Planar maps: bijections and applications

Éric Fusy (CNRS/LIX)

AEC summer school, Hagenberg, 2018

Universality phenomena for maps

For 'any' standard family $\mathcal{M} = \cup_n \mathcal{M}_n$ of rooted maps

(p-angulations, loopless, 2-connected, 3-connected, etc.)

- $m_n = \mathrm{Card}(\mathcal{M}_n)$ satisfies $m_n \sim c\,\gamma^n\, n^{-5/2}$ for some constants c, γ

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- scaling limit point of view:
  for M_n a random map in $\mathcal{M}_n$ and v_1, v_2 two random vertices in M_n
  let $X_n = \mathrm{distance}(v_1, v_2)$

Then $\dfrac{X_n}{n^{1/4}} \to$ universal proba. dist. & $(M_n, \frac{d}{n^{1/4}}) \to$ Brownian map

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- local limit point of view

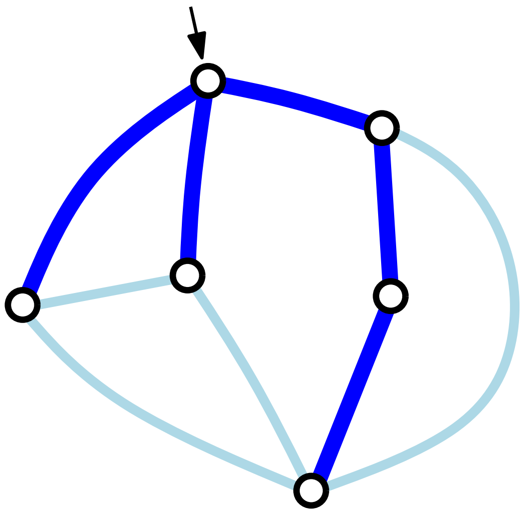
  let $Y_n^{(r)} = \#$(vertices at distance $\leq r$ from root-vertex in M_n)

  let $B^{(r)} := \lim_{n\to\infty} \mathbb{E}(Y_n^{(r)})$ Then $B^{(r)} \sim \kappa \cdot r^4$ as $r \to \infty$

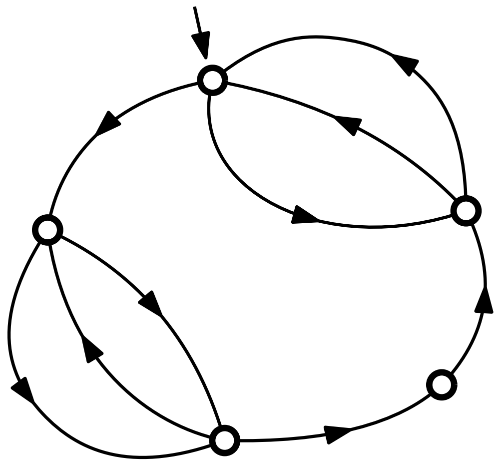
Looking for other universality classes

Structured planar map = pair (M, X), with M a rooted map
and X a combinatorial structure on M

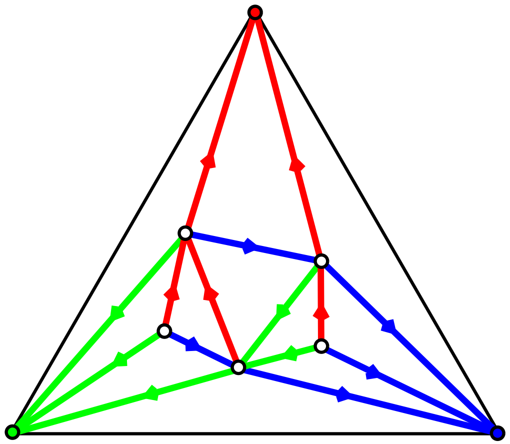
We can consider some natural families $\mathcal{S} = \cup_n \mathcal{S}_n$ of structured maps



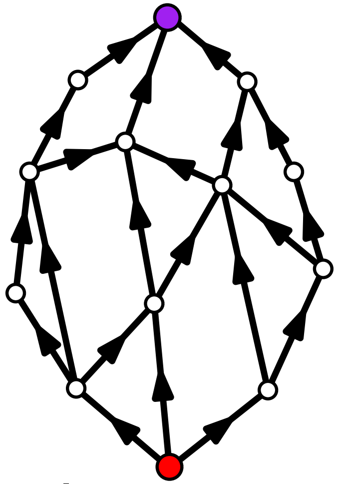
spanning tree



eulerian orientation



Schnyder wood



bipolar orientation

Watabiki predictions

[Watabiki'93]

If a model of maps gives asymptotic behaviours of the form $\kappa\, \gamma^n\, n^{-\alpha}$

then the central charge of the model is $c = -\dfrac{(3\alpha-5)(2\alpha-5)}{\alpha-1}$

prediction: $B^{(r)} \sim \text{constant} \times r^{\beta}$ with $\beta = 2\dfrac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$

	α	c	β	$1/\beta$
no structure	$5/2$	0	4	0.25
spanning tree	3	-2	$\frac{3+\sqrt{17}}{2}$	≈ 0.28
Bipolar ori.	4	-7	$\frac{4+2\sqrt{7}}{3}$	≈ 0.32
Schnyder wood	5	$-\frac{25}{2}$	$\frac{5+\sqrt{41}}{4}$	≈ 0.35

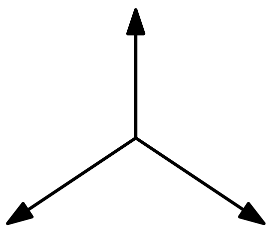
upper/lower bounds for β (consistent with prediction) [Gwynne, Holden, Sun'17]

Plan for today

review of bijective links (and discuss some connections/applications)

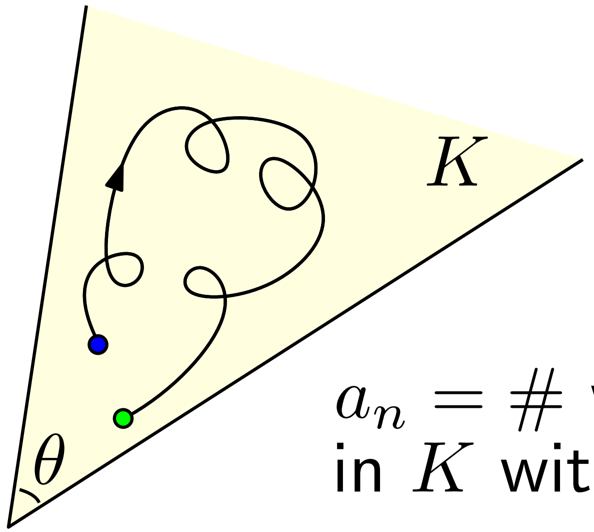
structured maps ⟷ lattice walks in quadrant (or in a 2d cone)

explains asymptotic behaviour, cf [Denisov-Wachtel'2015]



$\mathcal{S}$ step-set

with covariance matrix $= \mathrm{Id}_2$



$a_n = \#$ walks of length n in K with fixed endpoints

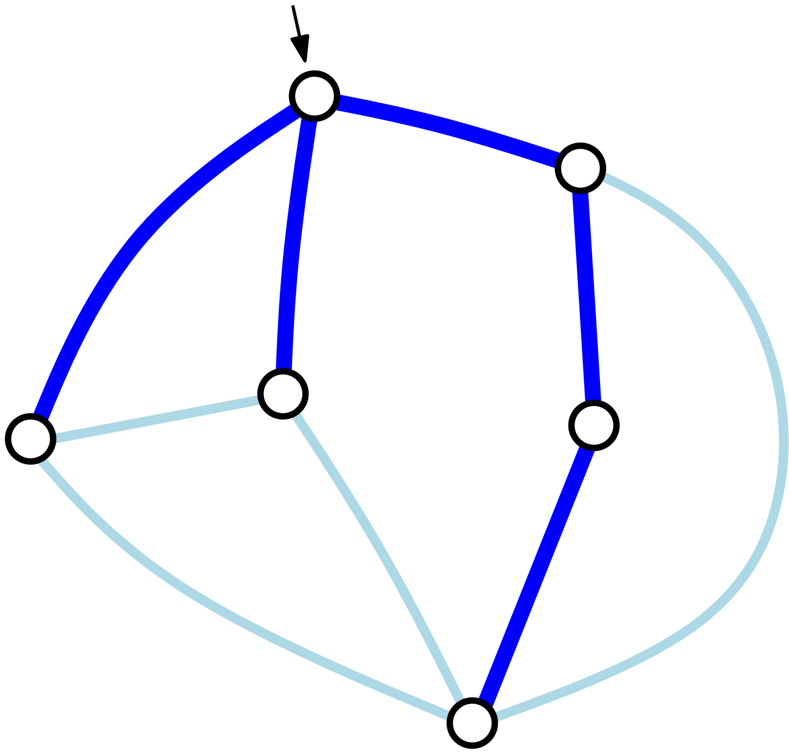
Then $a_n \sim \kappa\, \gamma^n\, n^{-p-1}$, with $p = \frac{\pi}{\theta}$

$\theta = \pi/2$ for spanning trees, $\pi/3$ for bipolar orientations, $\pi/4$ for Schnyder woods

Tree-rooted maps

(map + spanning tree)

Contour encoding of a tree-rooted map [Mullin'67]



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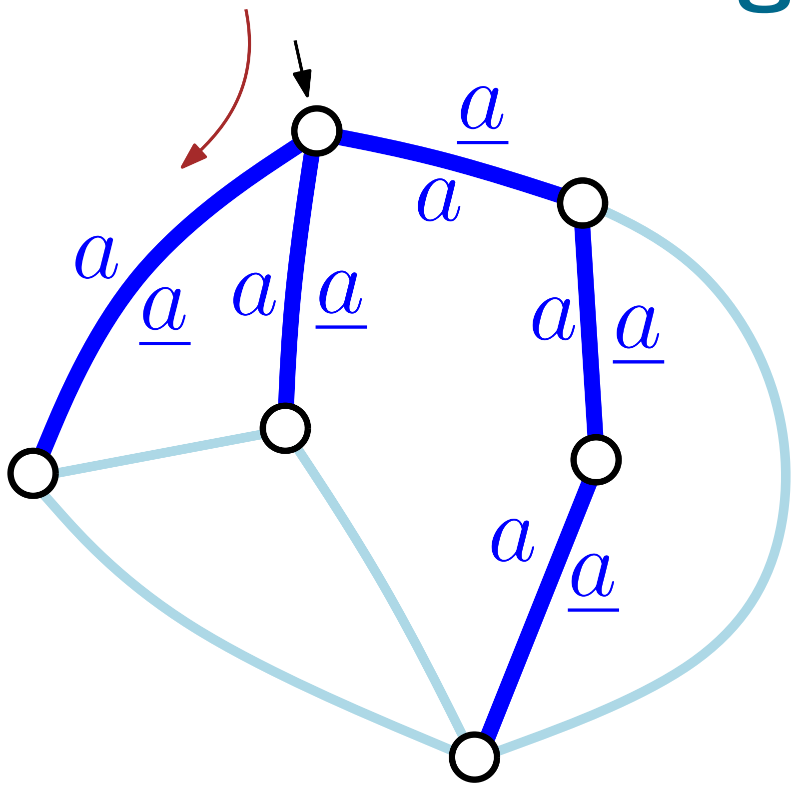

contour encoding of the tree T:

$a\,\underline{a}\,a\,\underline{a}\,a\,a\,a\,\underline{a}\,\underline{a}\,\underline{a}$

Dyck word

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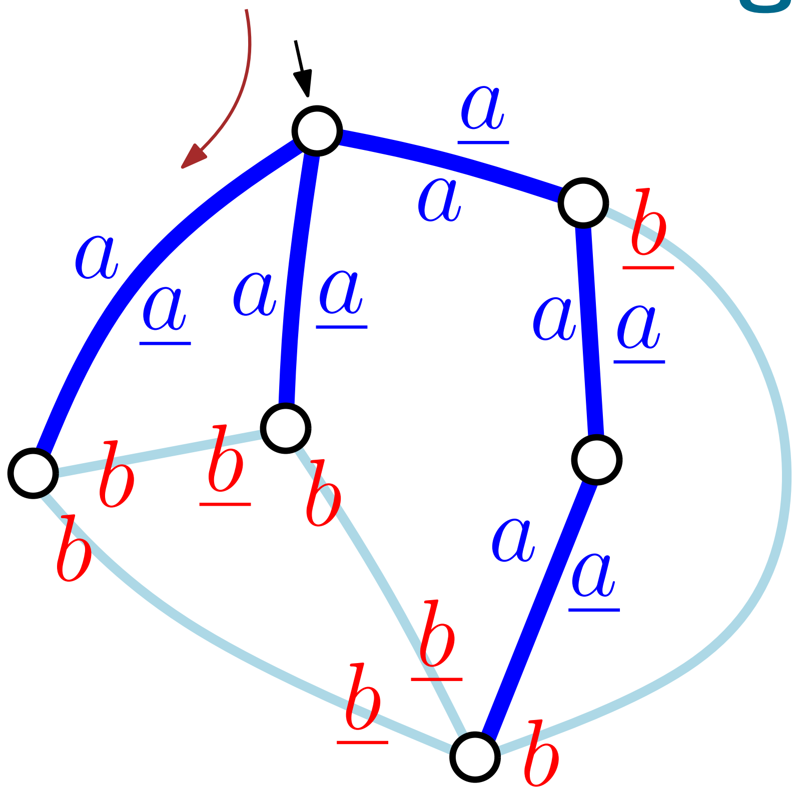

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shuffle of two Dyck words

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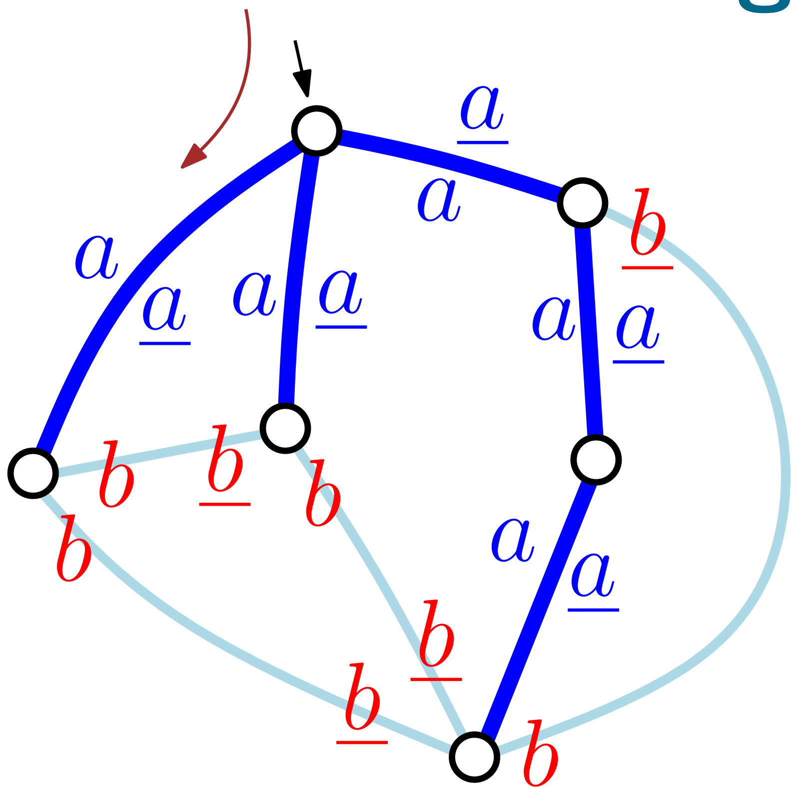

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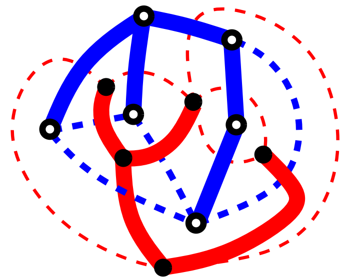
Rk: red word is the contour word for the dual spanning tree



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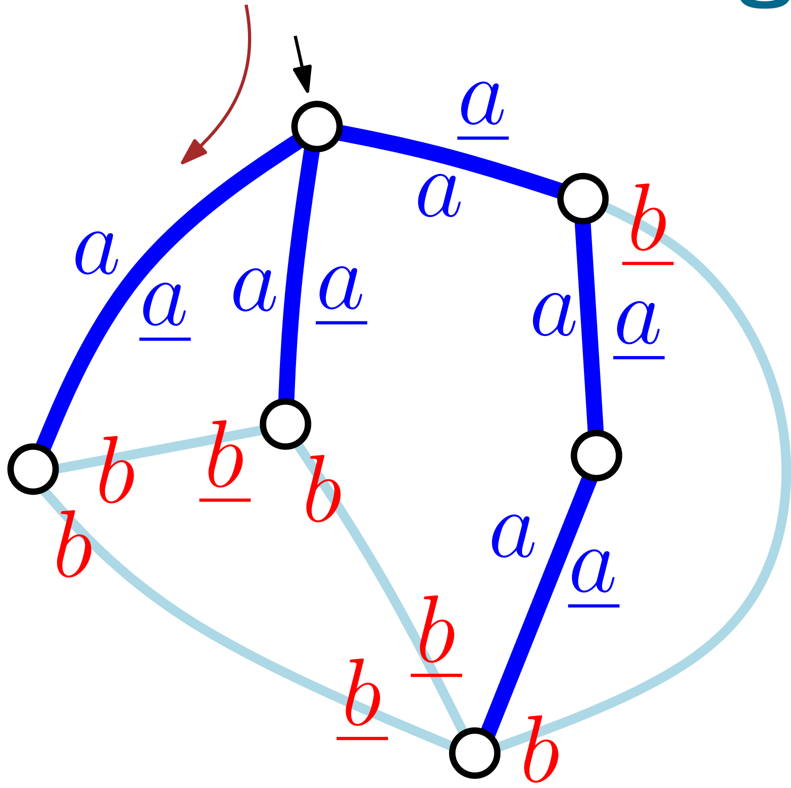

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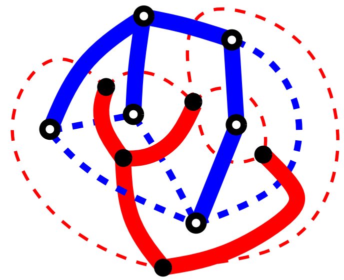
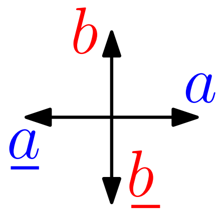


$\Rightarrow$ excursion in quadrant, with steps



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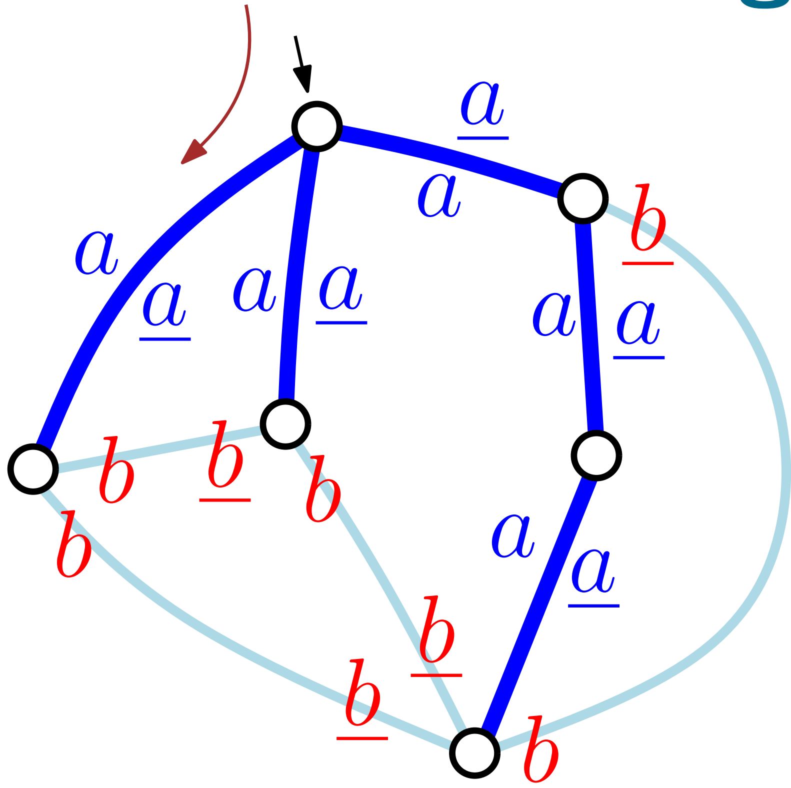

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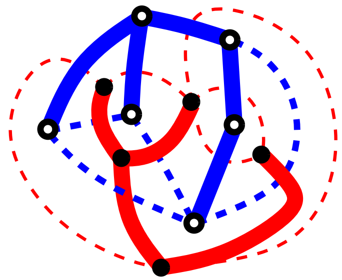

$t_n = \#$ tree-rooted maps with n edges satisfies

$$t_n = \sum_{k=0}^{n} \binom{2n}{2k} \mathrm{Cat}_k \mathrm{Cat}_{n-k}$$

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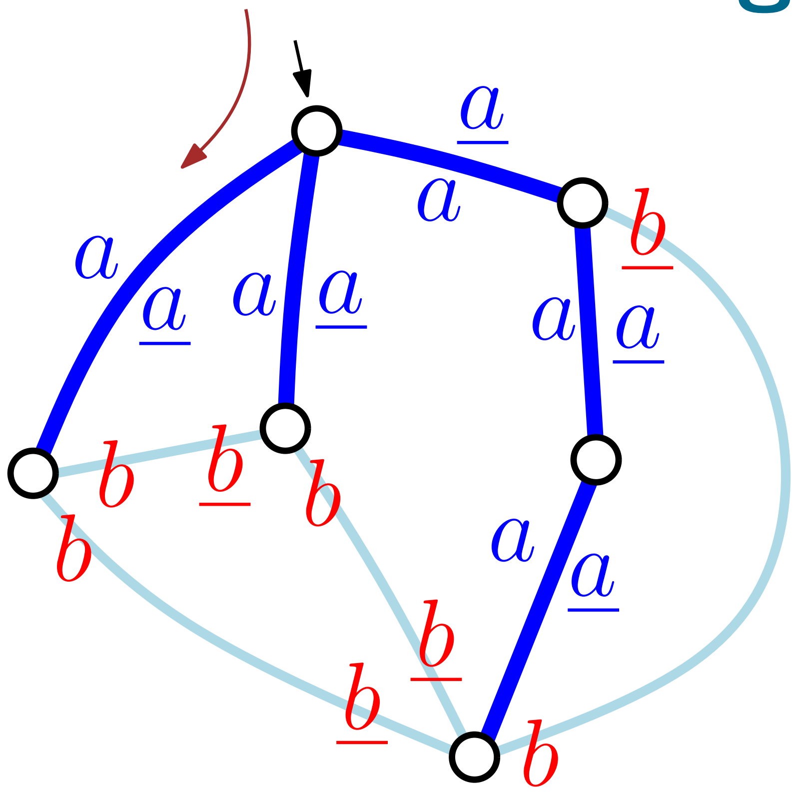

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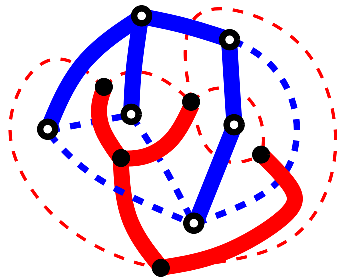

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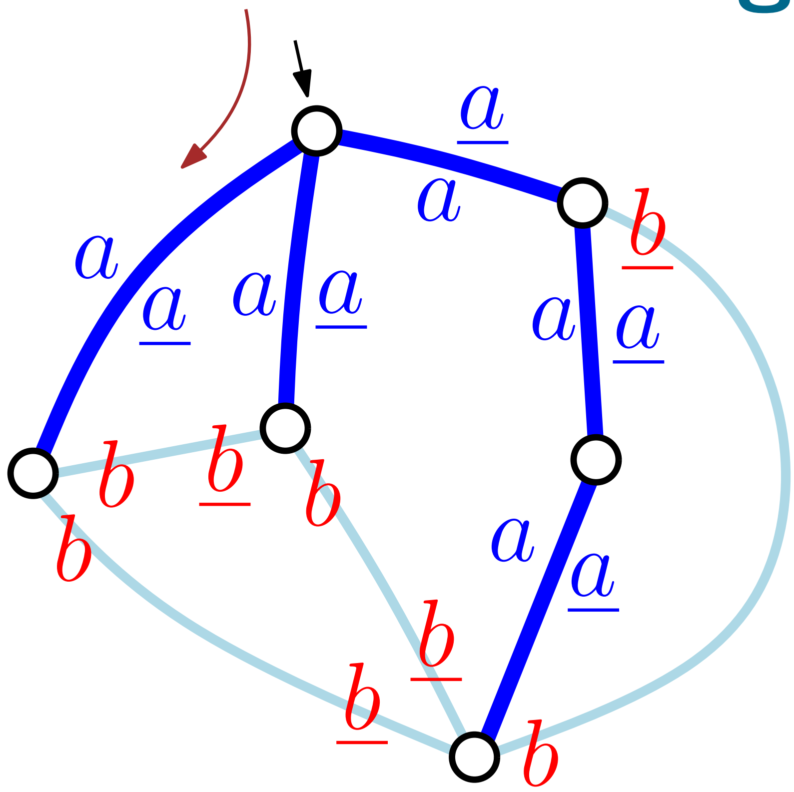

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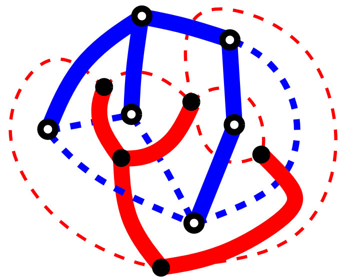

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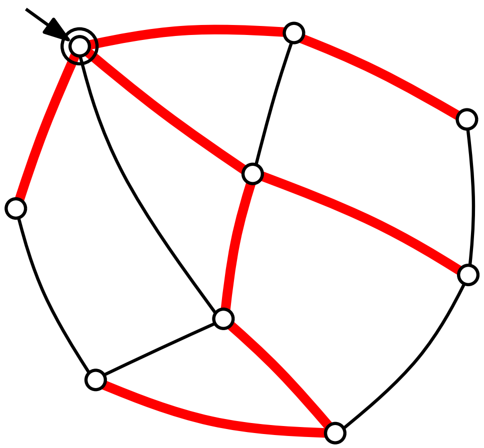
Hence $t_n \sim \frac{4}{\pi} 16^n n^{-3}$ with n^{-3} 'universal' for tree-rooted maps (cf exercise)

Direct proof that $t_n = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$ [Bernardi'07]

- First step:



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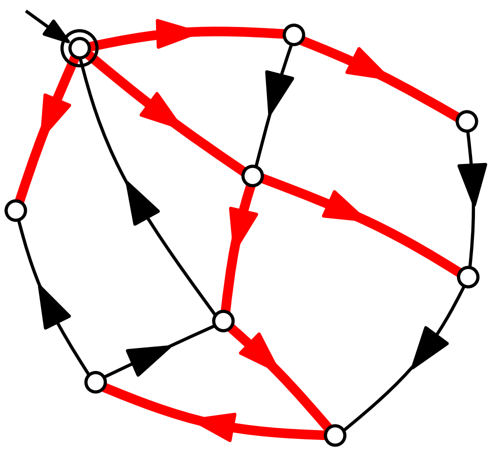
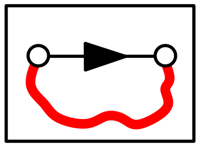
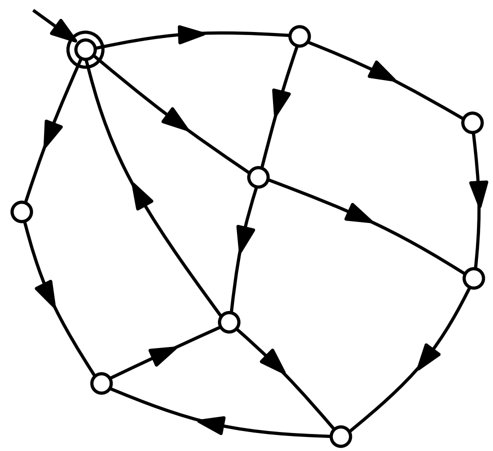


local rule





oriented rooted map
(root-accessible & no ccw cycle)

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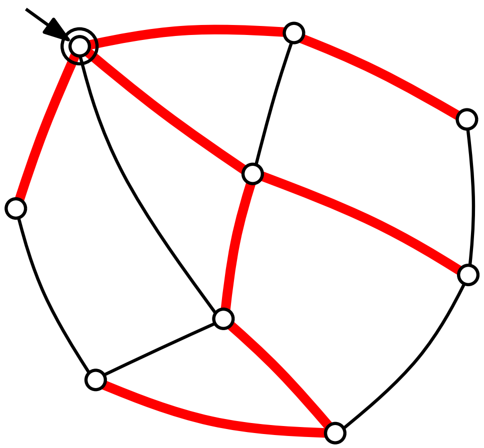

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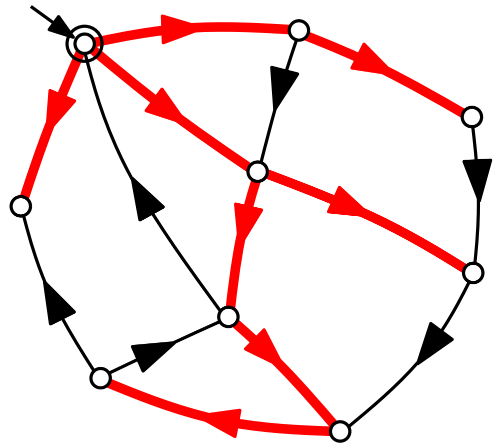

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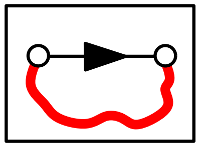

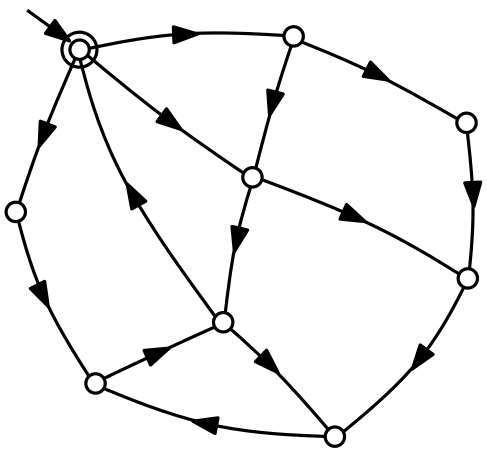

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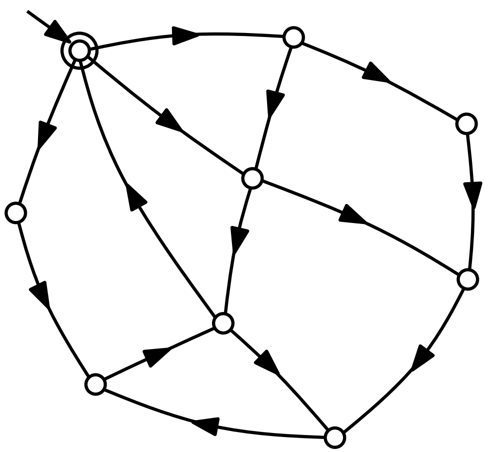

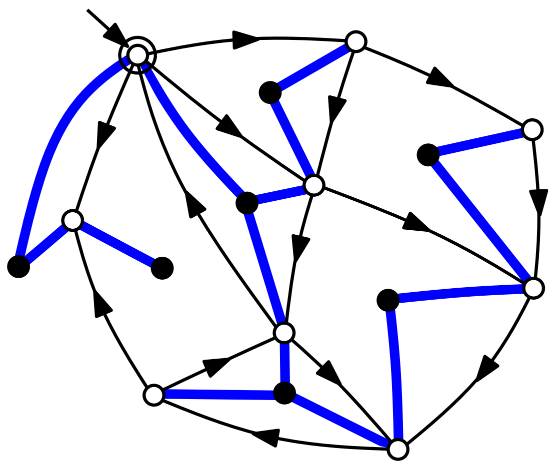

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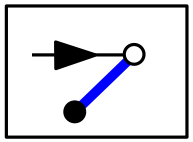

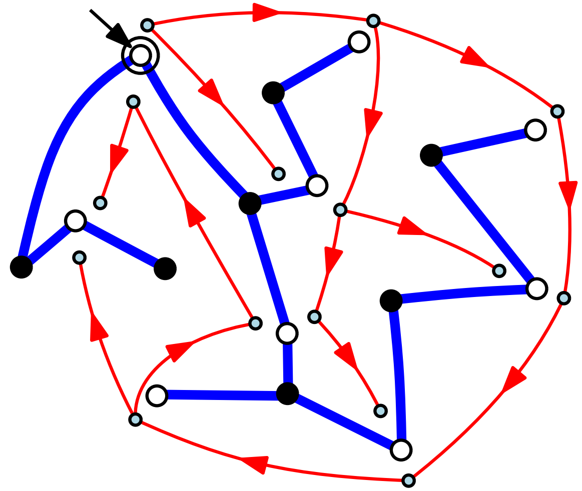

blue tree has $n+1$ edges
red tree has n edges

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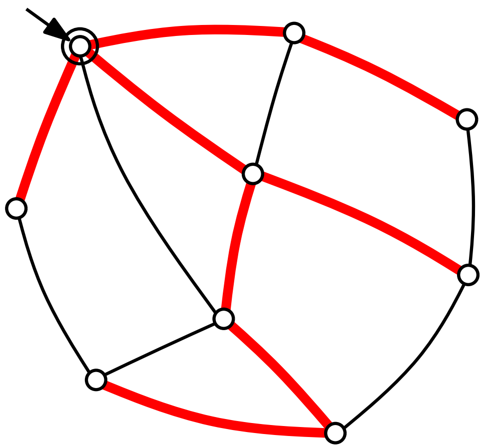
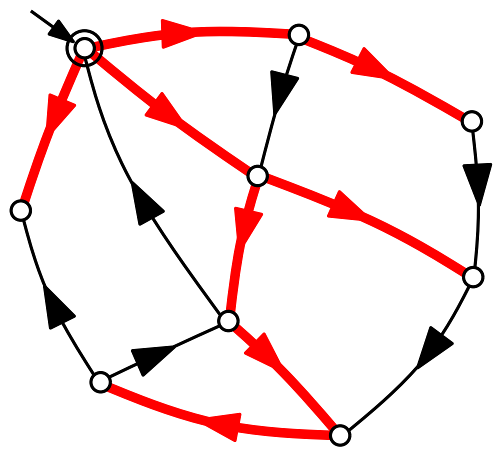


tree-rooted map



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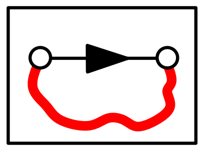

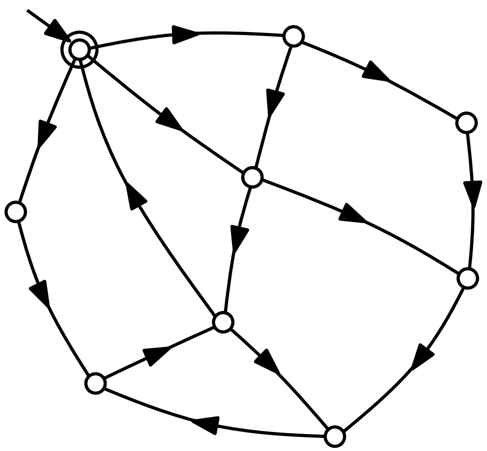

oriented rooted map
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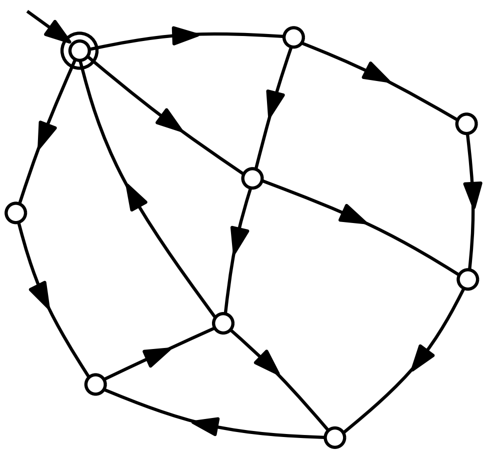

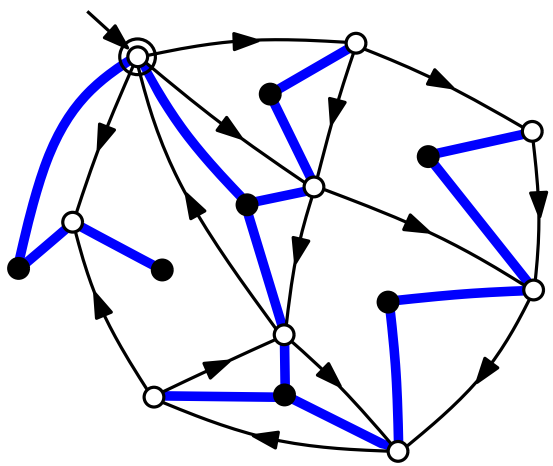

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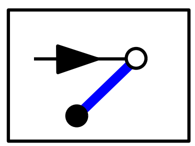
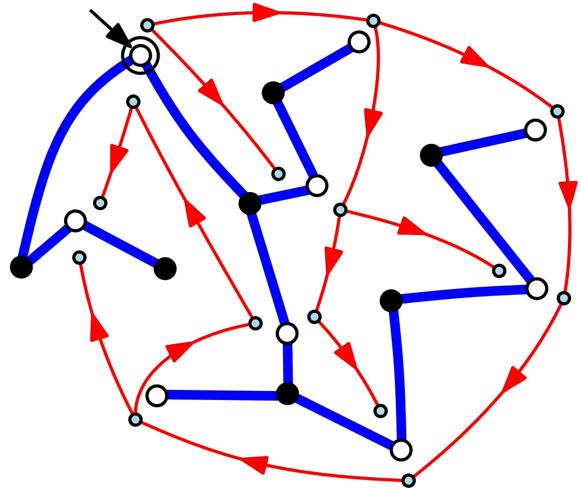


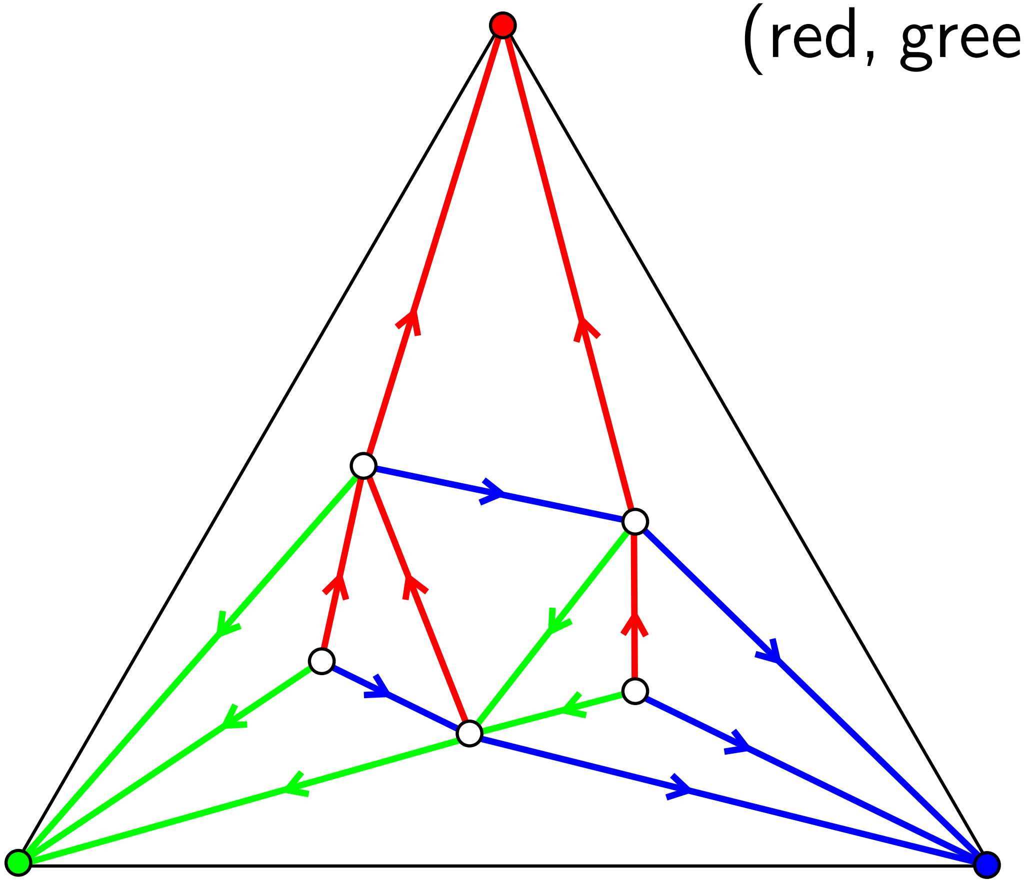


blue tree has $n+1$ edges
red tree has n edges

(the bijection Φ used previously this week is closely related to 2nd step)

Schnyder woods

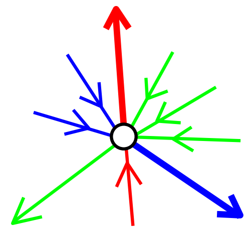
Schnyder woods on triangulations [Schnyder'89]



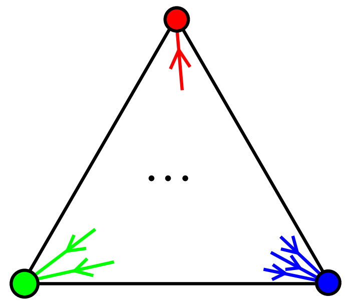
Schnyder wood = choice of a direction and color (red, green, or blue) for each inner edge, such that:

Local conditions:

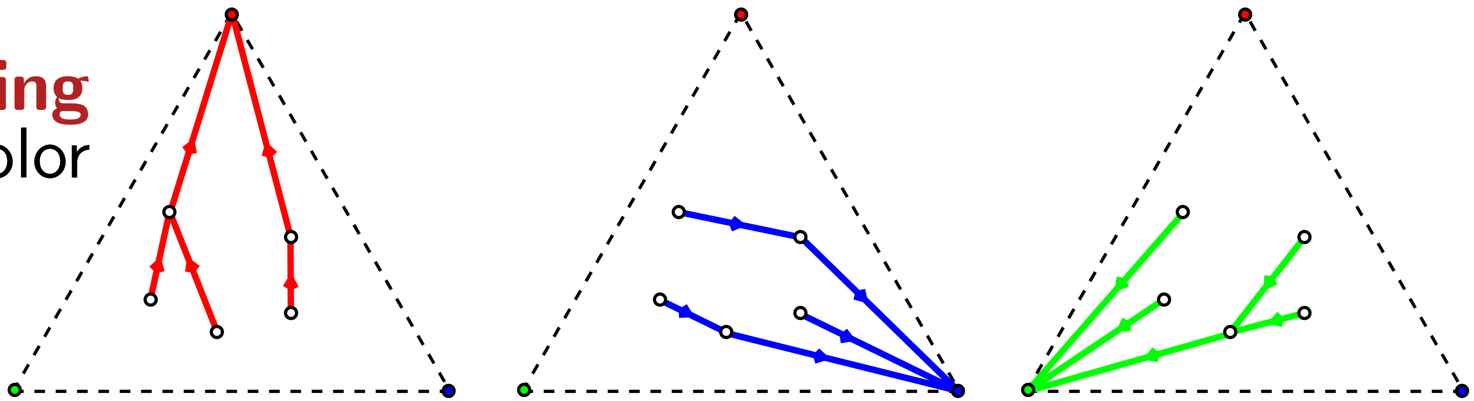
at each inner vertex



at the outer vertices



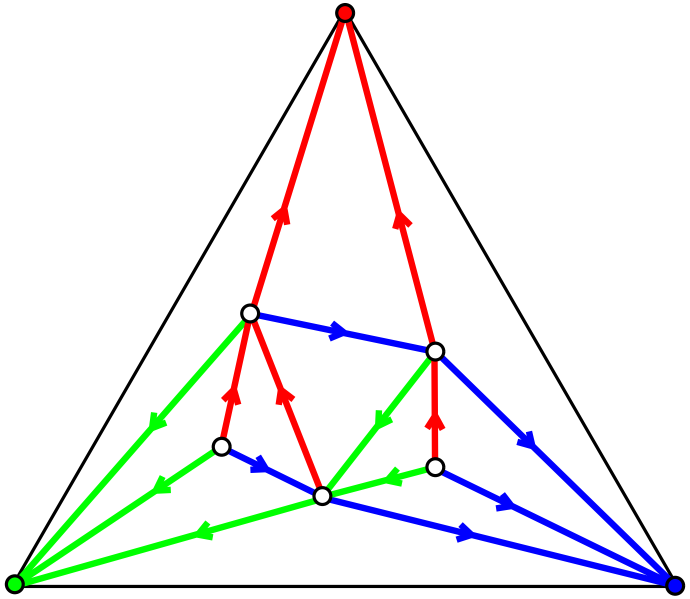
yields a **spanning tree** in each color



Equivalence with 3-orientations

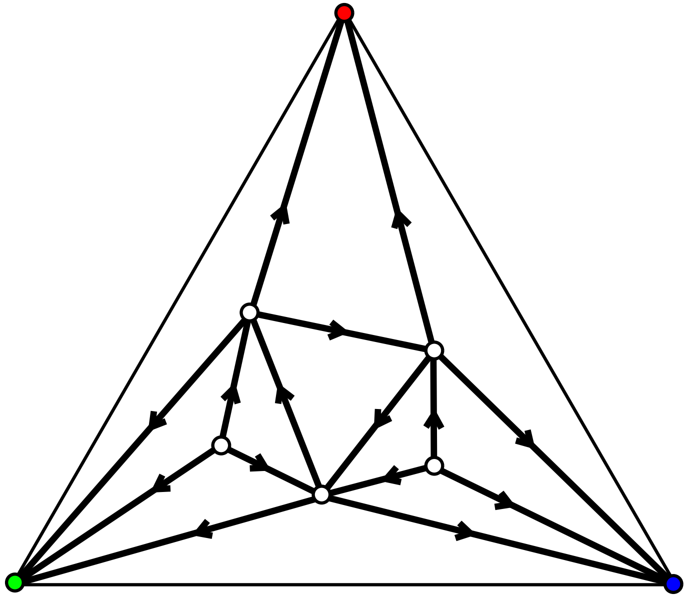
can propagate the colors (uniquely) from any 3-orientation

Schnyder wood

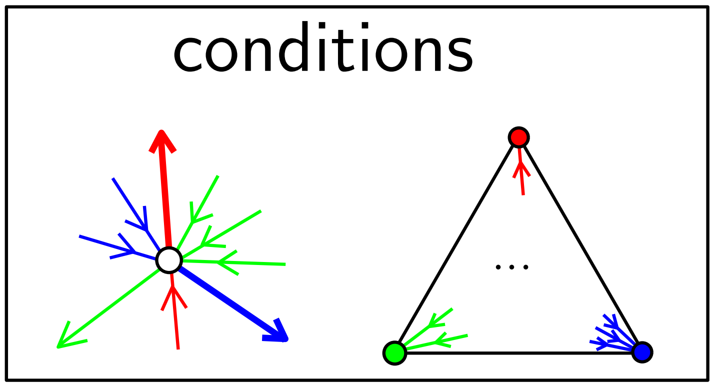


$\Leftrightarrow$

3-orientation



outdegree 3 at inner vertices
outdegree 0 at outer vertices



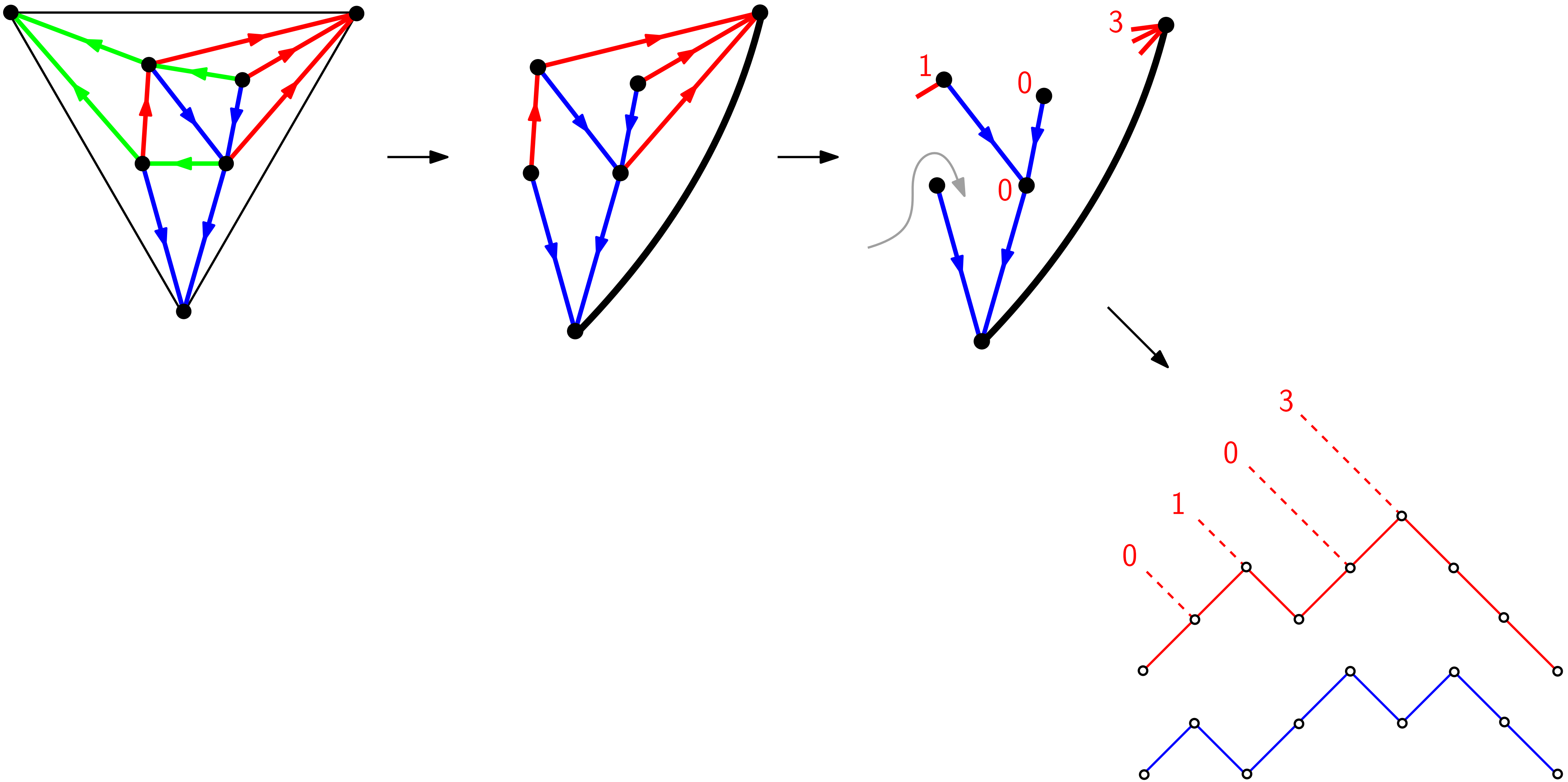
Bijective encoding of Schnyder woods

[Bernardi, Bonichon'09]

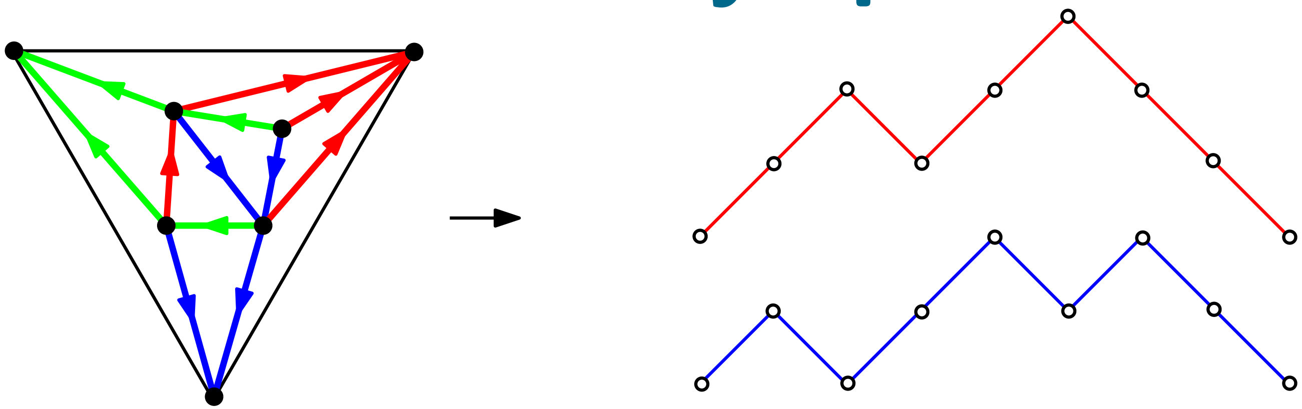
Schnyder woods on $n+3$ vertices

↕

non-intersecting pairs of Dyck paths of lengths $2n$



Enumerative formula, asymptotics



Let s_n = total number of Schnyder woods over triangulations with $n+3$ vertices

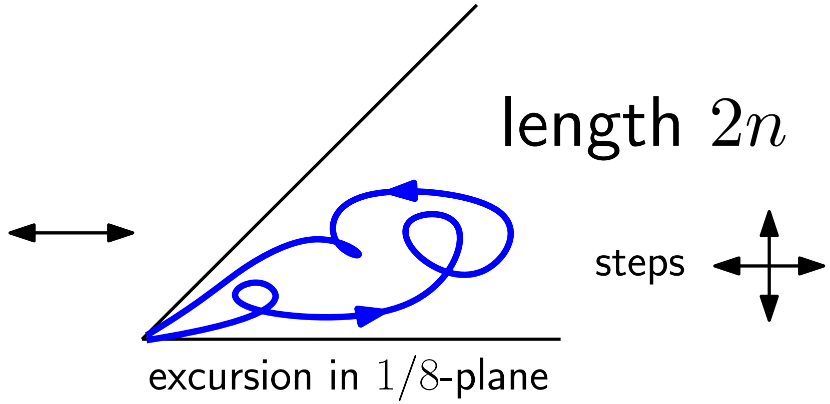
- Exact formula:

$$s_n = \mathrm{Cat}_n \mathrm{Cat}_{n+2} - \mathrm{Cat}_{n+1}\mathrm{Cat}_{n+1} = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

- Asymptotic formula: $s_n \sim \frac{24}{\pi} 16^n n^{-5}$

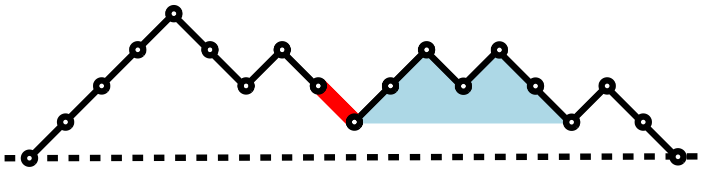
n^{-5} cf bijection

non-crossing pair
Dyck paths lengths $2n$

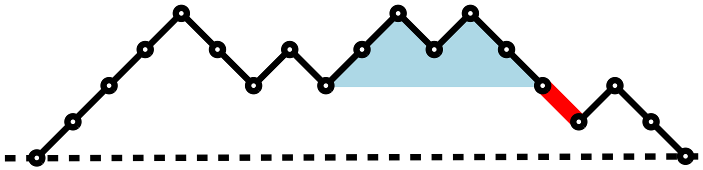


The Tamari lattice

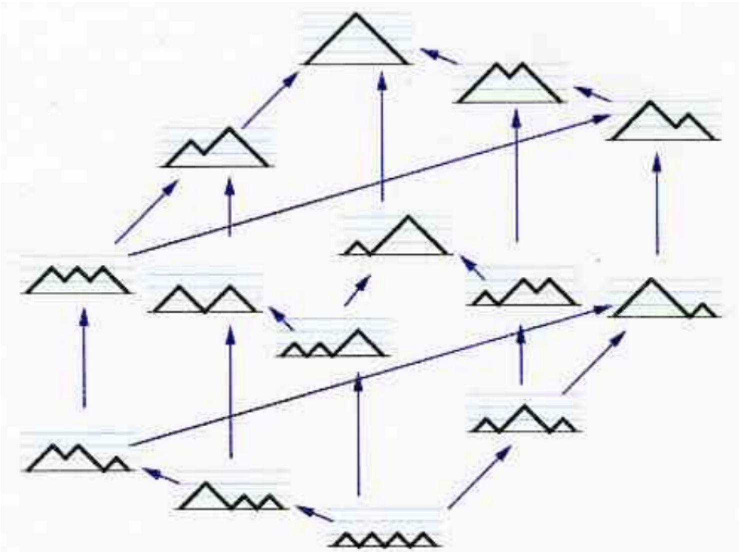
The Tamari lattice $\mathcal{L}_n$ is the partial order on Dyck paths of length $2n$ for the covering relation



$\prec$



(amounts to right rotation in corresponding binary trees)



the Tamari lattice for $n = 4$

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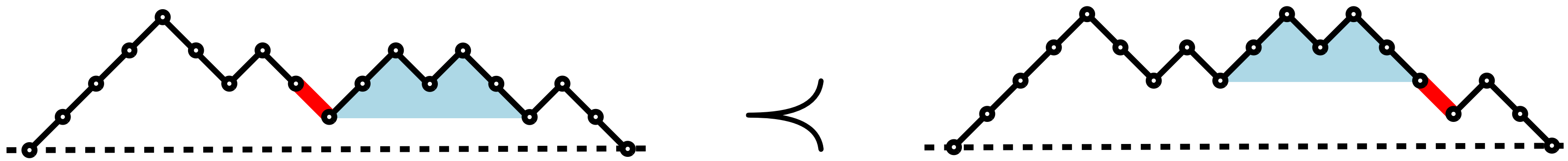

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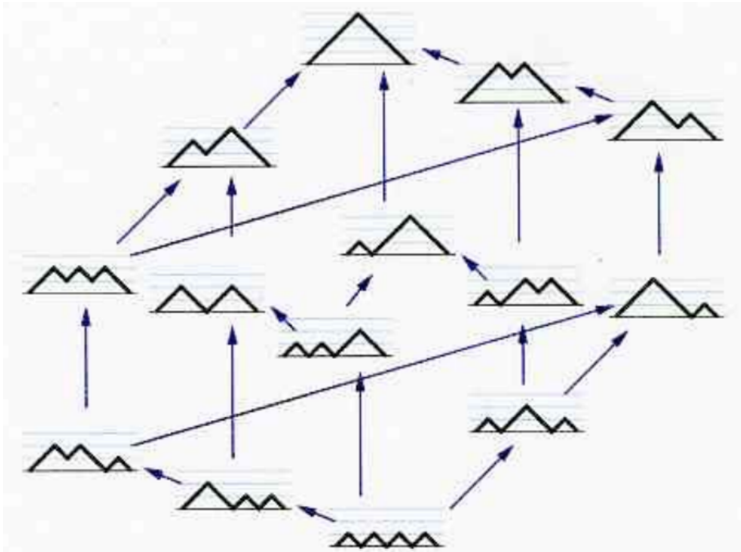

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it has 68 intervals

Interval in $\mathcal{T}_n$ = pair (t, t') such that $t \leq t'$

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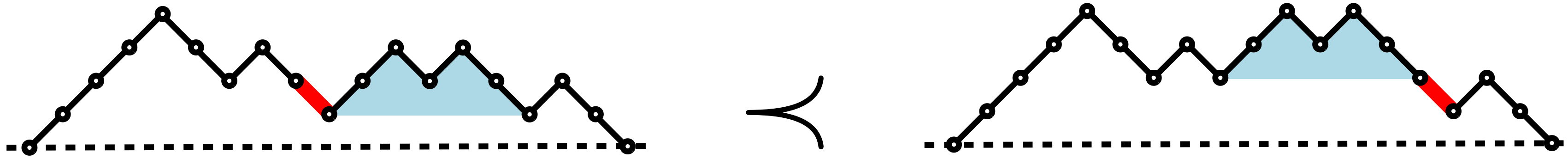

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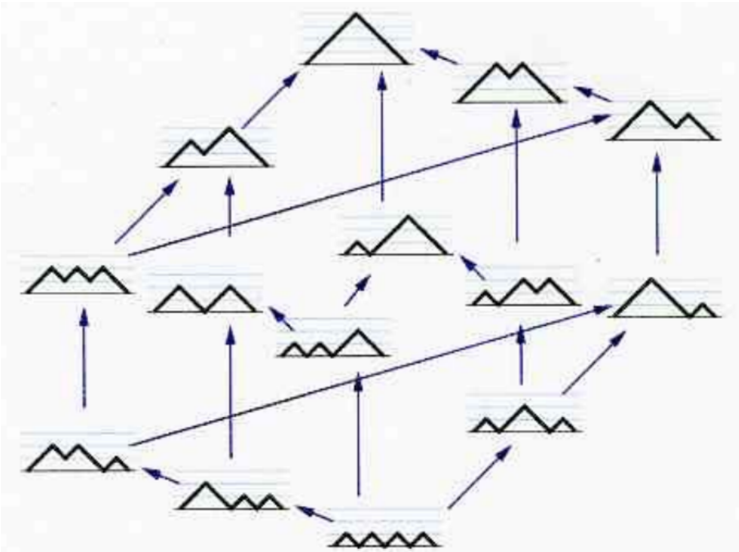

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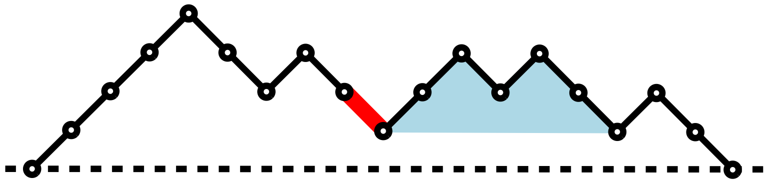
Interval in $\mathcal{T}_n$ = pair (t, t') such that $t \leq t'$

Theorem [Chapoton'06]: there are $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ intervals in $\mathcal{L}_n$

Rk: This is also the number of simple triangulations with $n + 3$ vertices

Characterization of intervals by length-vectors



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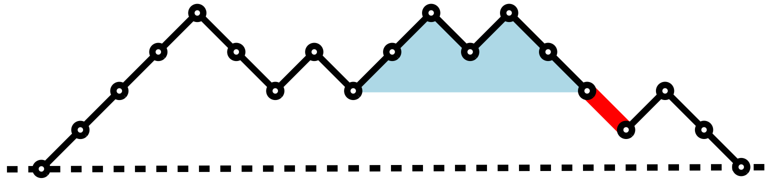

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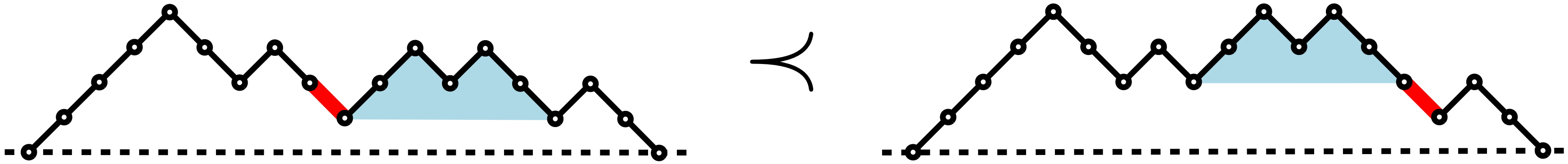
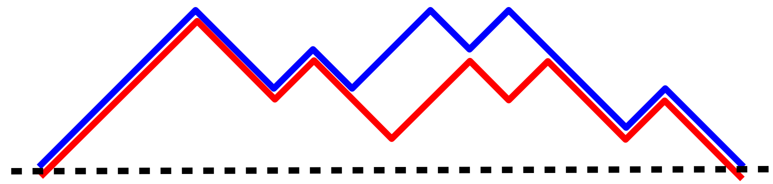


Rk: if $t \leq t'$ in $\mathcal{L}_n$, then t is below t'



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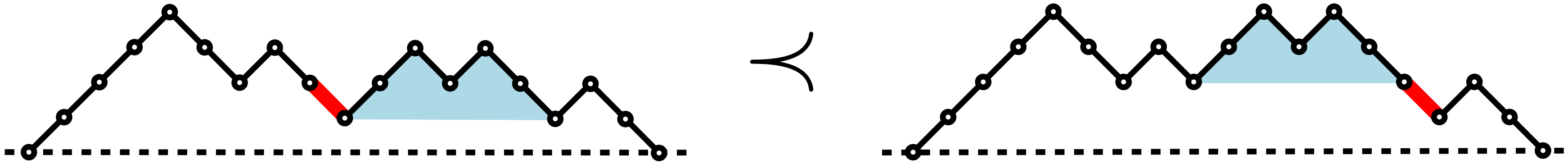

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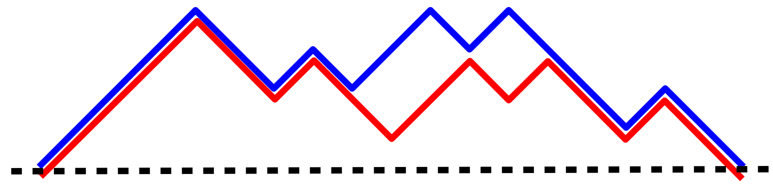

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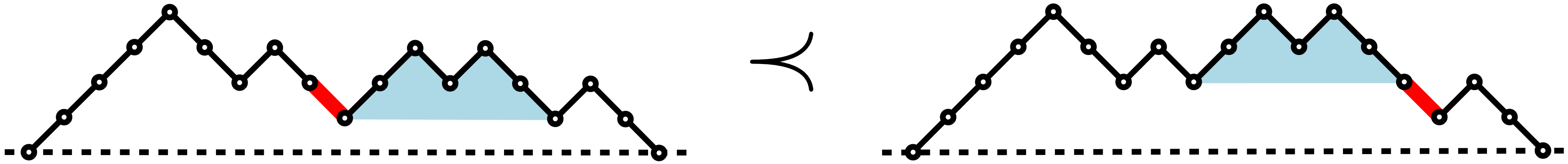

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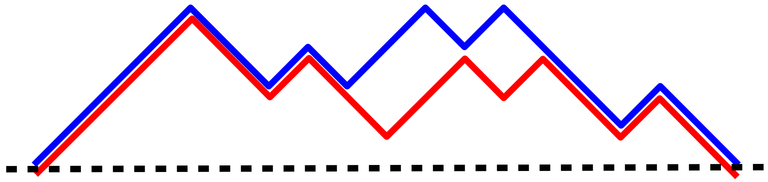
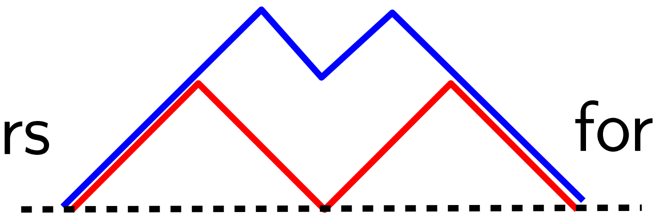


Q: How to characterize pairs  forming an interval in $\mathcal{L}_n$?

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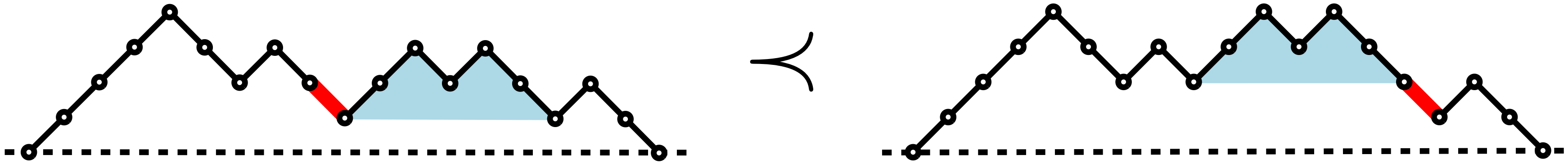

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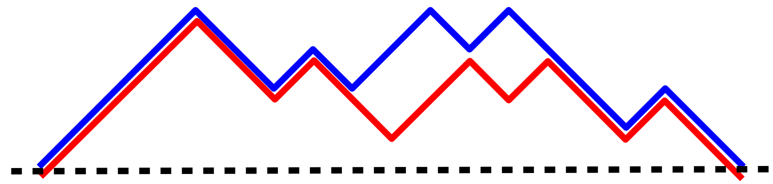

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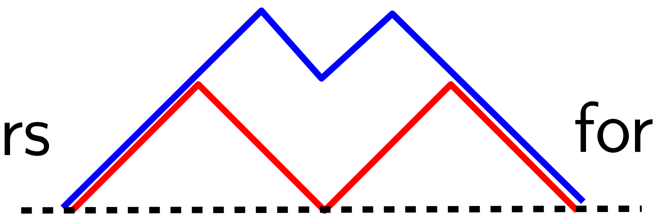

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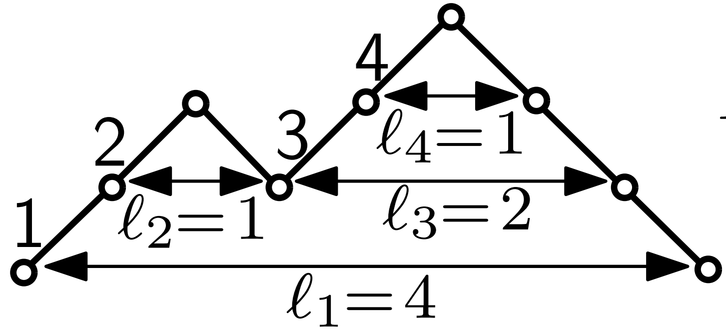
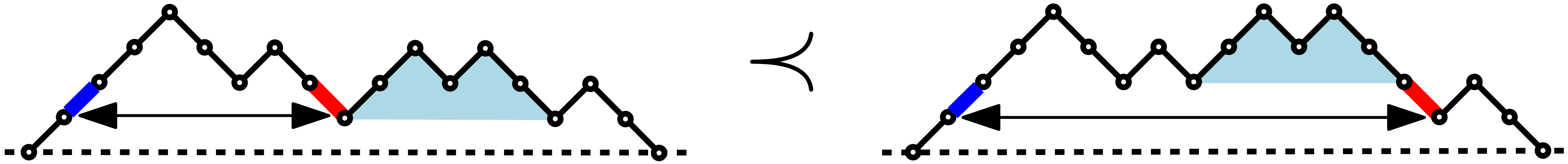
Length-vector L_D of D:



$L_D = (4, 1, 2, 1)$

Characterization of intervals by length-vectors



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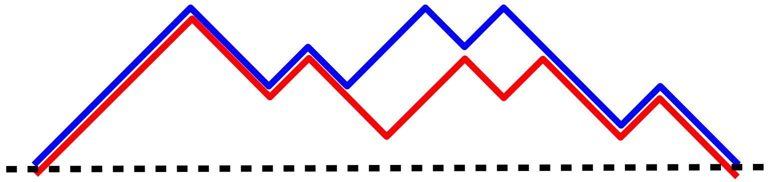
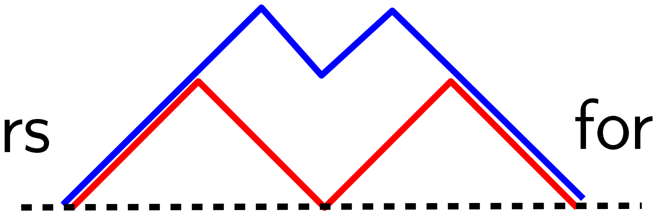


Q: How to characterize pairs  forming an interval in $\mathcal{L}_n$?

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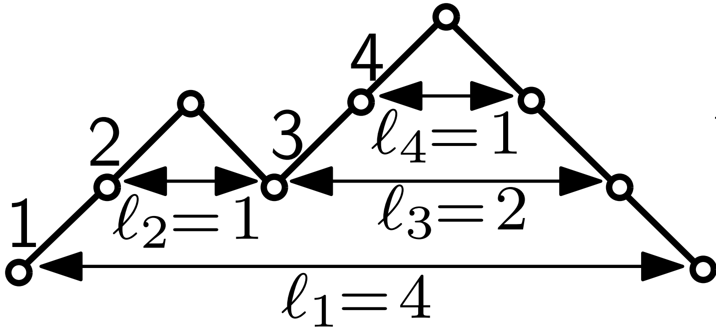


$L_D = (4, 1, 2, 1)$

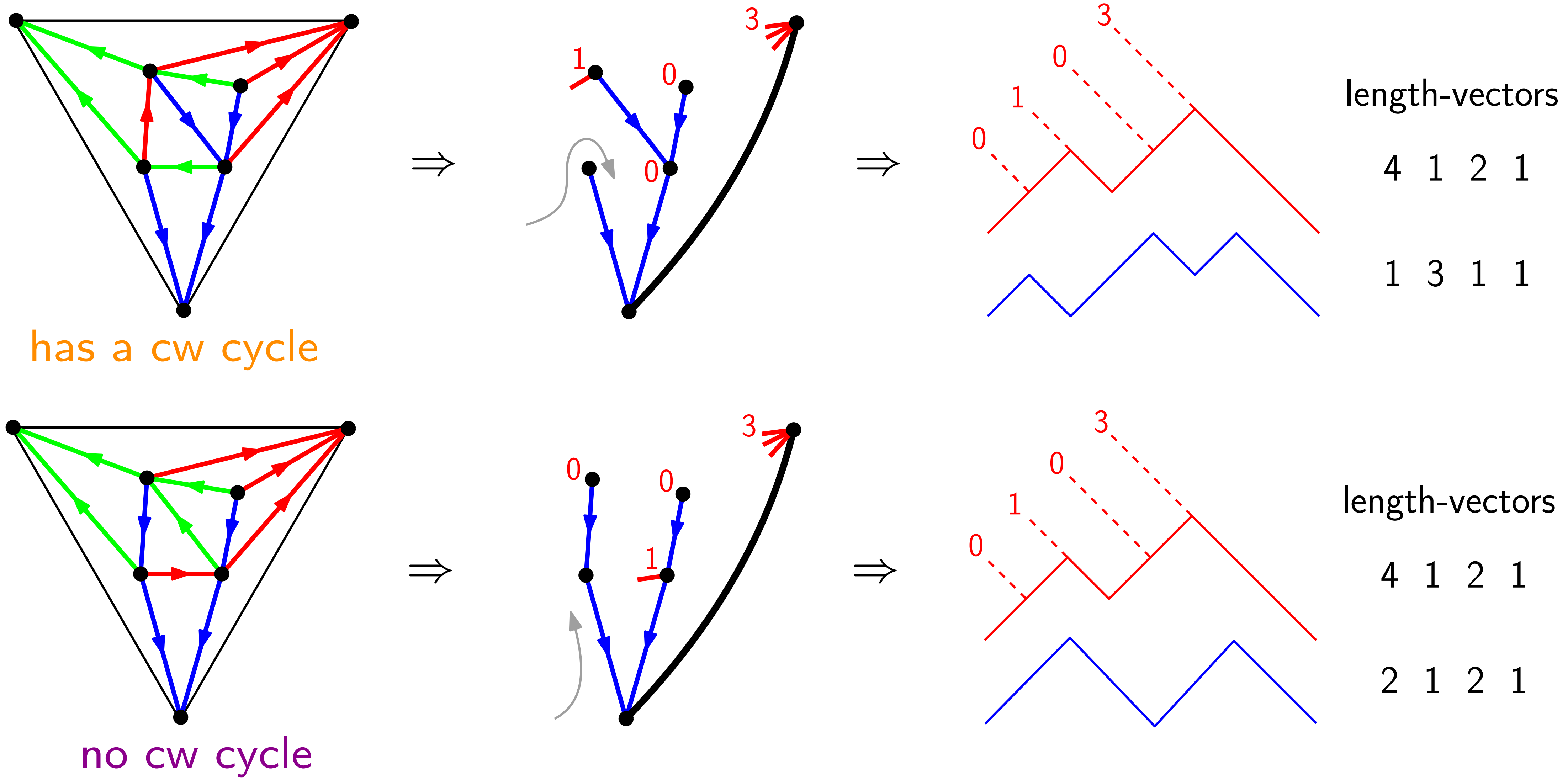
Lem: $D \leq D'$ in $\mathcal{L}_n$ iff $L_D \leq L_{D'}$

Specializing the bijection for Schnyder woods

[Bernardi, Bonichon'09]

Property: A triangulation has a unique Schnyder wood with no cw cycle

Property: A non-crossing pair of Dyck paths is an interval in $\mathcal{L}_n$ iff the corresponding Schnyder wood has no cw cycle



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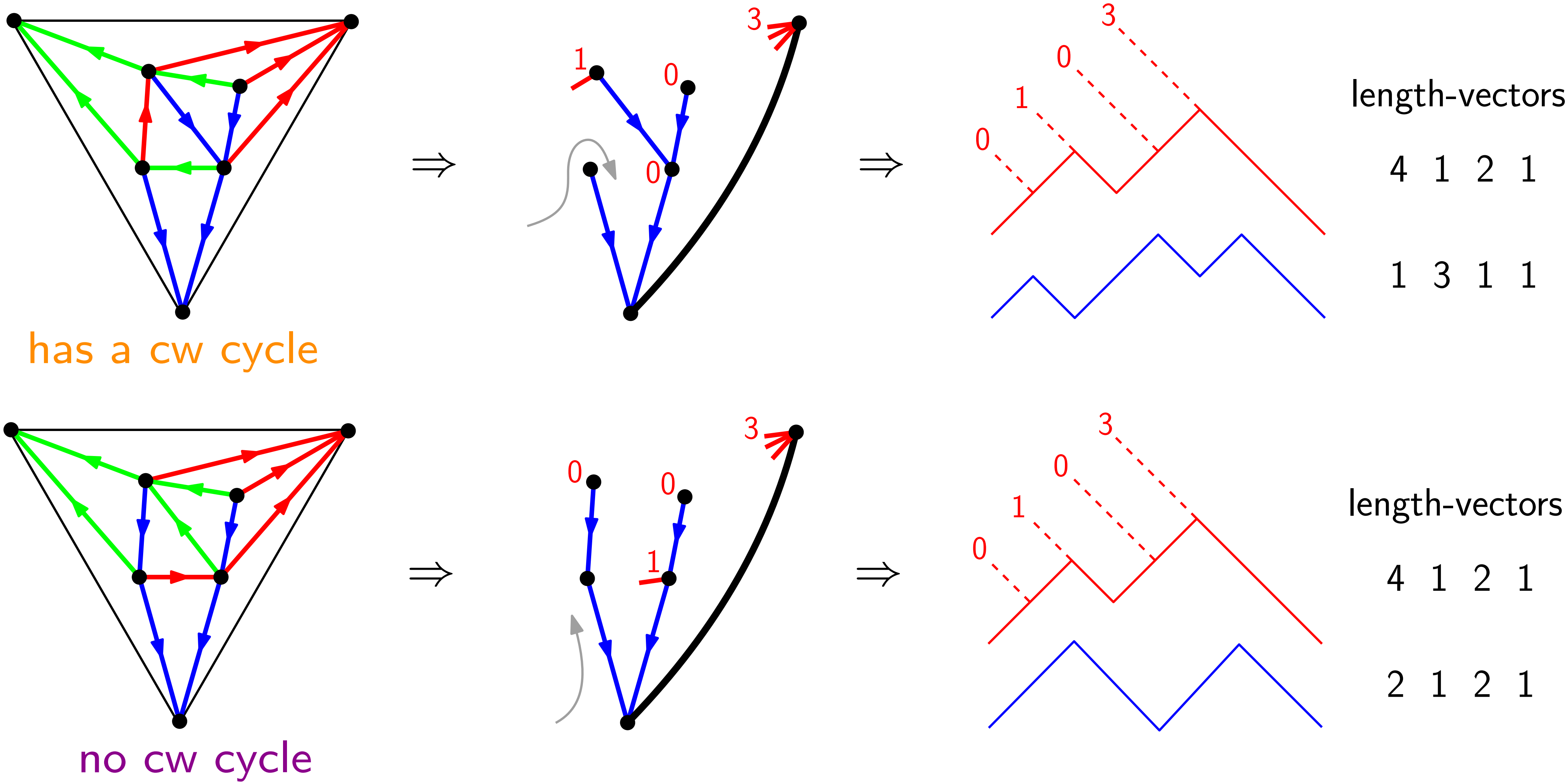


⇒ intervals in $\mathcal{L}_n$ are in bijection with simple triangulations with $n+3$ vertices

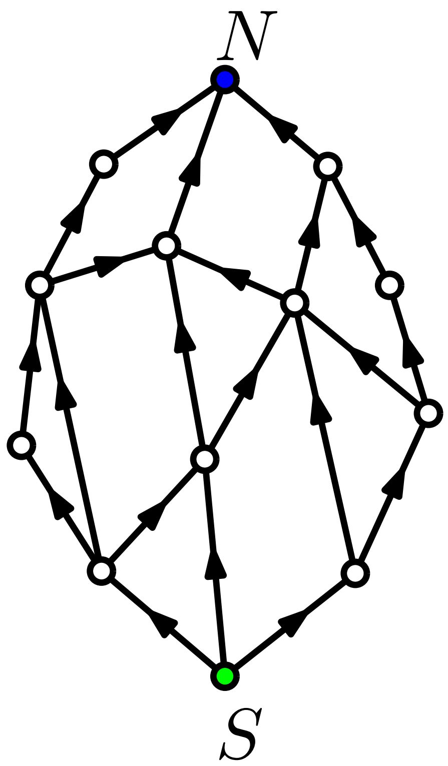
Bipolar orientations

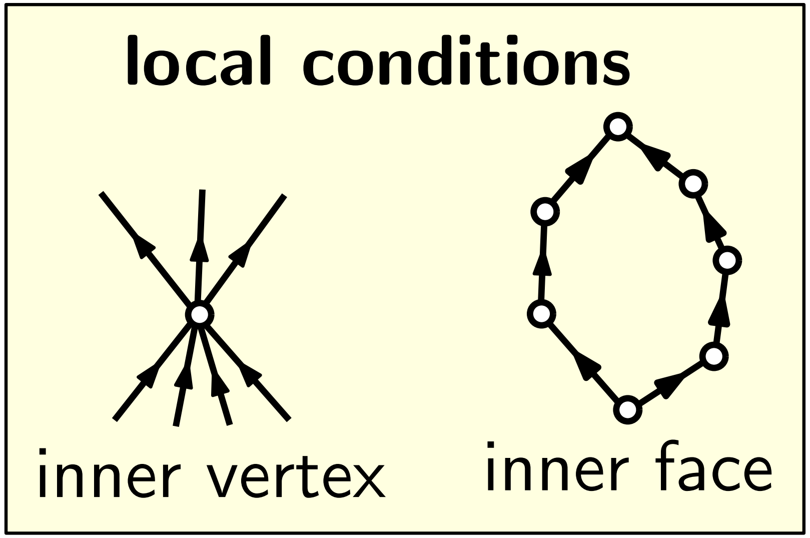
Definition

Let M be a planar map with two marked outer vertices S, N

Bipolar orientation of M = acyclic orientation of M

with S the unique source
and N the unique sink





Enumeration by edges

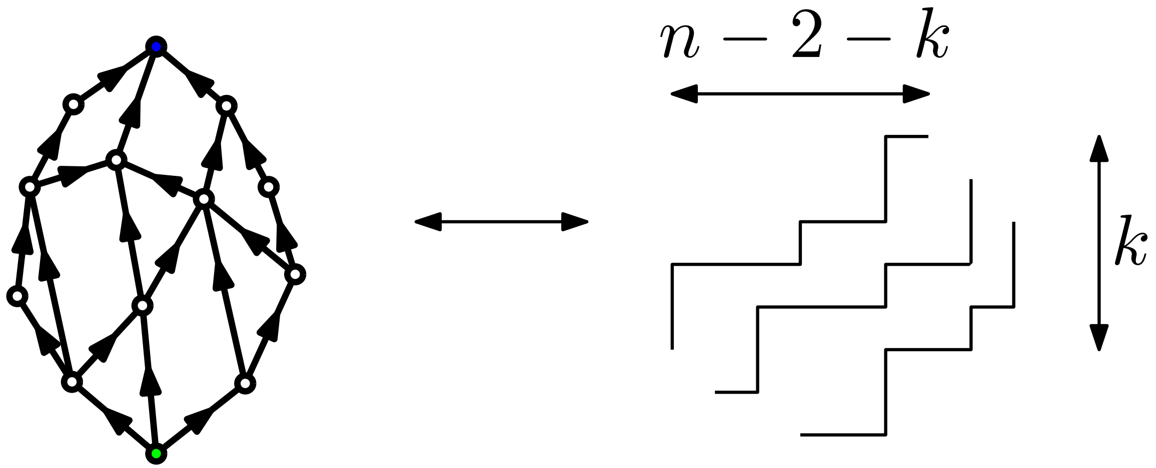
The number b_n of bipolar orientations with $n-1$ edges is

$$b_n = \frac{2}{n^2(n+1)} \sum_{k=0}^{n-1} \binom{n+1}{r-1}\binom{n+1}{r}\binom{n+1}{r+1}$$

Baxter numbers

cf bijections

$k+2$ vertices
$n-k$ faces



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b_n also counts many other classes (pattern-avoiding permutations, square tilings, etc.)

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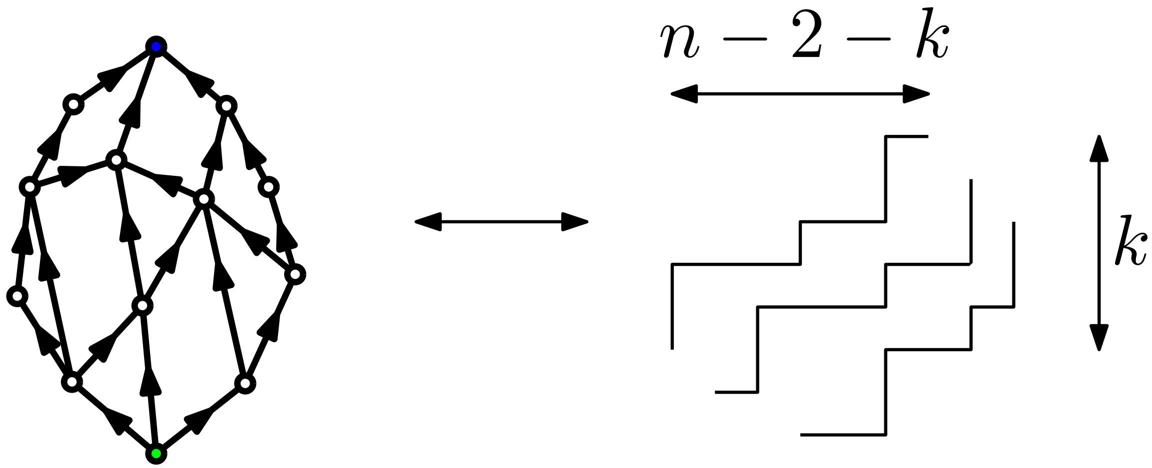

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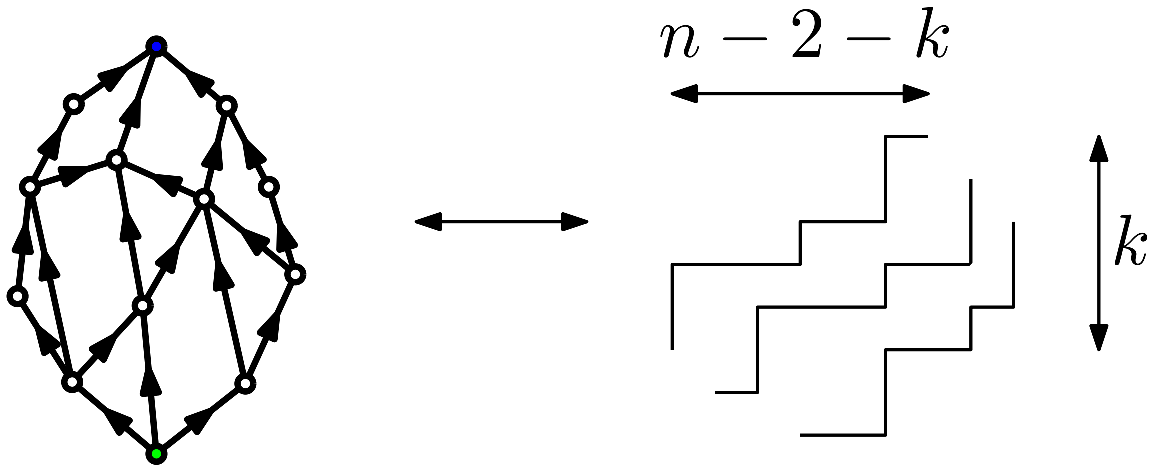

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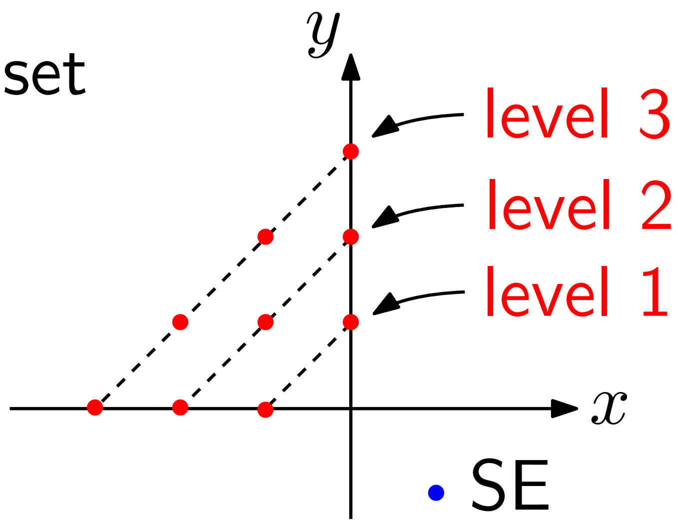
We show a bijection by Kenyon, Miller, Sheffield and Wilson
with lattice walks in quadrant (+control on face degrees)

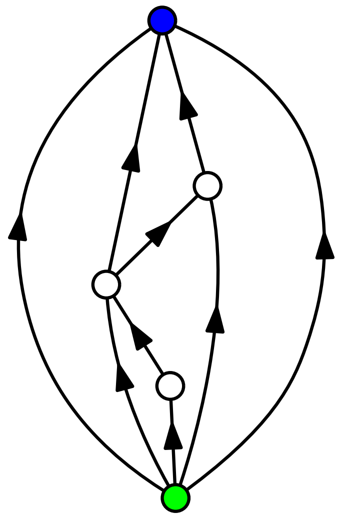
explains universality of n^{-4} for bipolar ori. + appli. to lattice walk enumeration

The Kenyon et al. bijection

tandem walks

step set

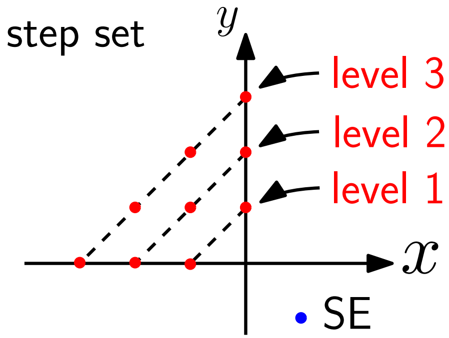


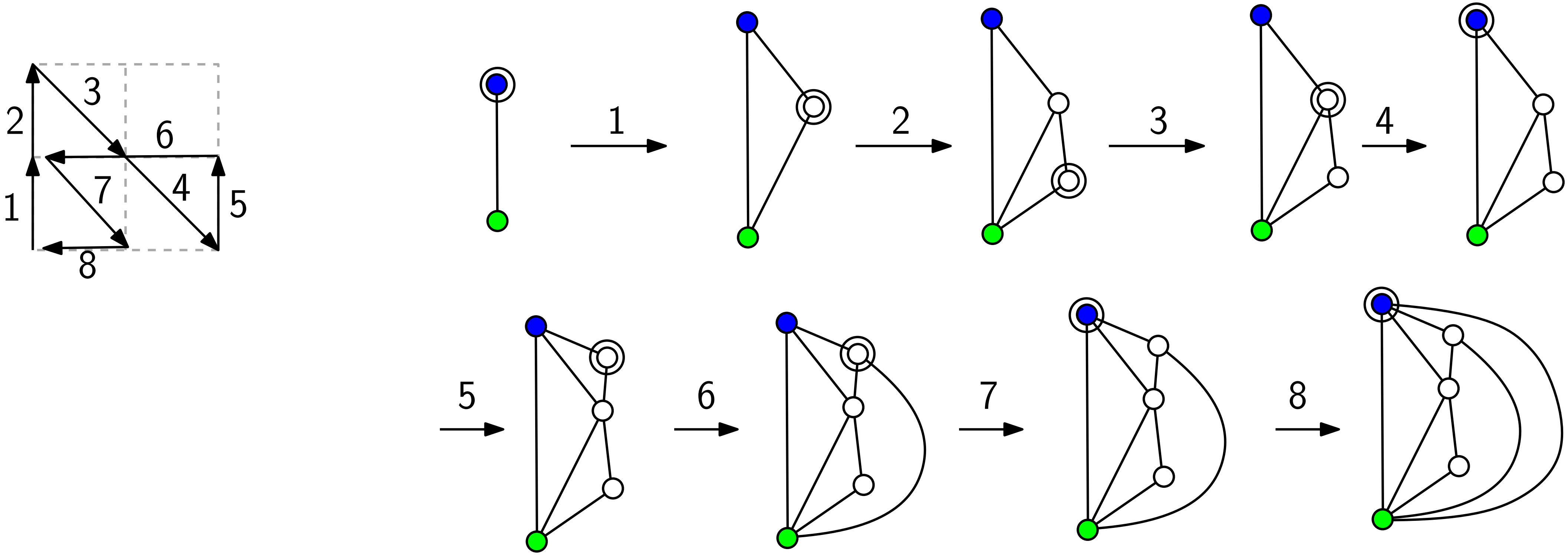


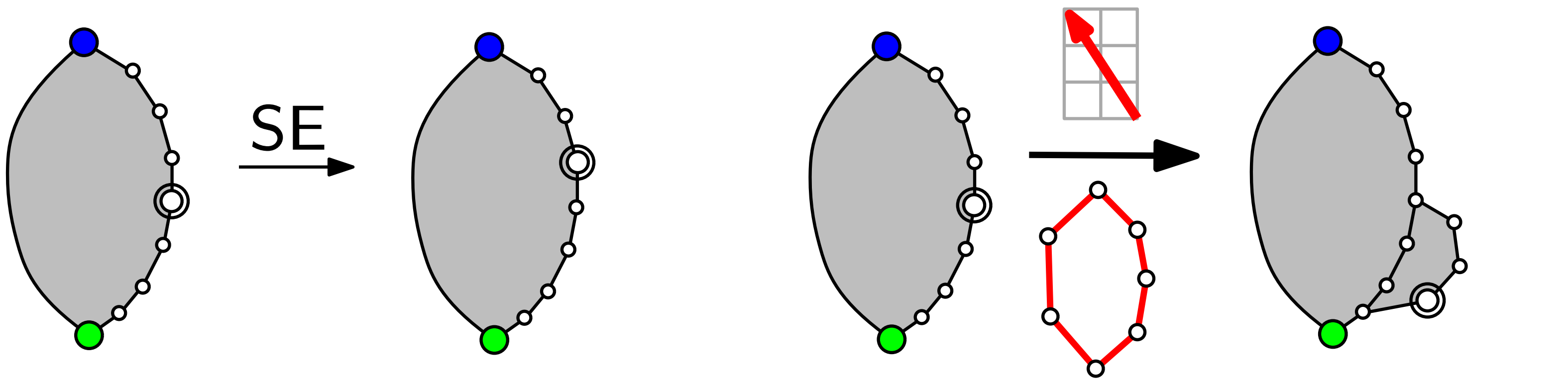
Tandem walks in quadrant (start & end at 0) $\xleftrightarrow{\text{bijection}}$ bipolar orientations inside bi-gon

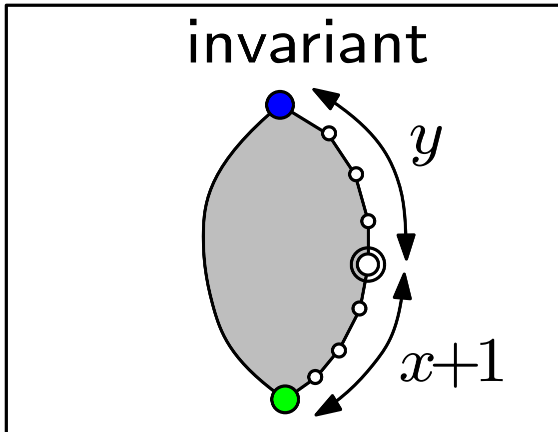
length n	$\longleftrightarrow$	$n+1$ edges
step level r	$\longleftrightarrow$	inner face of degree $r+2$
SE step	$\longleftrightarrow$	vertex $\notin \{S, N\}$

The Kenyon et al. bijection

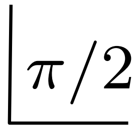


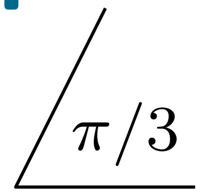






Consequences of the bijection

- The linear mapping that sends 

to 

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$\Rightarrow$ universality of the subexponential order n^{-4} for bipolar orientations

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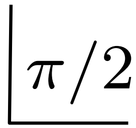
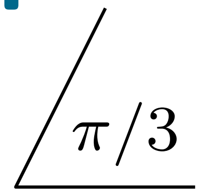
- The linear mapping that sends 

to 

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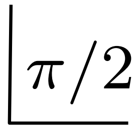
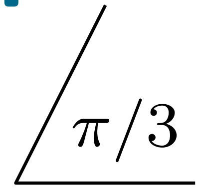
- The linear mapping that sends 

to 

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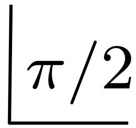
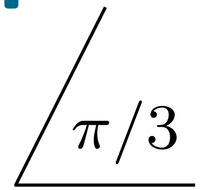
- The linear mapping that sends 

to 

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proof using the extended version of the bijection

(also possible by kernel method for walks with large steps [Bostan, Bousquet-Mélou, Melczer'18])