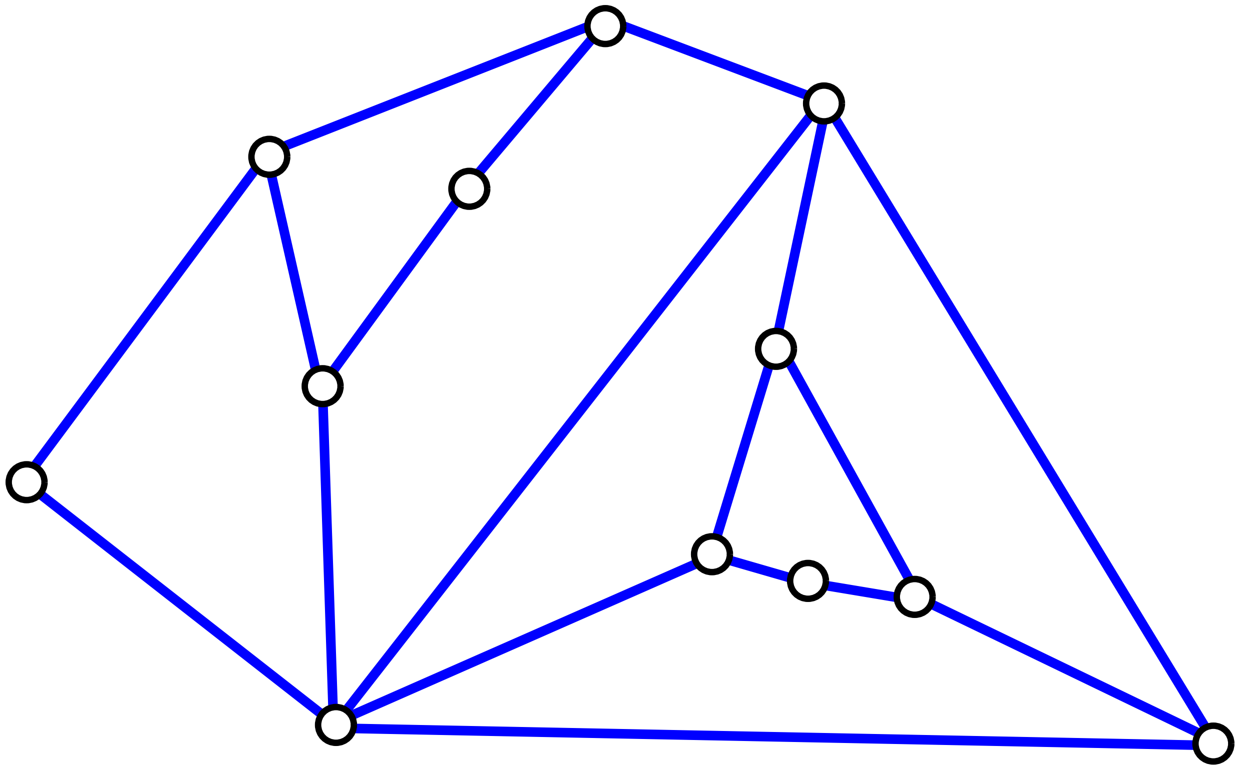
- **Planar maps**:
  - simpler enumeration formulas
  - can control distance parameters
  - **bijections!**

The girth parameter

The **girth** of a graph is the length of a shortest cycle within the graph



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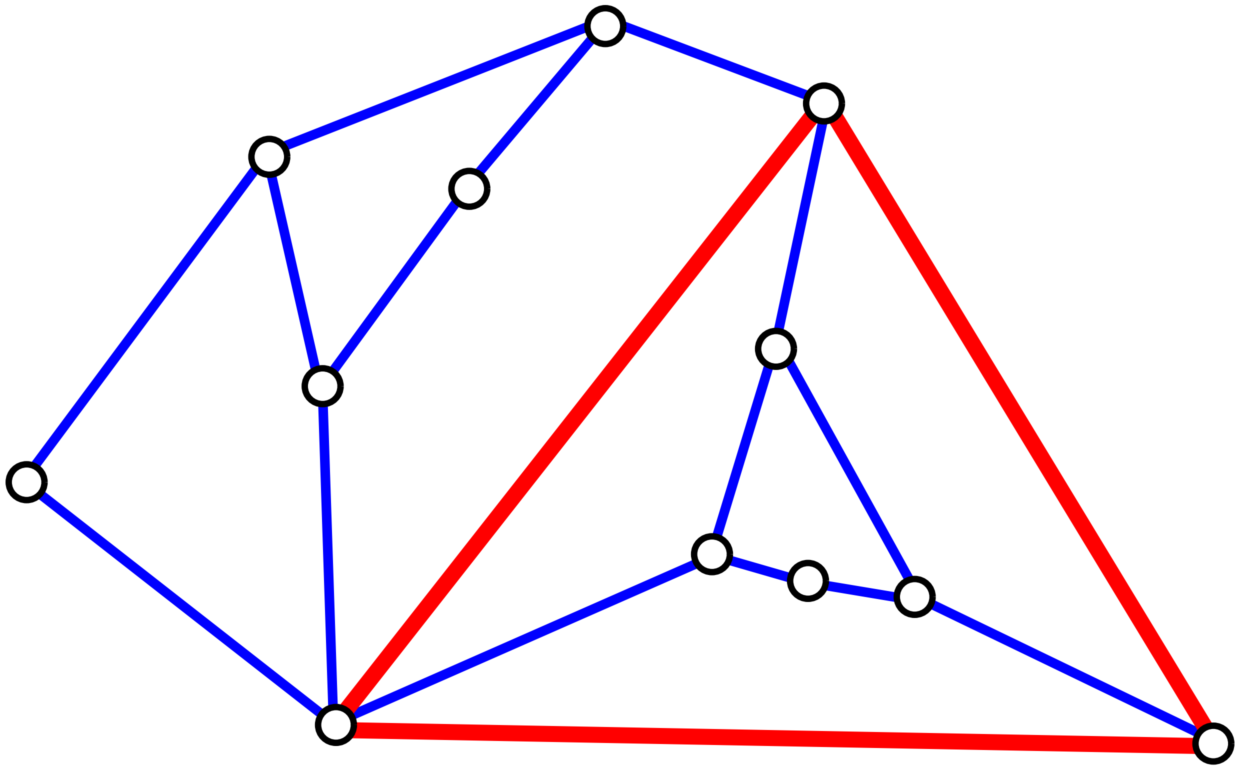

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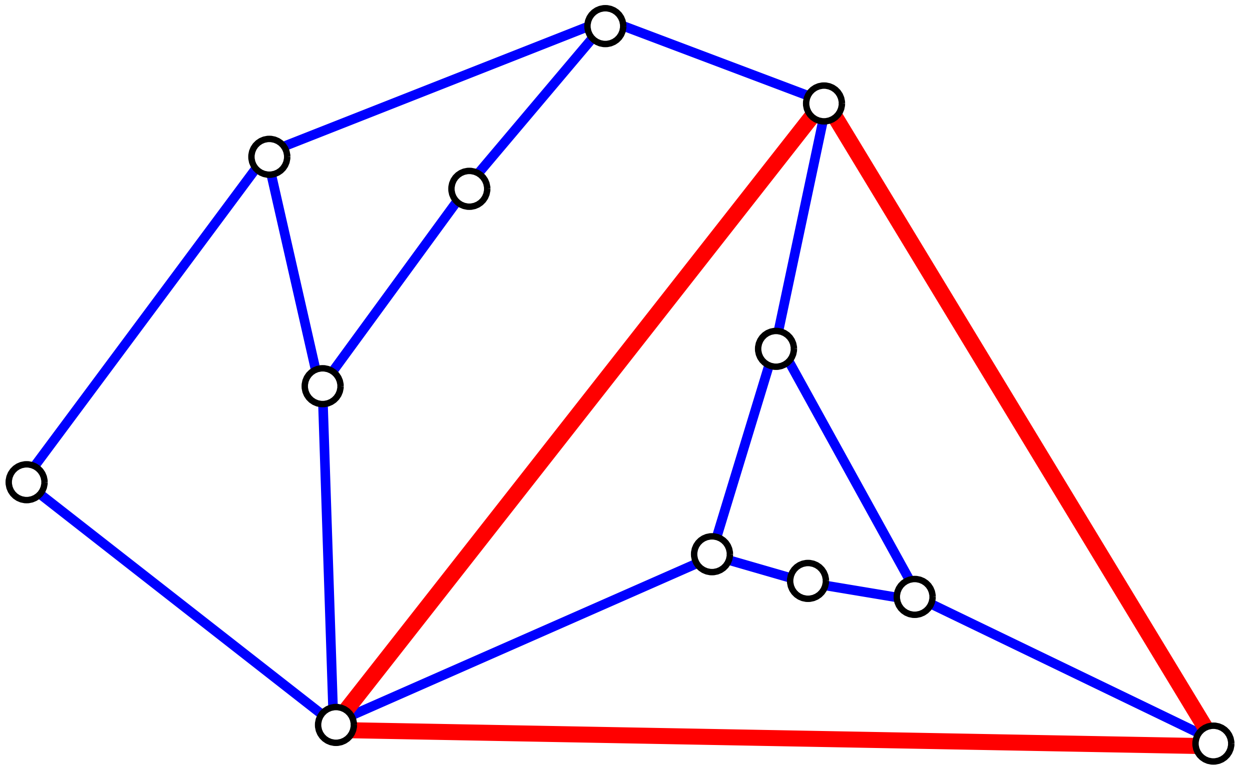

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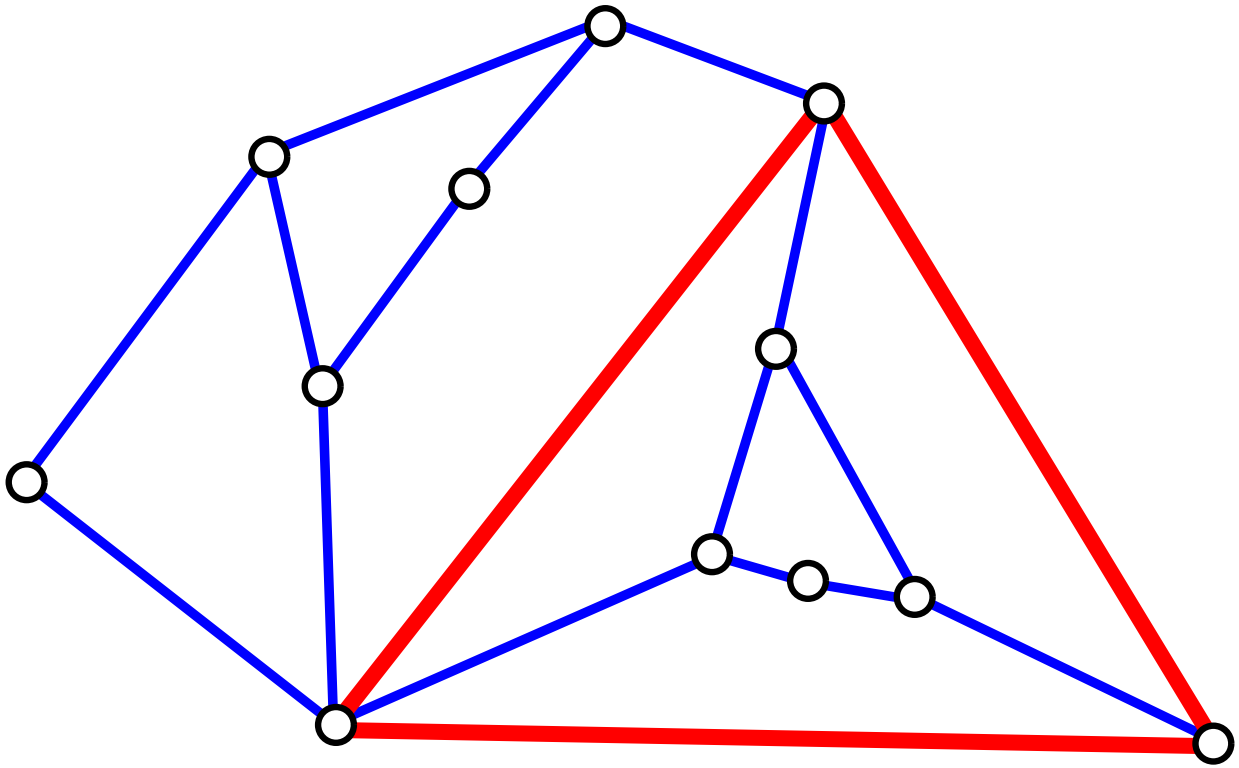

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Loopless $\Leftrightarrow$ girth ≥ 2
Simple $\Leftrightarrow$ girth ≥ 3
Triangle-free $\Leftrightarrow$ girth ≥ 4

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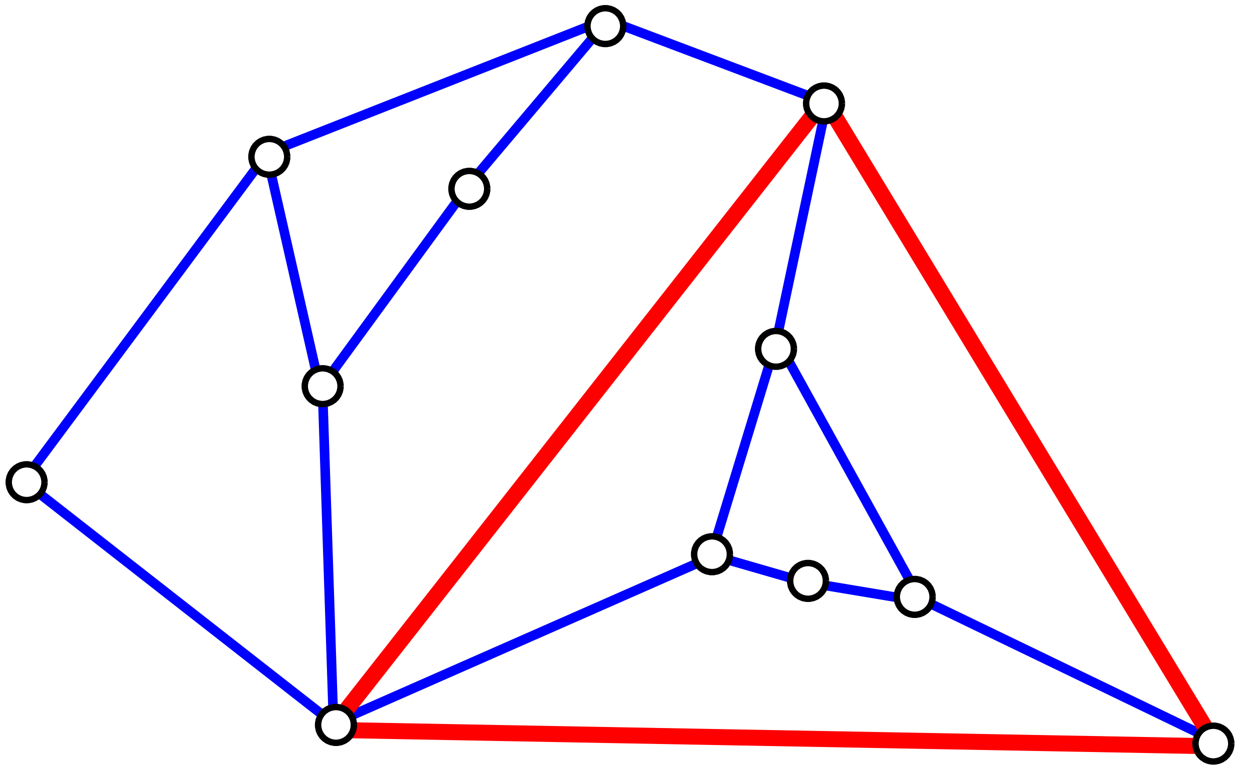

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Many natural map families are specified by constraints on the **girth** and on the **face-degrees** (loopless triangulations, simple quadrangulations,...)

Planar maps. Exact counting results

- Triangulations ($2n$ faces)

Loopless: $$\frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$$ Simple: $$\frac{1}{n(2n-1)}\binom{4n-2}{n-1}$$

- Quadrangulations (n faces)

General: $$\frac{2\cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$ Simple: $$\frac{2}{n(n+1)}\binom{3n}{n-1}$$

- Bipartite maps (n_i faces of degree $2i$)

$$\frac{2\cdot(\sum i\, n_i)!}{(2+\sum(i-1)n_i)!}\prod_i \frac{1}{n_i!}\binom{2i-1}{i}^{n_i}$$

Planar maps. Counting methods

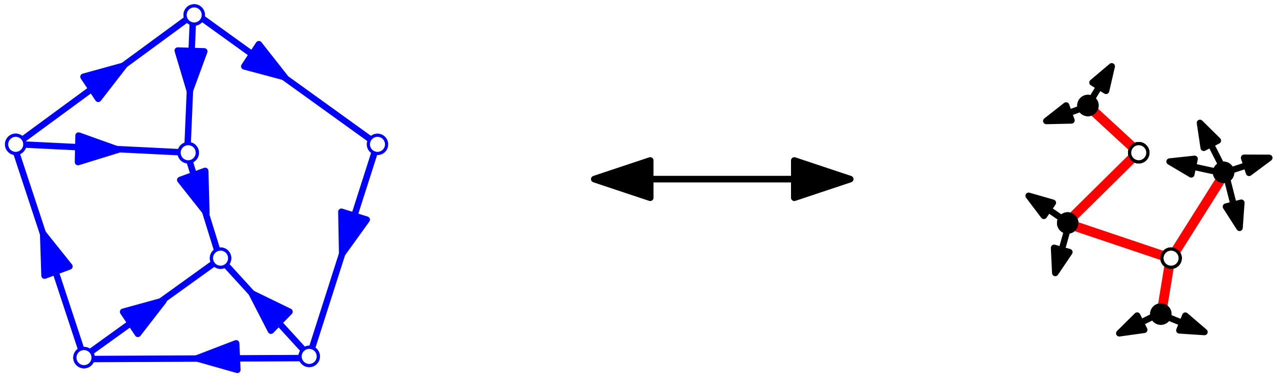
- **Generating functions** [Tutte 63]
  Recursive description of maps $\rightsquigarrow$ recurrences.

- **Matrix Integrals** ['t Hooft 74, Brézin et al'78]
  Feynmann Diagram $\approx$ maps.

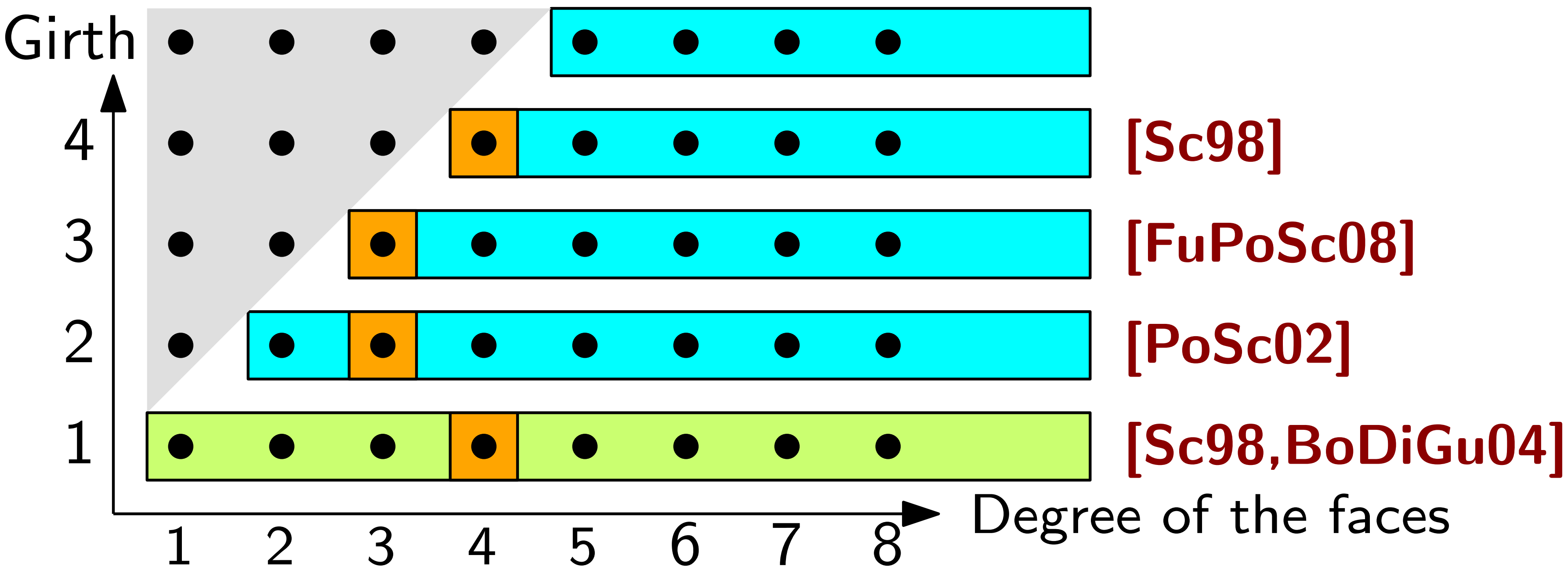
- **Bijections** [Cori-Vauquelin 81, Schaeffer 98]
  Maps $\rightsquigarrow$ decorated trees.

Outline

1. **Master bijection** between a class of **oriented maps** and a class of bicolored **decorated trees** (which are called mobiles).

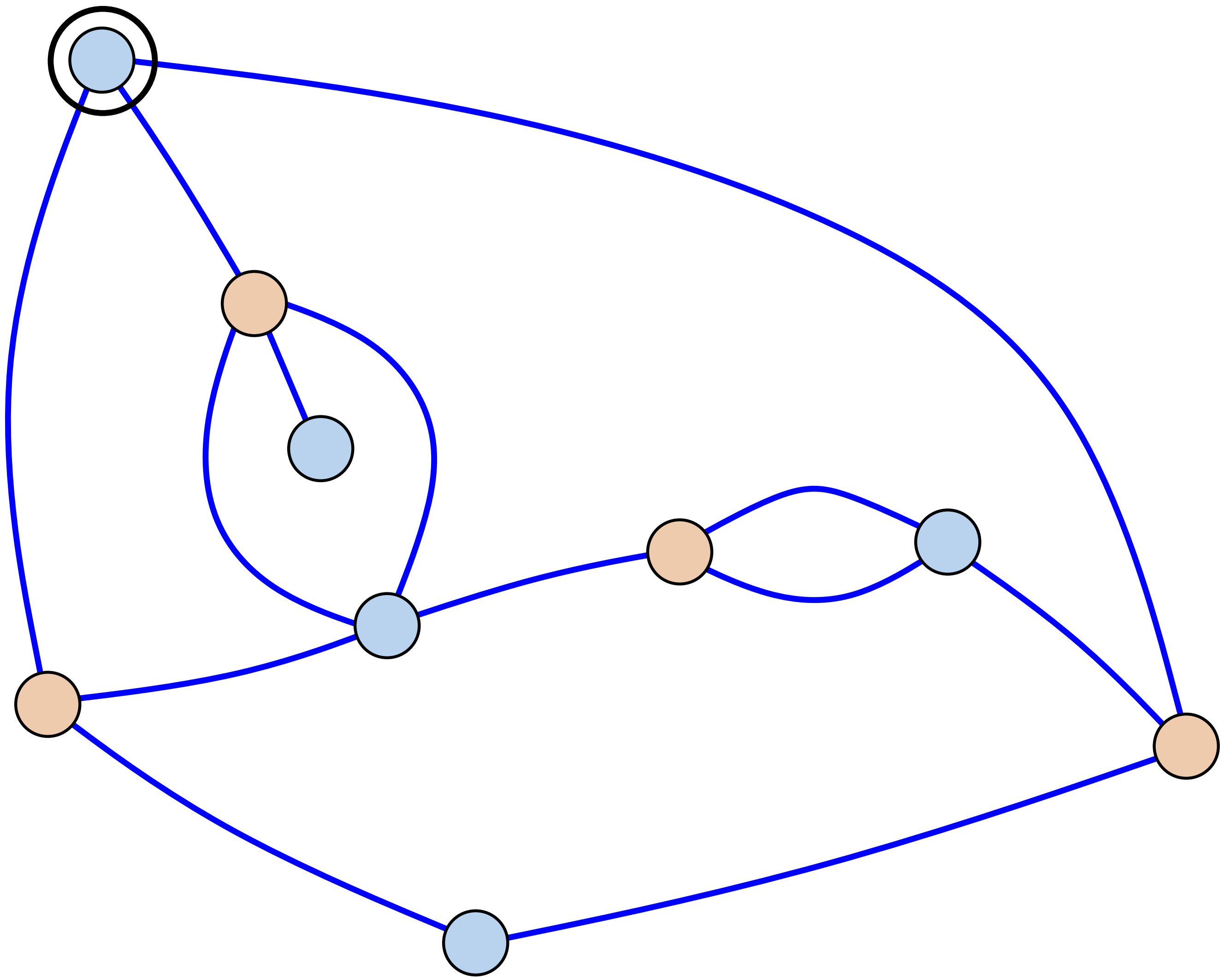


2. **Specializations** to classes of maps (via canonical orientations).



From oriented maps to mobiles

Pointed bipartite map → labelled mobile. [Sc98] [BoDiGu04]



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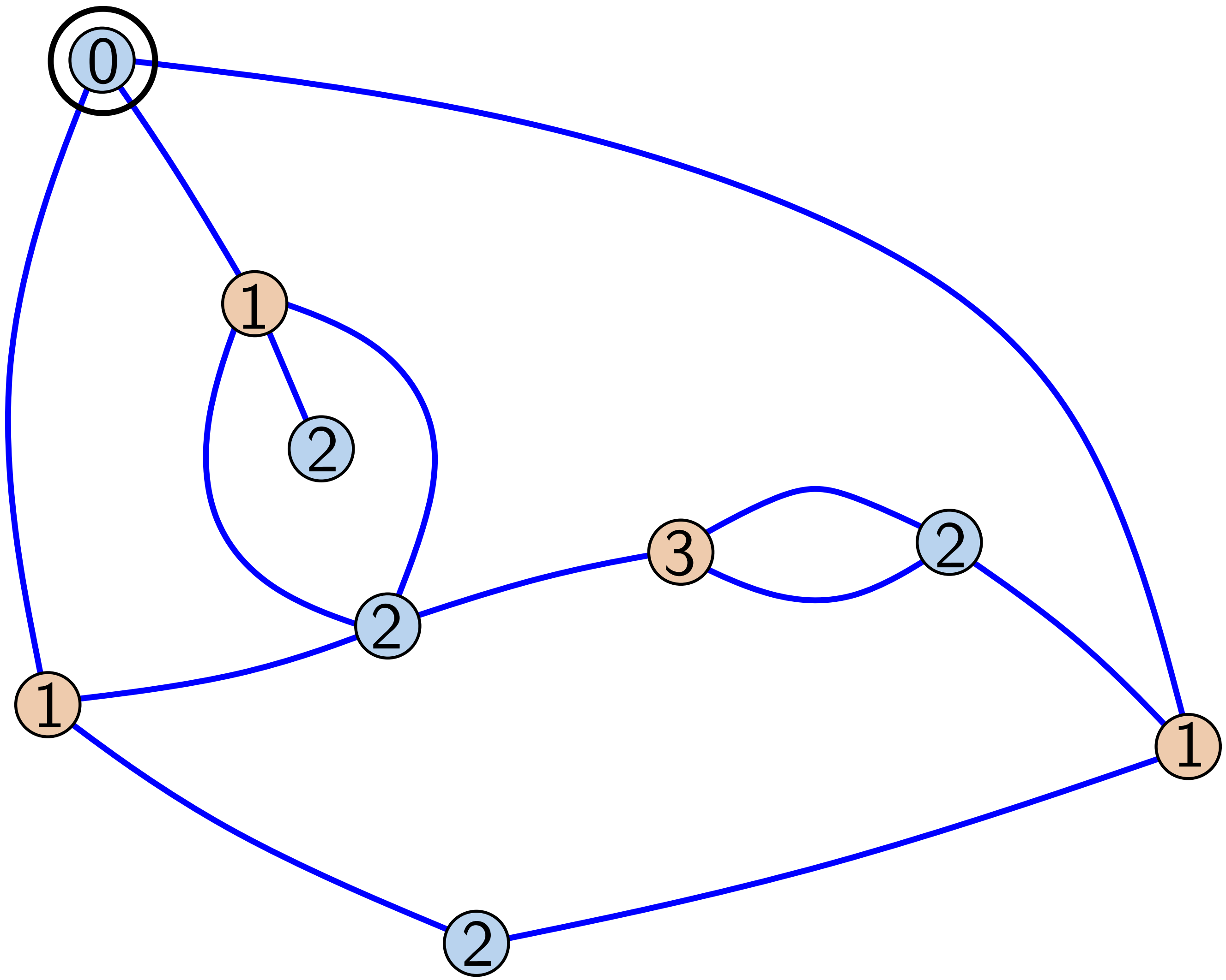
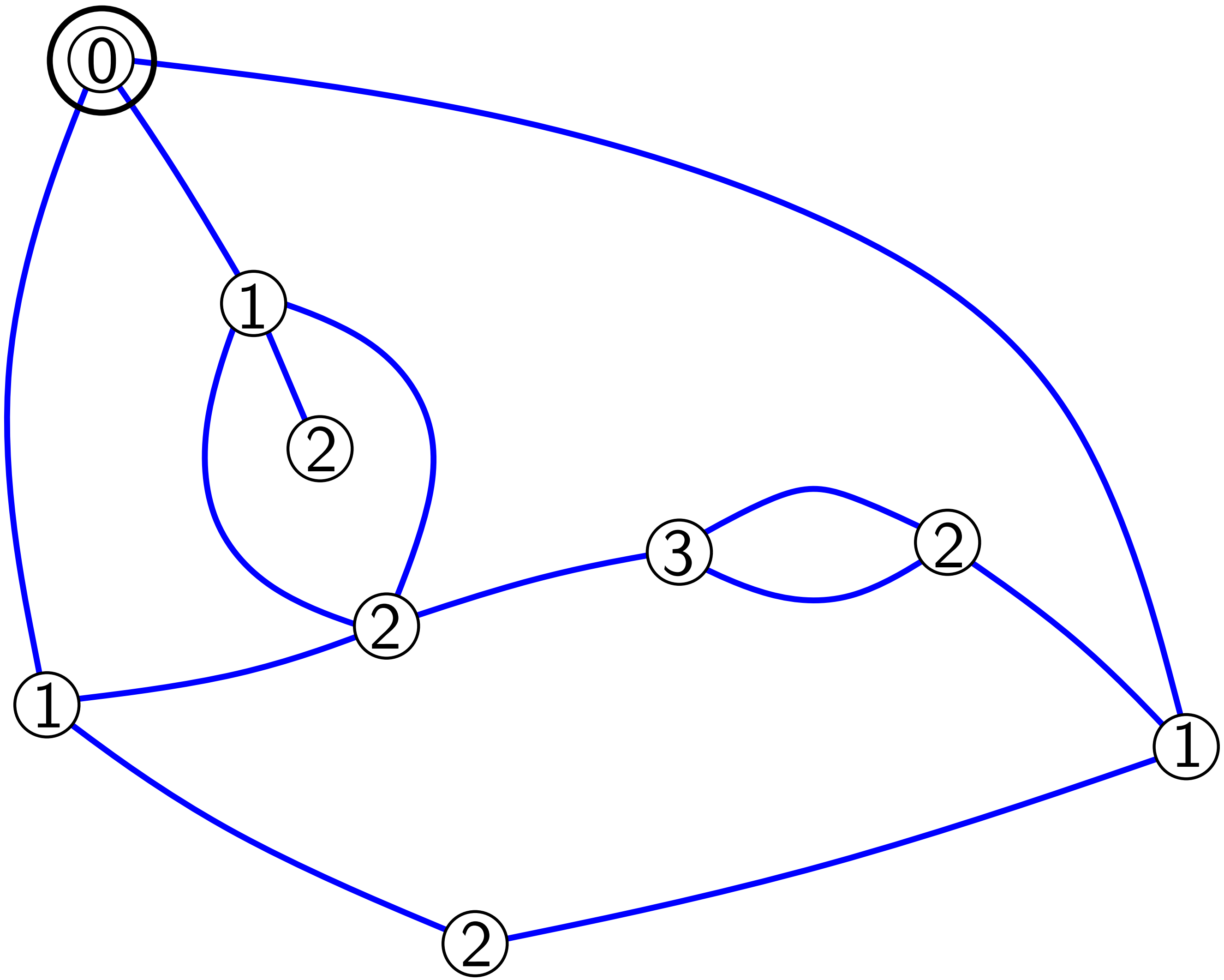


Label the vertices by the distance from pointed vertex

Pointed bipartite map → labelled mobile. [Sc98] [BoDiGu04]

Construct the labelled mobile



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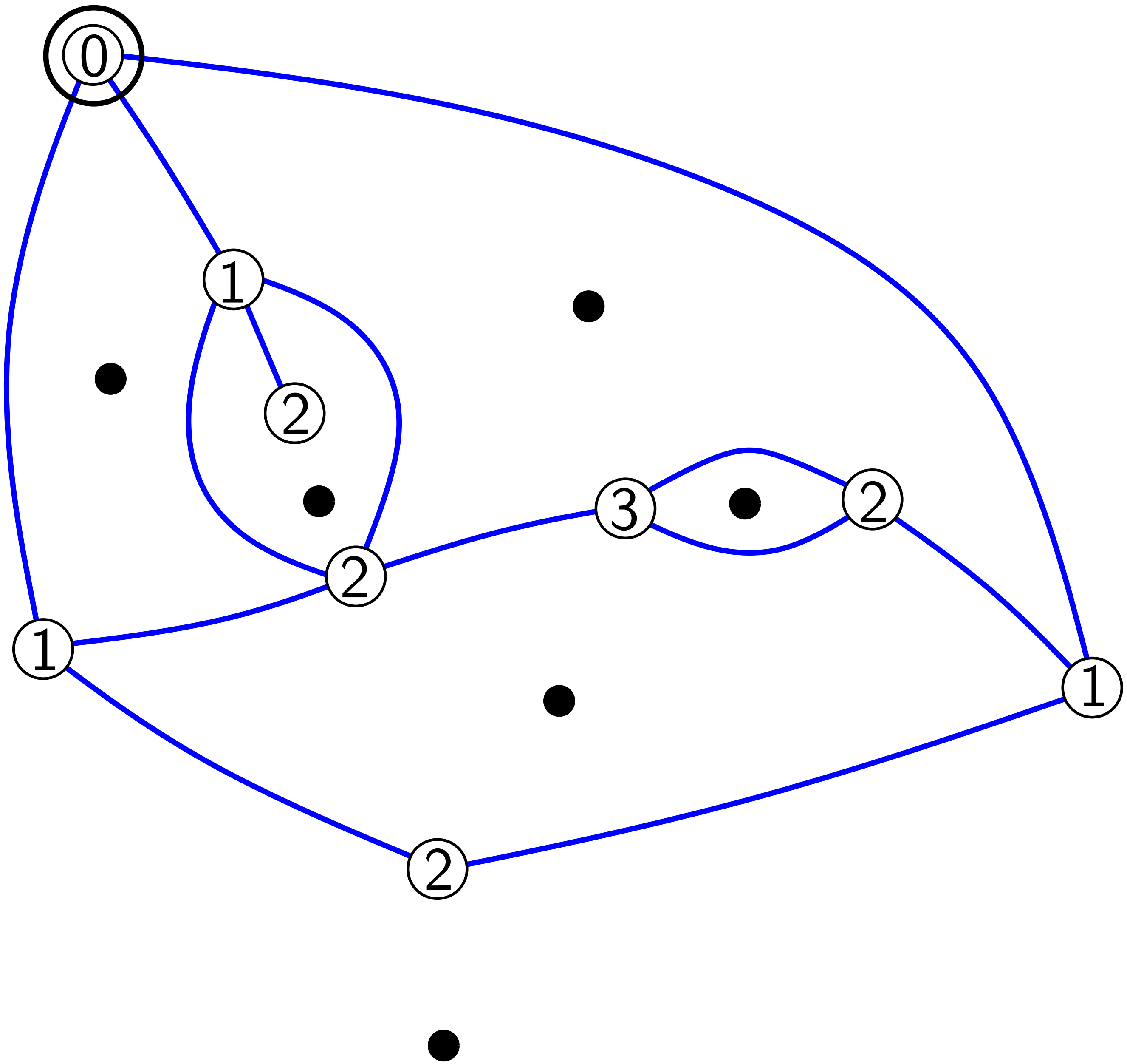

Construct the labelled mobile

(i) put one black vertex in each face

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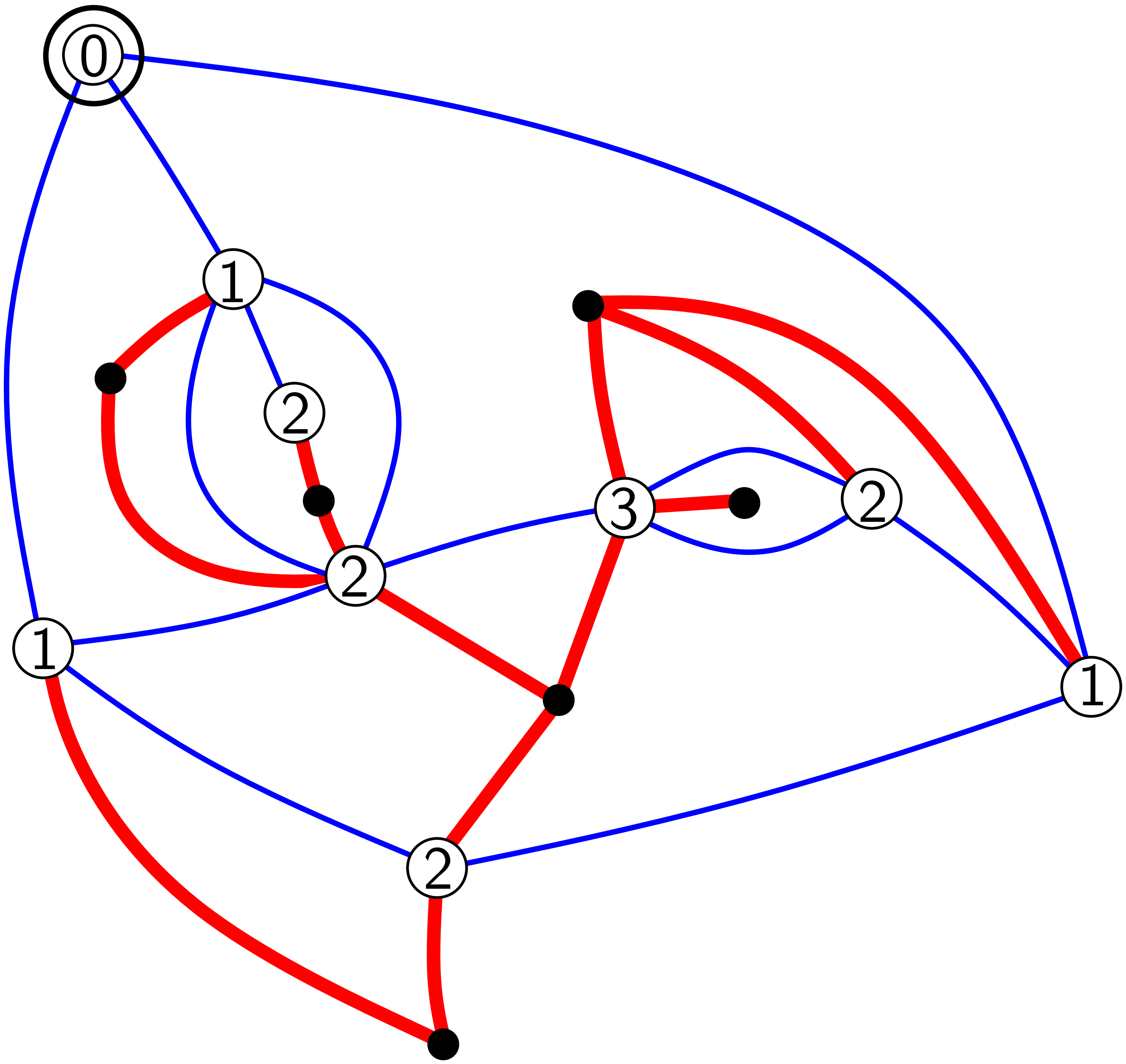
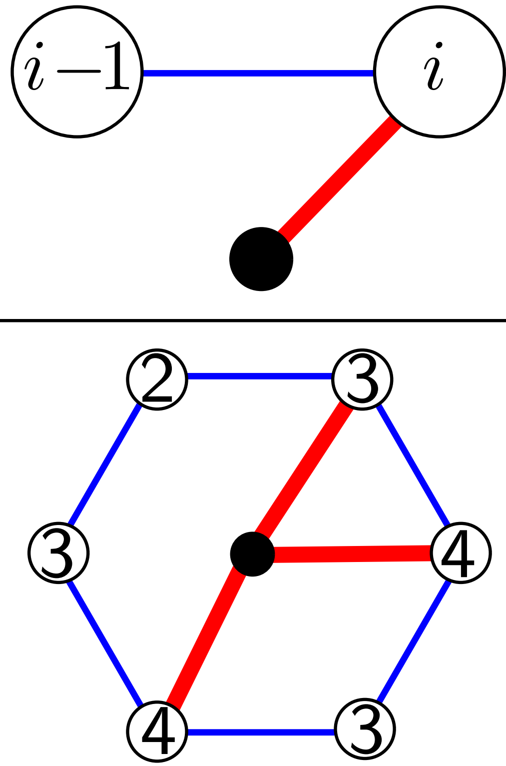


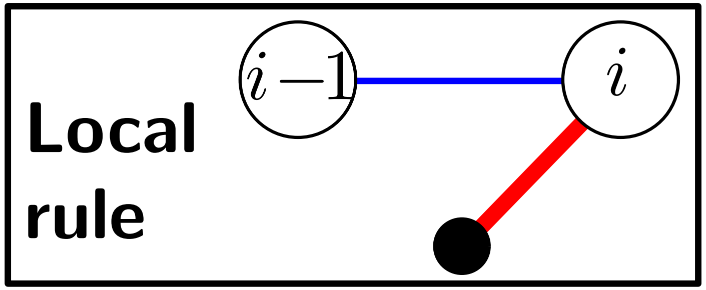
Construct the labelled mobile

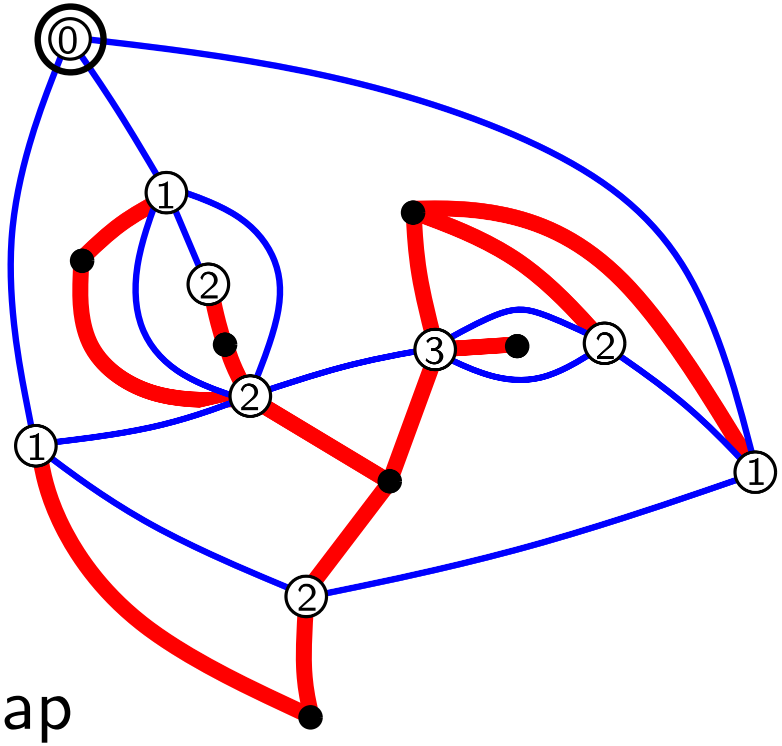
(i) put one black vertex in each face

(ii) each edge of the map gives one edge in the mobile



Pointed bipartite map $\rightarrow$ labelled mobile. [Sc98] [BoDiGu04]





Proof that the mobile is a tree

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Let T be the associated mobile

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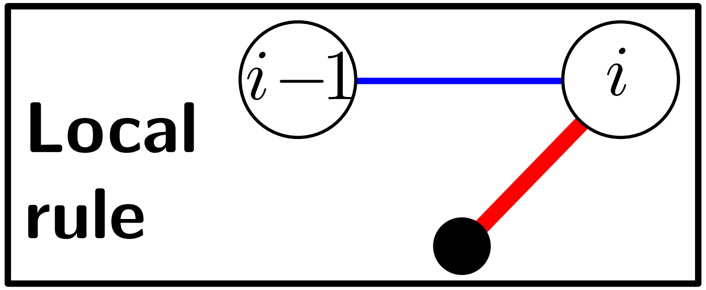
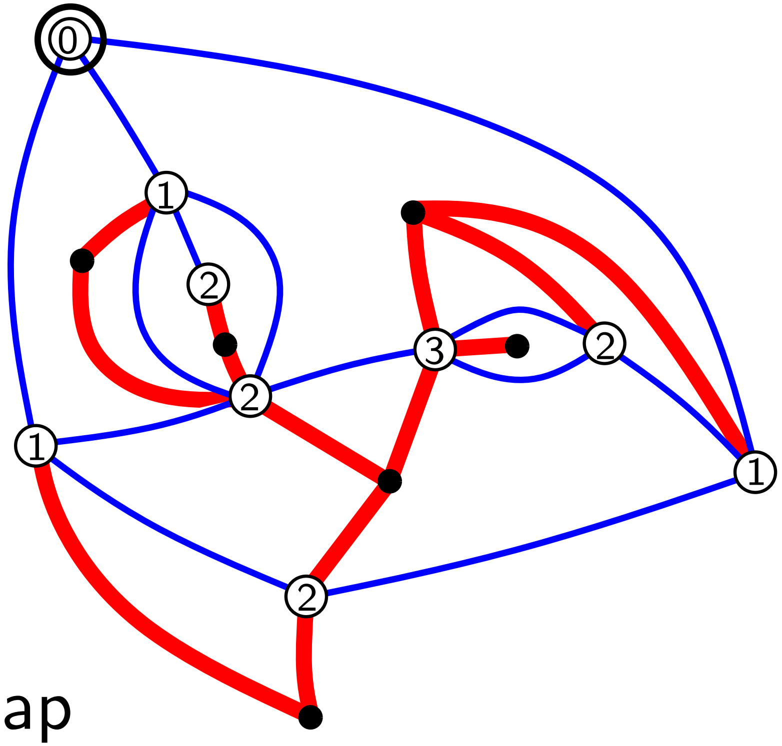




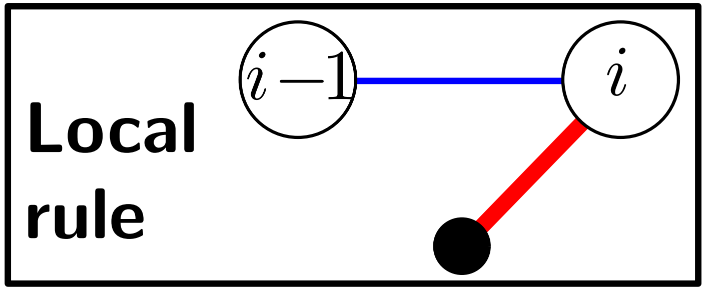
Proof that the mobile is a tree

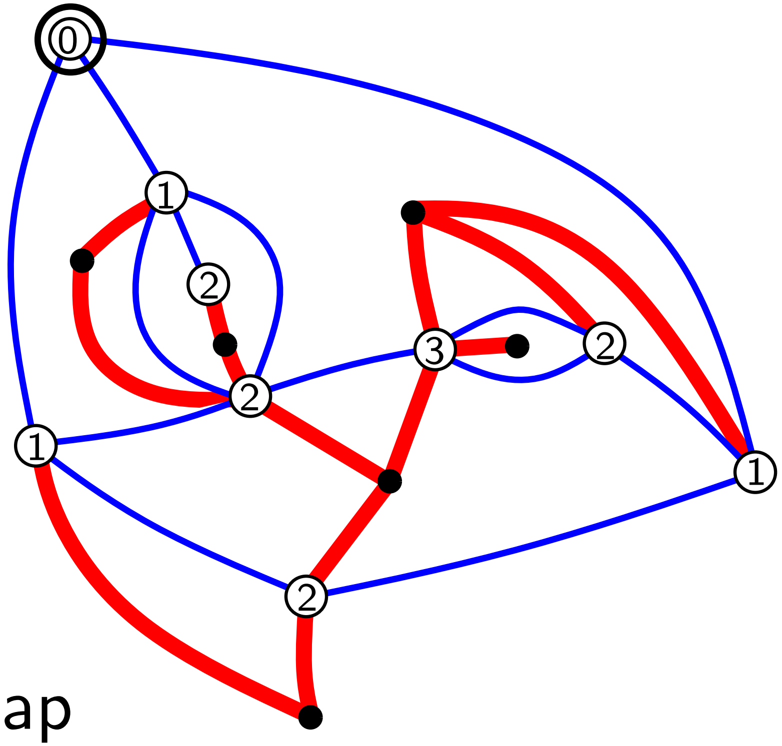
Let $G = (V, E, F)$ be a pointed bipartite map

Let T be the associated mobile

T has $|E|$ edges, and has $|V|+|F|-1=|E|+1$ vertices (Euler relation)

Pointed bipartite map → labelled mobile. [Sc98] **[BoDiGu04]**





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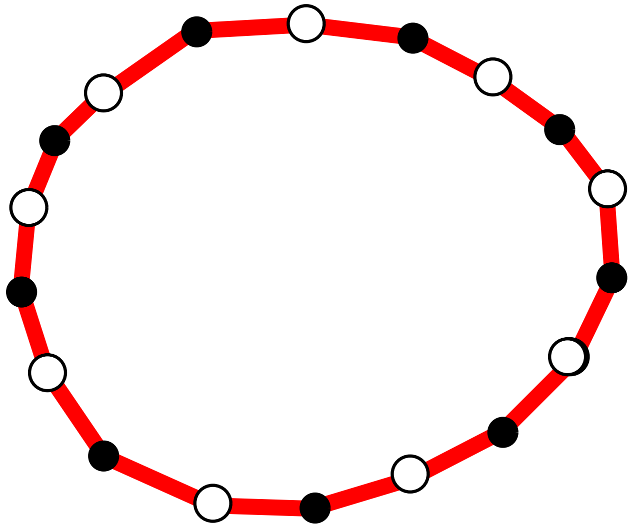

Assume that T has a cycle C

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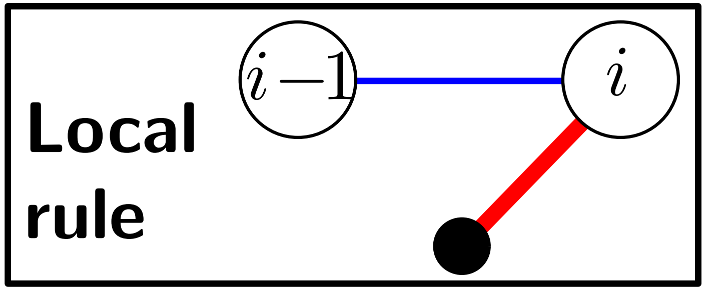

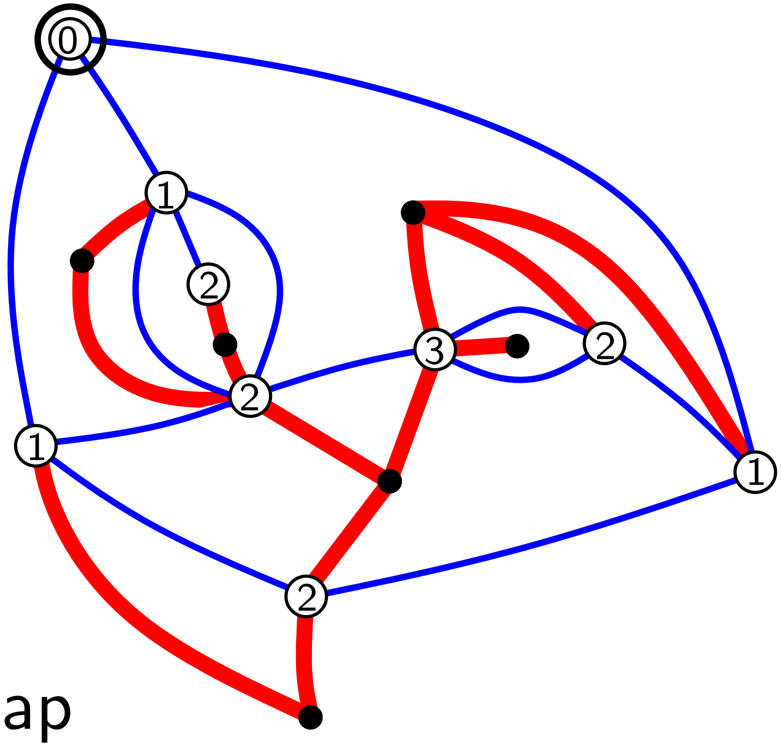

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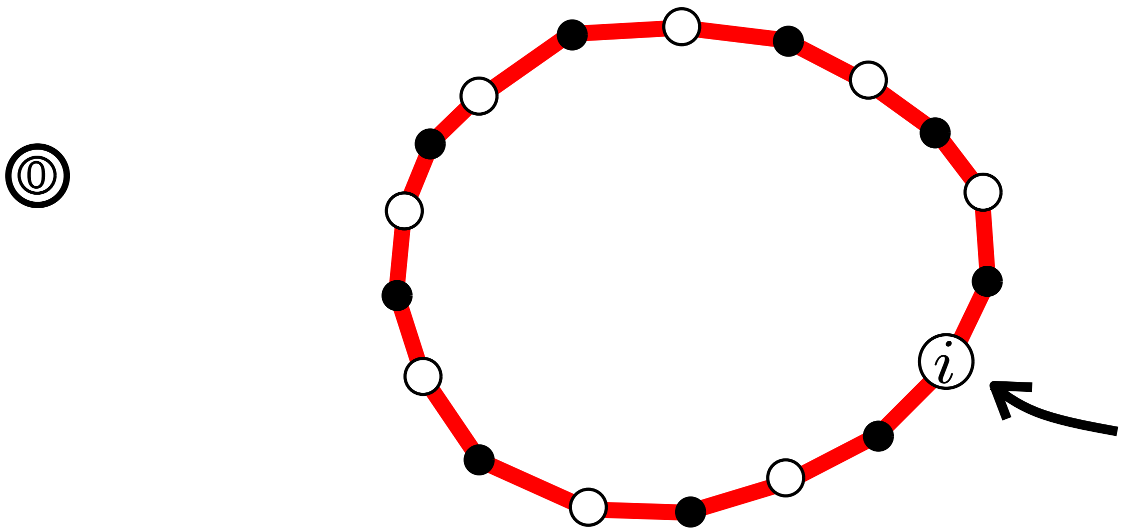
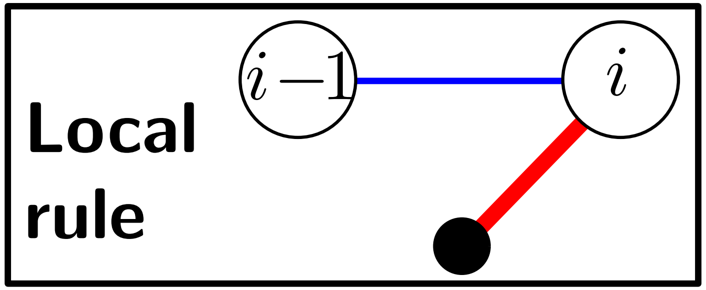
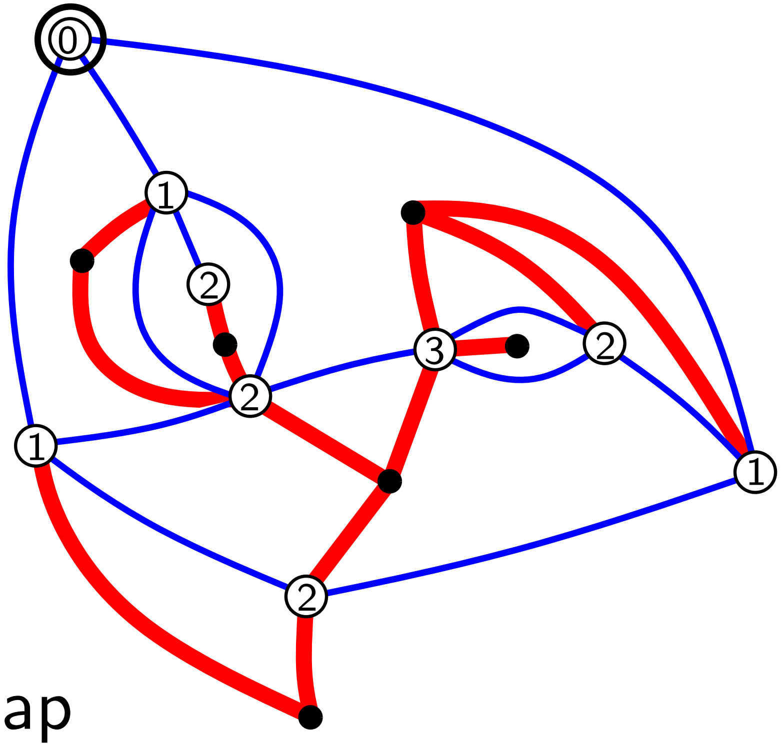


Assume that T has a cycle C

smallest label on C

Pointed bipartite map → labelled mobile. [Sc98] **[BoDiGu04]**





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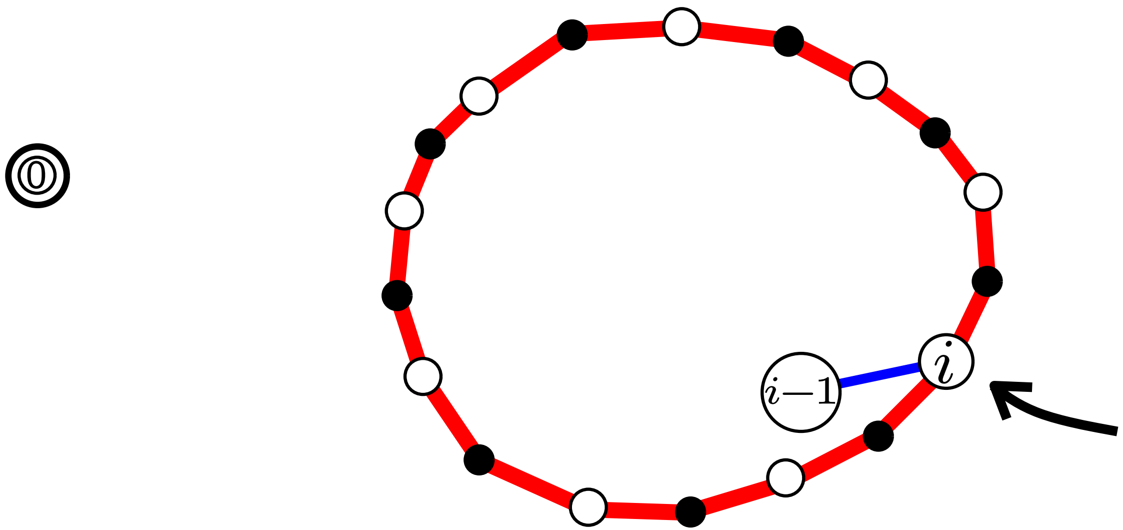

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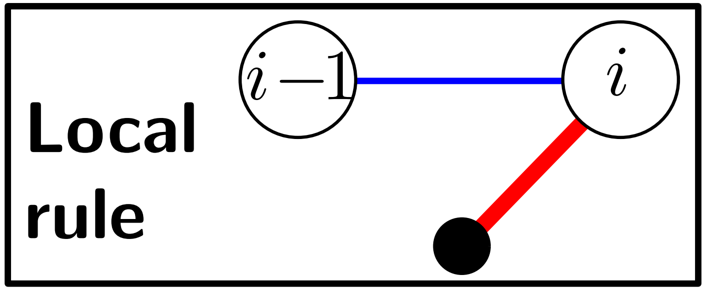

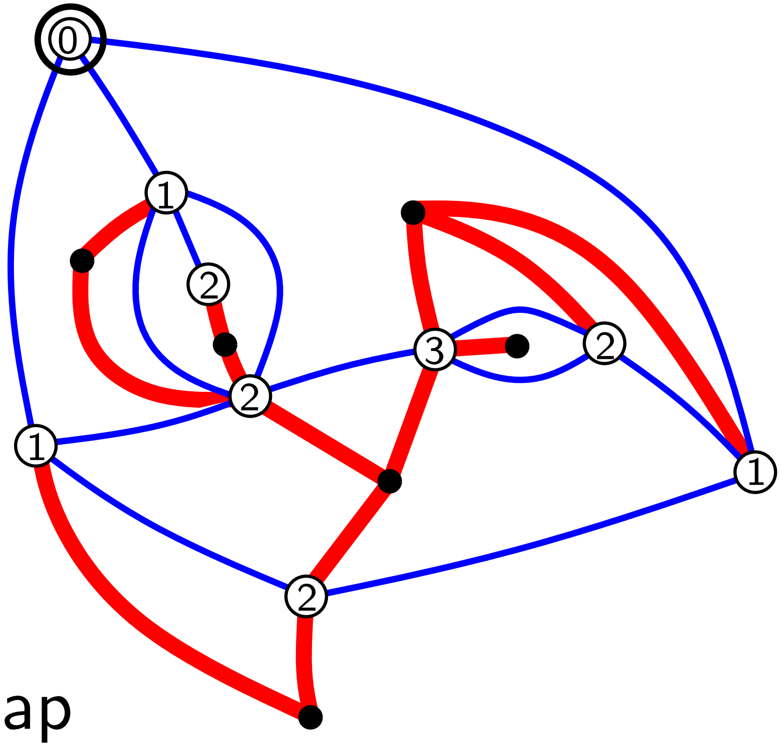

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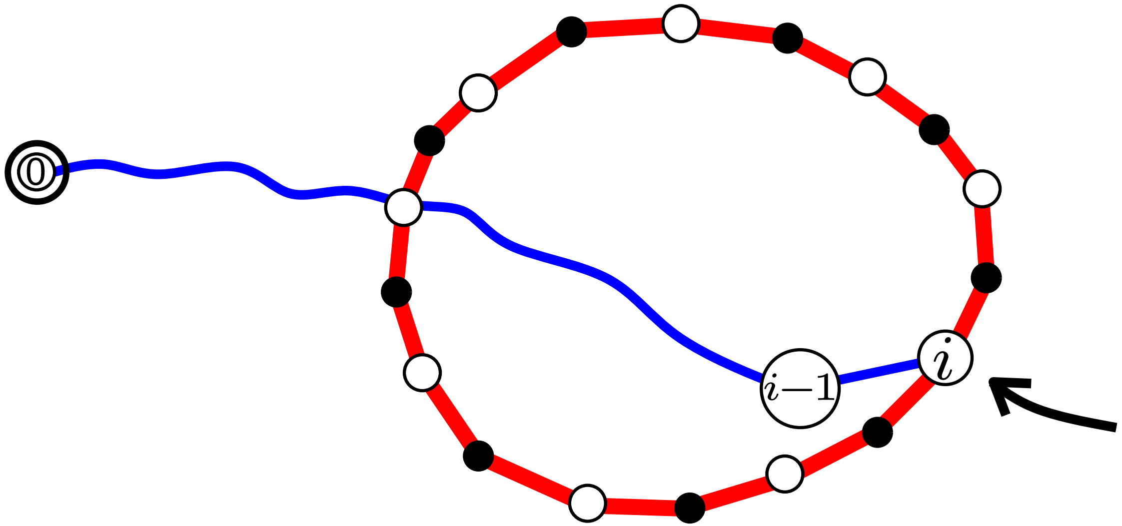

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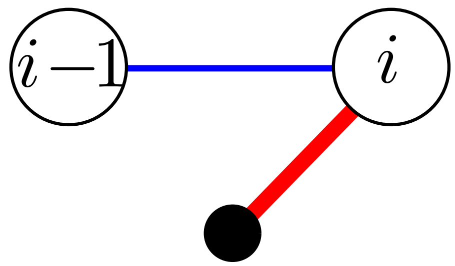
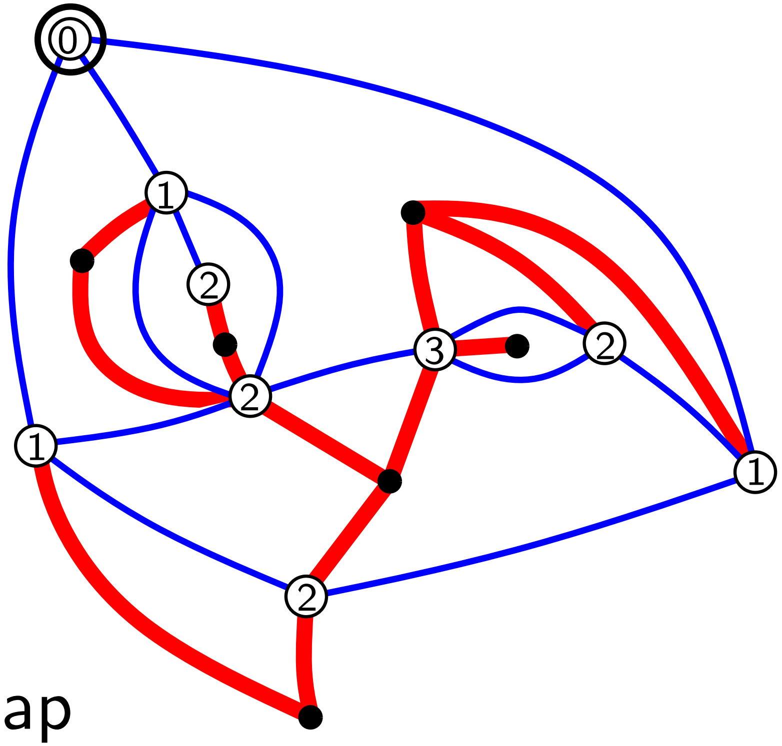
Pointed bipartite map → labelled mobile. [Sc98] [BoDiGu04]

Local rule





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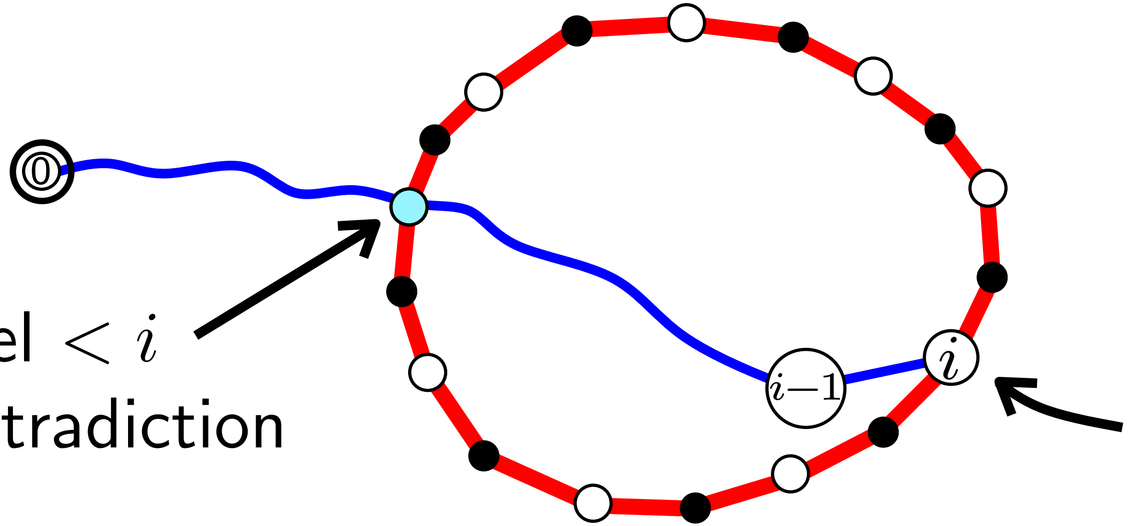
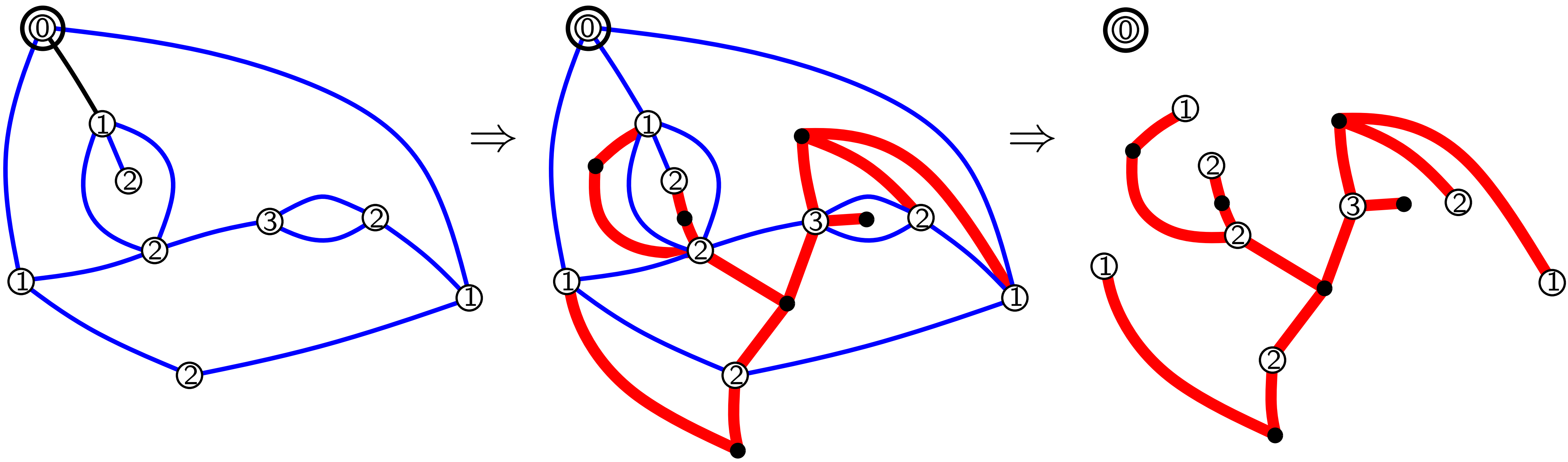
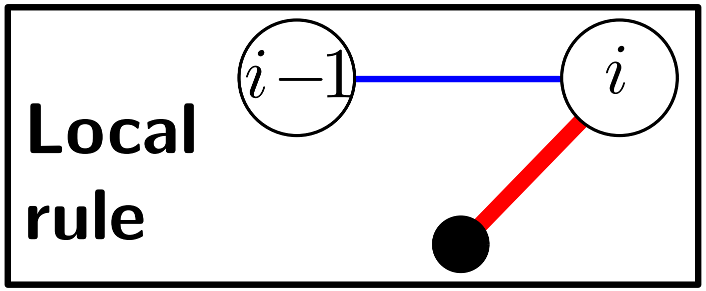


label $< i$ contradiction

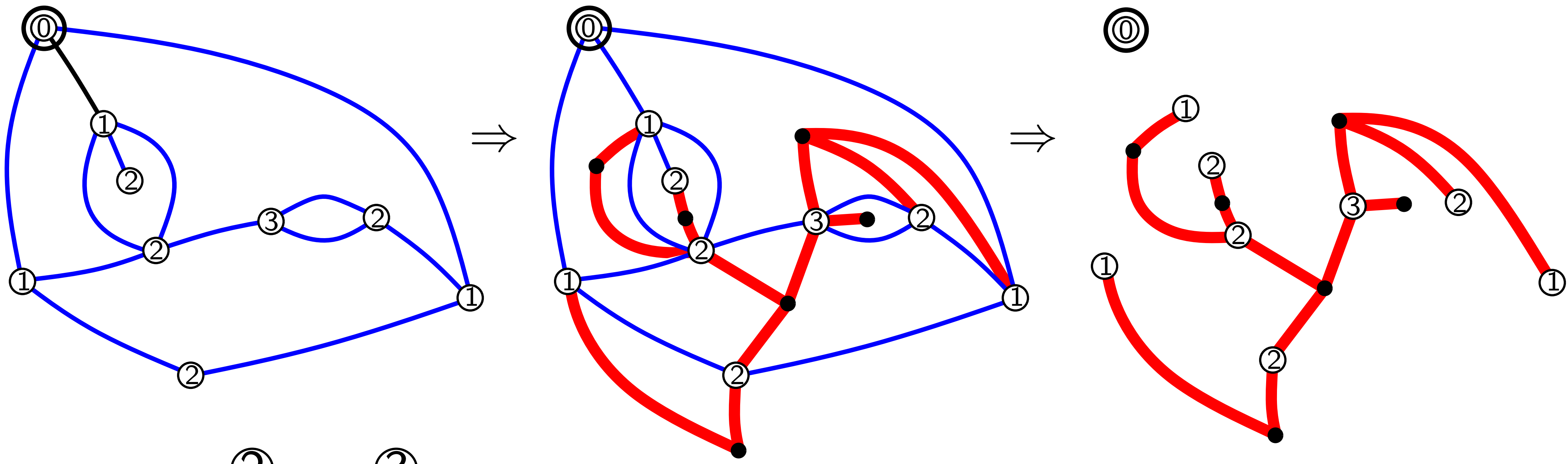
smallest label on C

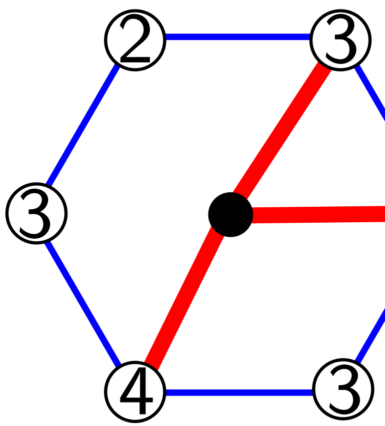
Pointed bipartite map $\rightarrow$ labelled mobile. [Sc98] [BoDiGu04]



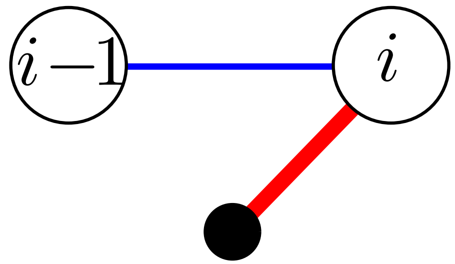


Pointed bipartite map $\to$ labelled mobile. [Sc98] [BoDiGu04]





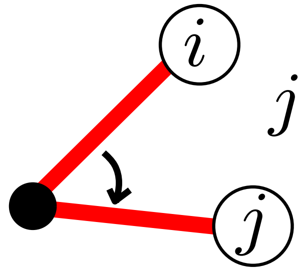
Local rule



Conditions:

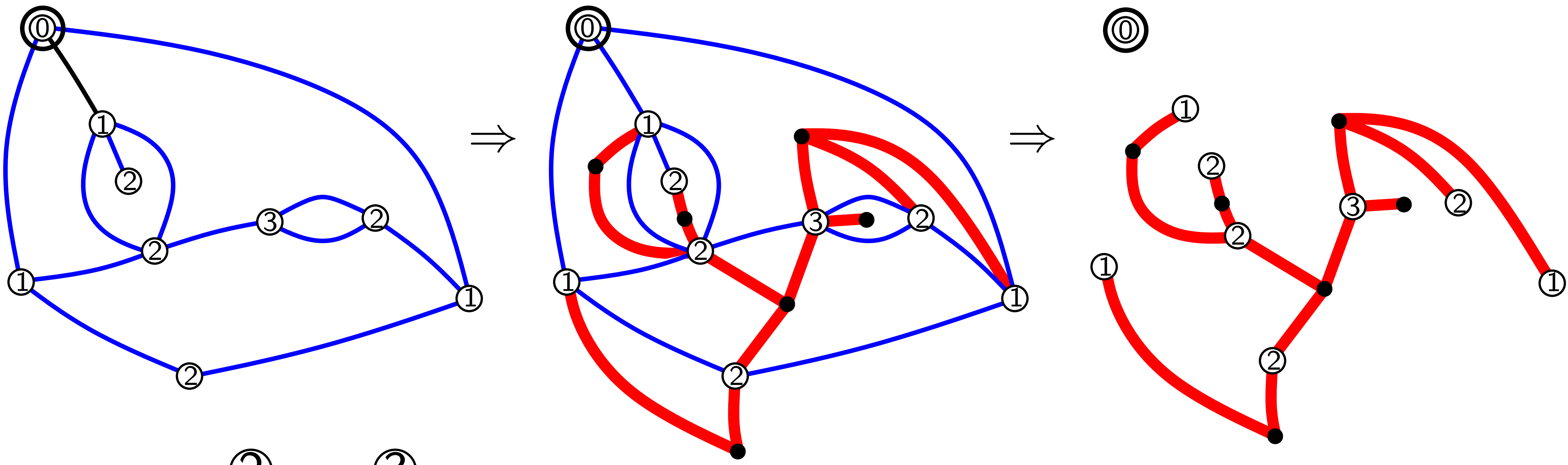
(i) $\exists$ vertex label 1

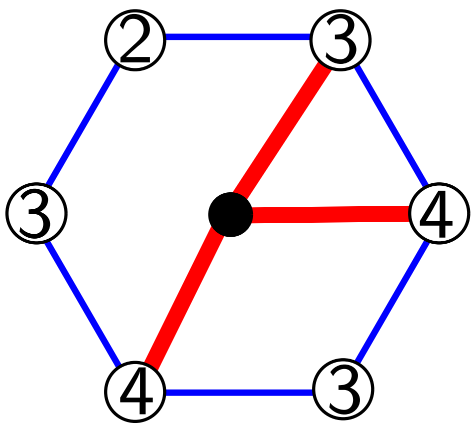
(ii)

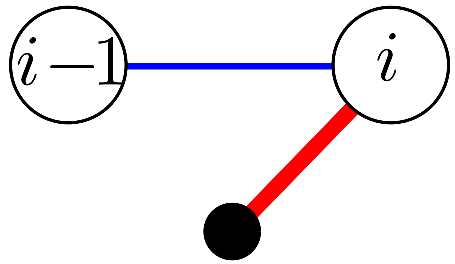


$j \le i+1$

Pointed bipartite map $\to$ labelled mobile. [Sc98] [BoDiGu04]







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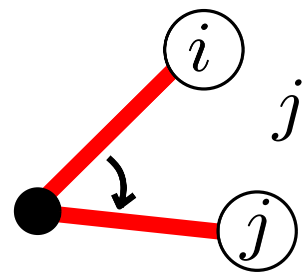
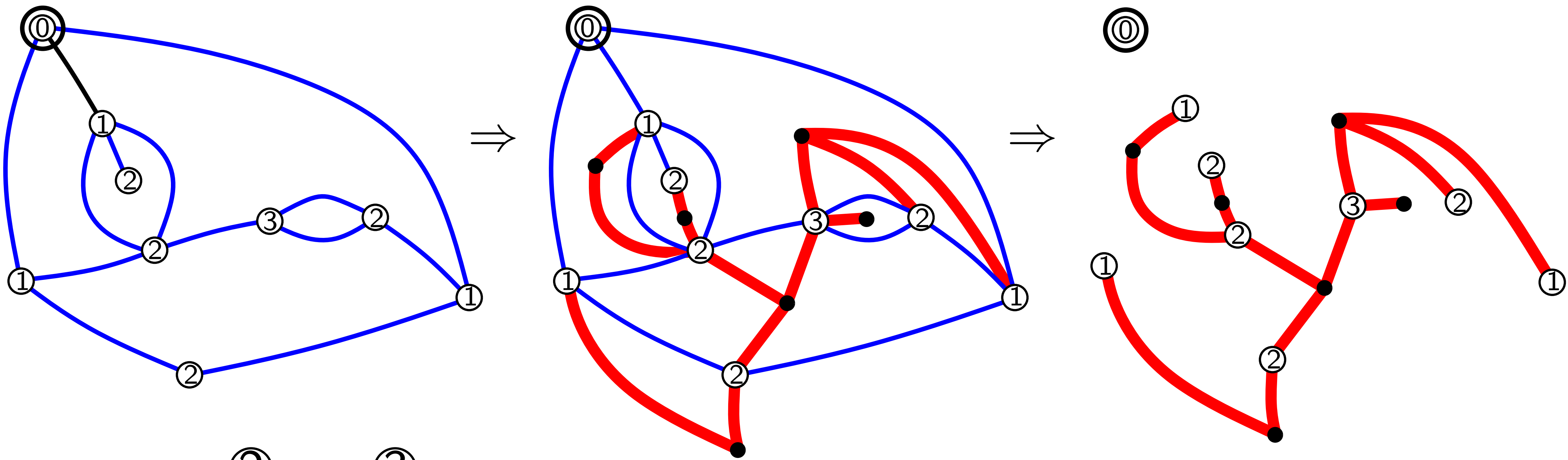
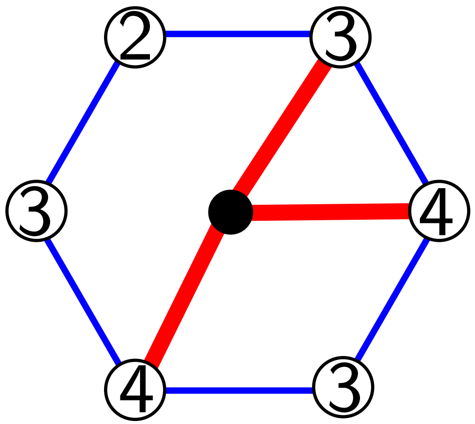
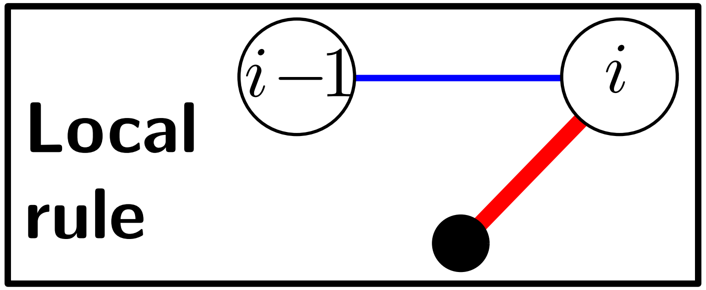


Theorem: The mapping is a **bijection**. Each **face of degree $2i$** of the bipartite map corresponds to a **black vertex of degree i** in the mobile

Pointed bipartite map → labelled mobile. [Sc98] [BoDiGu04]







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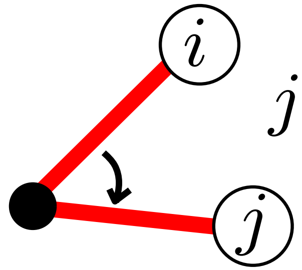

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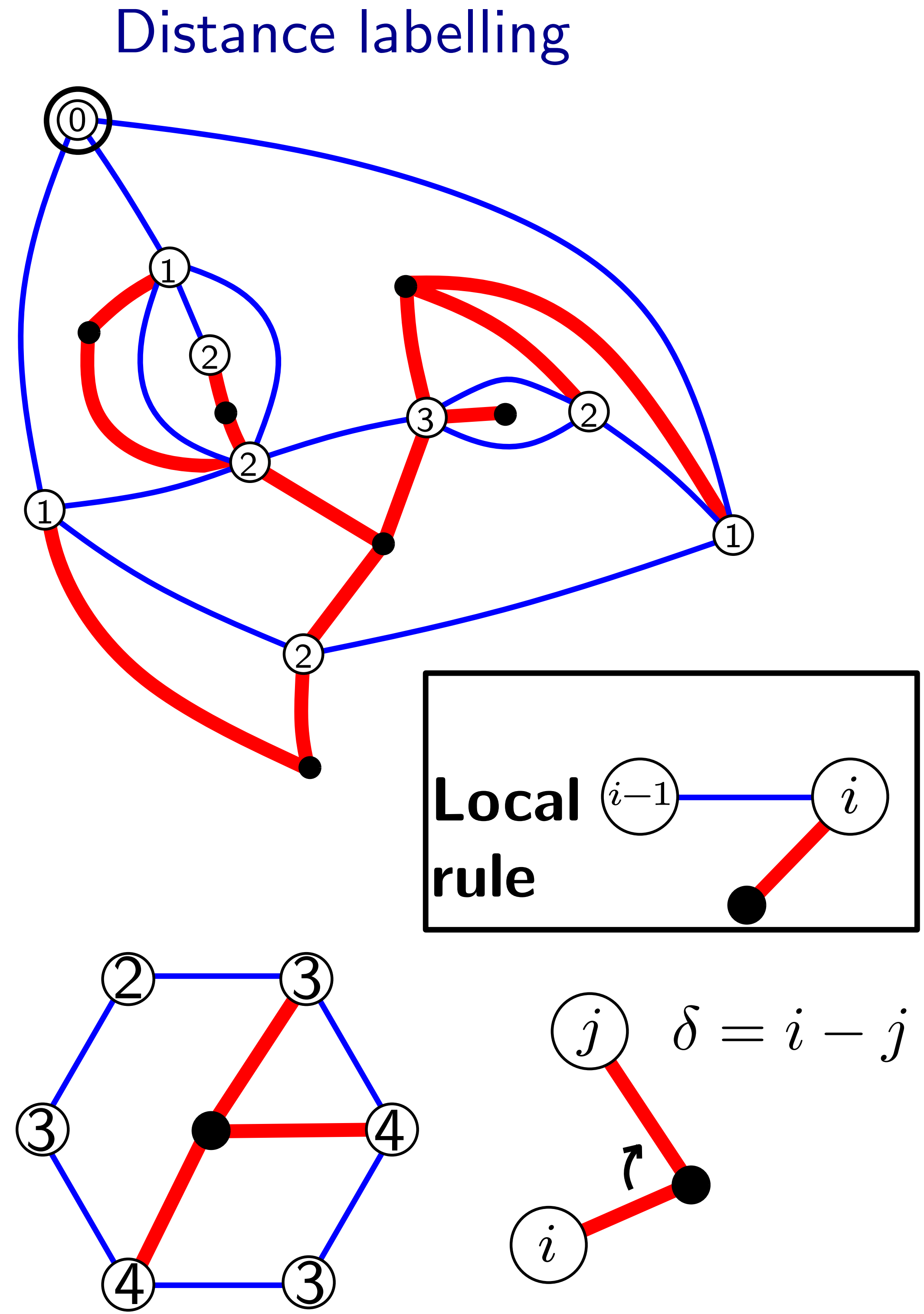
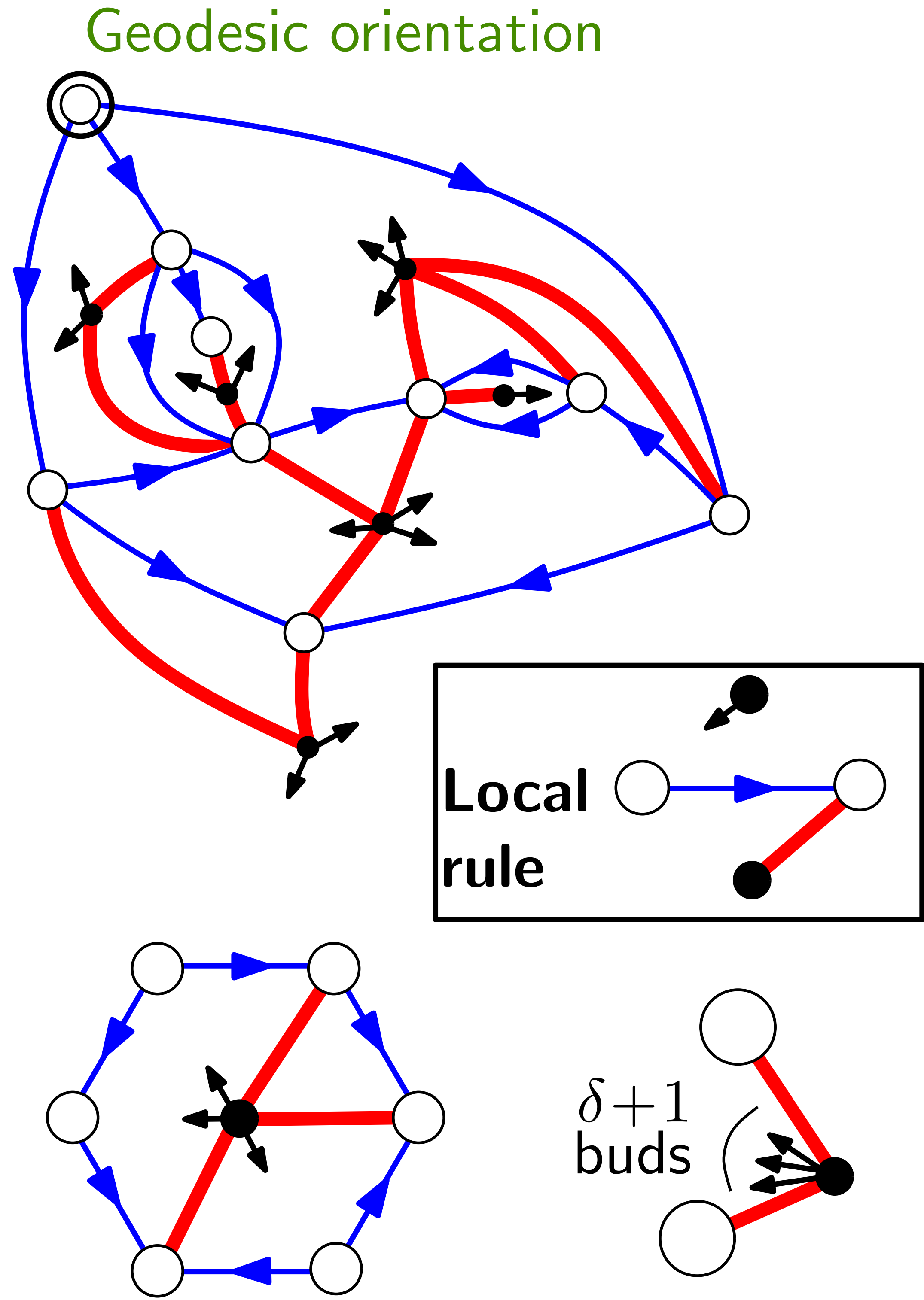
⇒ # rooted bipartite maps with n_i faces of degree $2i$ is

$$\frac{2\cdot(\sum i\, n_i)!}{(2+\sum(i-1)n_i)!}\prod_i \frac{1}{n_i!}\binom{2i-1}{i}^{n_i}$$

Reformulation with orientations.





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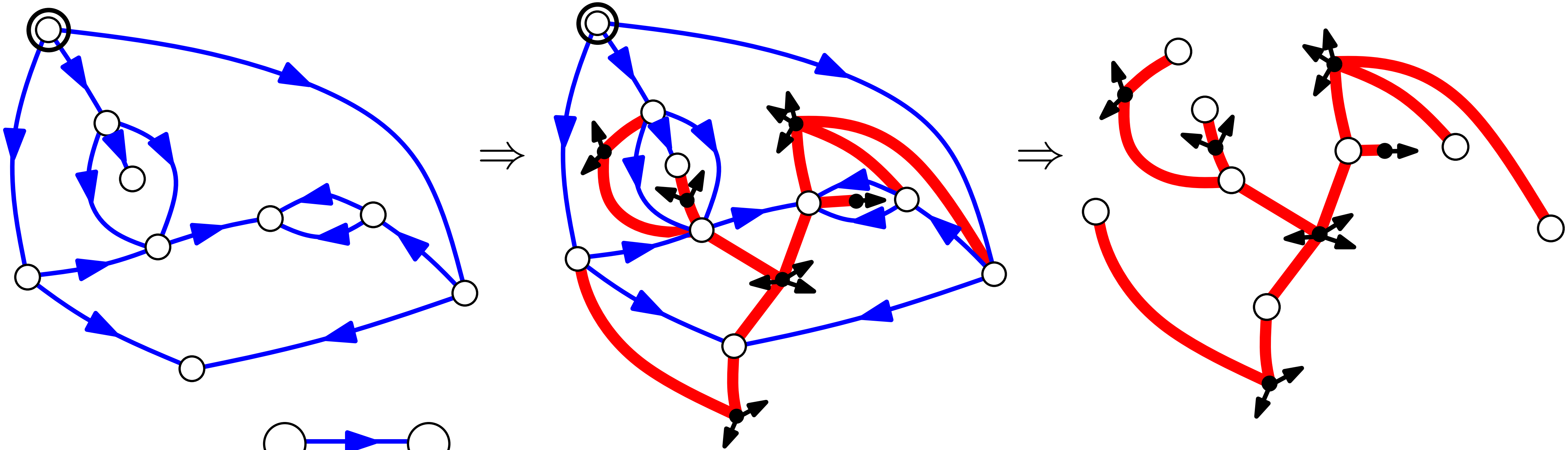

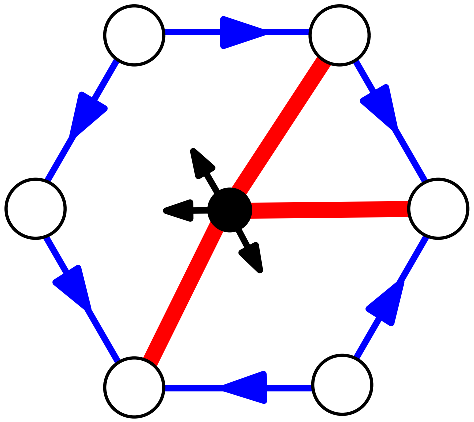

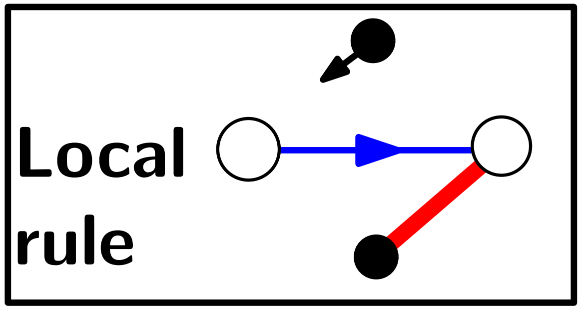

Condition:
At each black vertex, as many buds as white neighbours

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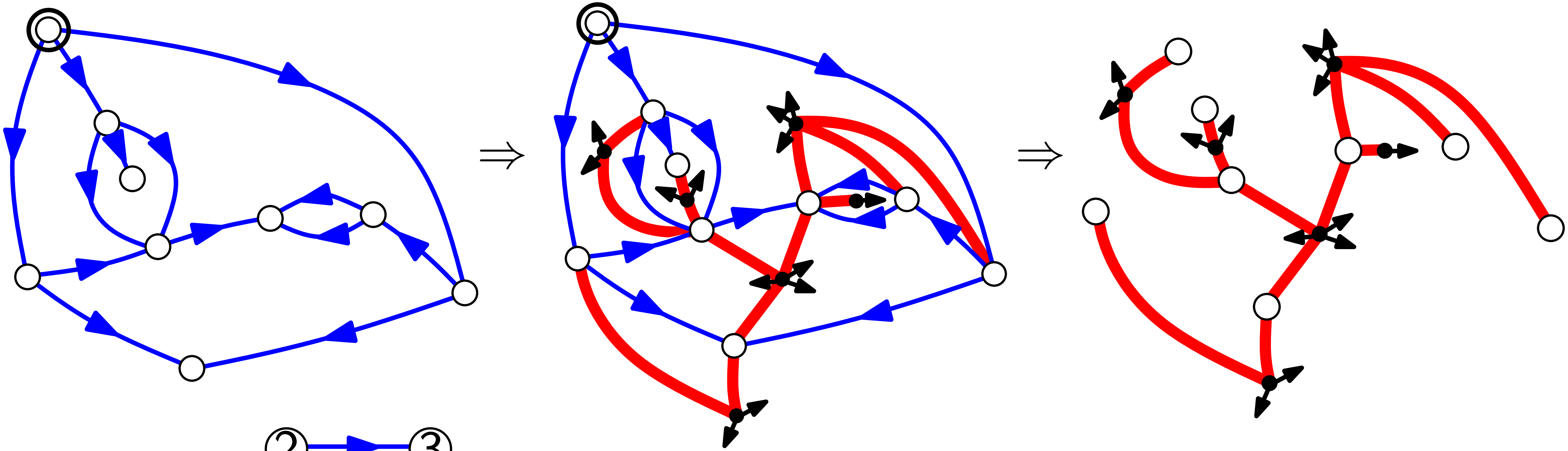

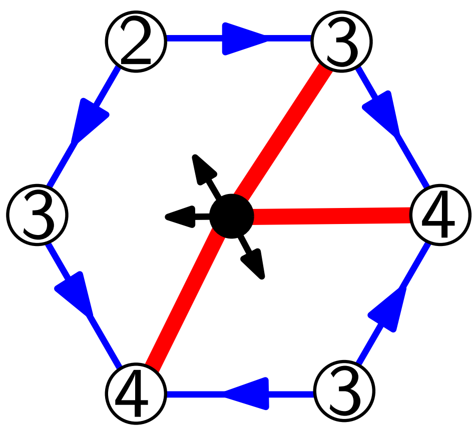

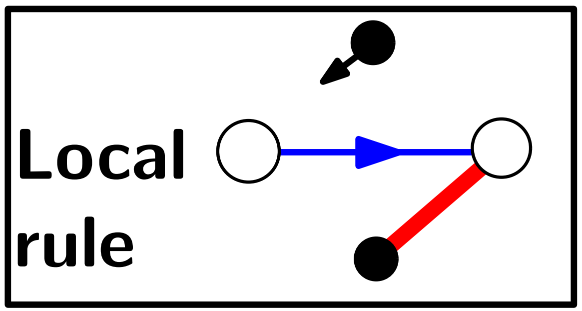

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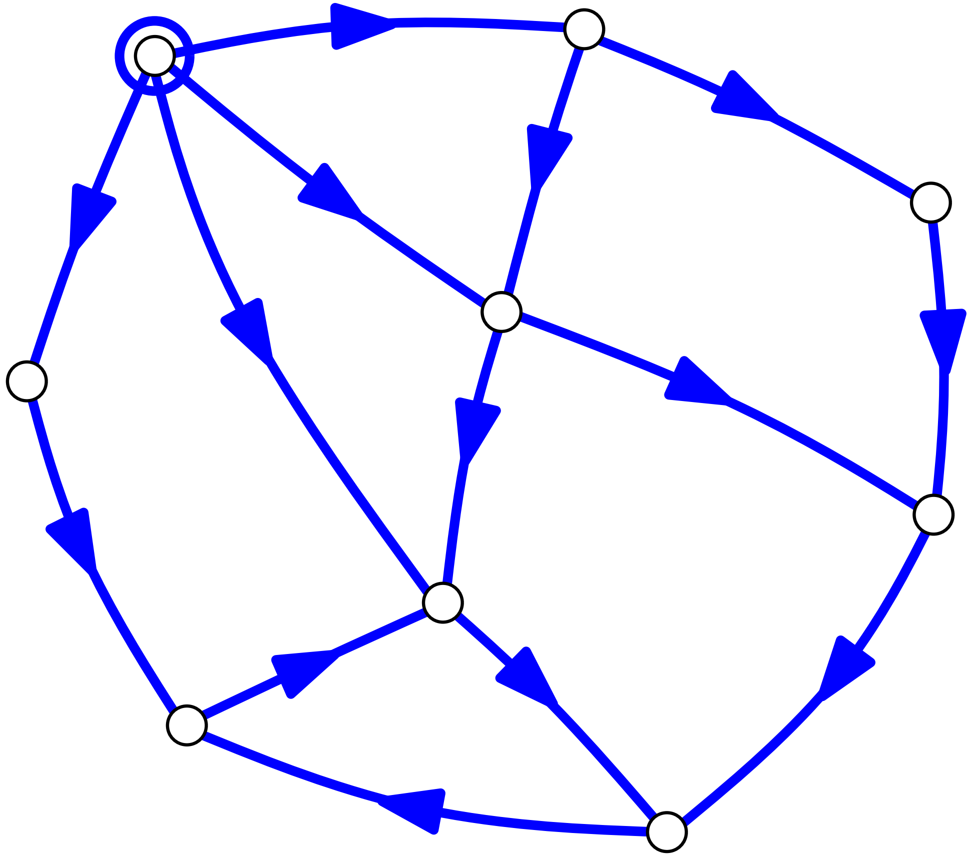
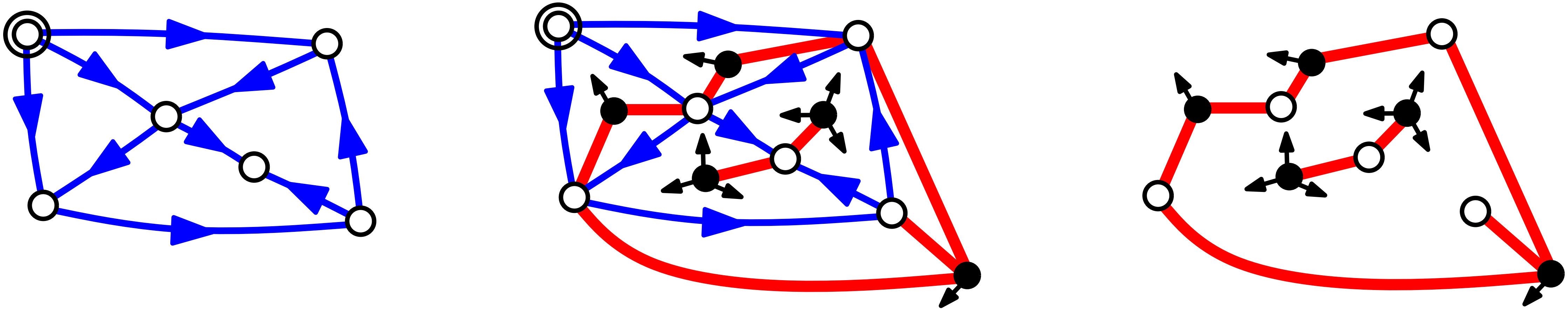
Source-orientations

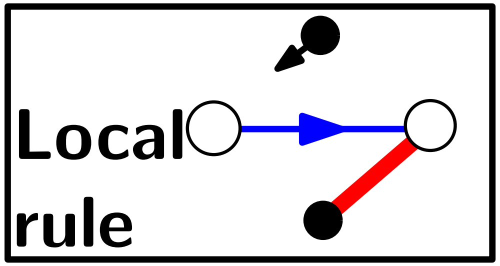
A **source-orientation** is an orientation of a pointed map such that

- The pointed vertex (called the source) has **only outgoing edges**
- **Accessibility**: Each vertex can be reached from the source



Mobile construction for source-orientations





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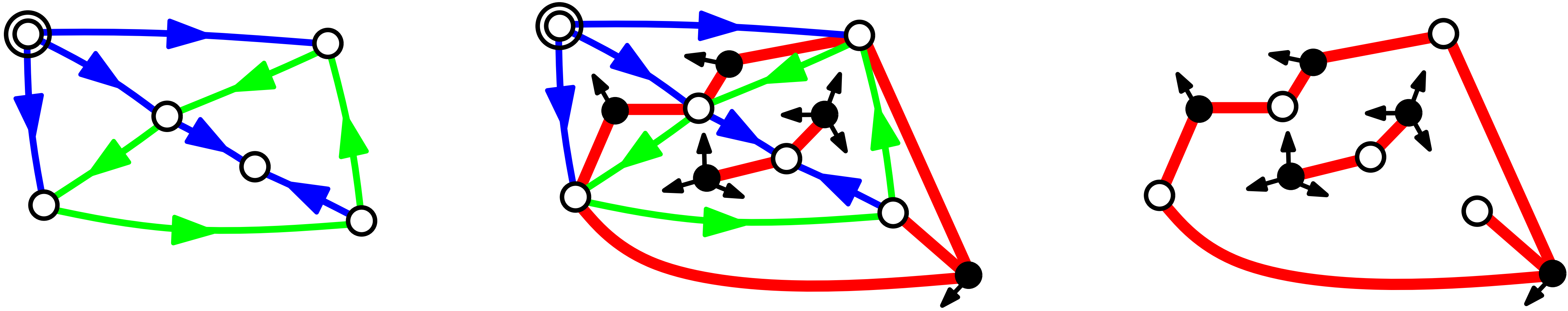

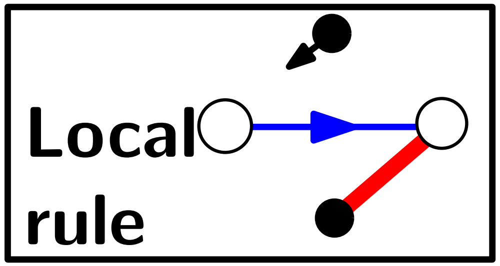

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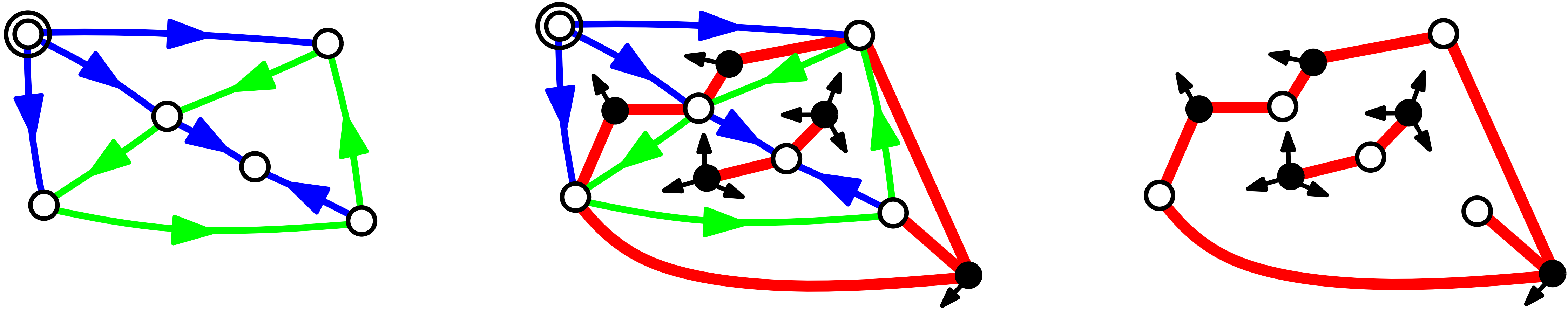
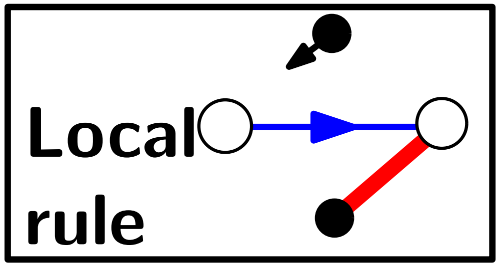
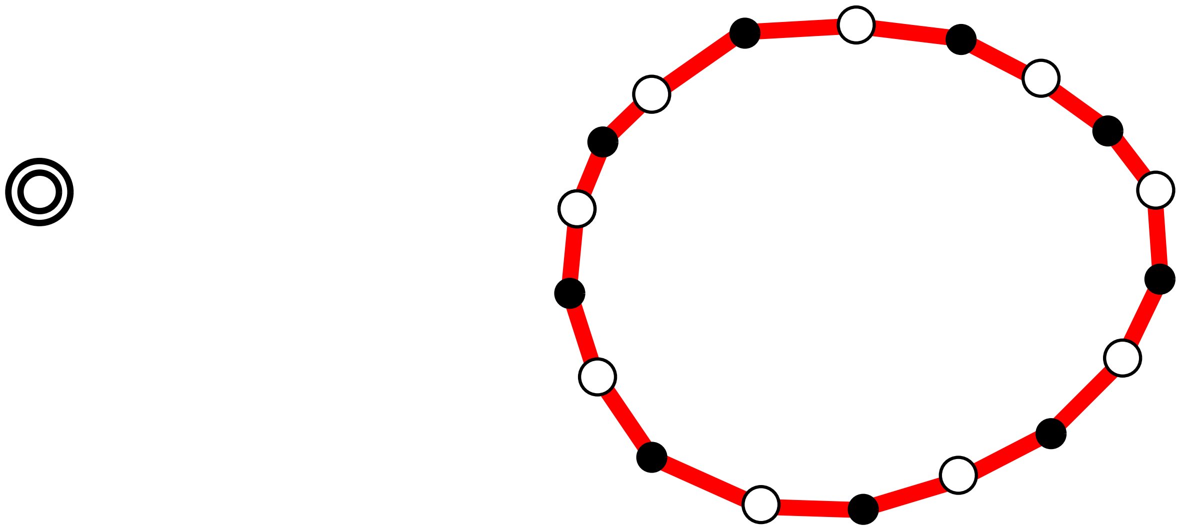




Cycle in mobile ⇒ ccw circuit in the source-orientation



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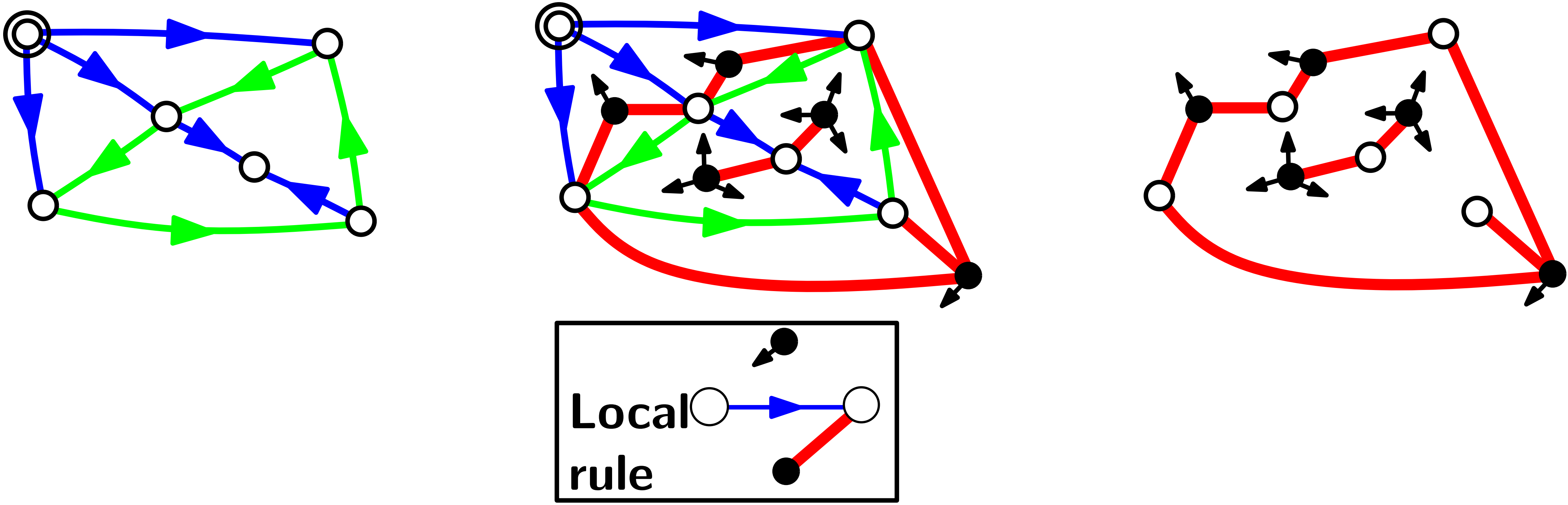

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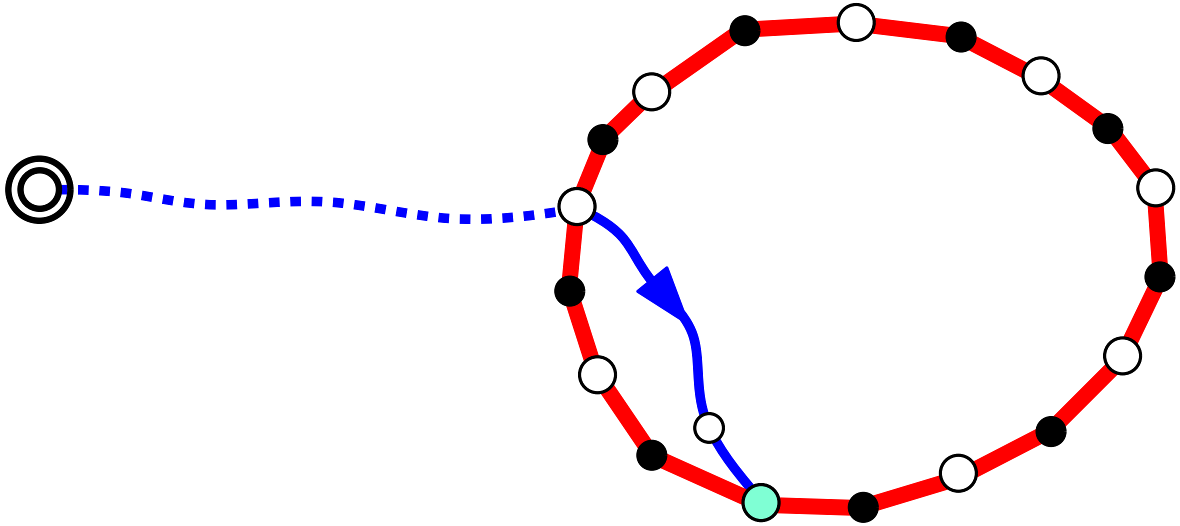

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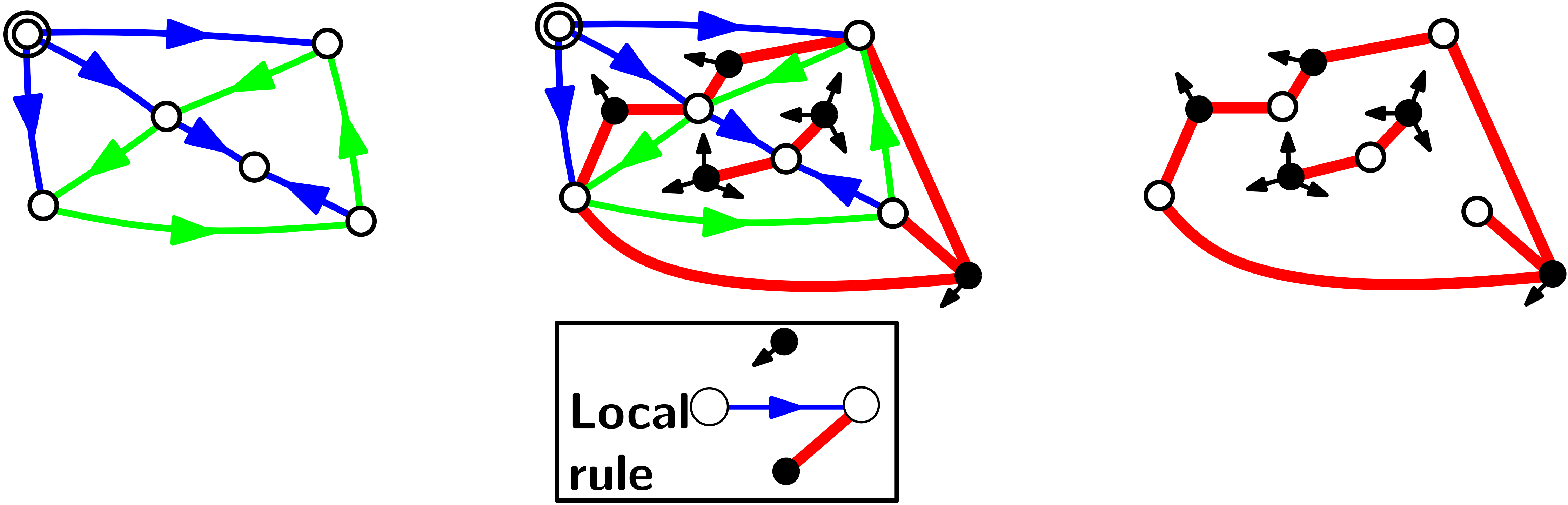

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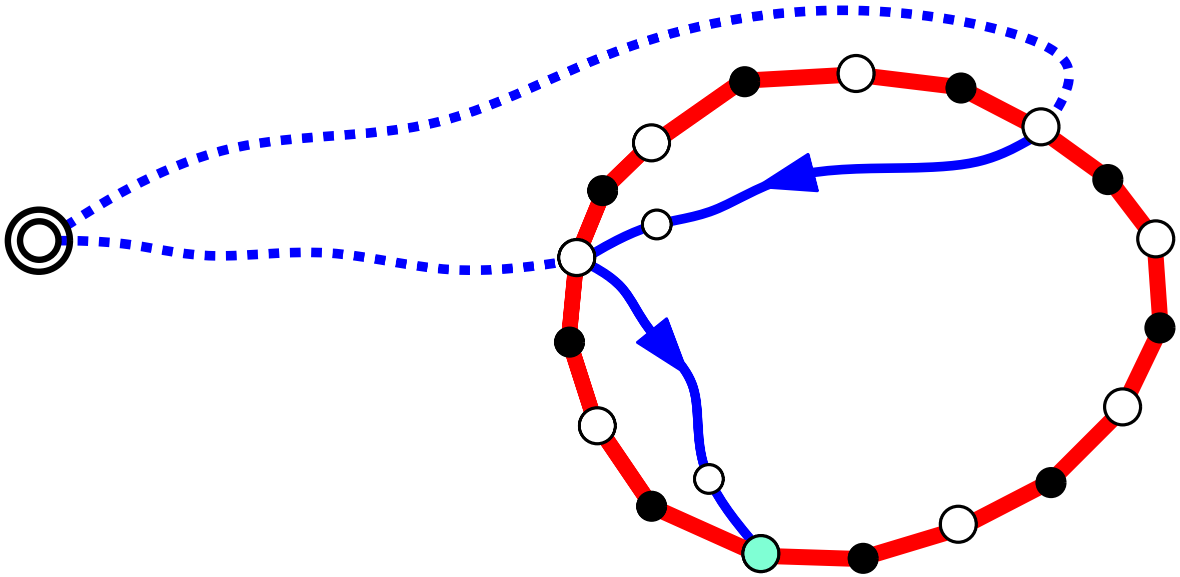

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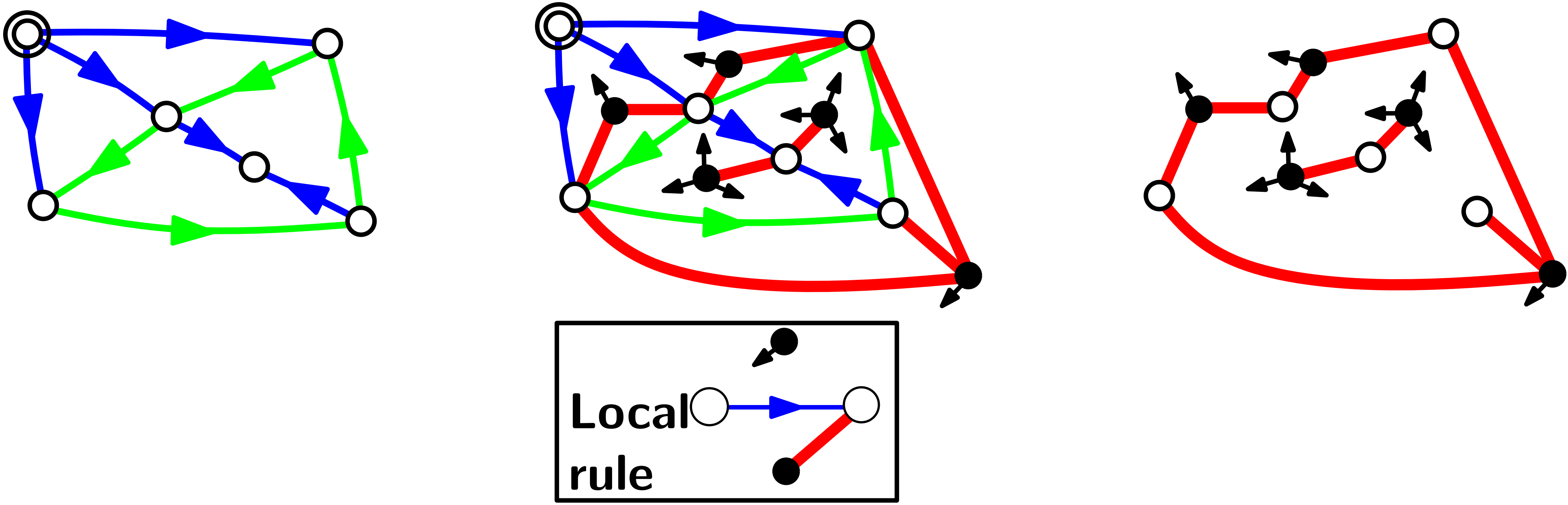

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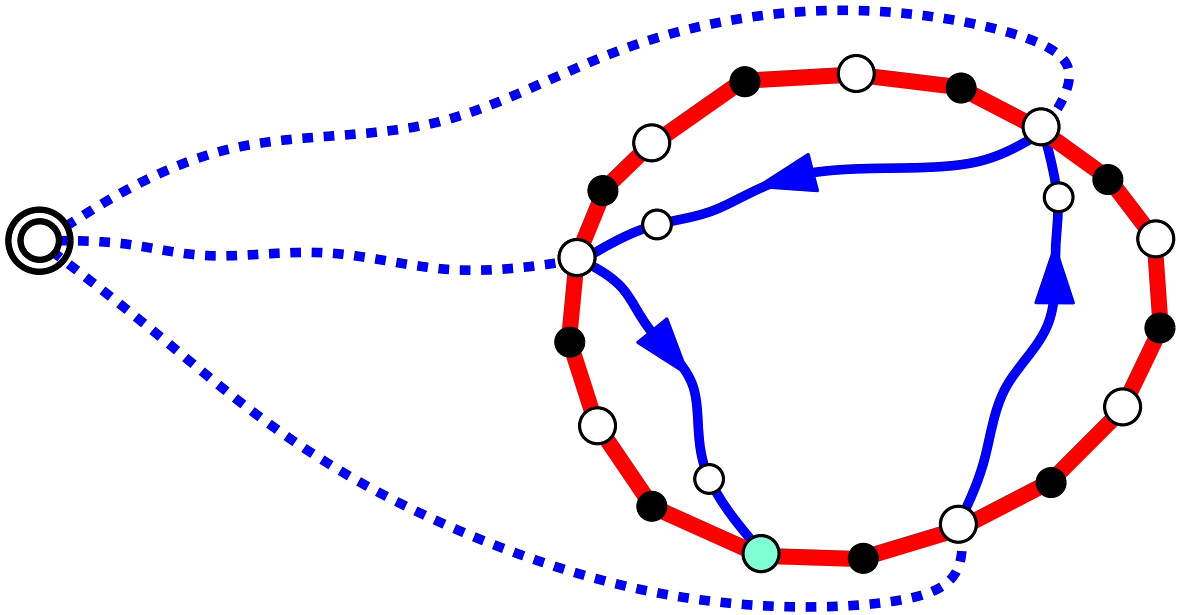

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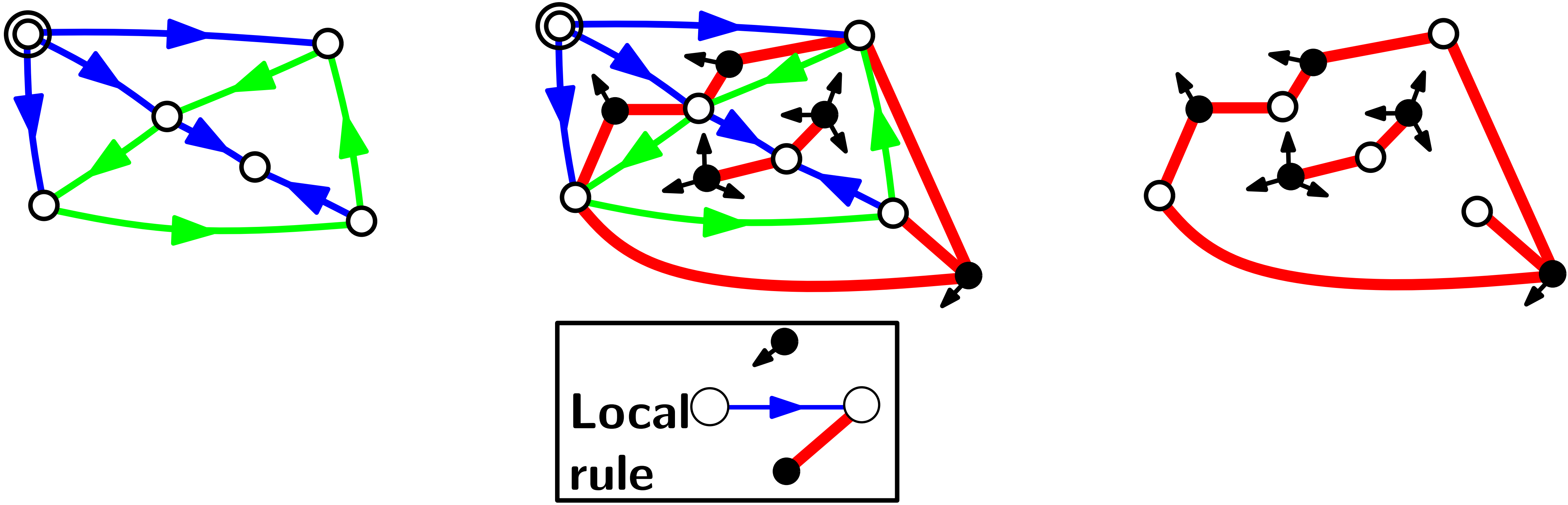

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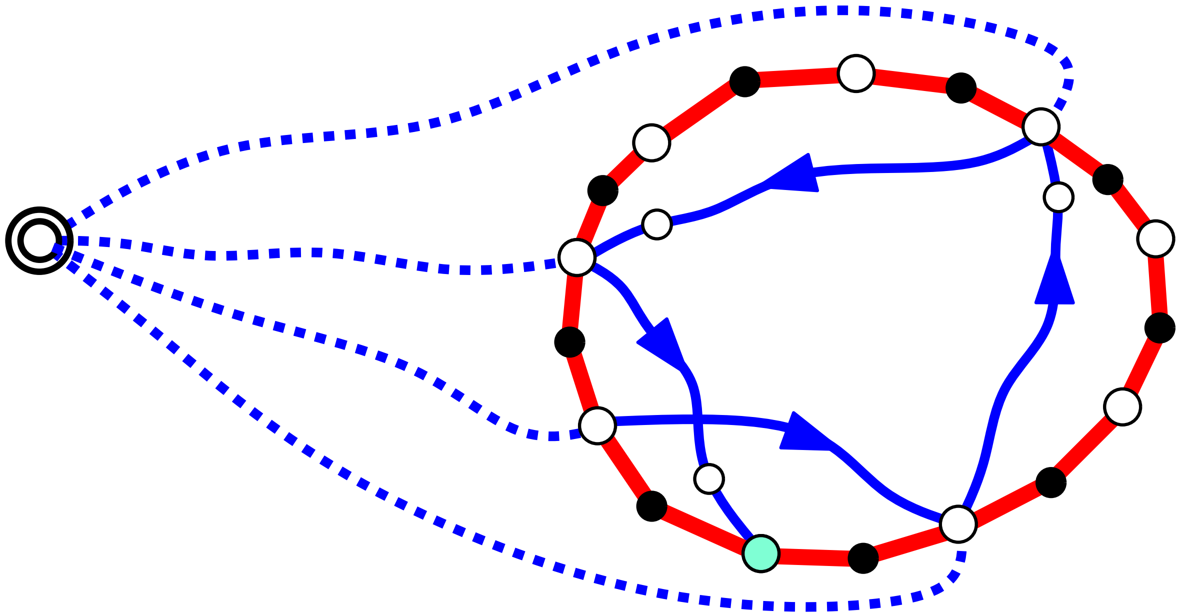

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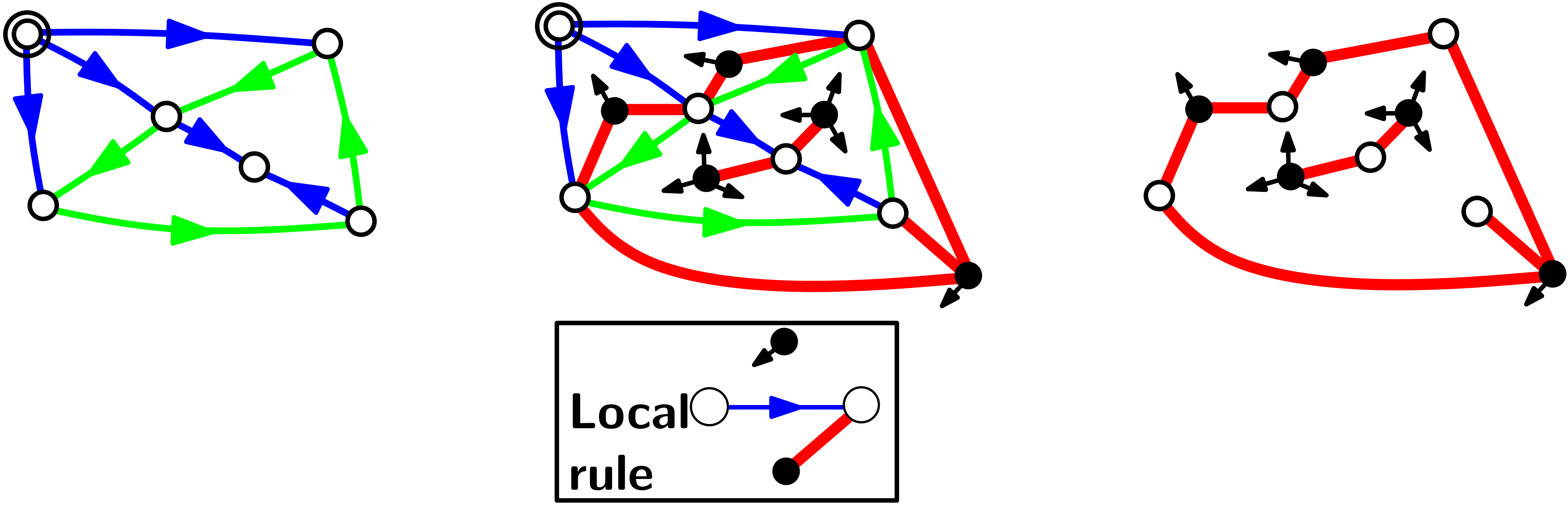

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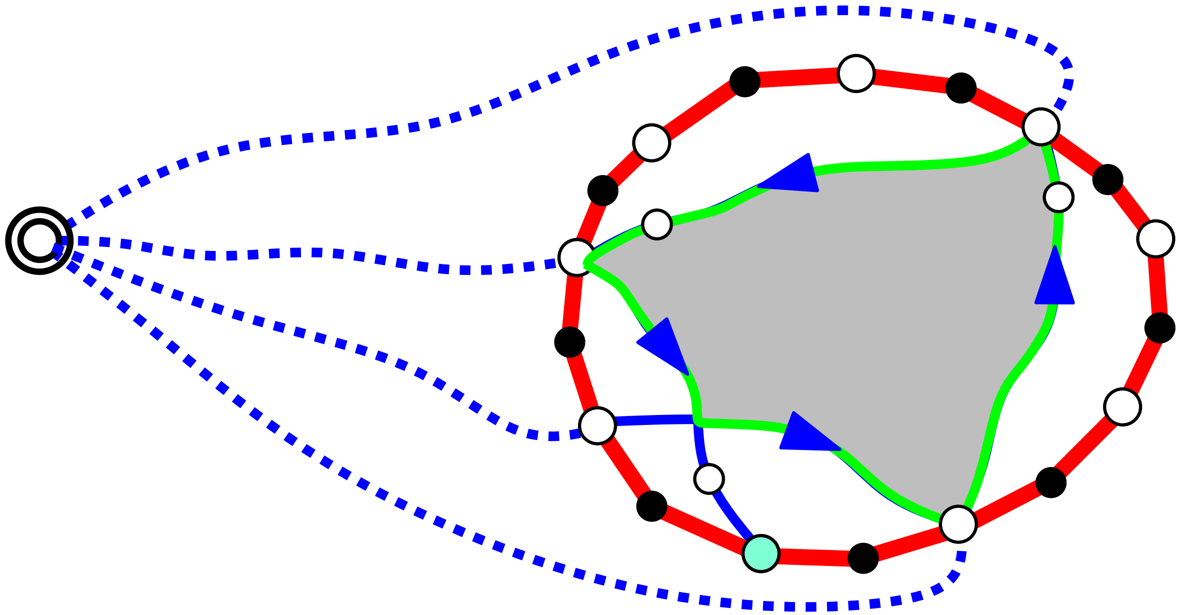
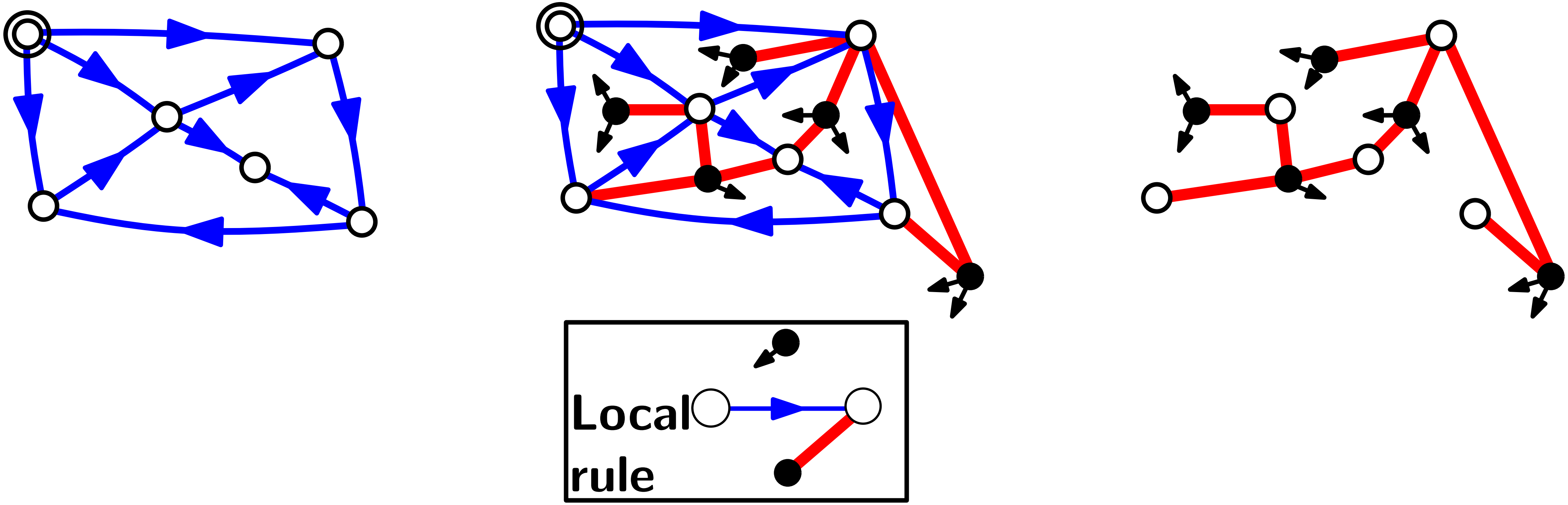
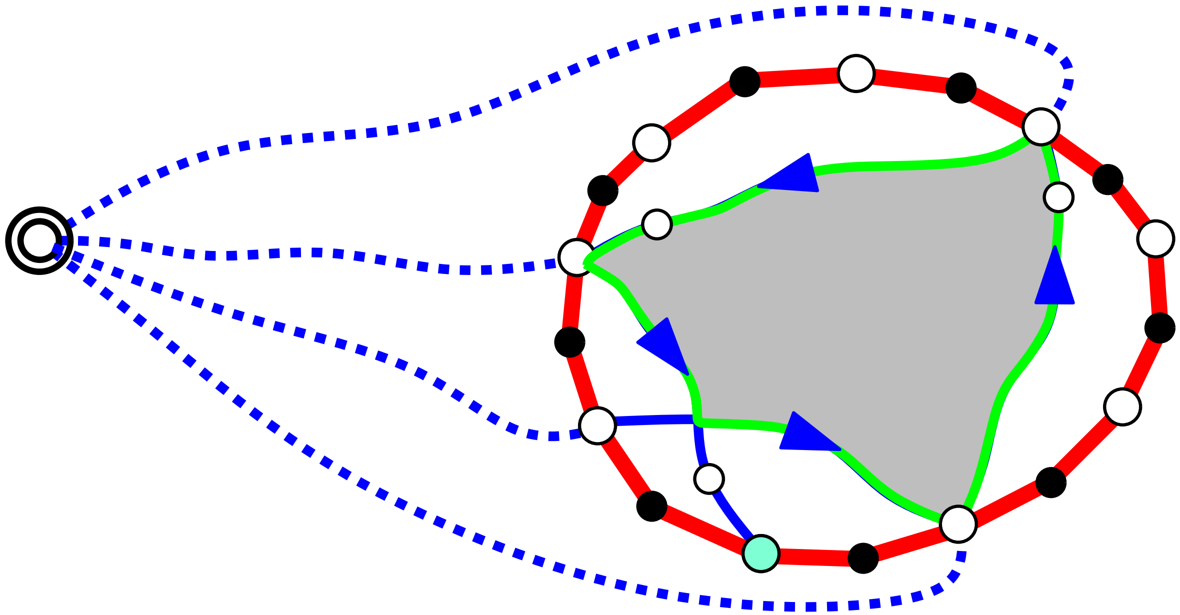


Prisoner
cycle lemma

Mobile construction for source-orientations



Cycle in mobile $\Rightarrow$ ccw circuit in the source-orientation



Prisoner
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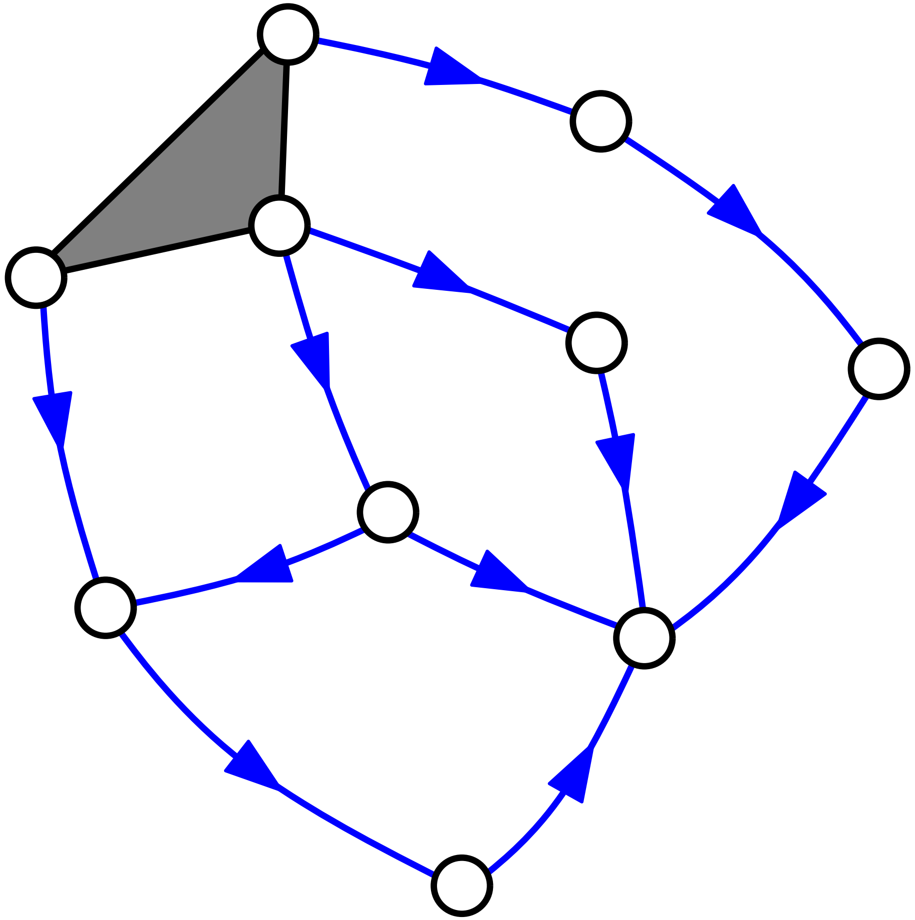
d-gonal source-orientations

We allow the source of the orientation to be a d-gon, with $d \geq 0$

Example for $d = 3$



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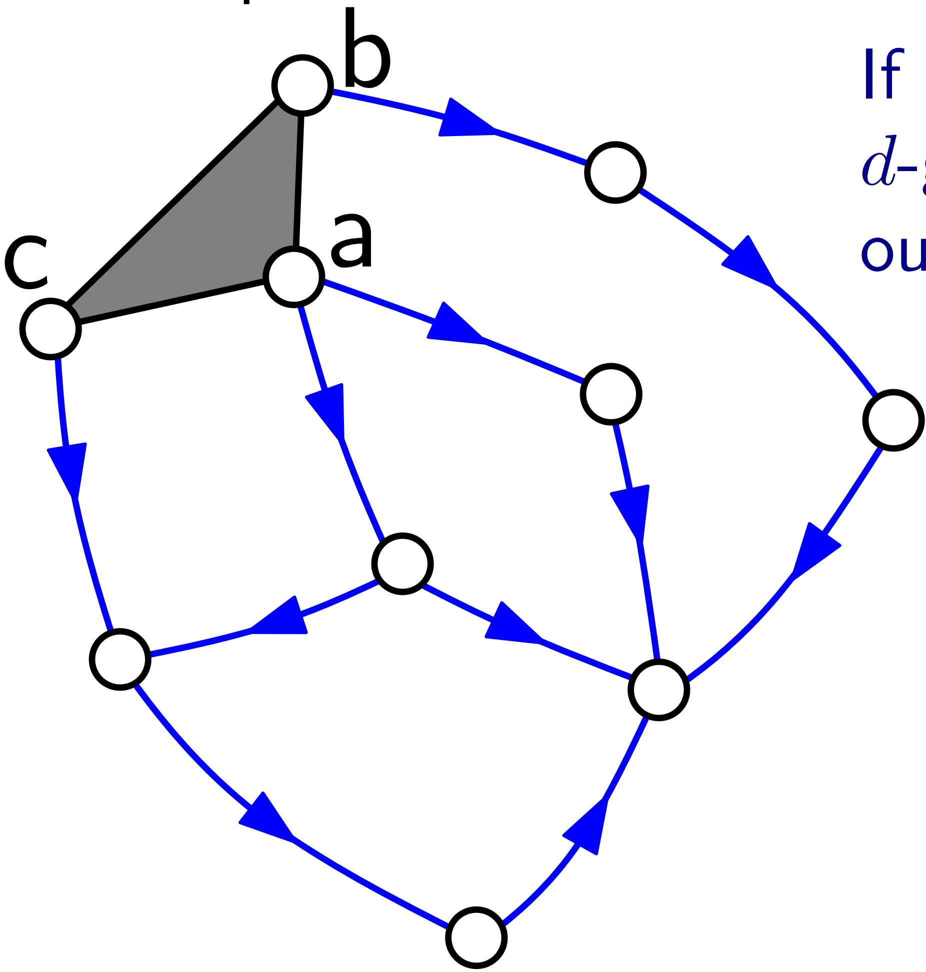
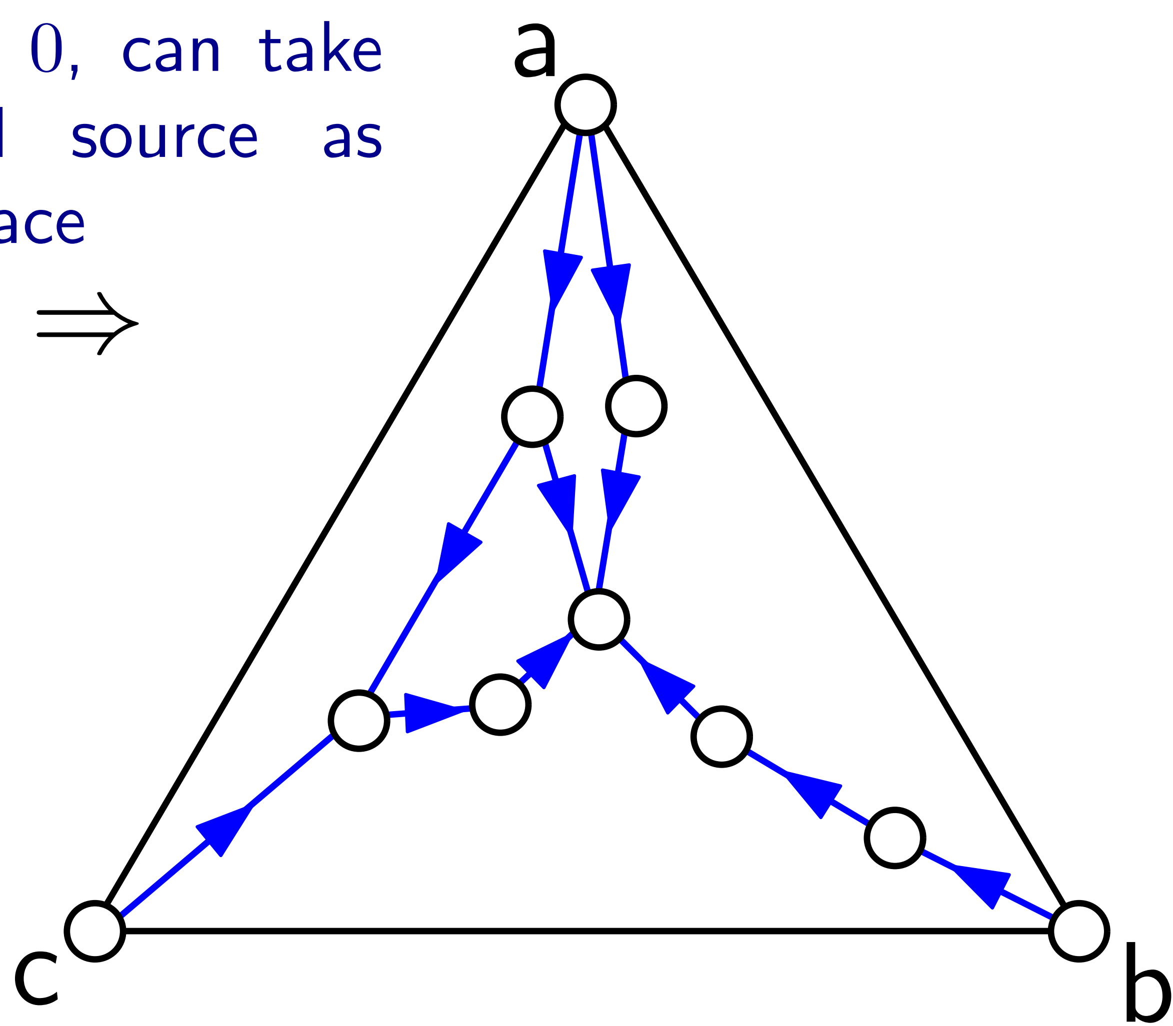


If $d > 0$, can take d-gonal source as outer face

$\Rightarrow$



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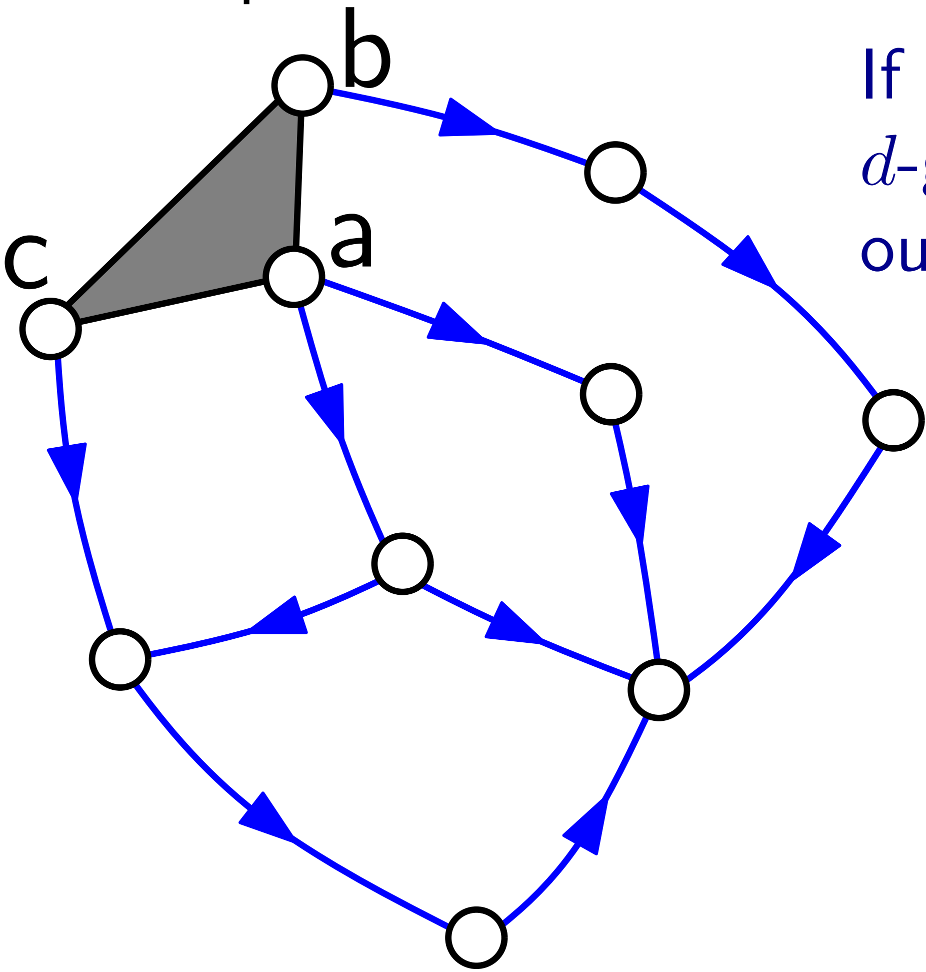

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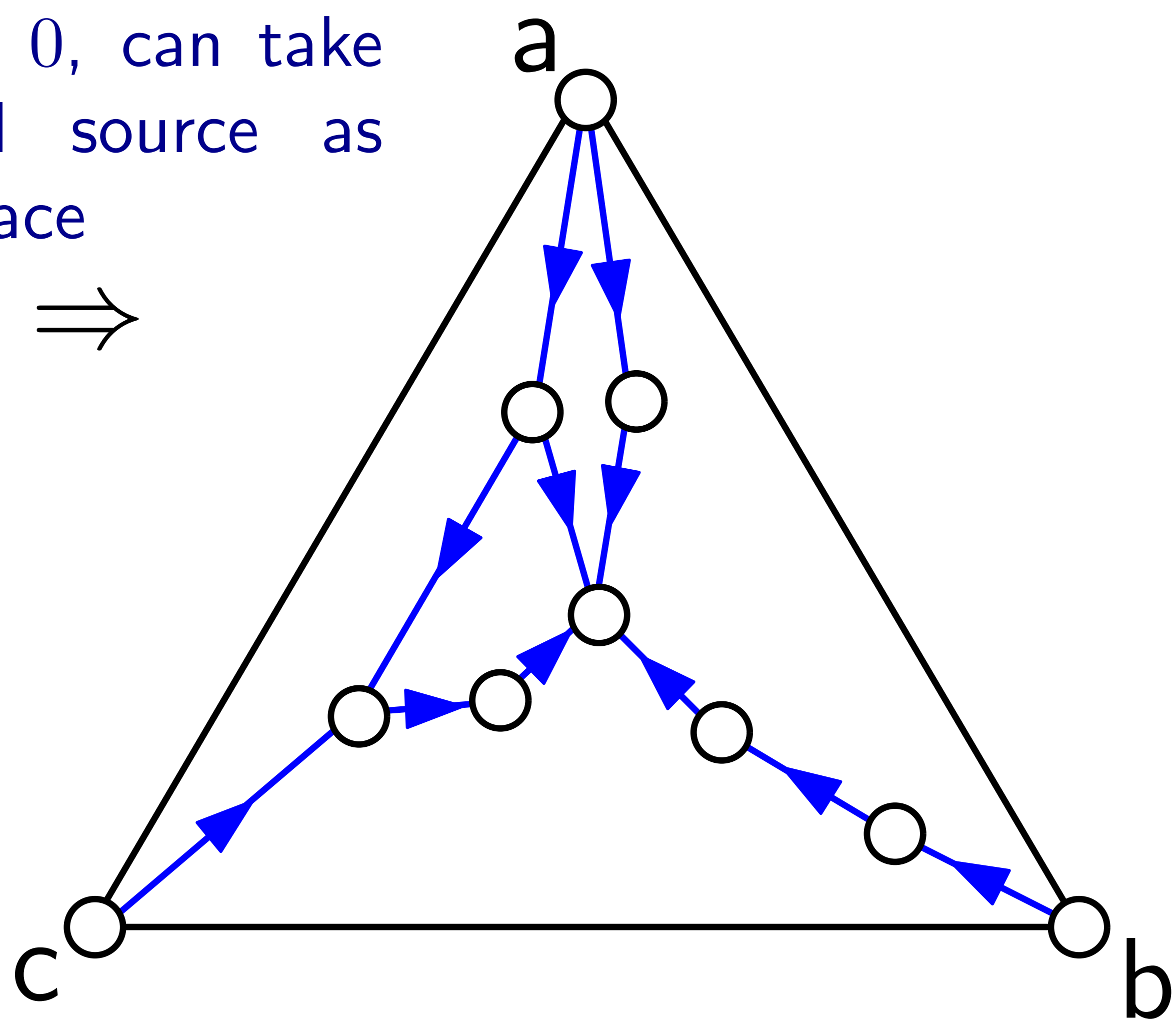
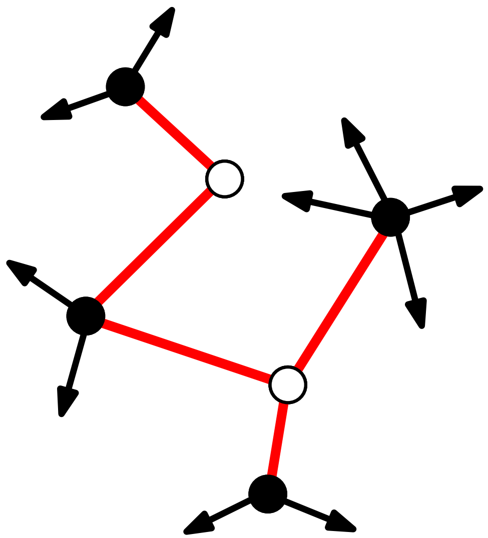


Let $\mathcal{O}_d$ be the set of d-gonal source-orientations with no ccw circuit

Let $\mathcal{O} = \cup_{d \geq 0} \mathcal{O}_d$

Mobiles

A **mobile** is a plane tree with vertices properly colored in black and white, together with buds (half-edges) incident to black vertices.



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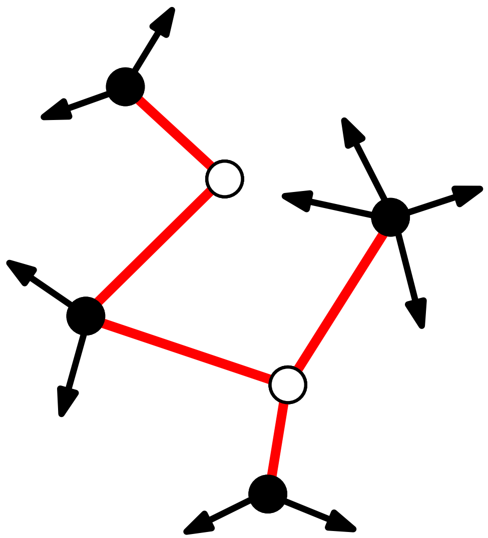

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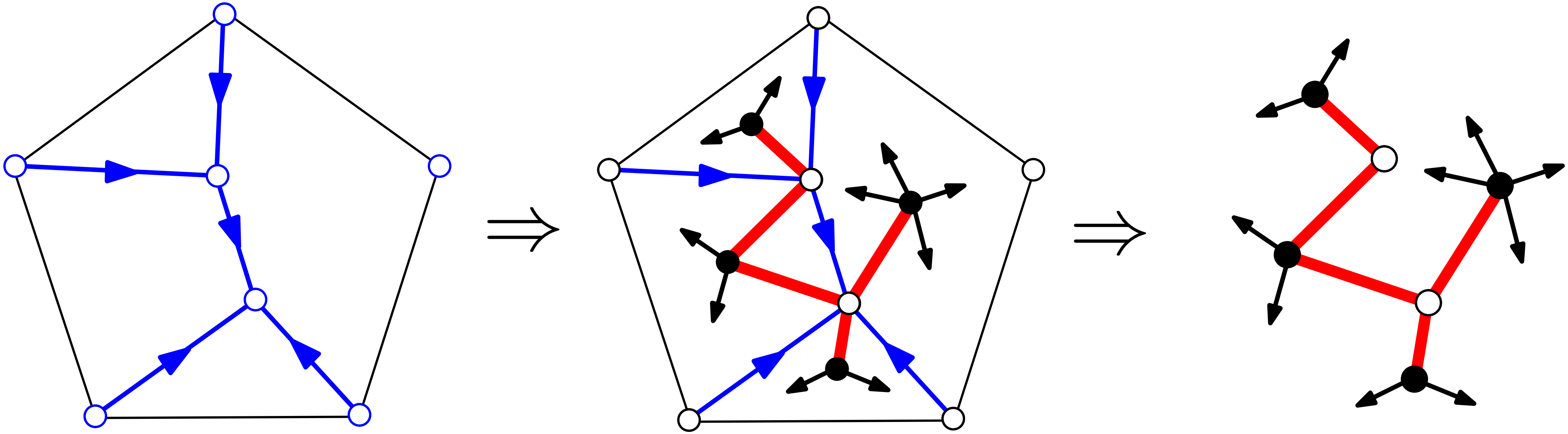
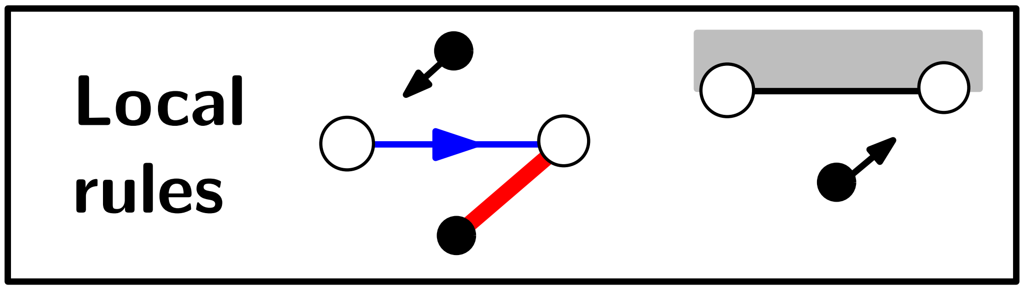
Let $\mathcal{M}$ be the set of mobiles of nonnegative excess

Master bijection Φ





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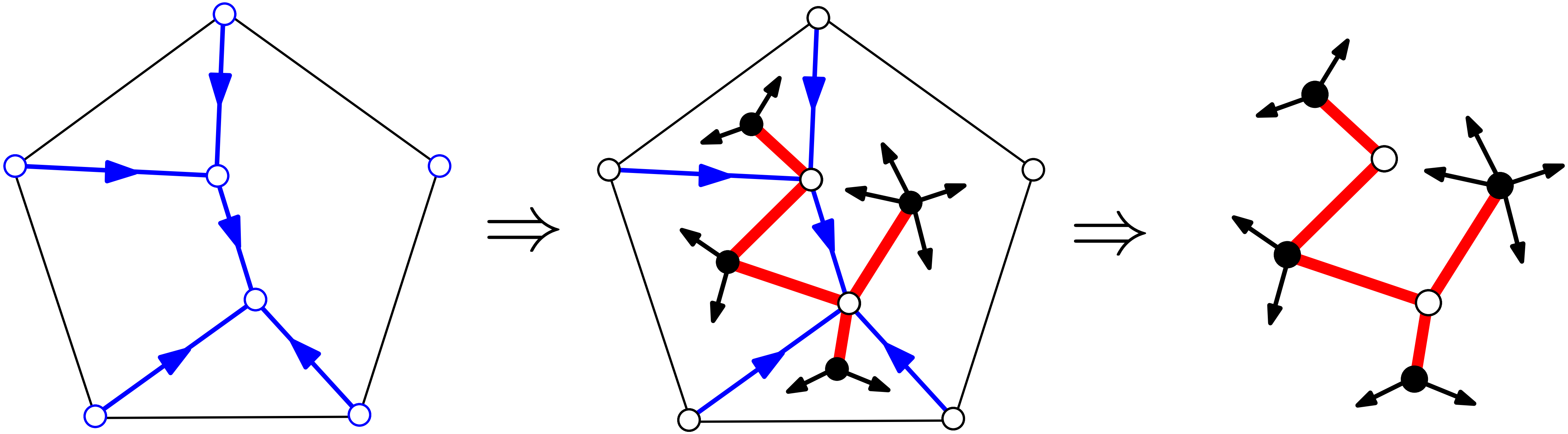

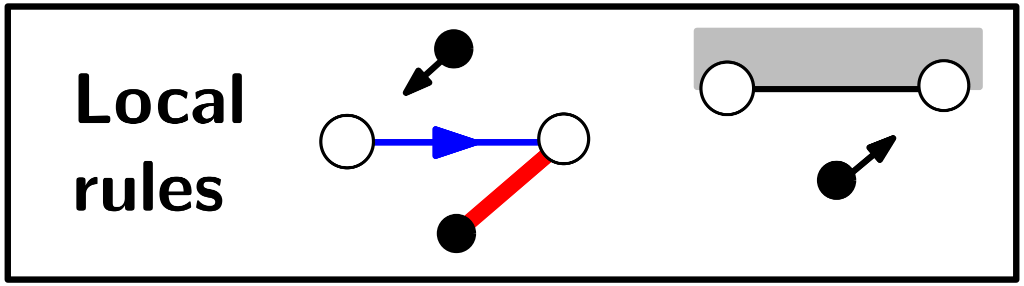

Theorem [Bernardi-F'10]: Φ is a **bijection** between $\mathcal{O}$ and $\mathcal{M}$.
Moreover,

degree of external face	$\longleftrightarrow$	excess
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indegree of internal vertices	$\longleftrightarrow$	degree of white vertices

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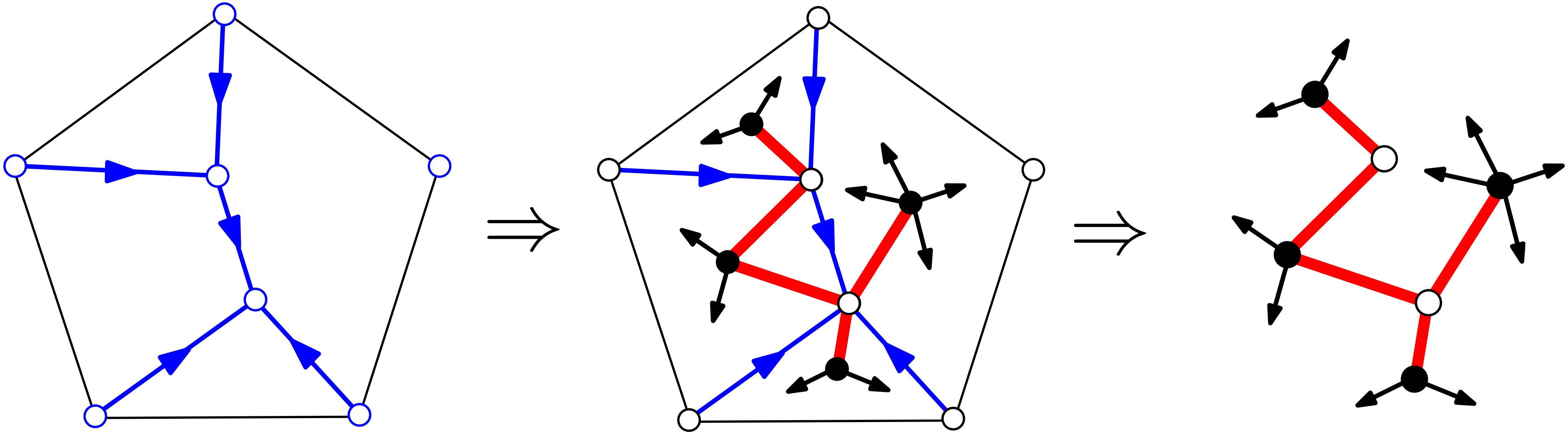

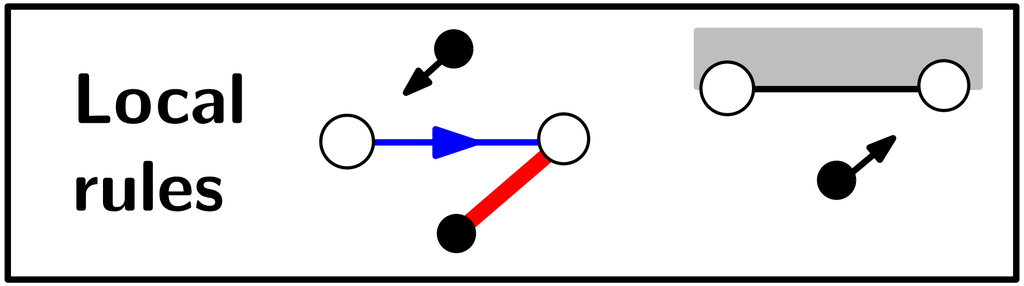

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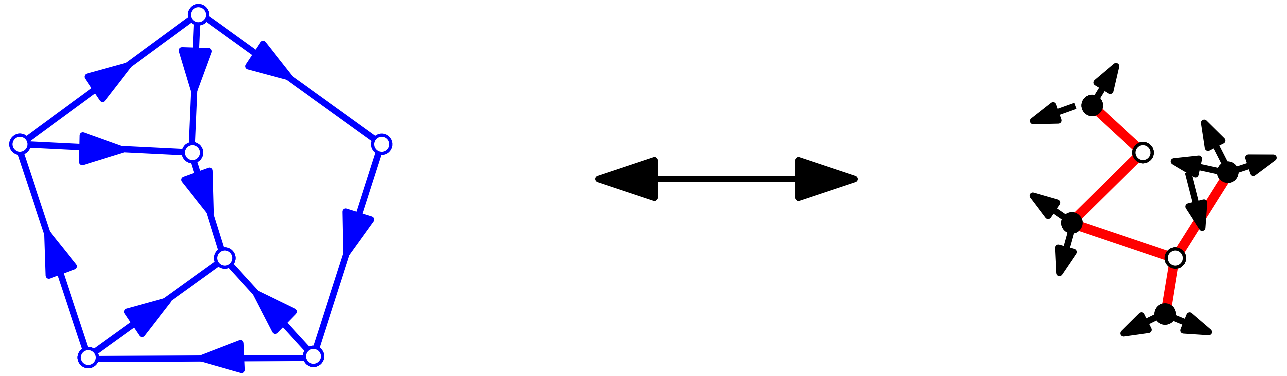
cf [Bernardi'07], [Bernardi-Chapuy'10]

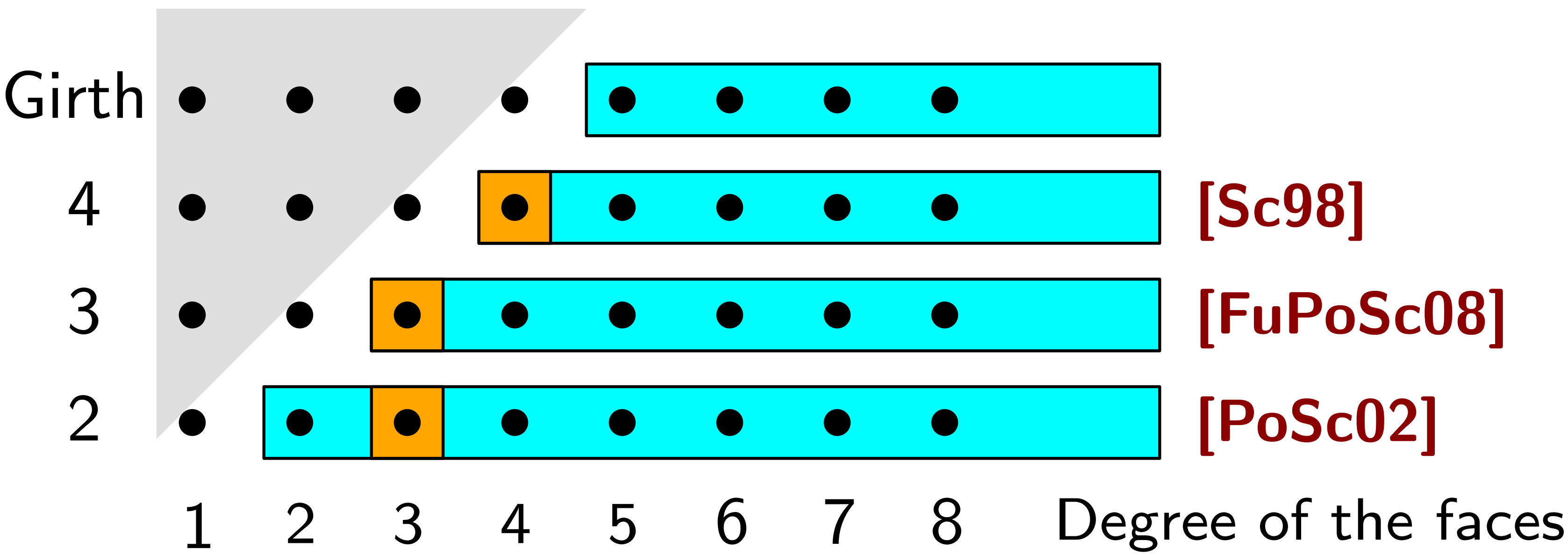
Using the master bijection
for map enumeration

Main new results

The **Master bijection** between $\mathcal{O}$ (orientations) and $\mathcal{M}$ (mobiles)



allows to count maps by girth & face-degrees (via canonical orientations).



Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a **subfamily** $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:

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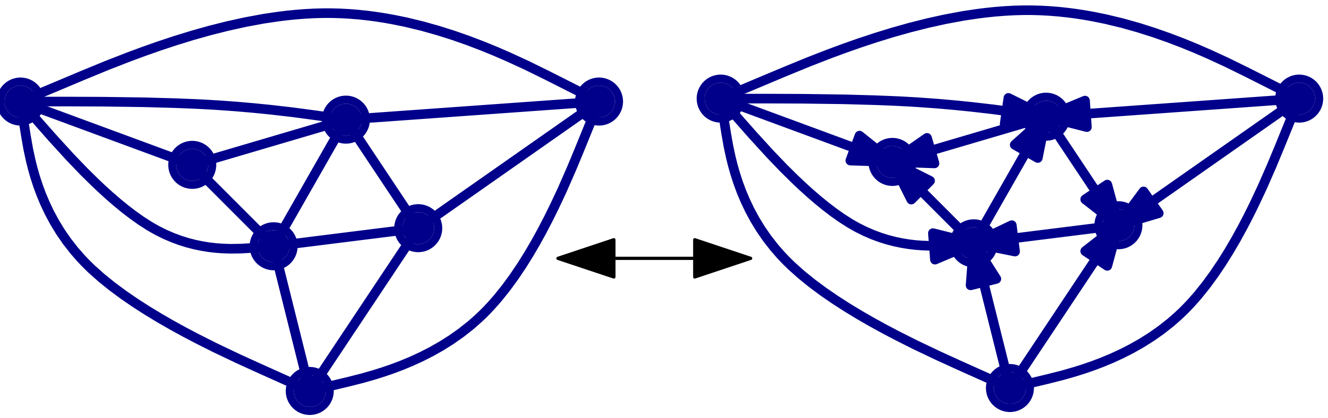

$\mathcal{C} \simeq$ subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with

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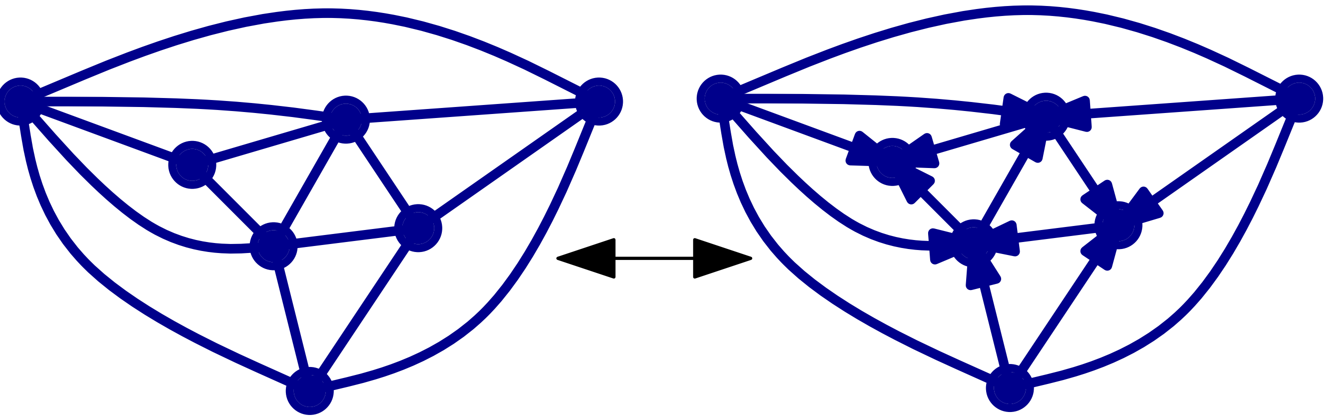

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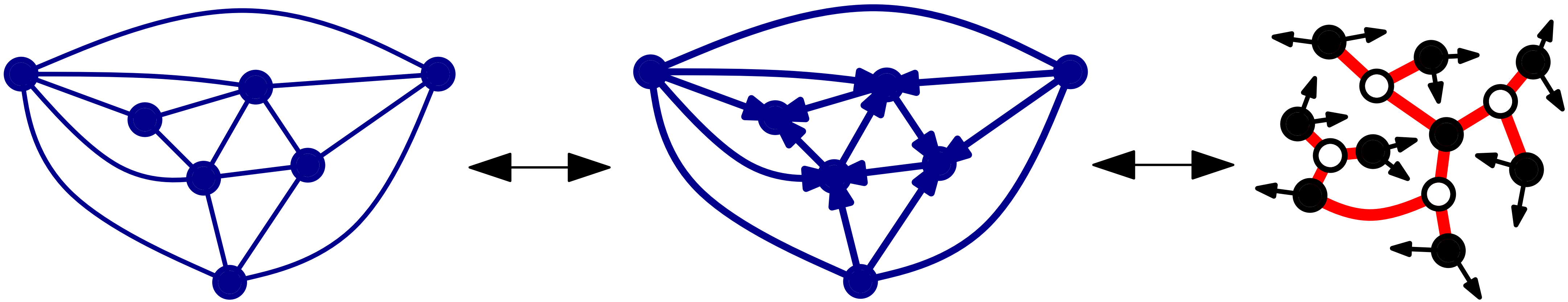
(2) **Specialize** the master bijection to the subfamily $\mathcal{O}_C$



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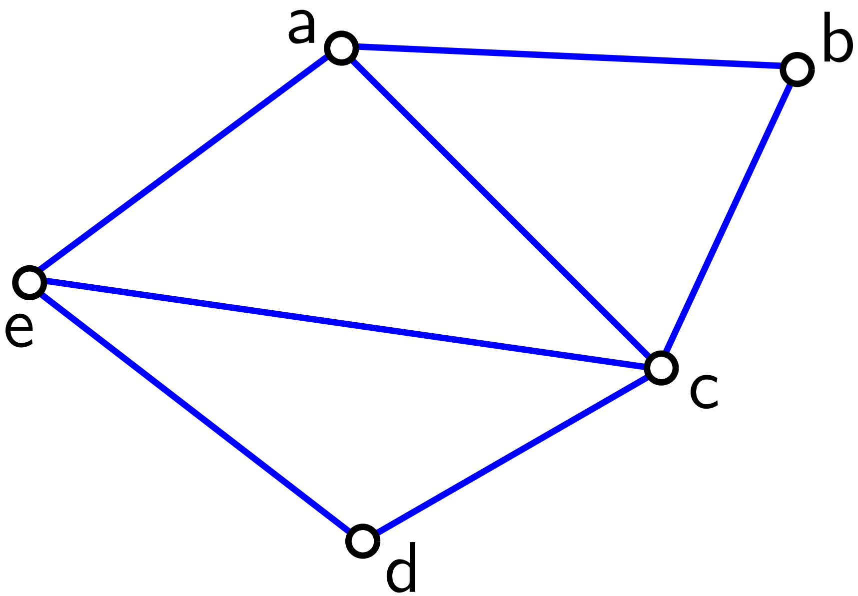
α-orientations

Let $G = (V, E)$ be a graph

Let α be a function from V to $\mathbb{N}$



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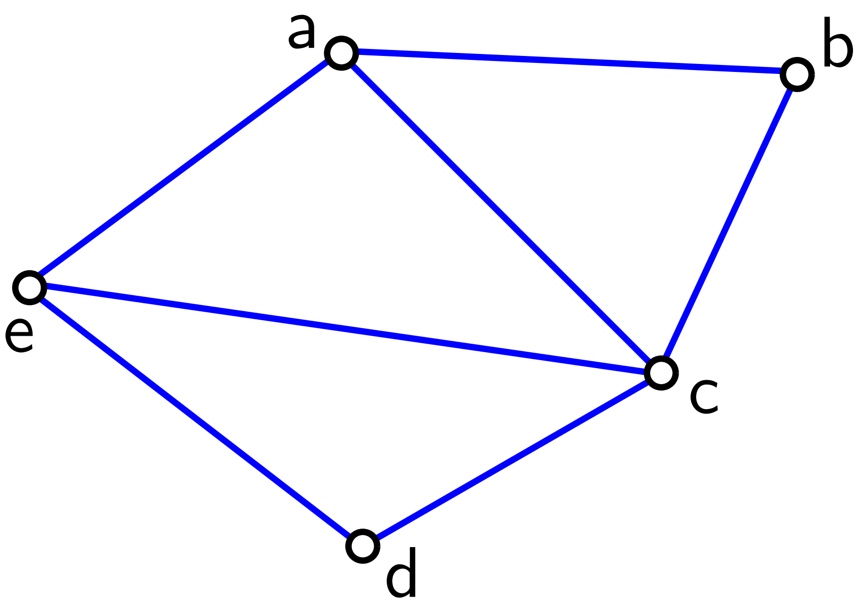

$$\alpha: \begin{array}{l} a \to 2 \\ b \to 1 \\ c \to 2 \\ d \to 0 \\ e \to 2 \end{array}$$

Def: An ***α*-orientation** is an orientation of G where for each $v \in V$

$$\boxed{\text{indegree}(v) = \alpha(v)}$$

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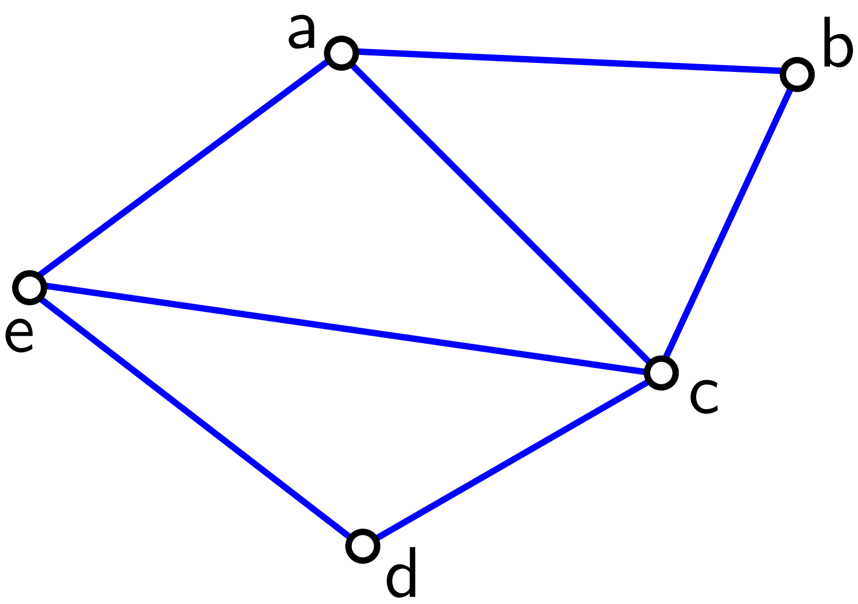

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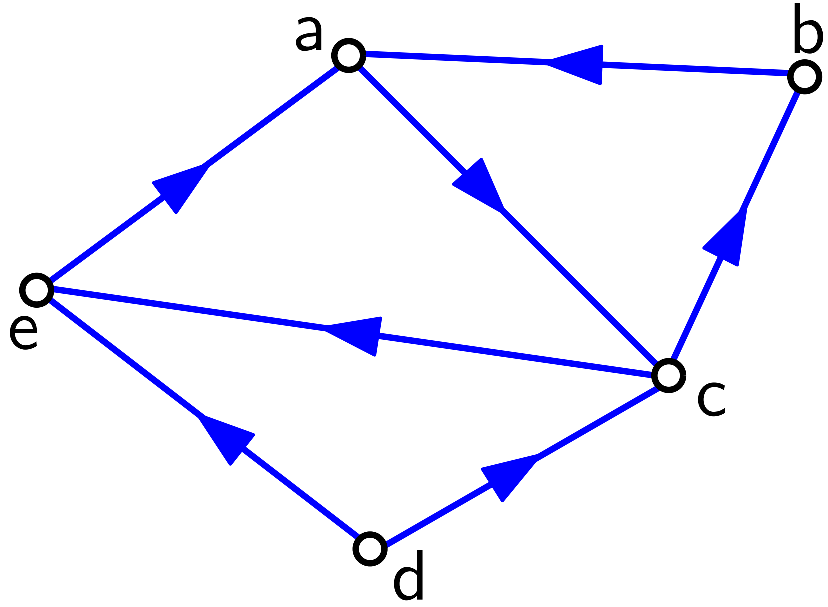

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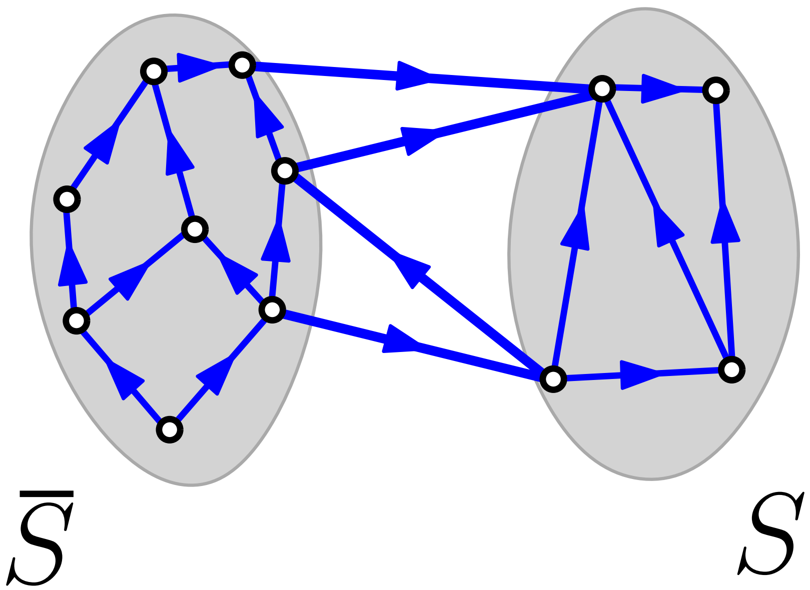
α-orientations: criteria for existence and accessibility

- If an α-orientation **exists**, then



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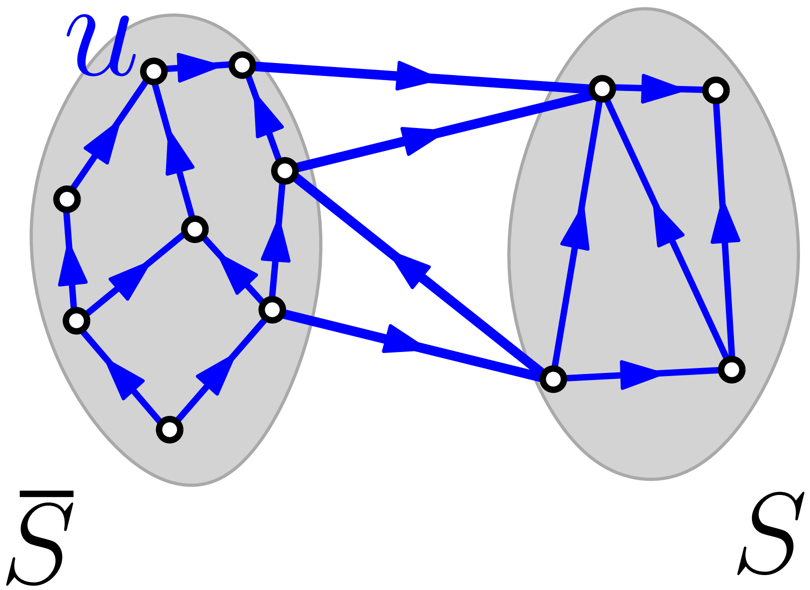

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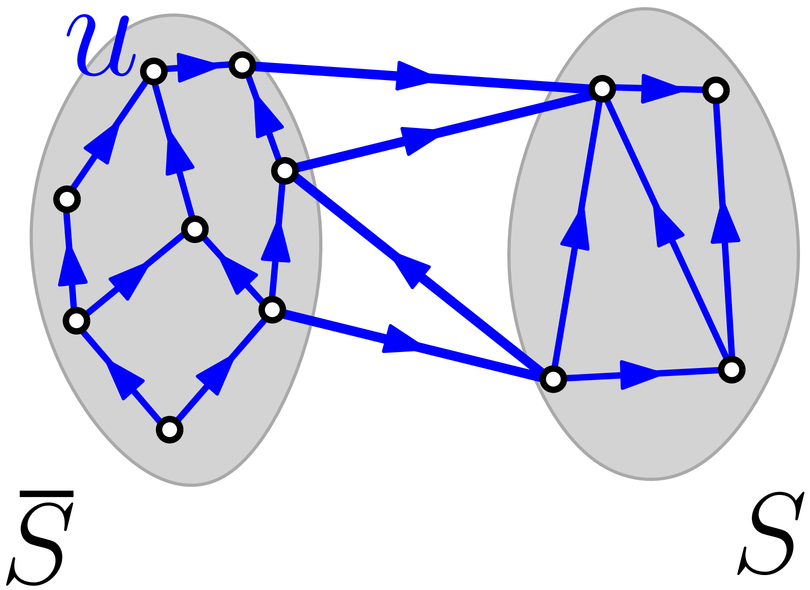

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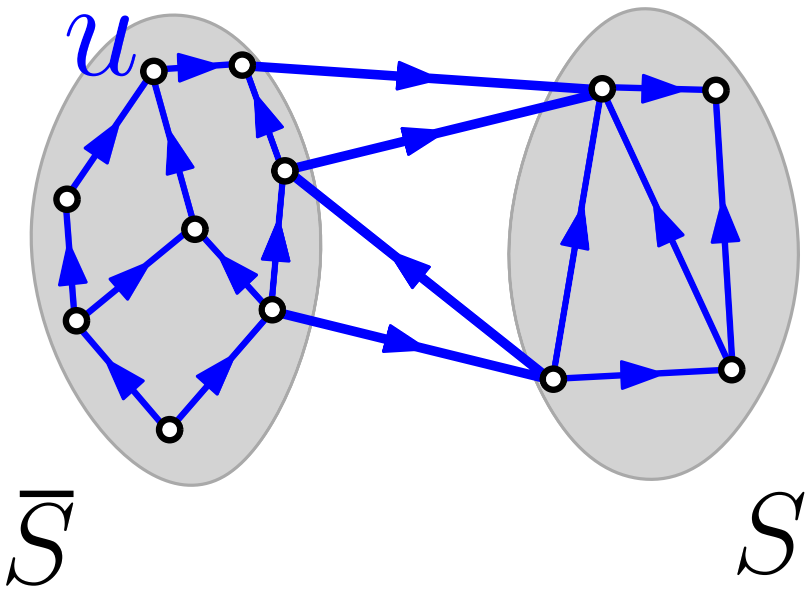

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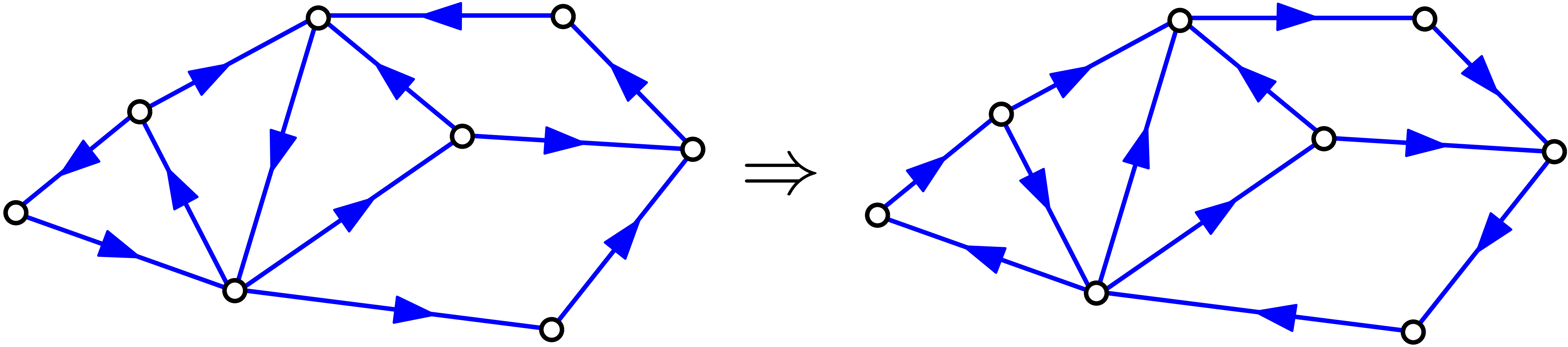
$\Rightarrow$ accessibility from $u \in V$ just depends on α (not on which α-orientation)

α-orientations for plane maps

Fundamental lemma: If a plane map admits an α-orientation, then it admits a **unique** α-orientation **without ccw circuit**, called **minimal**



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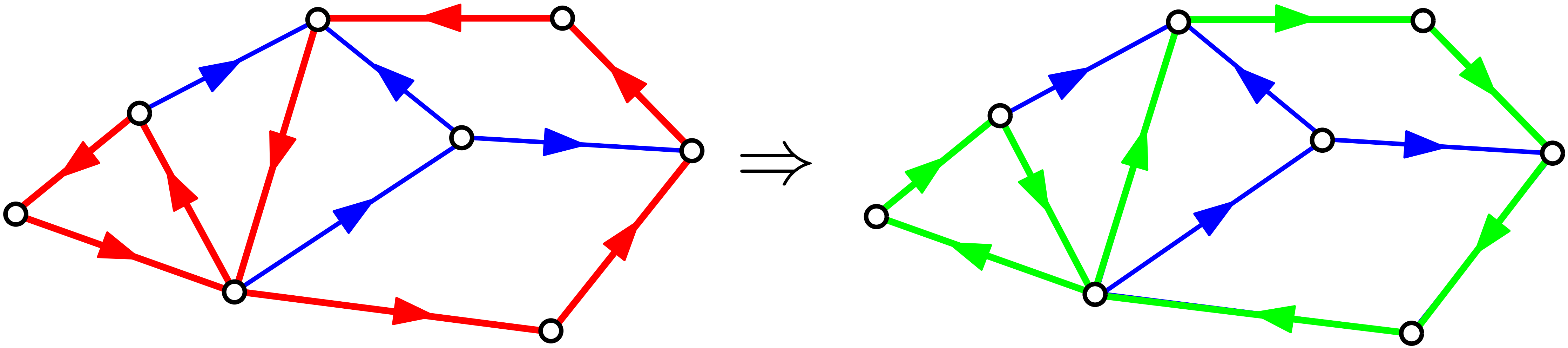

Uniqueness proof: if $O_1 \neq O_2$, edges where O_1 and O_2 **disagree** form an **eulerian suborientation** of $O_1 \Rightarrow$ contains a circuit (ccw in O_1 or O_2)

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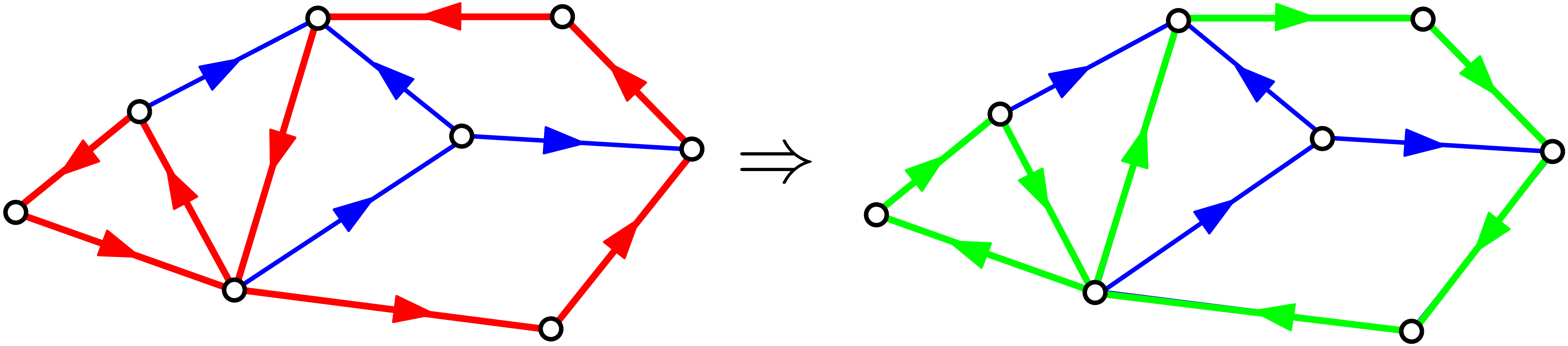

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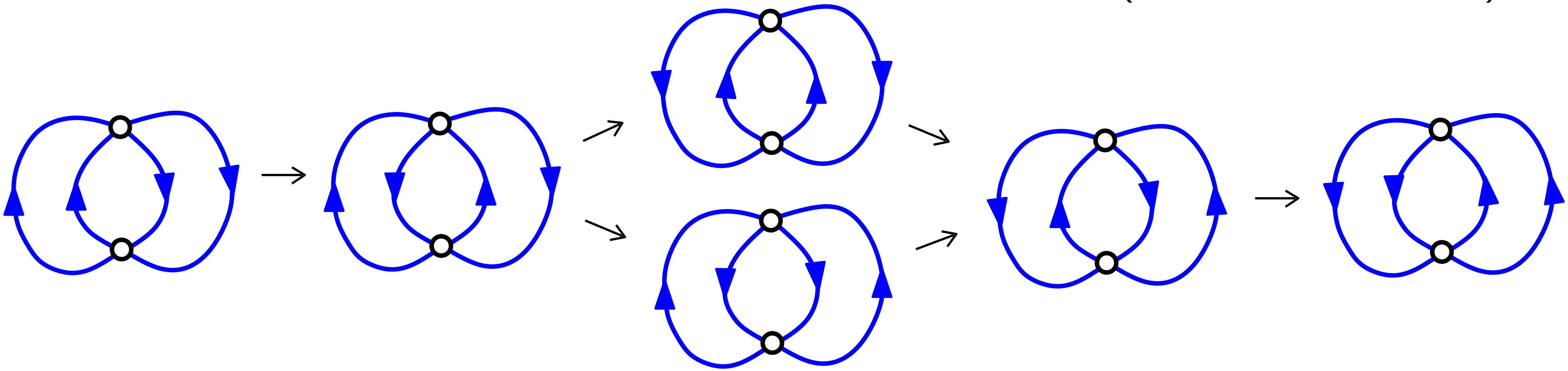

Set of α-orientations = **distributive lattice**

[Khueller et al'93], [Propp'93], [O. de Mendez'94], [Felsner'03]

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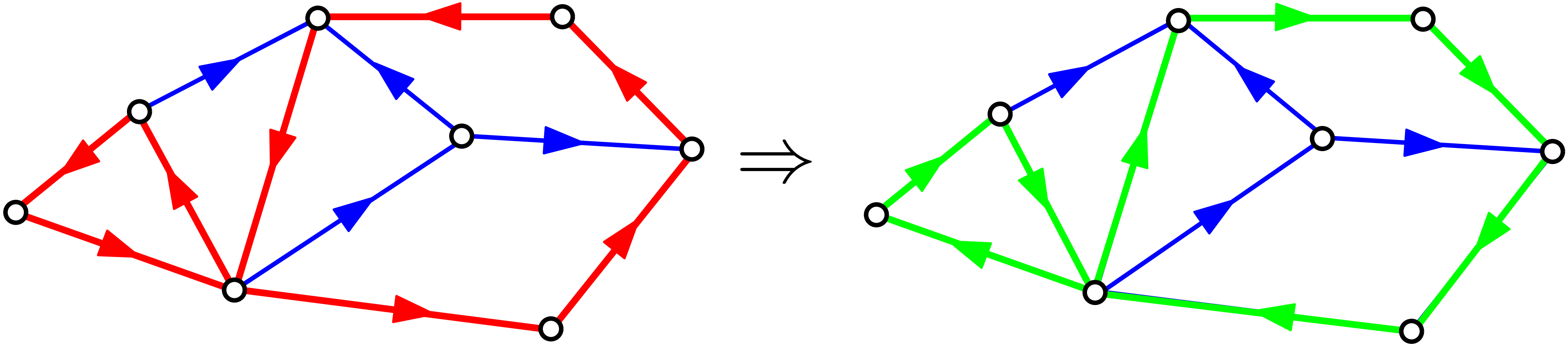

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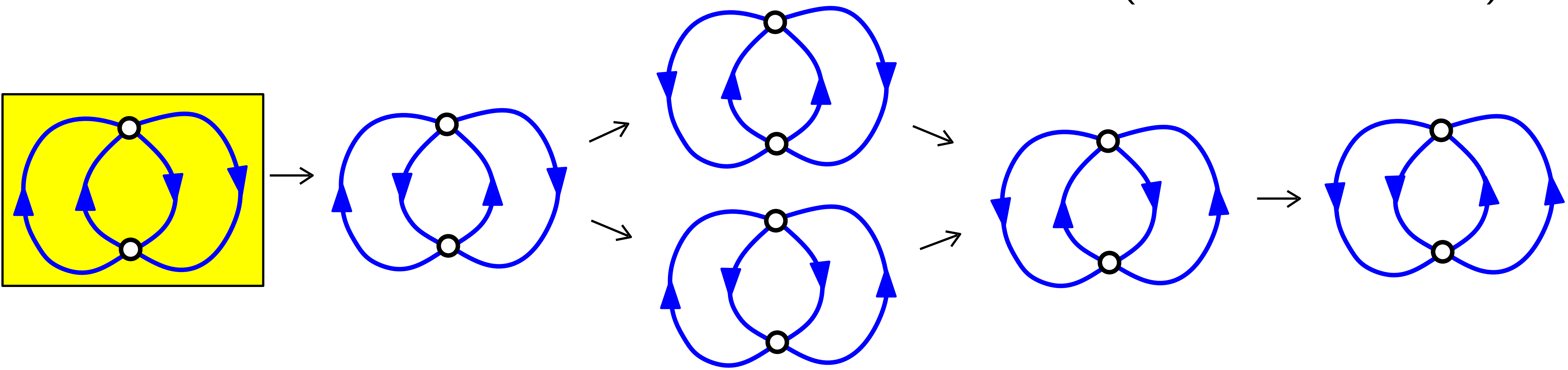

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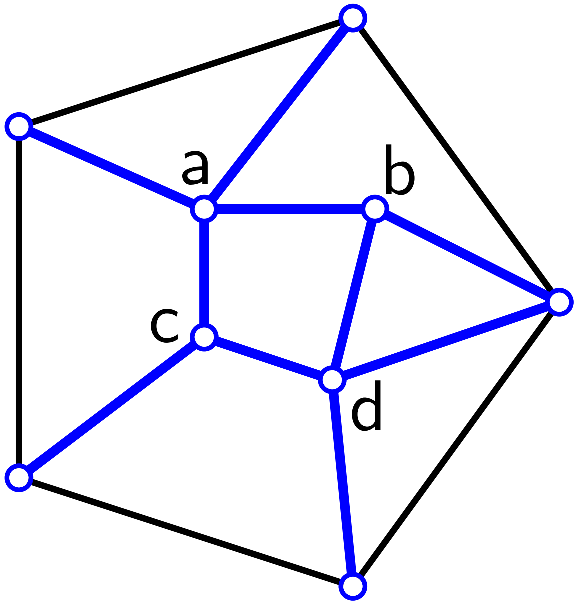
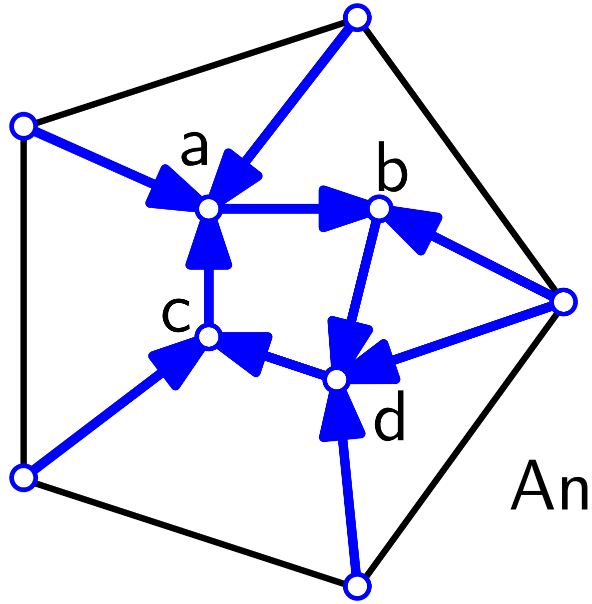
α-orientations for plane maps in our setting

- **External polygon** (the source) of the plane map is **unoriented**
- **Indegrees** are only on the **internal vertices**



$$\alpha: \quad a \to 3, \quad b \to 2, \quad c \to 2, \quad d \to 3$$



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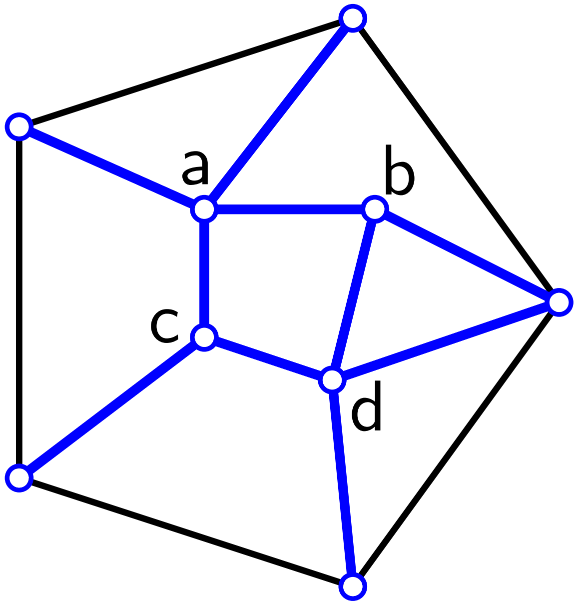

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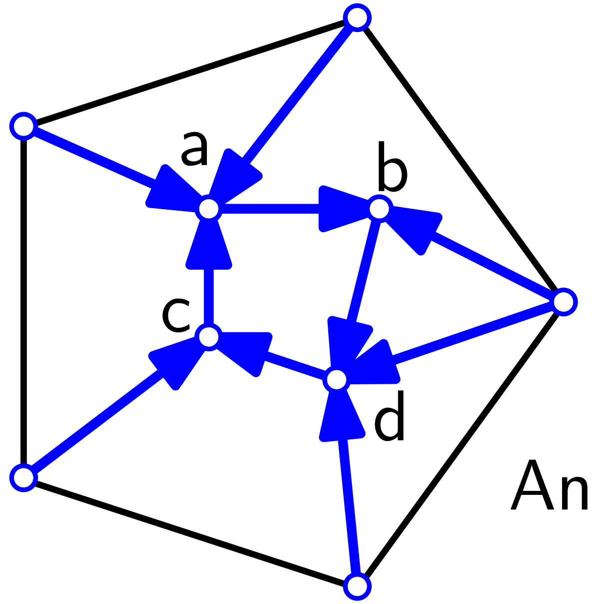
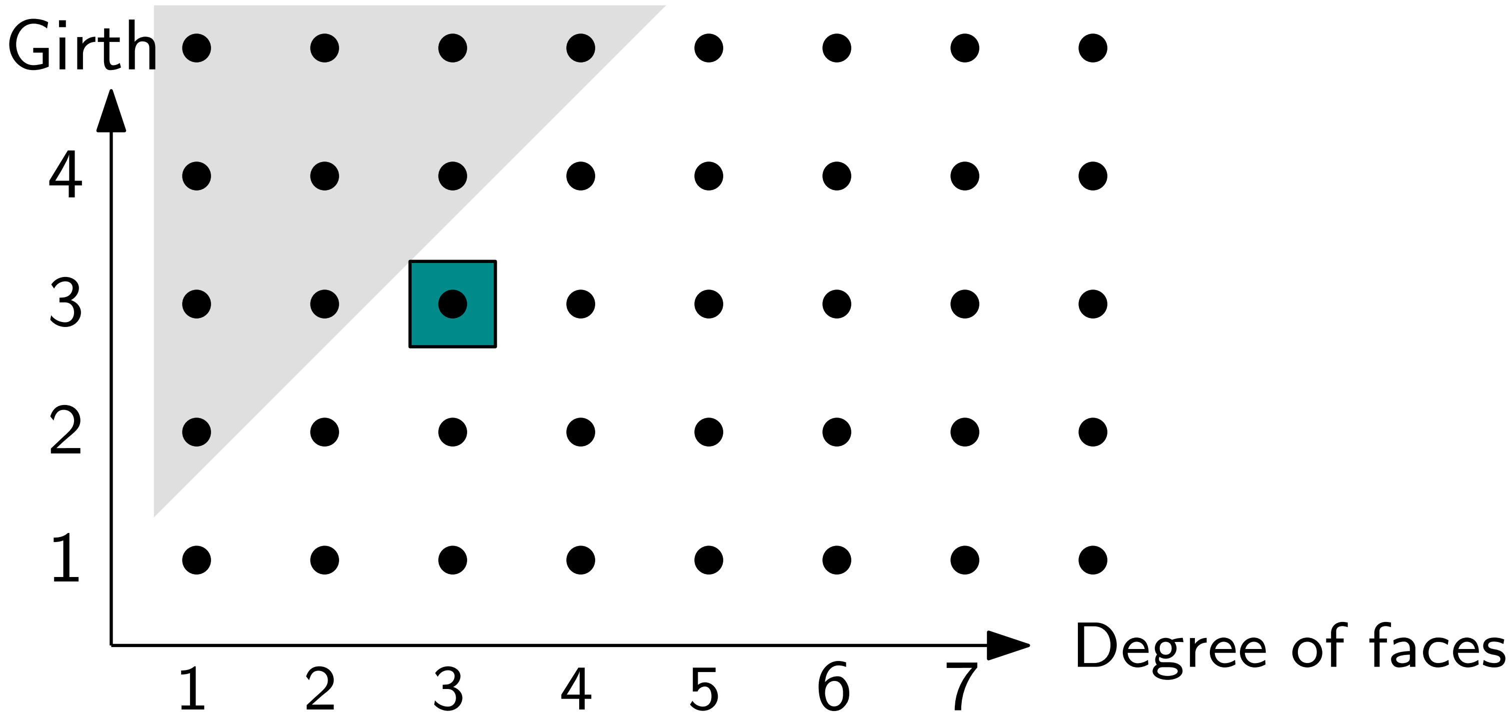


An α-orientation

Partition V (vertex-set) as $V_i \cup V_e$ and E (edge-set) as $E_i \cup E_e$

- **Existence**: (i) $\sum_{v \in V_i} \alpha(v) = |E_i|$
  (ii) $\forall S \subseteq V, \quad \sum_{v \in S \cap V_i} \alpha(v) \geq |E_S \cap E_i|$
- **Accessibility from outer face**: (iii) $\forall S \subseteq V_i, \; S \neq \emptyset \quad \sum_{v \in S \cap V_i} \alpha(v) > |E_S \cap E_i|$
- **Distributive lattice** structure

Example: simple triangulations

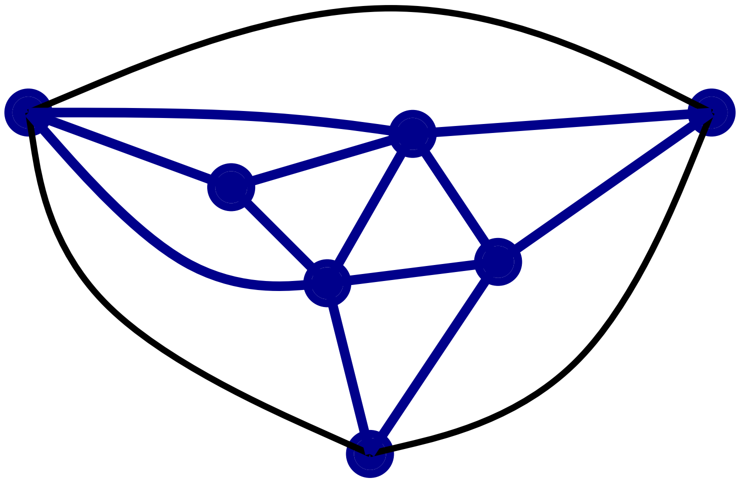


Triangulations

Fact: A triangulation with n internal vertices has $3n$ internal edges.

Proof: The numbers v, e, f of vertices edges and faces satisfy:

- Incidence relation: $3f = 2e$.
- Euler relation: $v - e + f = 2$. □



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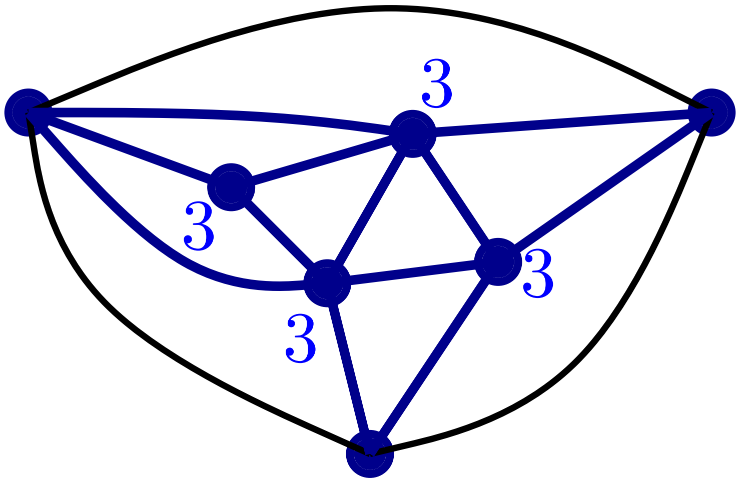
Fact: A triangulation with n internal vertices has $3n$ internal edges.

Natural candidate for indegree function:

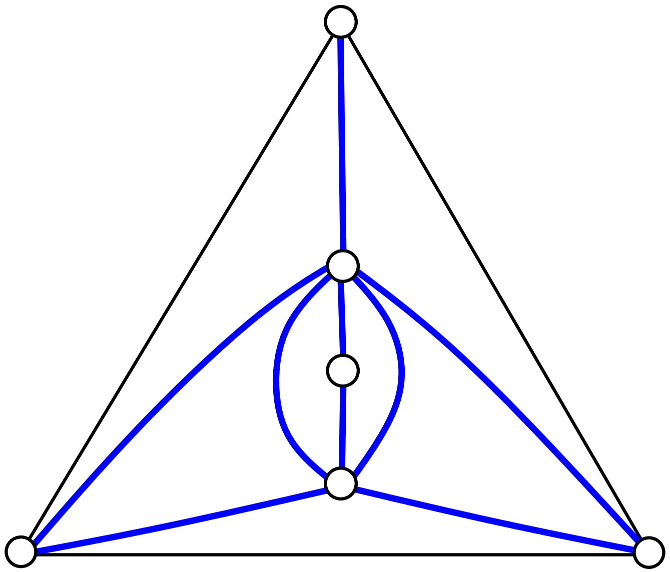
$$\alpha: \ v \mapsto 3 \text{ for each internal vertex } v.$$

call **3-orientation** such an α-orientation

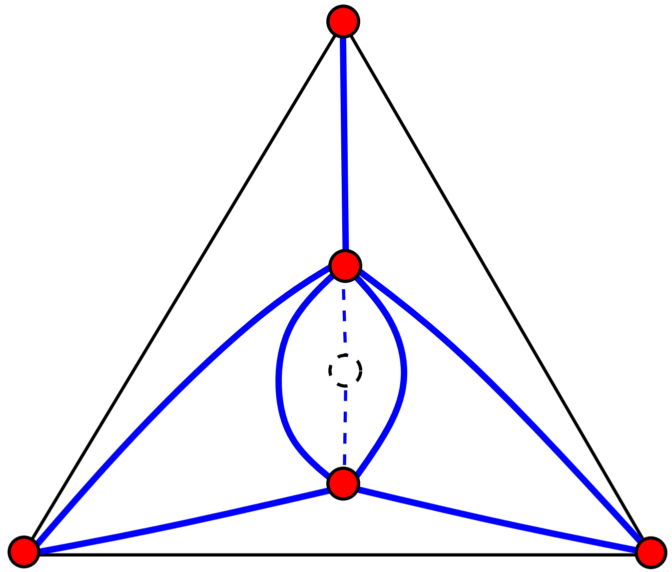


Triangulations

Fact: A triangulation admitting a 3-orientation is simple



$\Rightarrow$



k internal vertices
$3k+1$ internal edges

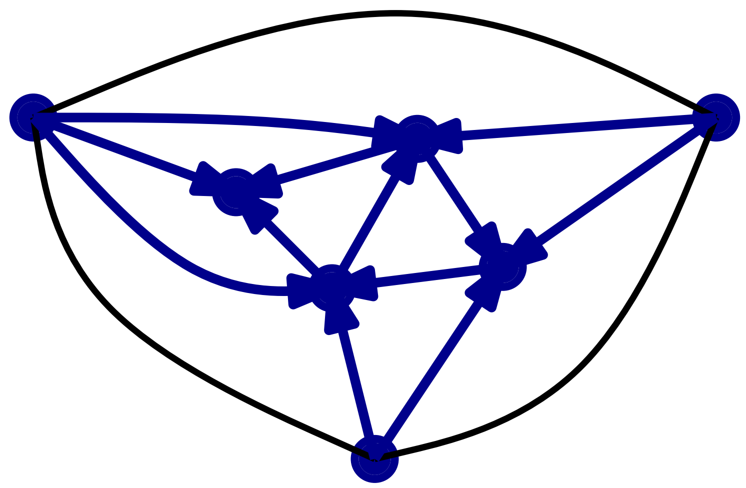
Triangulations

Thm [Schnyder 89]: A simple triangulation admits a 3-orientation.

New (easier) proof: Any simple planar graph $G = (V, E)$ satisfies

$$\frac{|E| - 3}{|V| - 3} \geq 3 \qquad \text{(Euler relation)}$$

hence the existence/accessibility conditions are satisfied.□

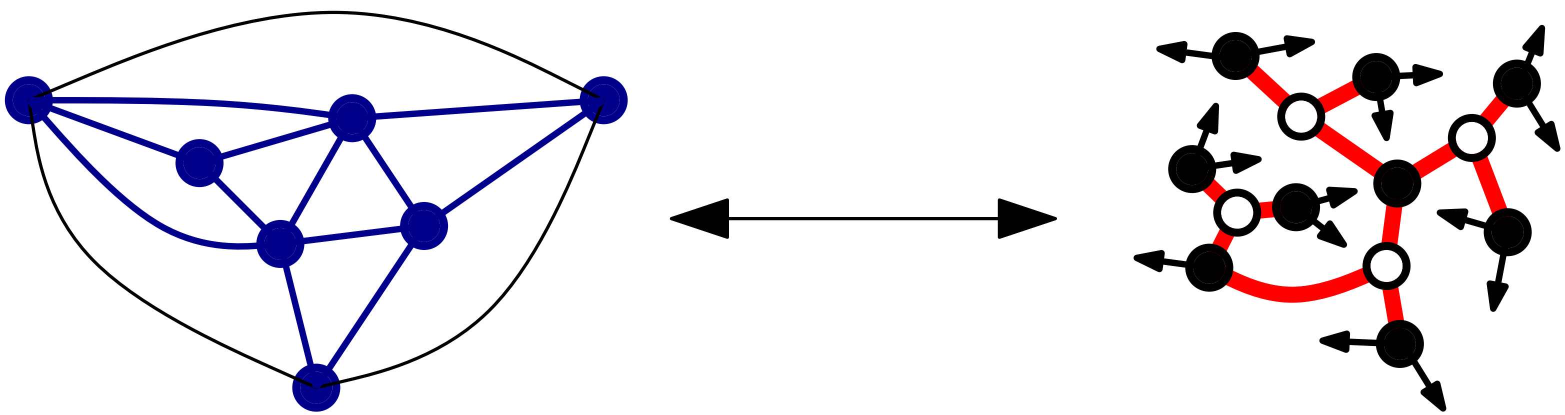


Triangulations

$\Rightarrow$ The class $\mathcal{T}$ of simple triangulations is identified with the class of plane orientation $\mathcal{O}_{\mathcal{T}} \subset \mathcal{O}$ with faces of degree 3, and internal vertices of indegree 3.

Thm [recovering FuPoSc08]: By specializing the master bijection Φ to $\mathcal{O}_{\mathcal{T}}$ one obtains a bijection between simple triangulations and mobiles such that

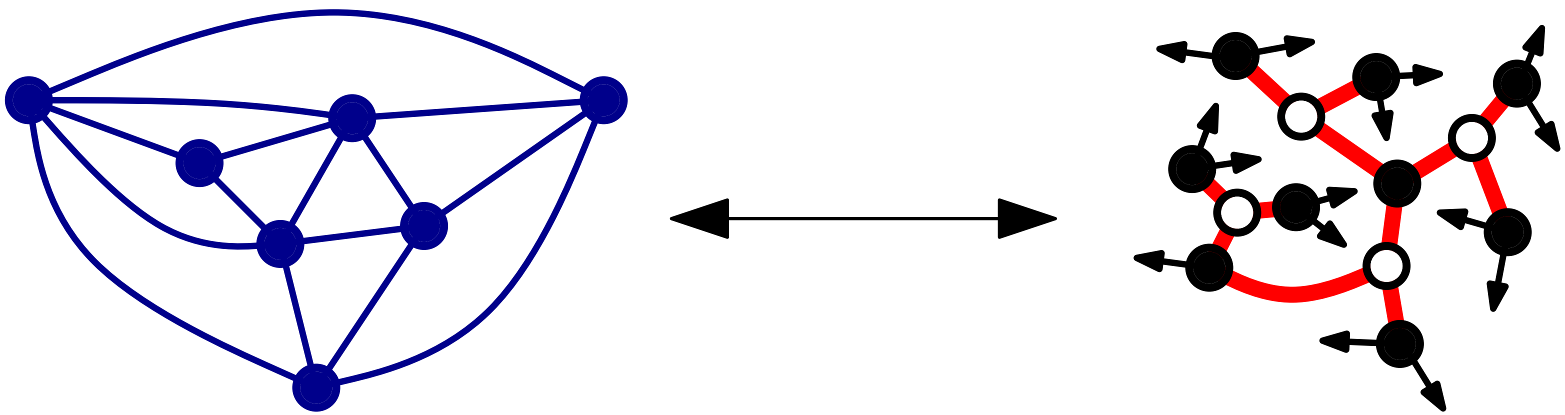
- black vertices have degree 3
- white vertices have degree 3
- the excess is $+3$ (redundant).



Triangulations

Counting: The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

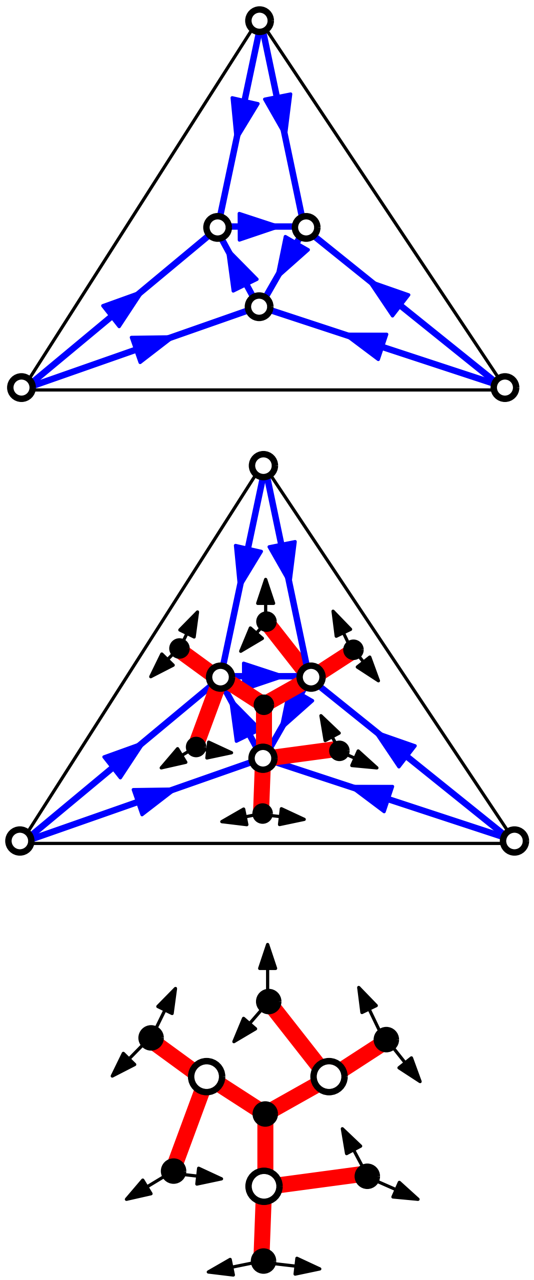
Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\dfrac{1}{n(2n-1)}\dbinom{4n-2}{n-1}$.



Triangulations: two constructions

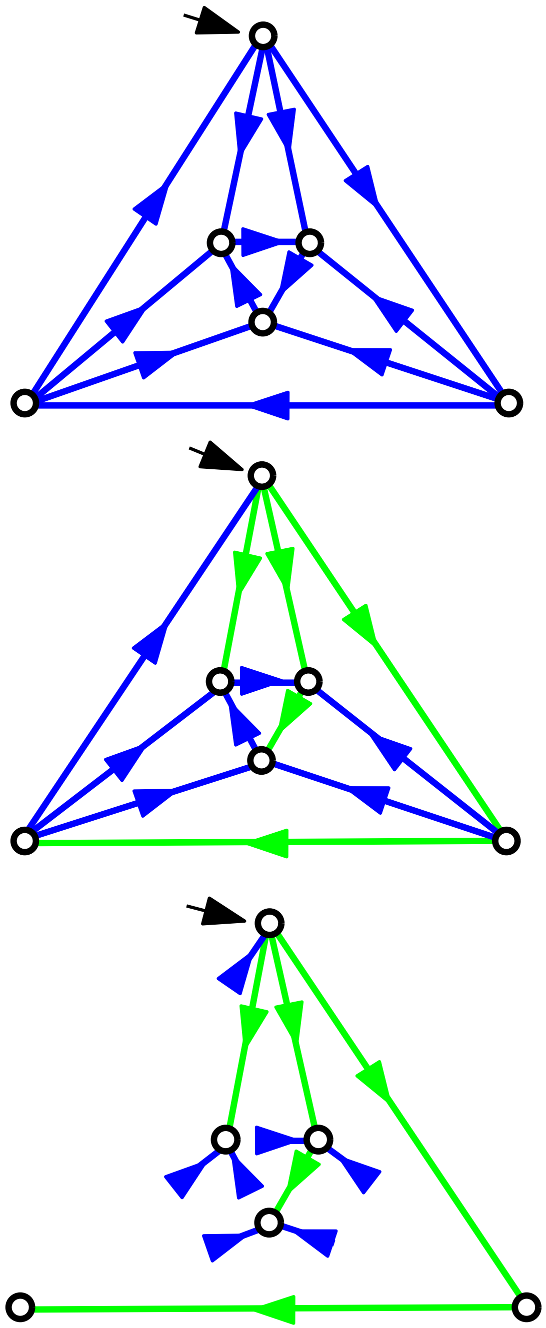
mobiles

[FuPoSc'08], [Bernardi-F'10]

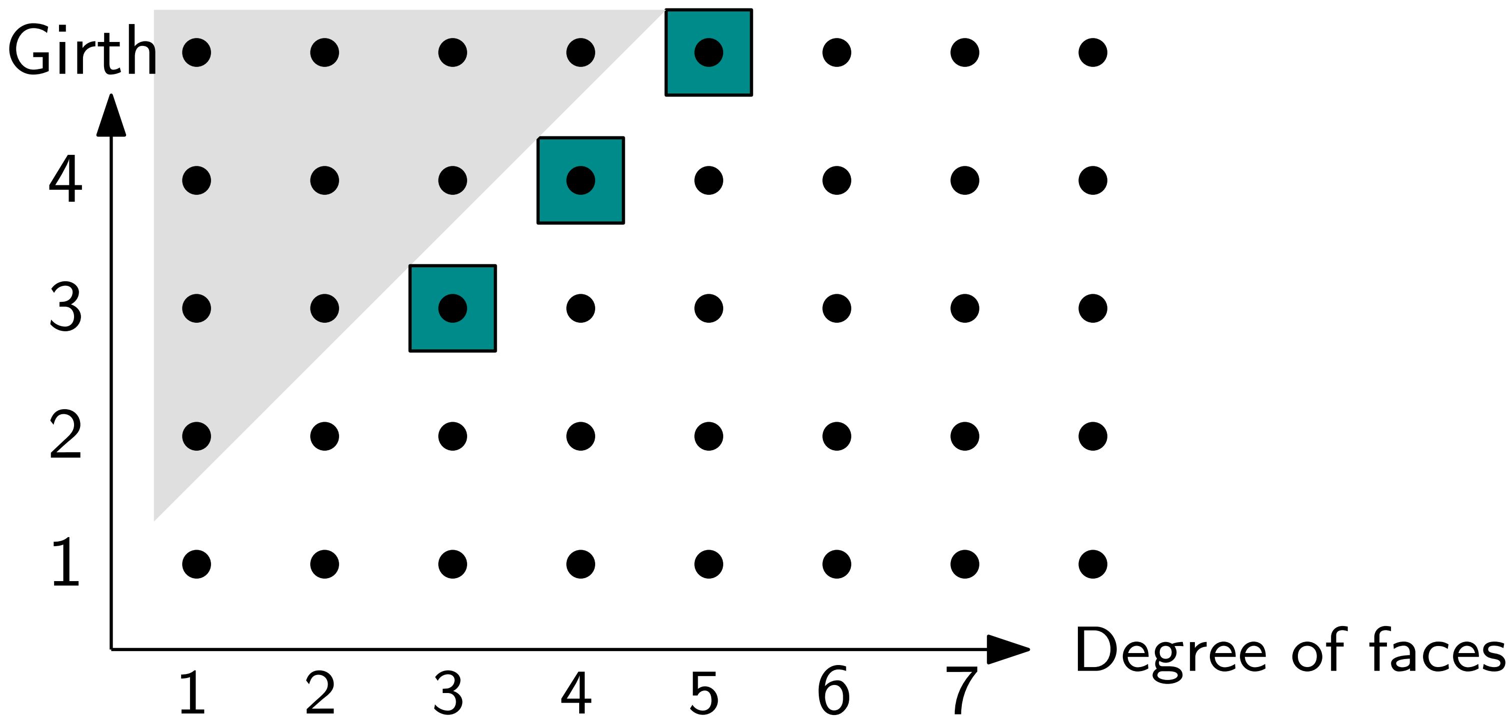


blossoming trees

[PoSc'03], [AlPo'11]

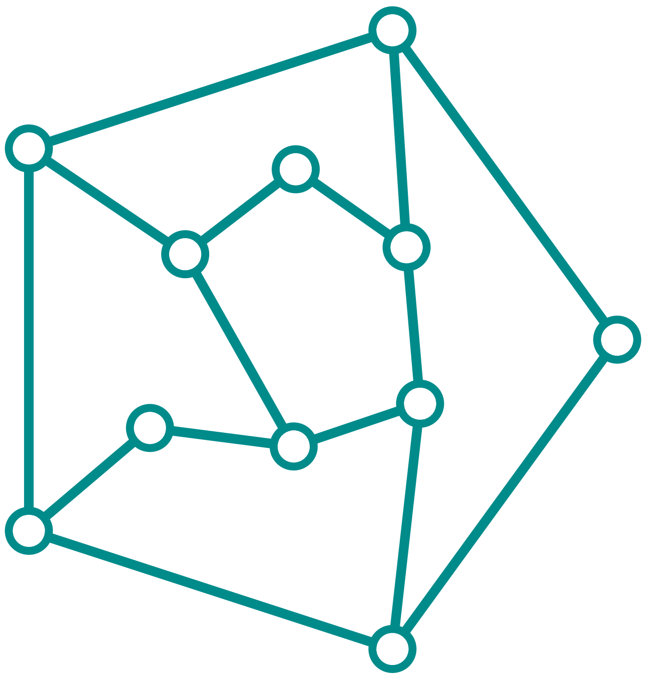


More specializations
d-angulations of girth d.



d-angulations of girth d

Fact: A d-angulation with $(d-2)n$ internal vertices has dn internal edges.



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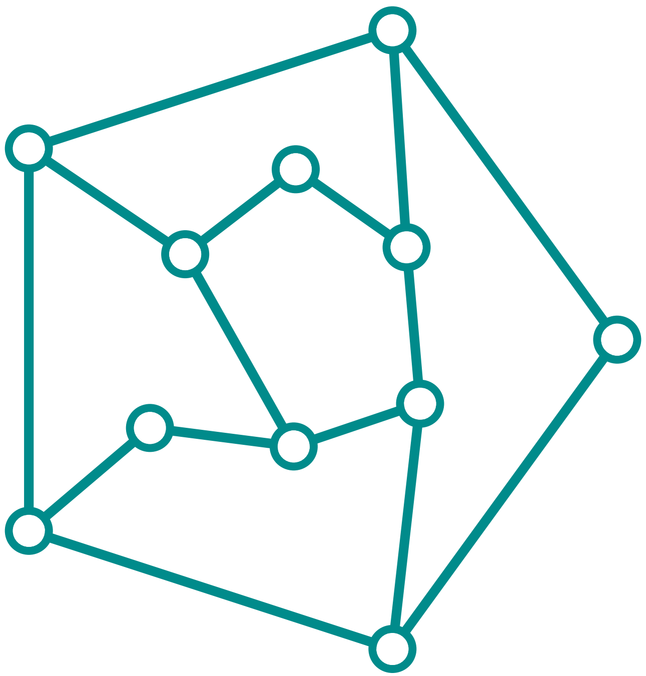
Fact: A d-angulation with $(d-2)n$ internal vertices has dn internal edges.

Natural candidate for indegree function:

$$\alpha : \ v \mapsto \frac{d}{d-2} \text{ for each internal vertex } v \ldots$$



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Fact: A d-angulation with $(d-2)n$ internal vertices has dn internal edges.

Idea: We can look for an orientation of $(d-2)G$ with indegree function

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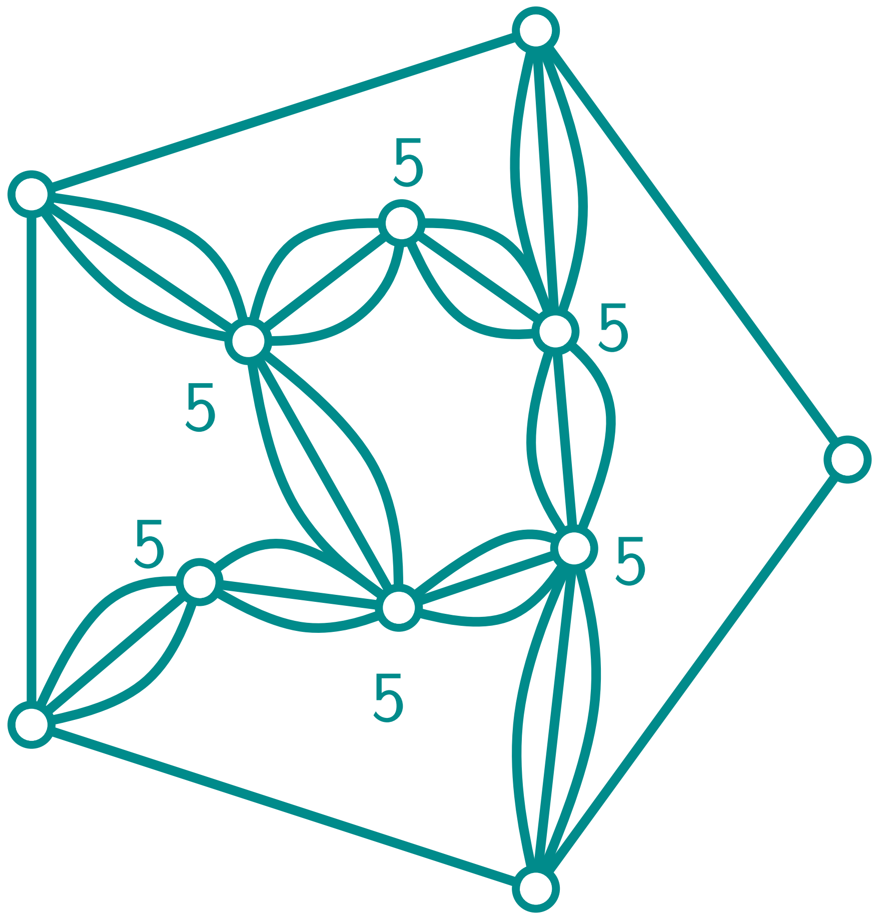

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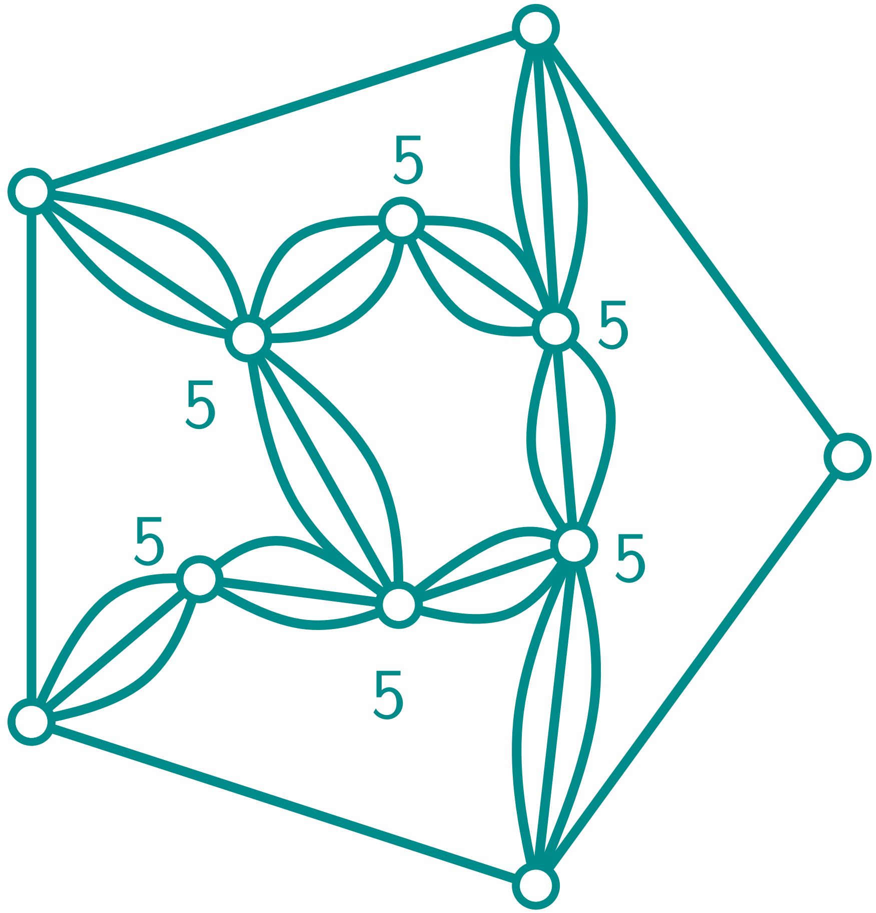
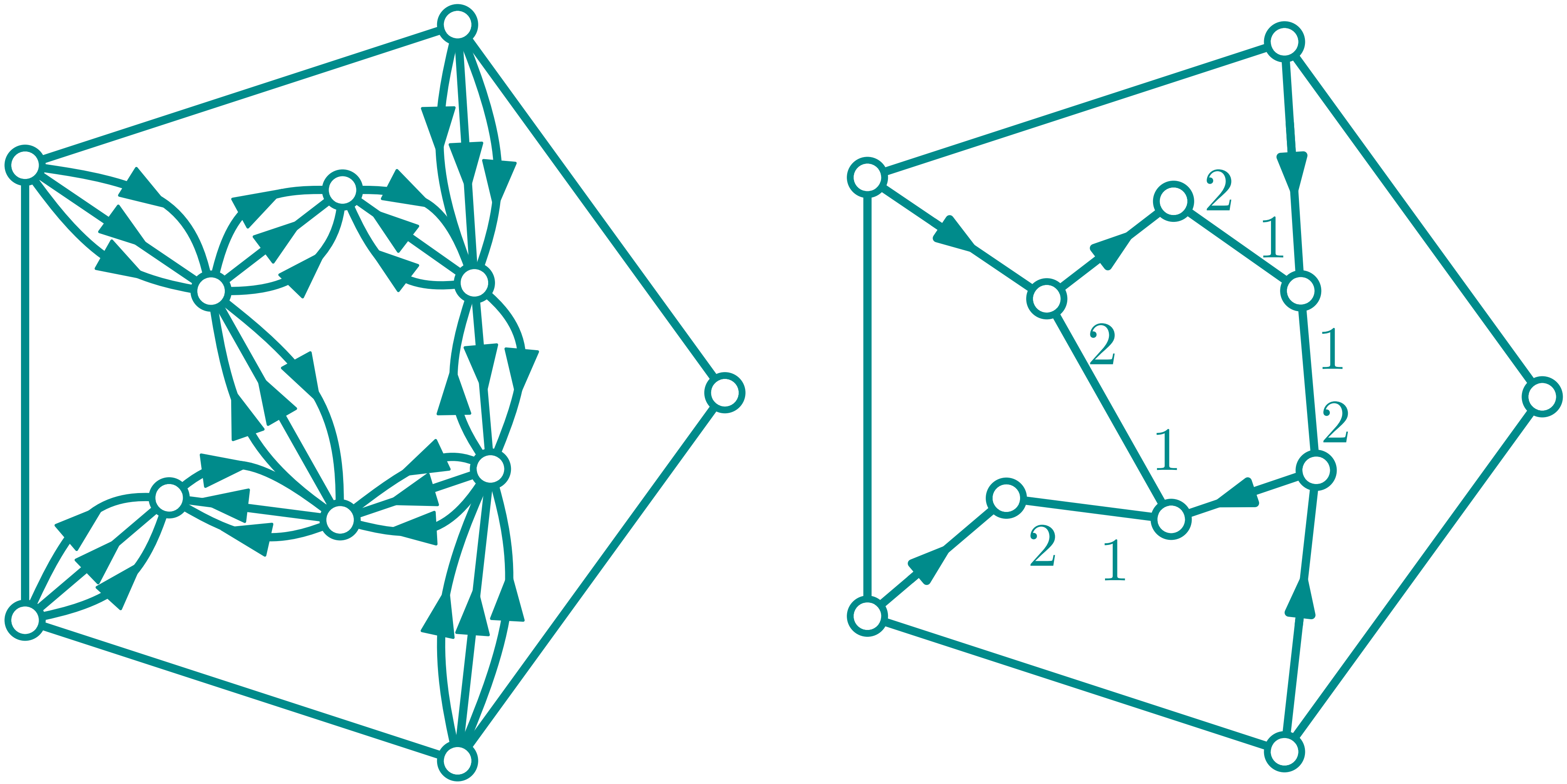
call $d/(d-2)$-orientation such an orientation



d-angulations of girth d

Thm [Bernardi-F'10]: Let G be a d-angulation. Then $(d-2)G$ admits a $d/(d-2)$- orientation if and only if G has girth d.

$d = 5$

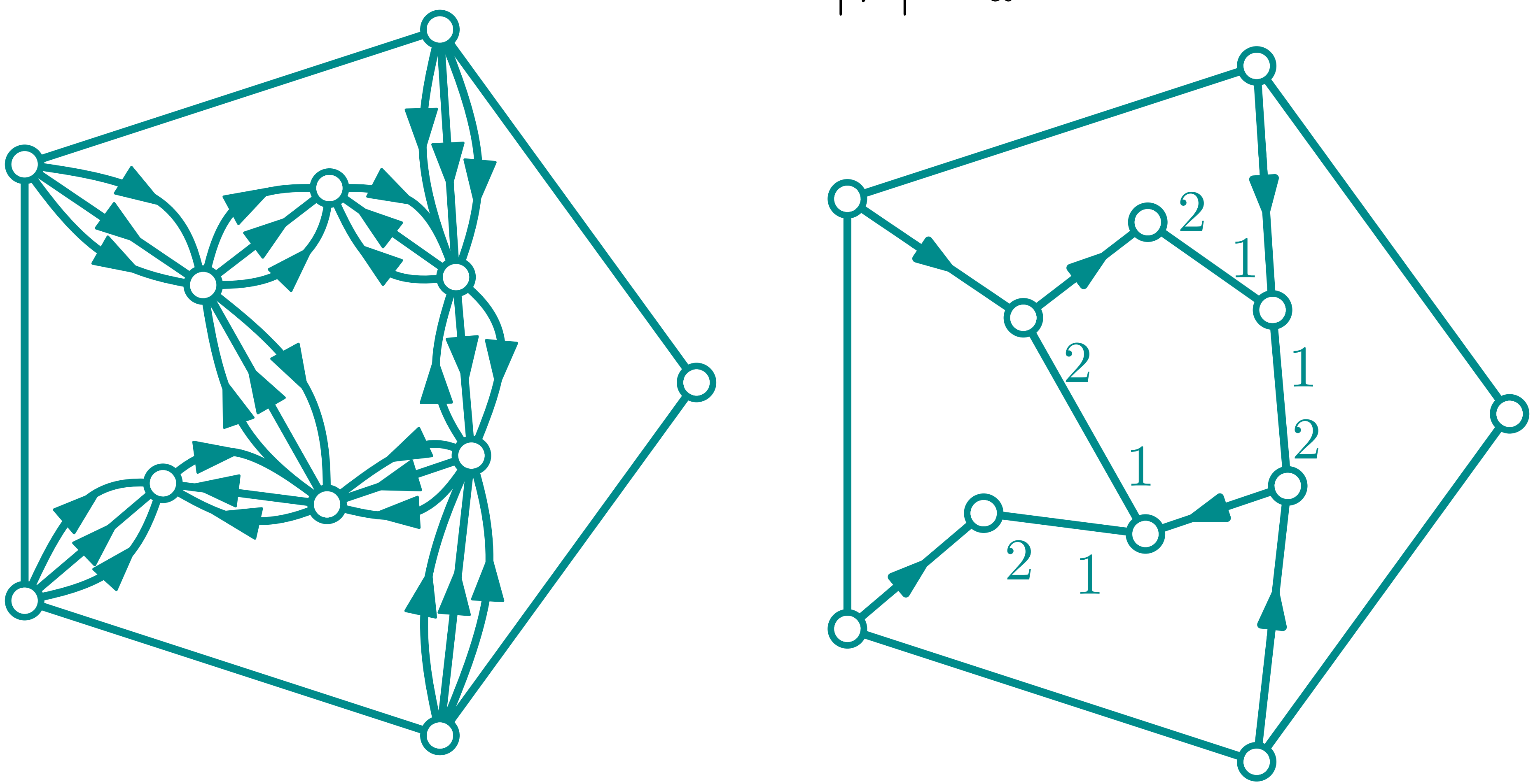


d-angulations of girth d

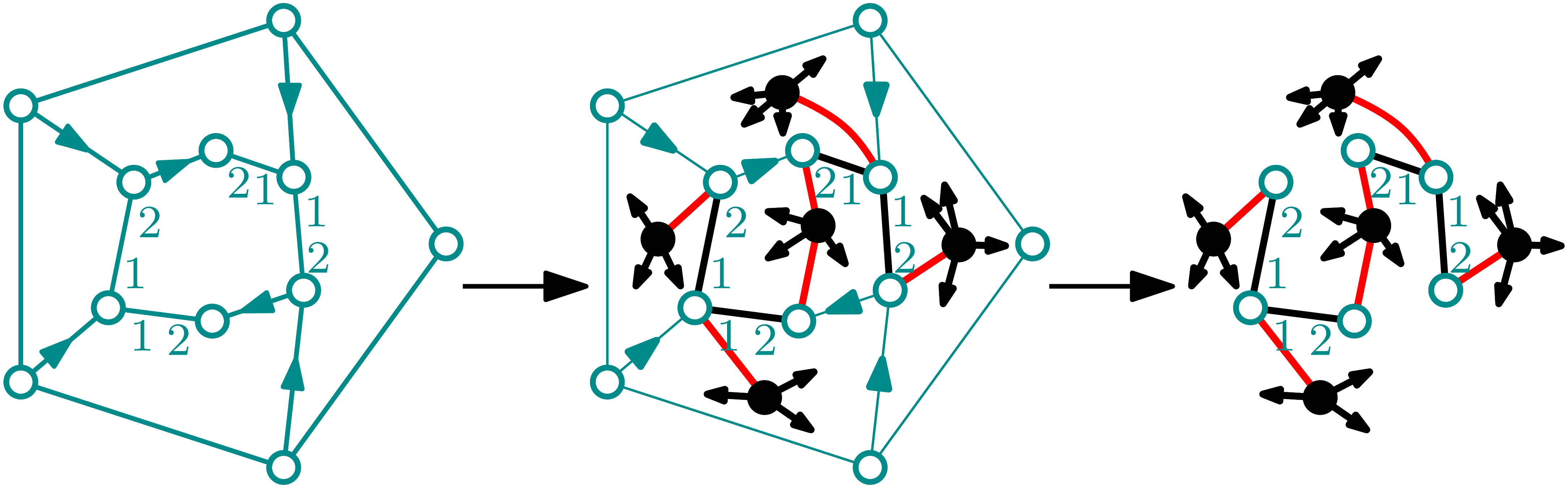
Thm [Bernardi-F'10]: Let G be a d-angulation. Then $(d-2)G$ admits a $d/(d-2)$- orientation if and only if G has girth d.

Proof: Similar to $d = 3$. Uses the fact that a planar graph $G = (V, E)$ of girth at least d satisfies $\dfrac{|E| - d}{|V| - d} \geq d$

$d = 5$

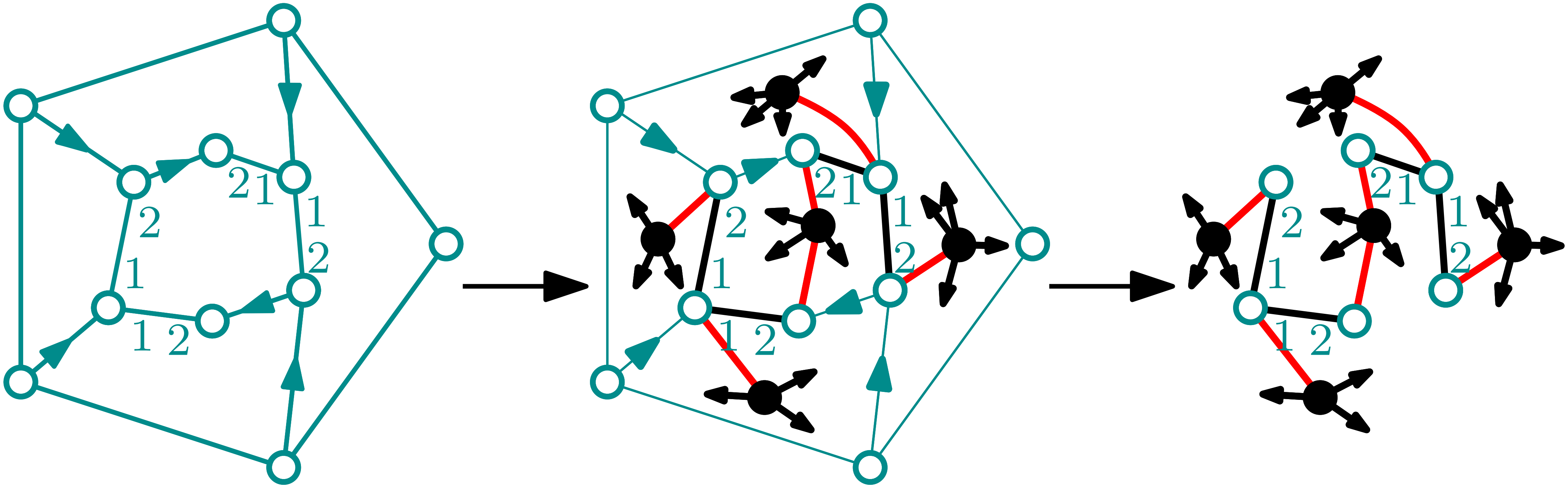


Master bijection for weighted orientations



There are now white-white edges in the mobile, with two positive weights summing to $d-2$.

Master bijection for weighted orientations



Theorem [Bernardi-F'10]: The master bijection can be expressed in the weighted setting:

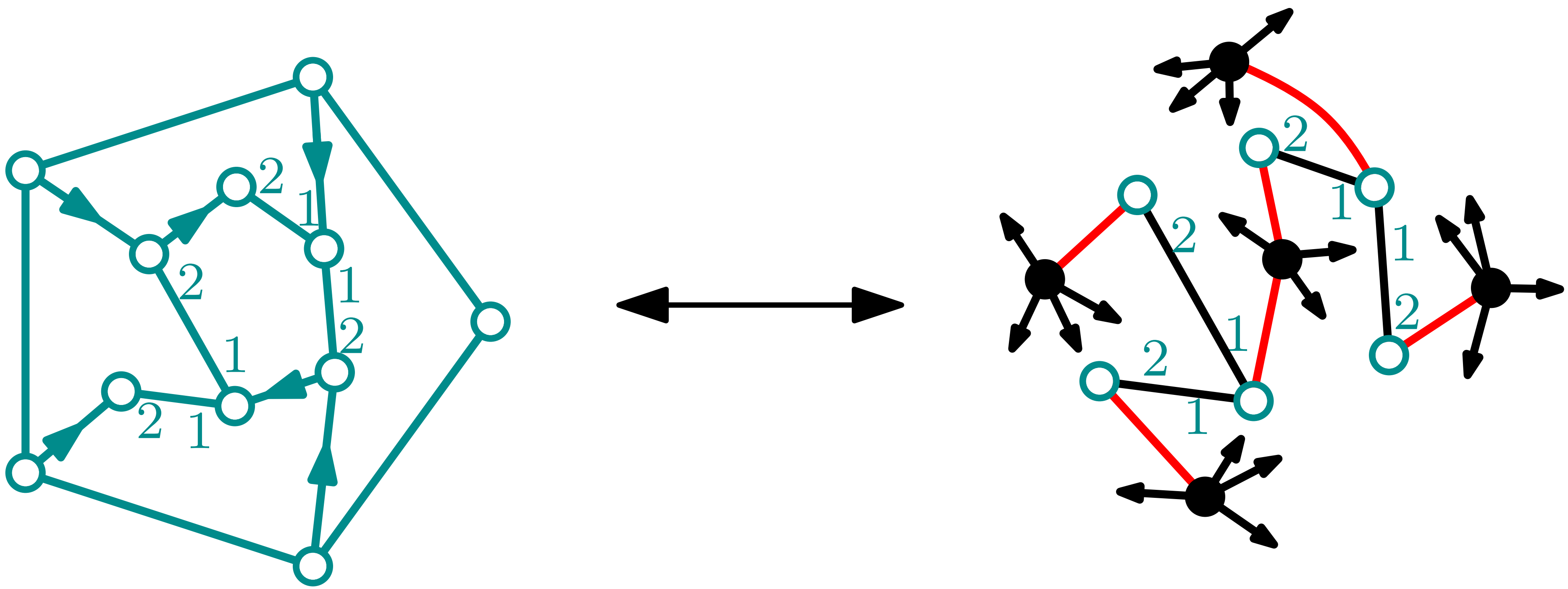
Moreover,

degree of internal faces	$\longleftrightarrow$	degree of black faces
indegree of internal vertices	$\longleftrightarrow$	indegree of white vertices
weights of internal edges	$\longleftrightarrow$	weights of edges
degree of external face	$\longleftrightarrow$	excess

d-angulations of girth d

Thm [Bernardi-F'10]: A d-angulation G admits a $d/(d-2)$-orientation if and only if G has girth d.

$\Rightarrow$ The class $\mathcal{T}_d$ of d-angulations of girth d can be identified with the class of weighted orientations in $\mathcal{O}$, with faces of degree d, edges of weight $d-2$, and internal vertices of indegree d.

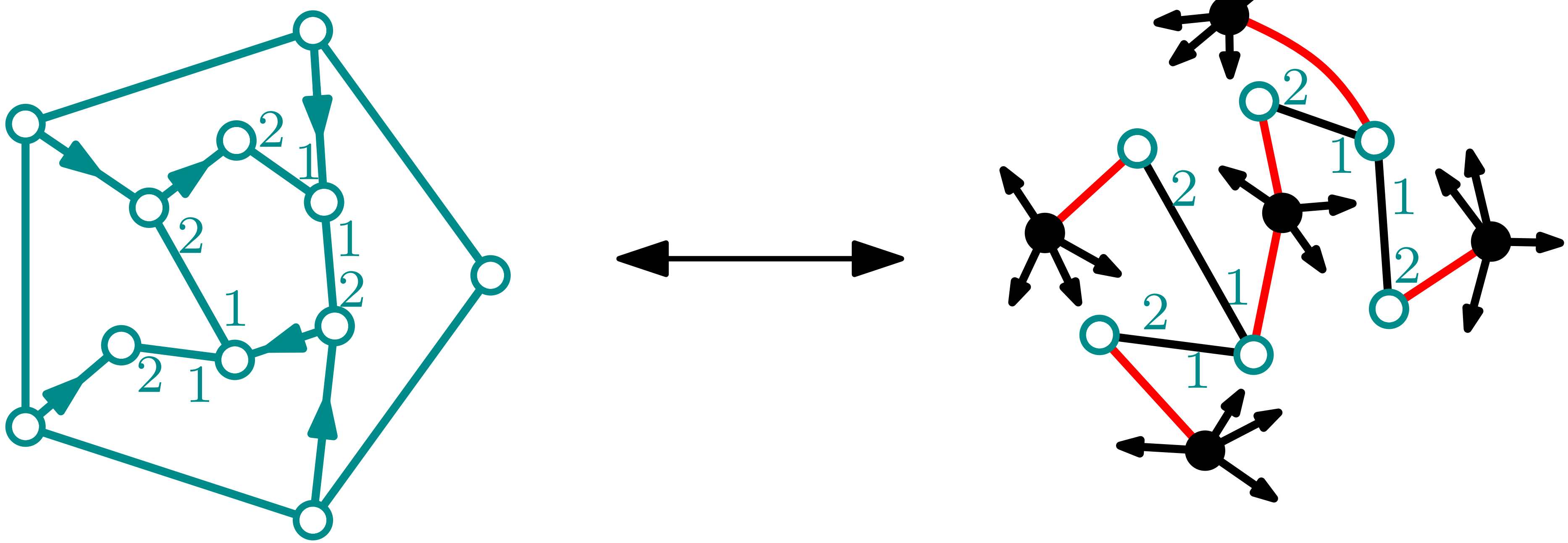


d-angulations of girth d

Thm [Bernardi-F'10]: A d-angulation G admits a $d/(d-2)$-orientation if and only if G has girth d.

Thm [Bernardi-F'10]: By specializing the master bijection one obtains a bijection between d-angulations of girth d and mobiles (with white-white edges having weights summing to $d-2$) such that

- black vertices have degree d
- white vertices have indegree d
- the excess is d (redundant).



d-angulations of girth d: counting

Thm[Bernardi-F'10]: Let $W_0, W_1, \ldots, W_{d-2}$ be the power series in x defined by: $W_{d-2} = x(1+W_0)^{d-1}$
and $\forall j < d-2, \quad W_j = \sum_r \sum_{\substack{i_1,\ldots,i_r>0 \\ i_1+\cdots+i_r=j+2}} W_{i_1} \cdots W_{i_r}$.

The generating function F_d of rooted d-angulations of girth d satisfies

$$\boxed{F_d'(x) = (1+W_0)^d.}$$

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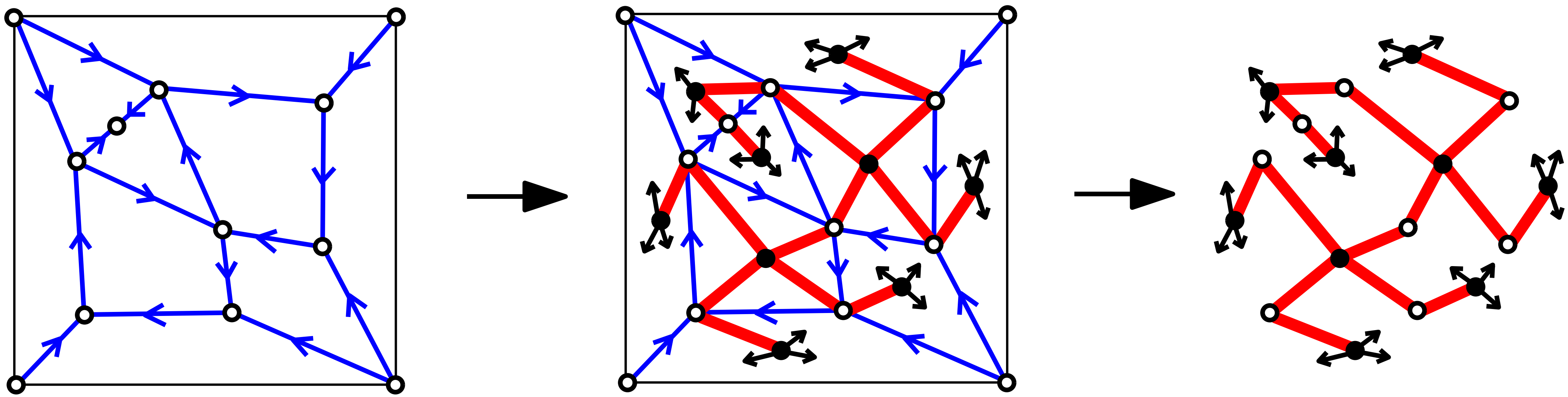
Example d=5:

$W_3 = x(1+W_0)^4$
$W_0 = W_1^2 + W_2$
$W_1 = W_1^3 + 2W_1W_2 + W_3$
$W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$

Simplification in the bipartite case

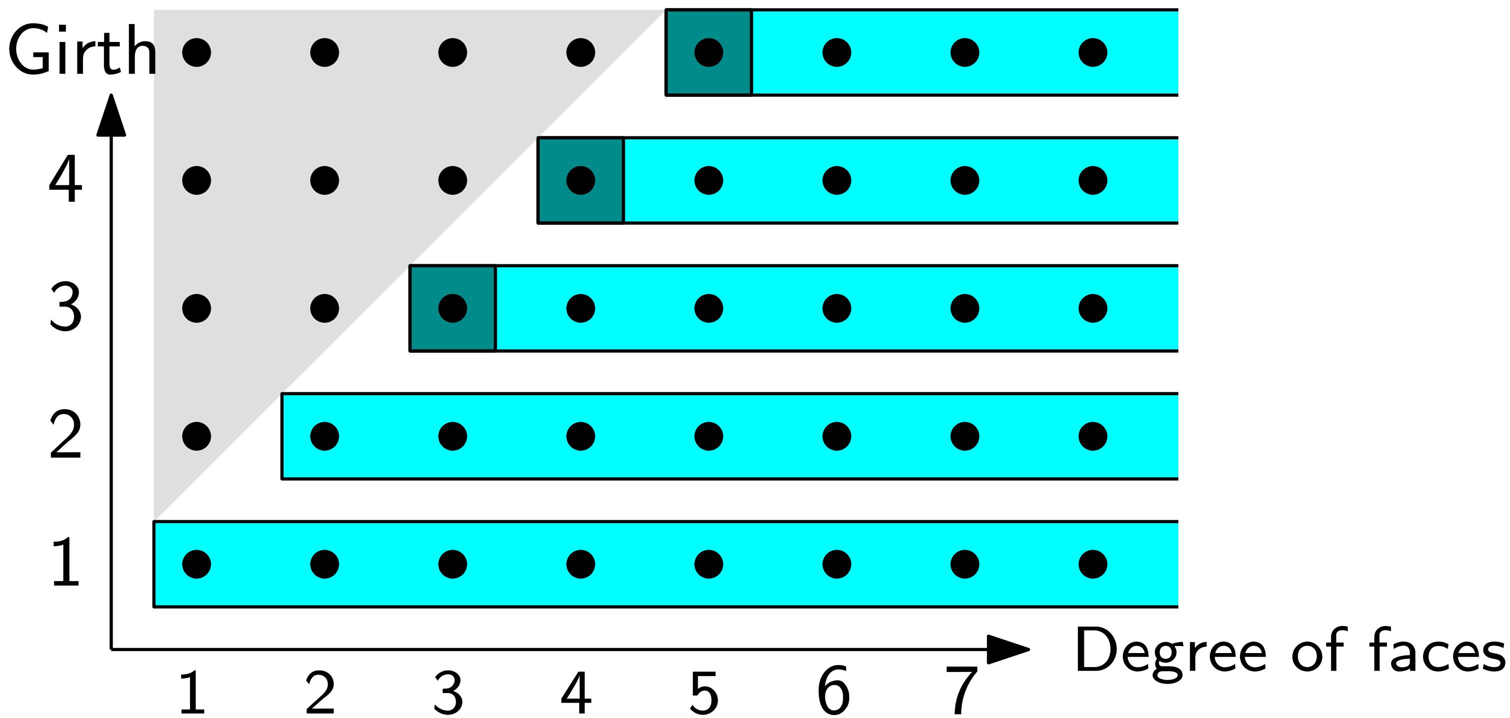
- For d even, $d = 2b$, we have $\dfrac{d}{d-2} = \dfrac{b}{b-1}$
- Can work with $b/(b-1)$-orientations:
  - edges have weight $b-1$
  - vertices have indegree b

Example: $b = 2$, simple quadrangulations



recover a bijection of Schaeffer (1999)

More specializations
Maps of girth d.



We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face

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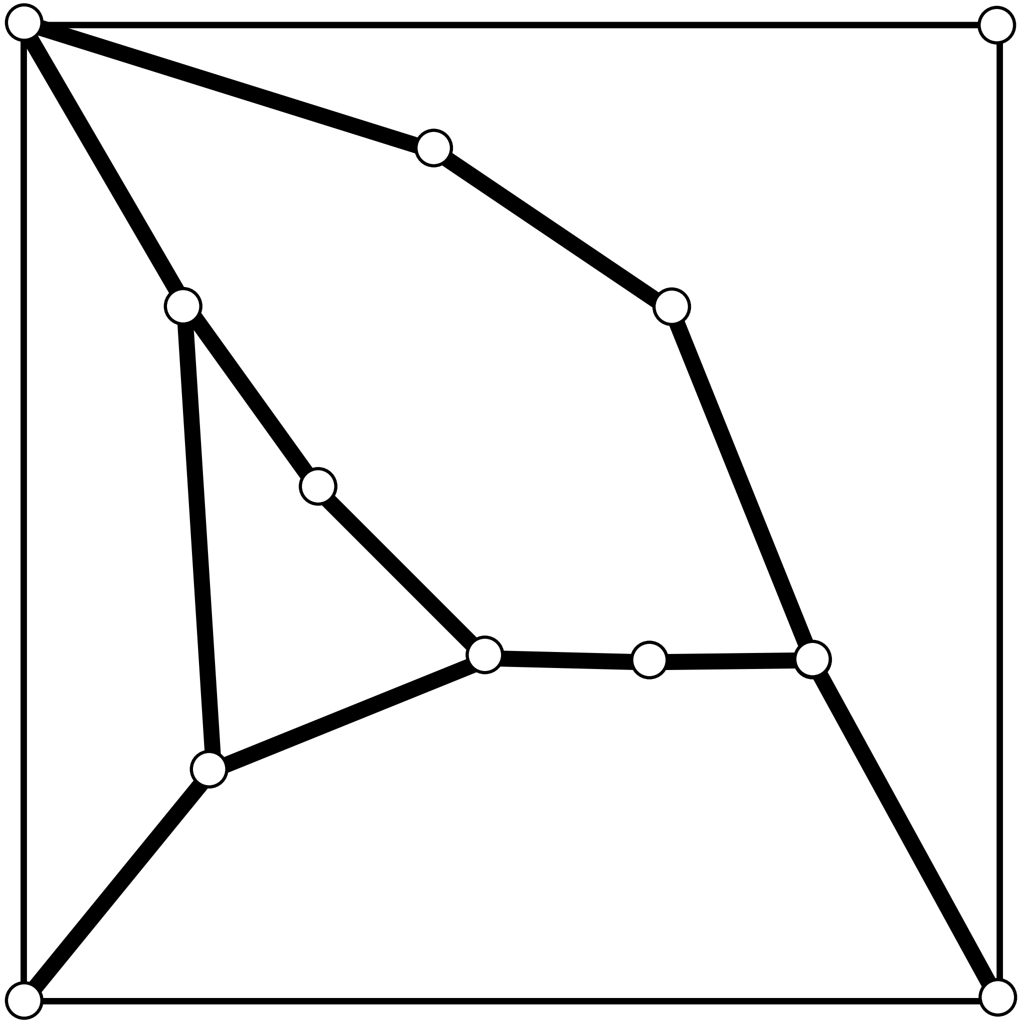

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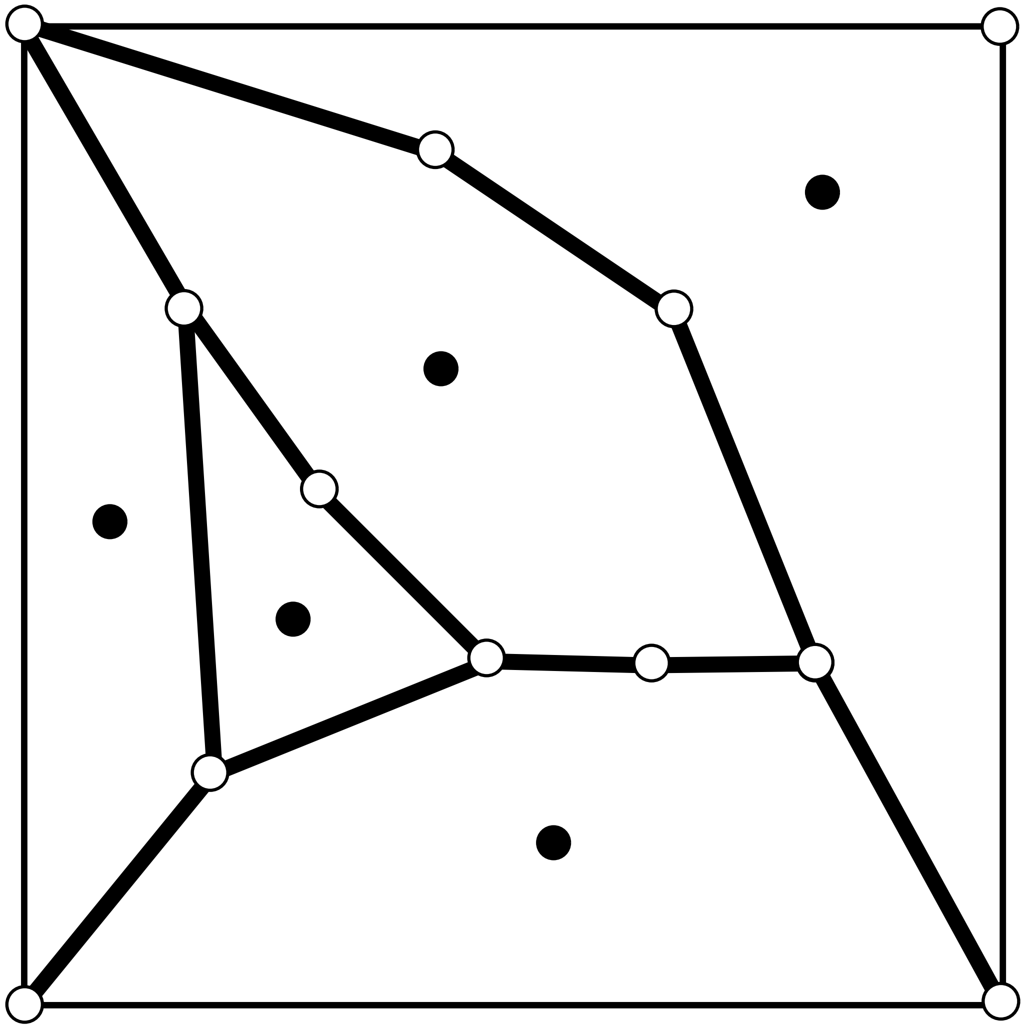

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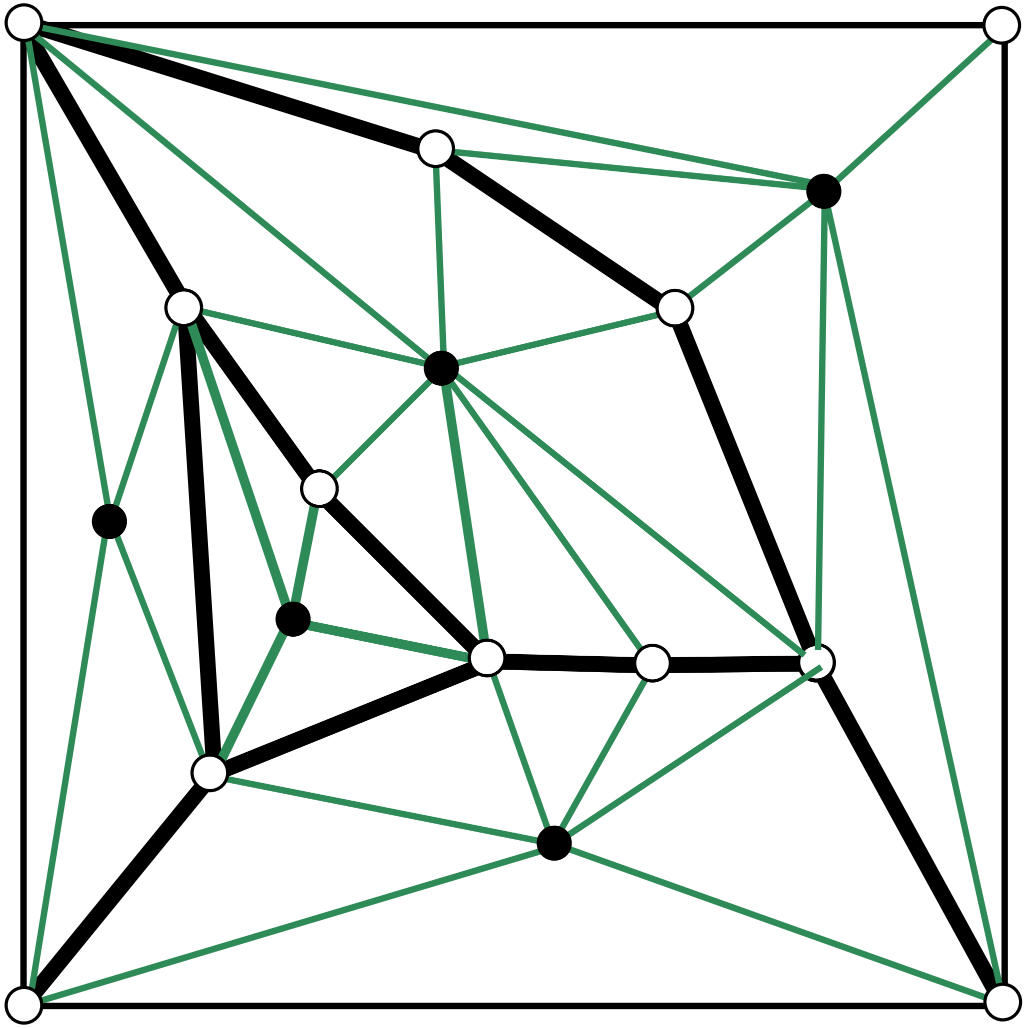
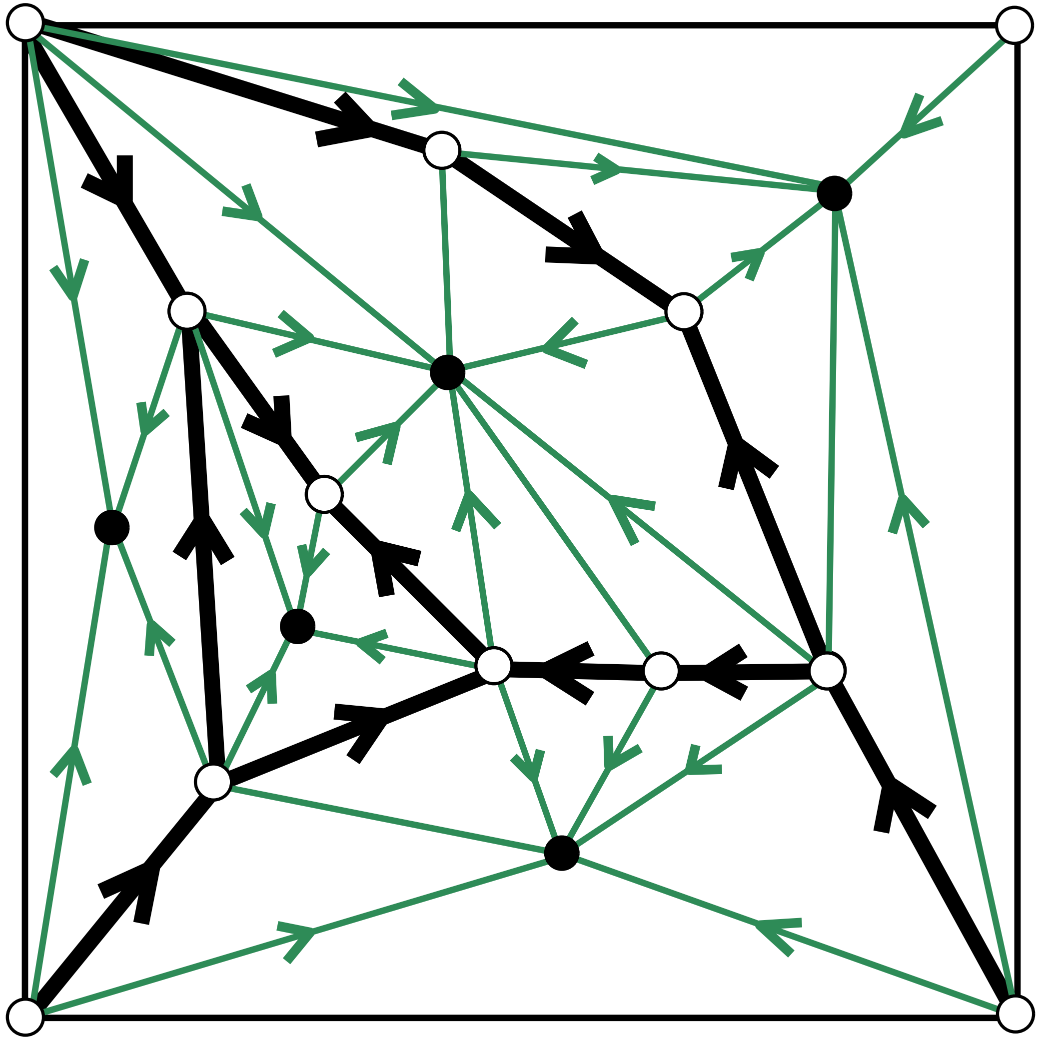


Insert a star
in each internal face

We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



Generalized 2-orientation

- Each internal white vertex has indegree 2
- Each black vertex of degree $2i$ has outdegree $i - 2$

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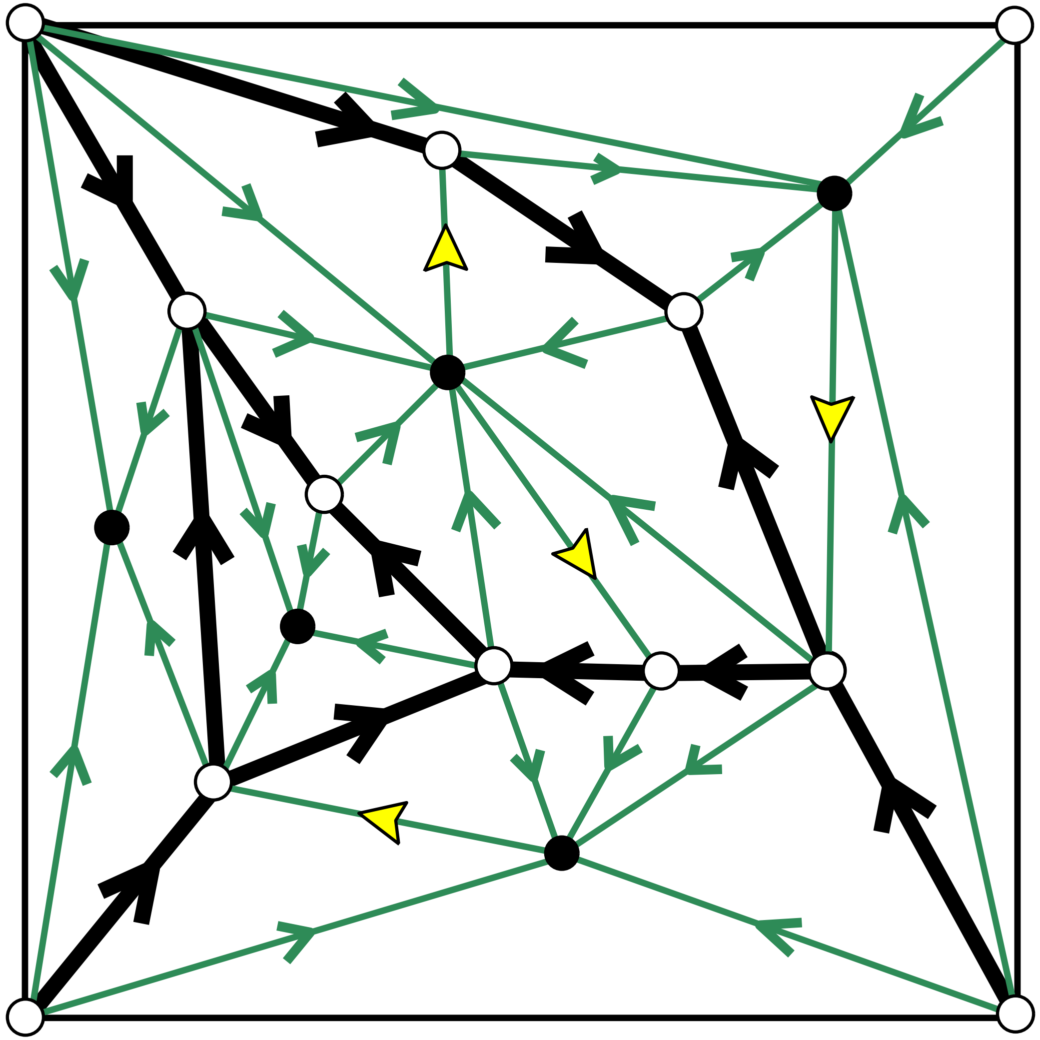
Case $b = 2$ (simple bipartite maps), with quadrangular outer face



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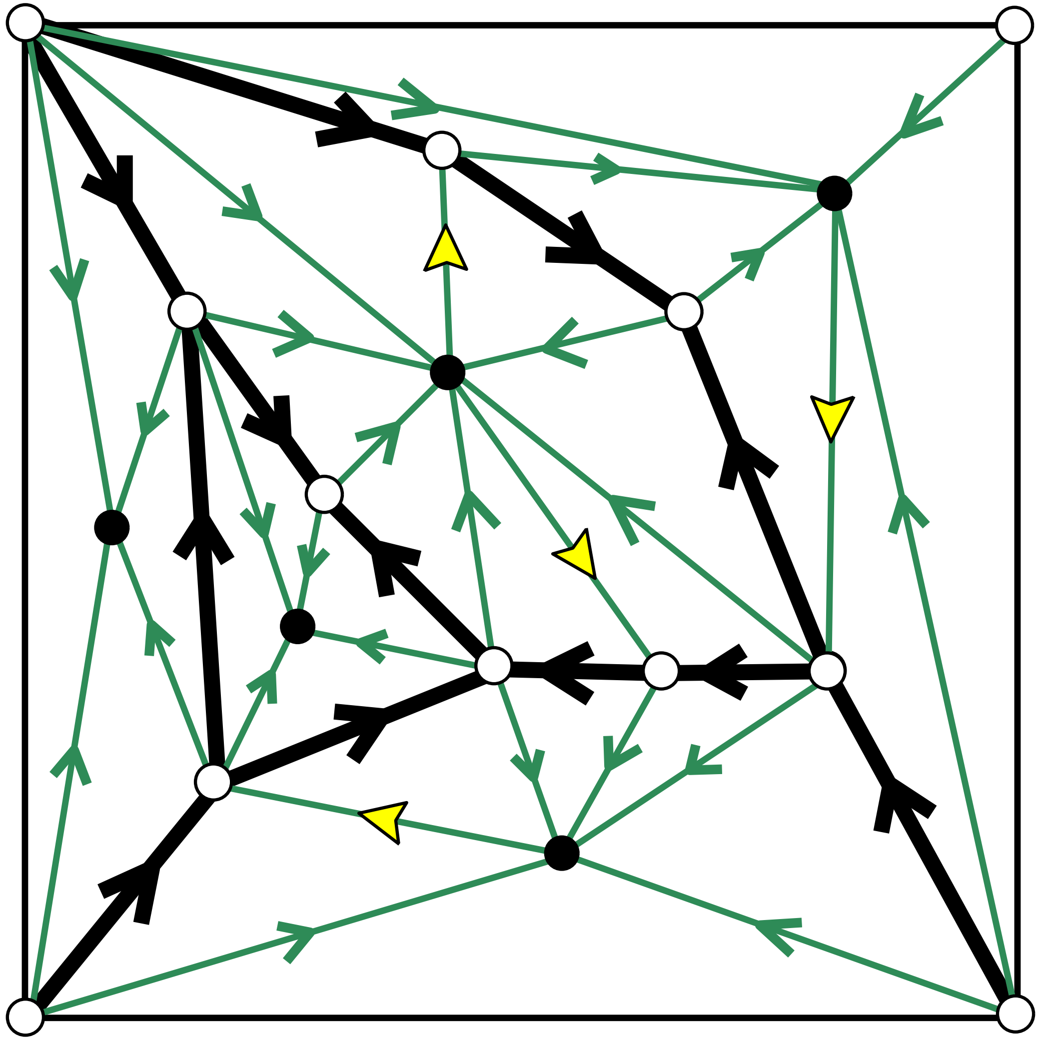

Generalized 2-orientation

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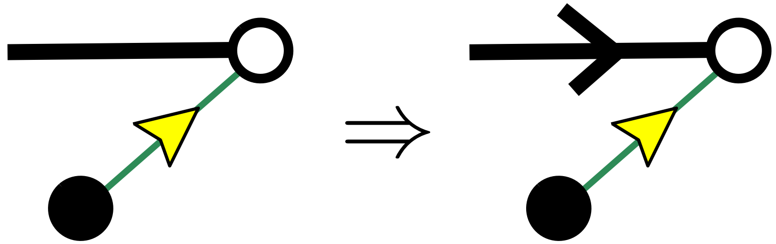
For the minimal one:



& still accessible after deleting the stars

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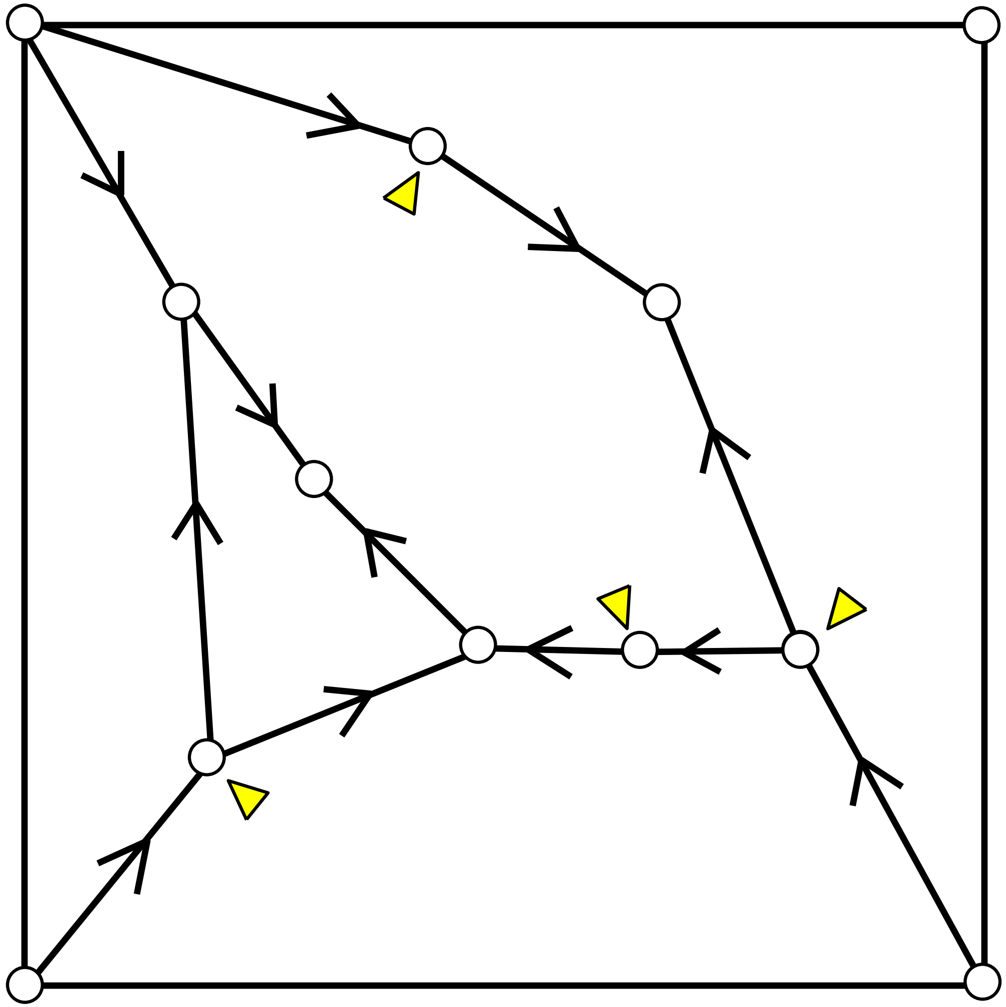
Case $b = 2$ (simple bipartite maps), with quadrangular outer face



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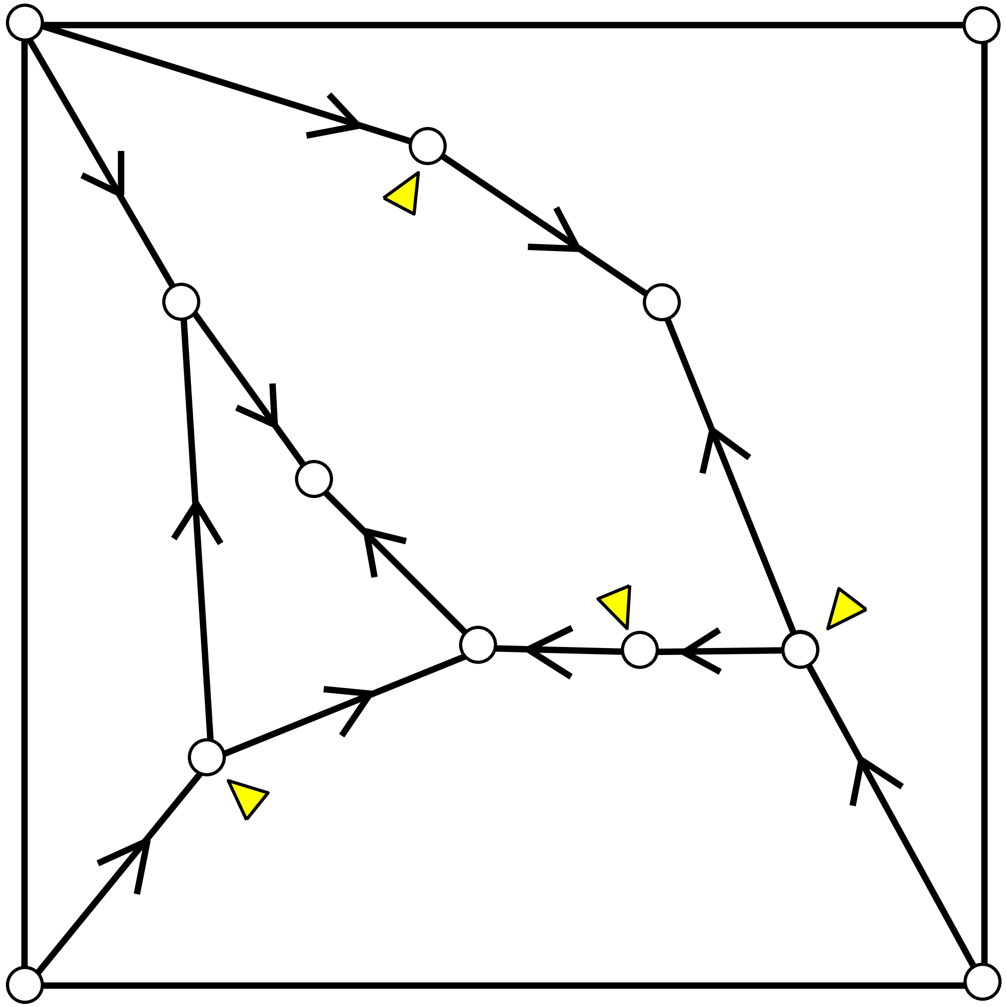

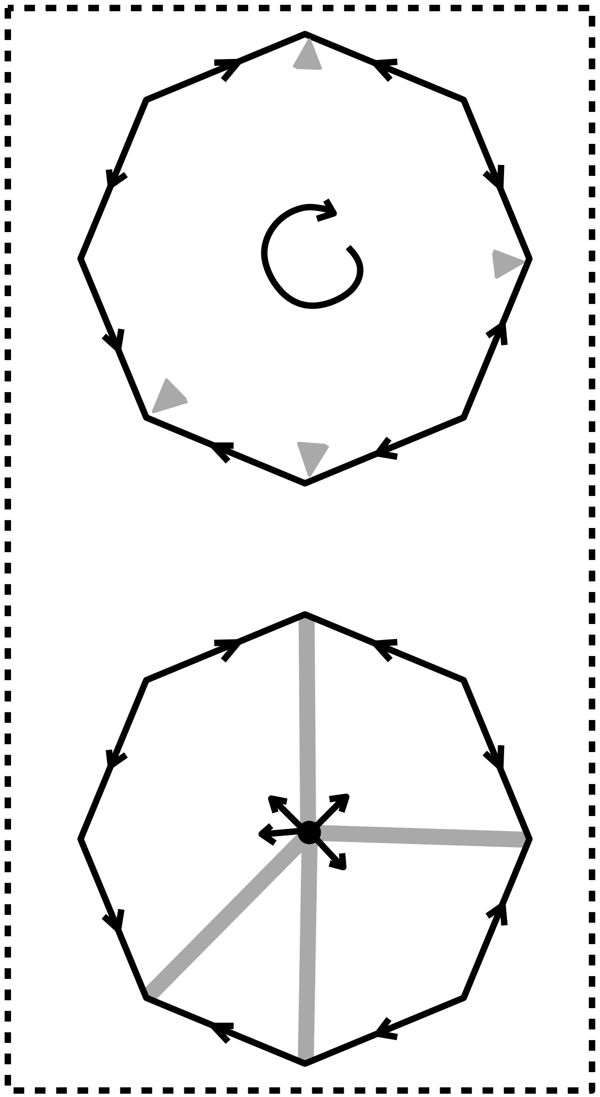

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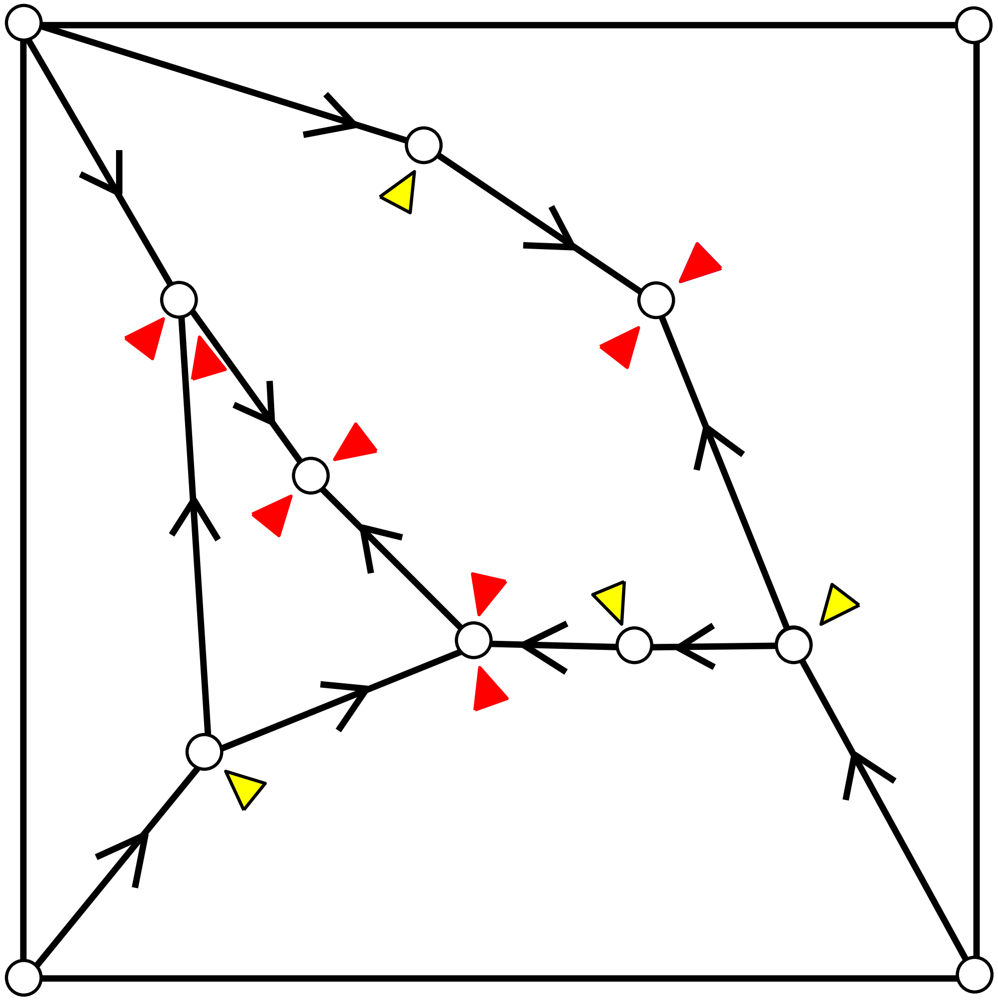

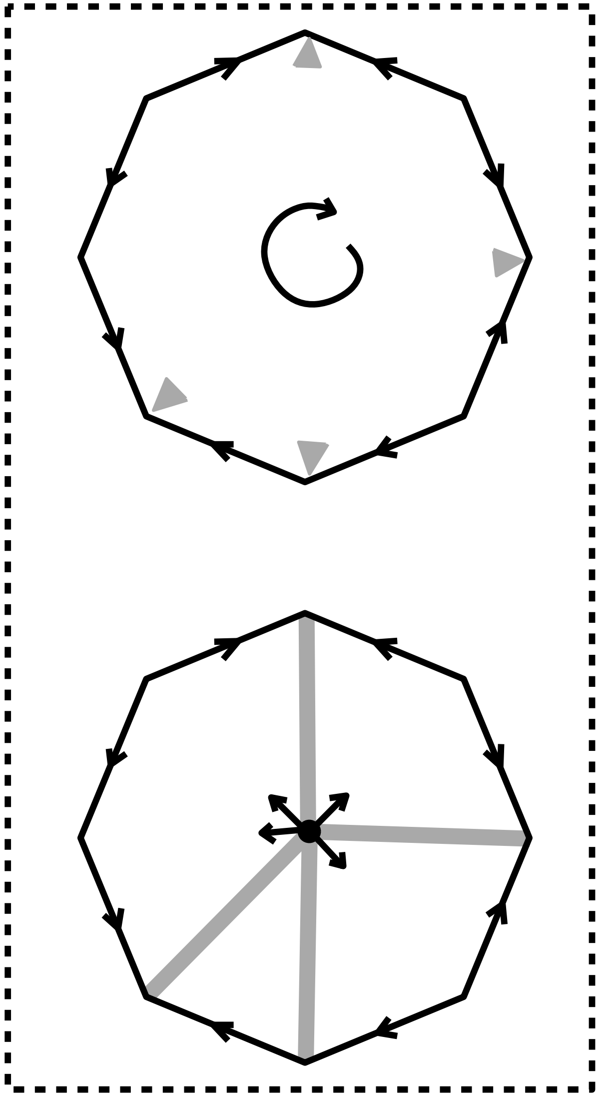

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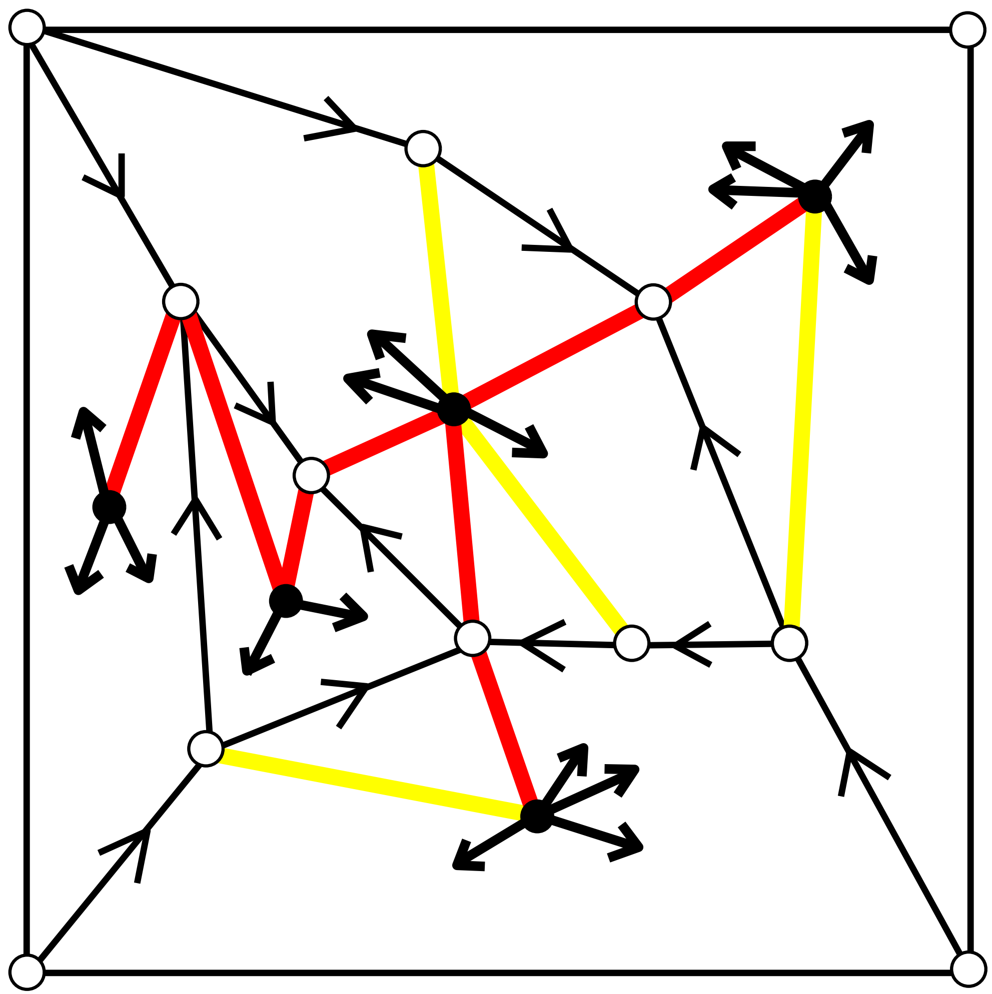

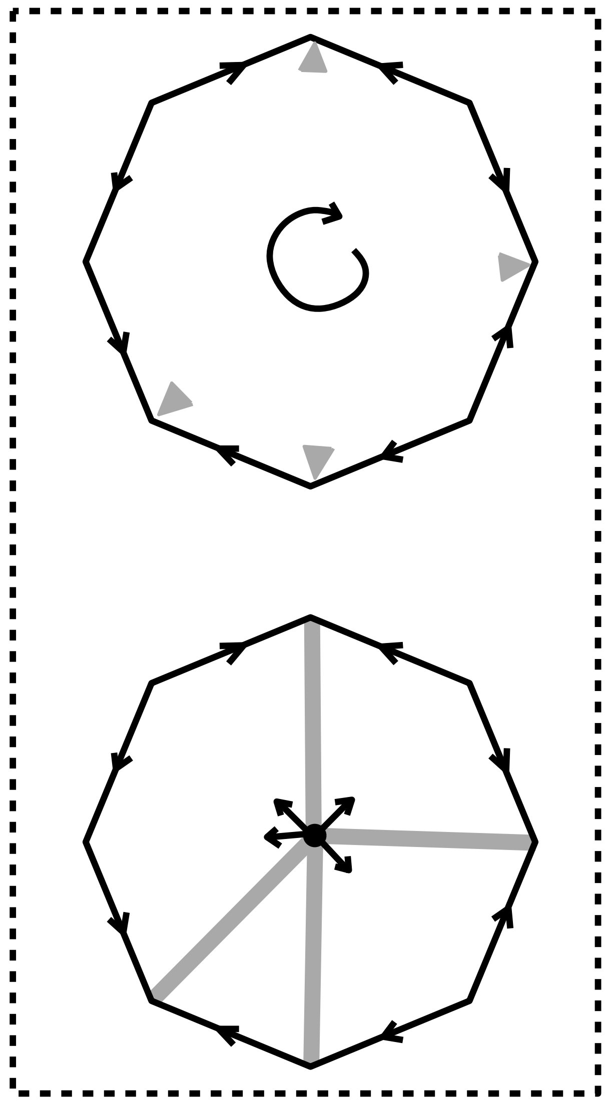

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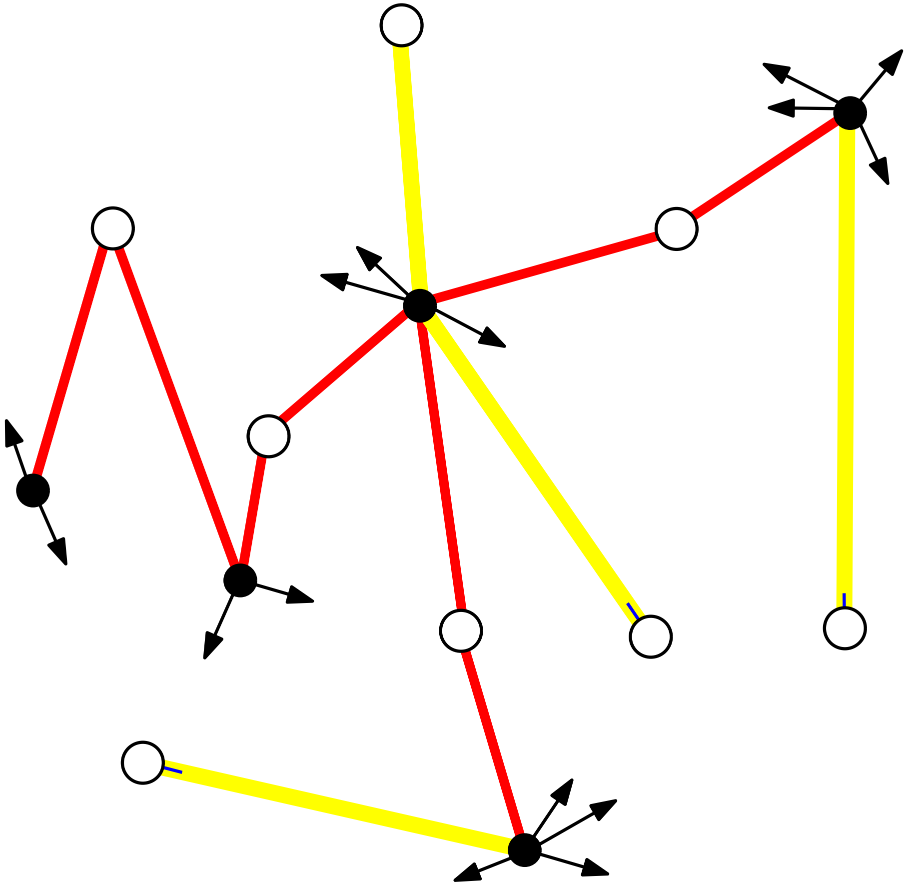


White vertices either have:

- indegree 2 (middle of red edge)
- indegree 1 (end of leg)

Each black vertex of degree $2i$ has $i - 2$ legs

Closed formulas

Prop [Bernardi-F'11]: The number of **rooted simple bipartite maps** with n_i faces of degree $2i$ is

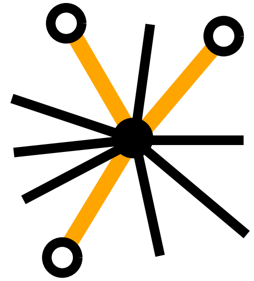
$$2\frac{(\sum(i+1)\,n_i-3)!}{(\sum in_i-1)!}\prod_{i\geq 2}\frac{1}{n_i!}\binom{2i-1}{i+1}^{n_i}$$

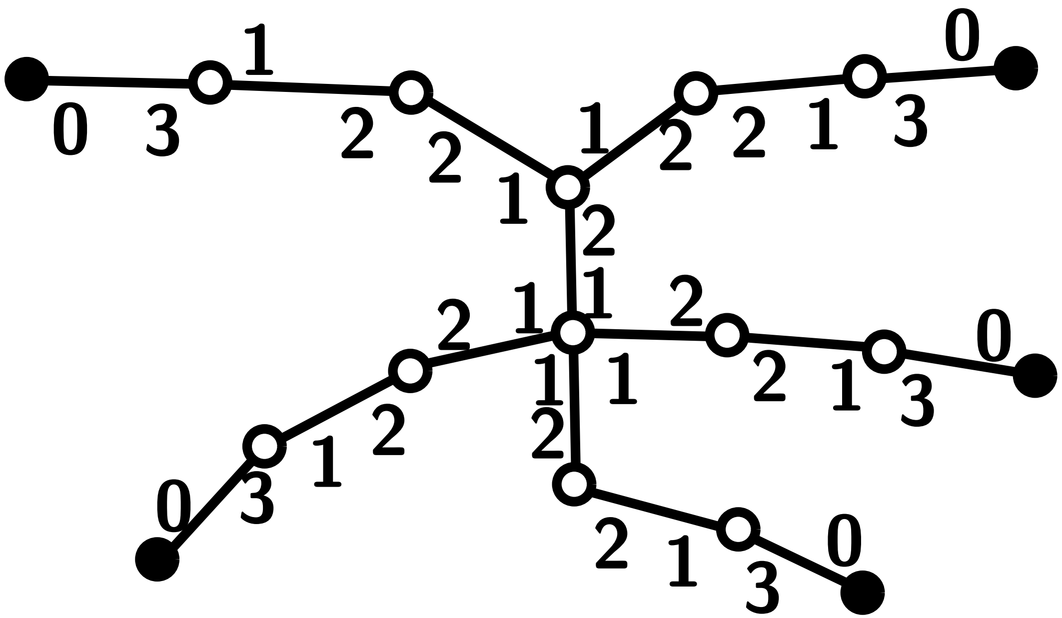
This can be compared with the formula obtained by Tutte (62) (recovered bijectively by Schaeffer) for **unconstrained rooted bipartite maps**:

$$2\frac{(\sum i\,n_i)!}{(\sum(i-1)n_i+2)!}\prod_{i\geq 1}\frac{1}{n_i!}\binom{2i-1}{i}^{n_i}$$

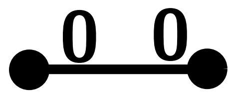
Shape of the mobile in higher (bipartite) girth

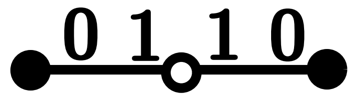
- Each black vertex of degree $2i$ has $i-b$ **legs**
- There are **connectors** between the black vertices





a connector for $b = 4$

Connectors, for $b = 1$: 

$b = 2$: 

$b = 3$: **binary trees**

Thanks.

On the ArXiv:

- A bijection for triangulations, quadrangulations, pentagulations, etc.
- Bijective counting of maps by girth and degree.