Planar maps: bijections and applications

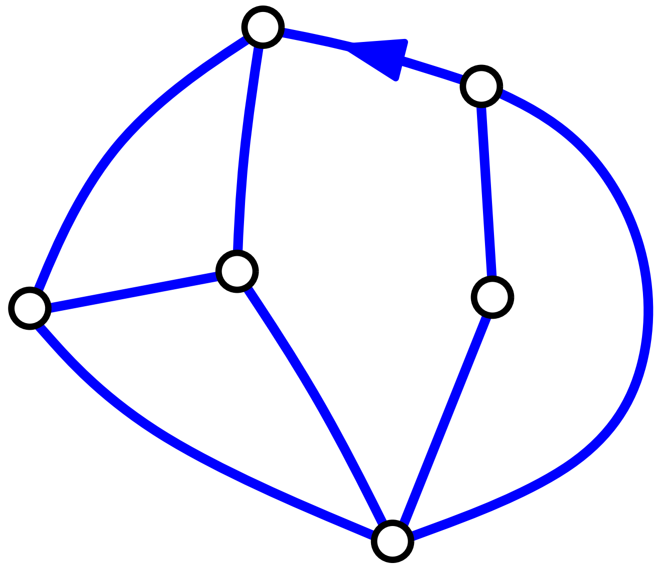
Éric Fusy (CNRS/LIX)

AEC summer school, Hagenberg, 2018

Rooted maps

A map is **rooted** by marking and orienting an edge

a rooted map

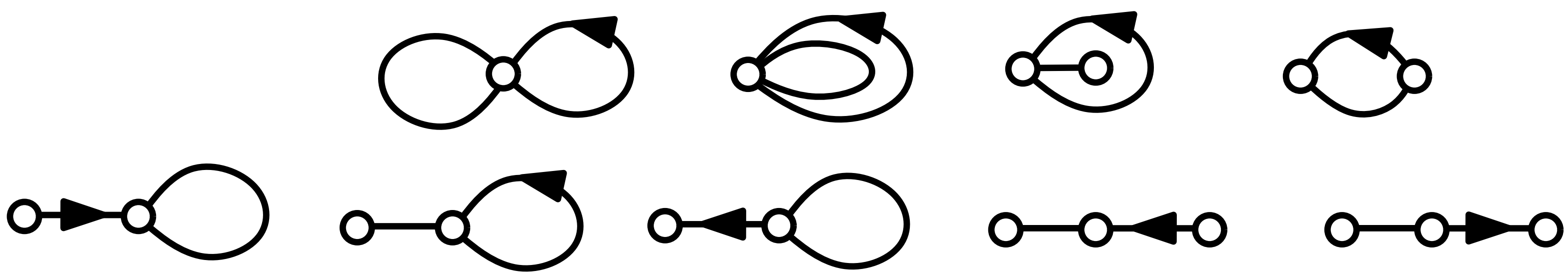


the face on the right of the root is taken as the outer face

Rooted maps are combinatorially easier than maps
(no symmetry issue, root gives starting point for recursive decomposition)

The 2 rooted maps with one edge

The 9 rooted maps with two edges



Counting rooted maps

Let a_n be the number of rooted maps with n edges

n	1	2	3	4	5	6	7	$\cdots$
a_n	2	9	54	378	2916	24057	208494	$\cdots$

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Theorem: (Tutte'63)

$$\frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$

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Not an isolated case:

- Triangulations ($2n$ faces)

Loopless: $\frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$ Simple: $\frac{1}{n(2n-1)}\binom{4n-2}{n-1}$

- Quadrangulations (n faces)

General: $\frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$ Simple: $\frac{2}{n(n+1)}\binom{3n}{n-1}$

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Bijective aspects of planar maps

Motivations for bijections

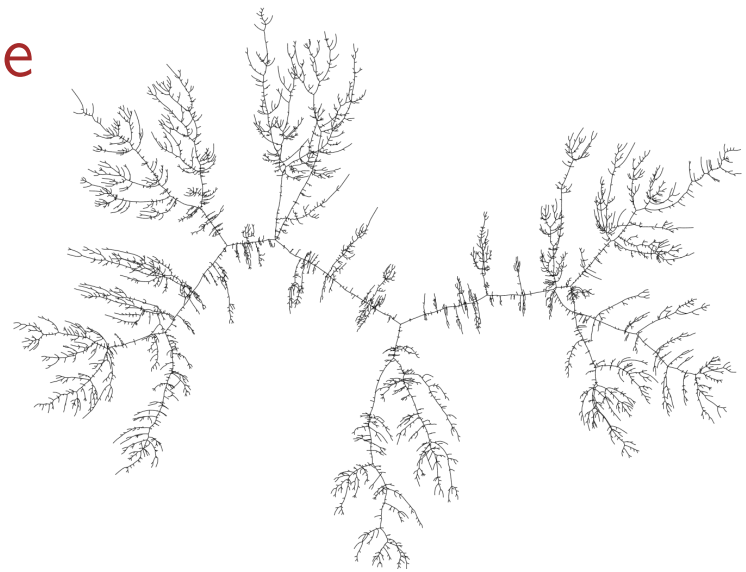
- efficient manipulation of maps (random generation algo.)

- key ingredient to study distances (diameter,...) in random maps
  - typical distances of order $n^{1/4}$ ($\neq n^{1/2}$ in random trees)
  - random map M with n edges $=$ random discrete metric space (M, d)

Theo: [Le Gall, Miermont'13]

$(M, \frac{1}{n^{1/4}} d)$ converges to a continuum random metric space called the Brownian map

large tree



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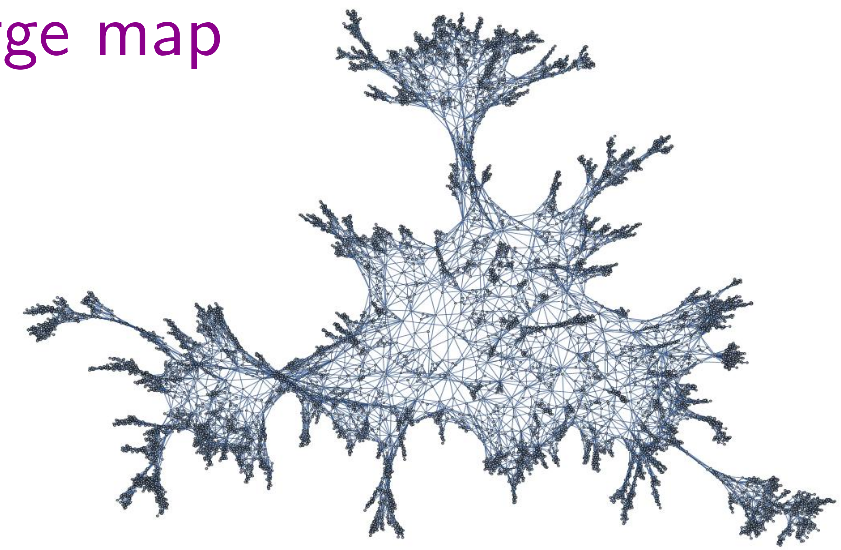
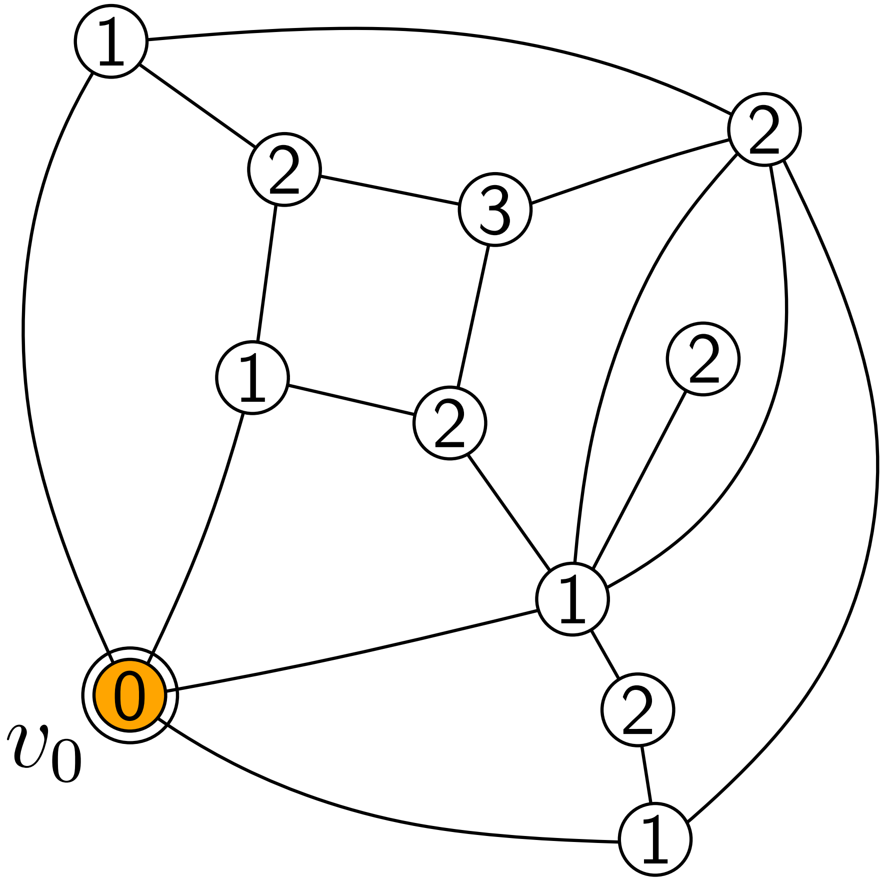


(analog for maps of the Continuous Random Tree)

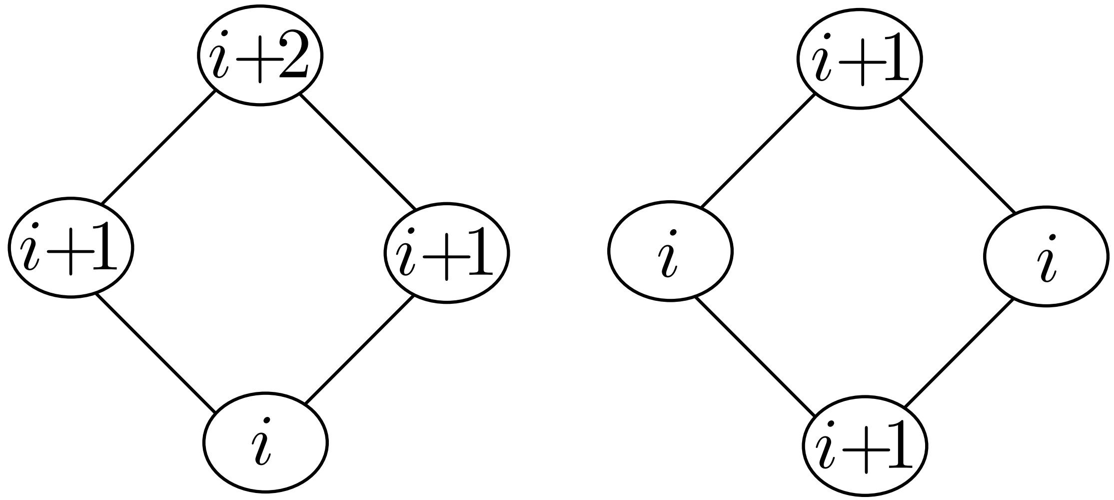
Pointed quadrangulations, geodesic labelling

Pointed quadrangulation = quadrangulation with a marked vertex v_0

Geodesic labelling with respect to v_0: $\ell(v) = \mathrm{dist}(v_0, v)$



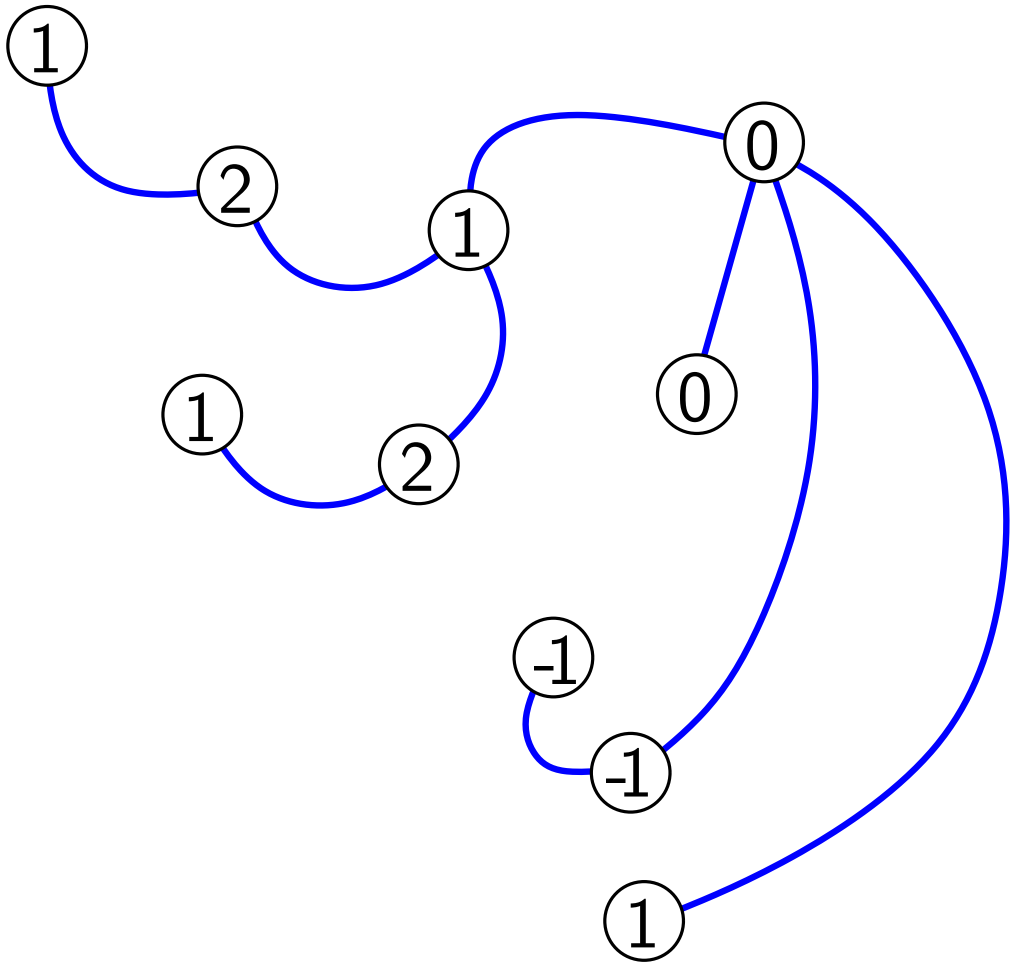
Rk: two types of faces



Well-labelled trees

Well-labelled tree = plane tree where

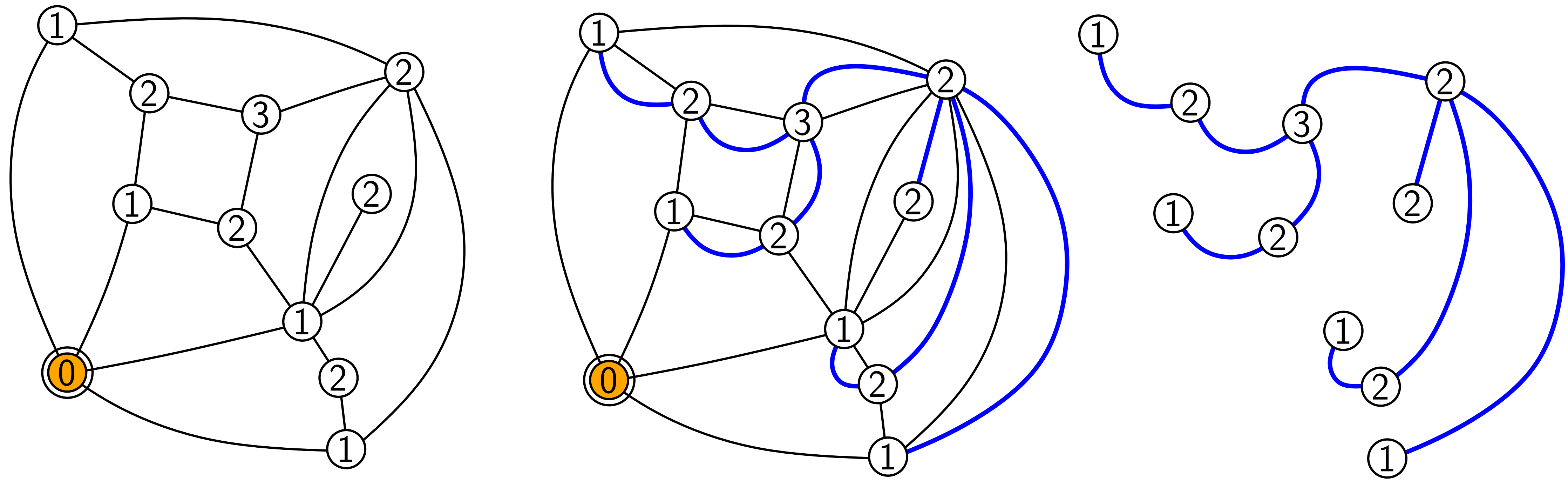
- each vertex v has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$



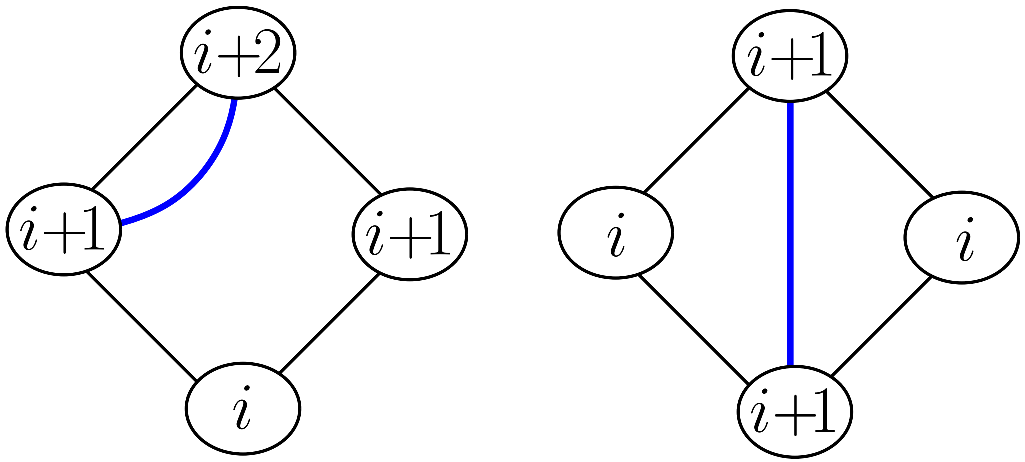
The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

Pointed quadrangulation ⇒ well-labelled tree with min-label=1

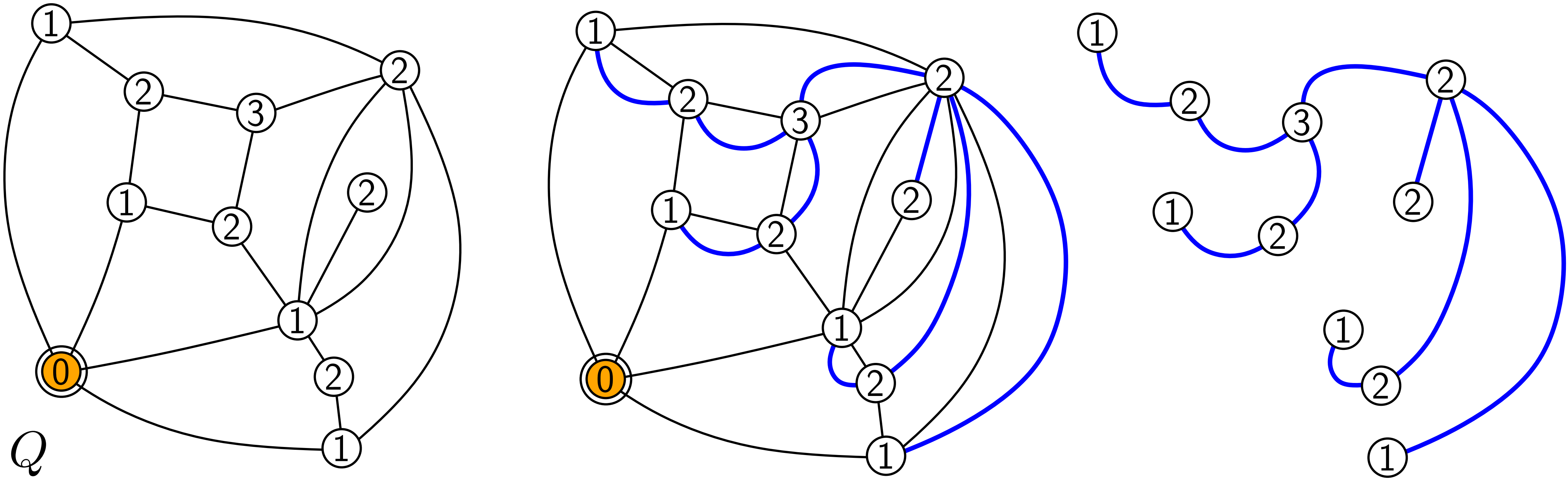
n faces ⇒ n edges



Local rule in each face:



Proof that it gives a tree



n faces
$n+2$ vertices

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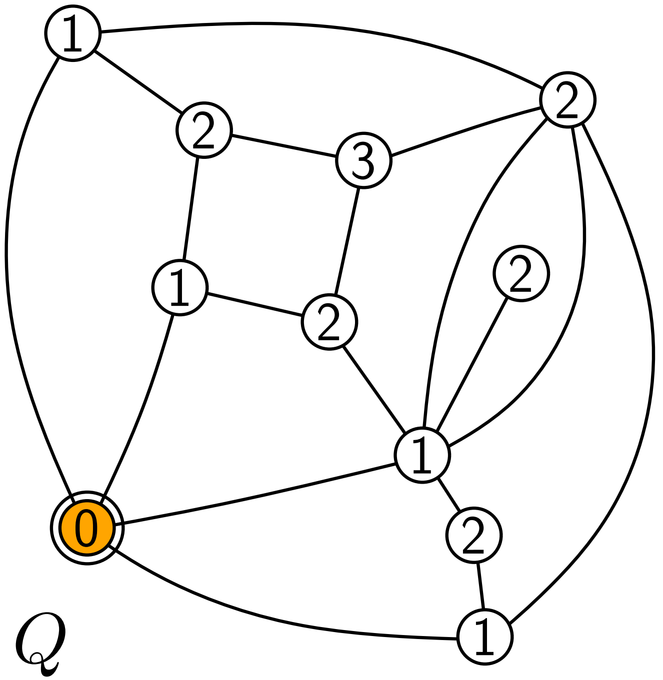

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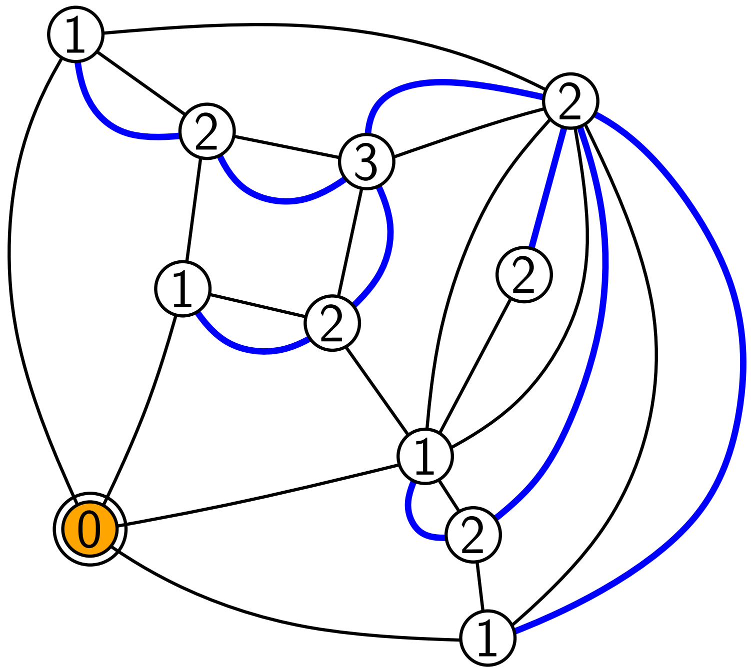

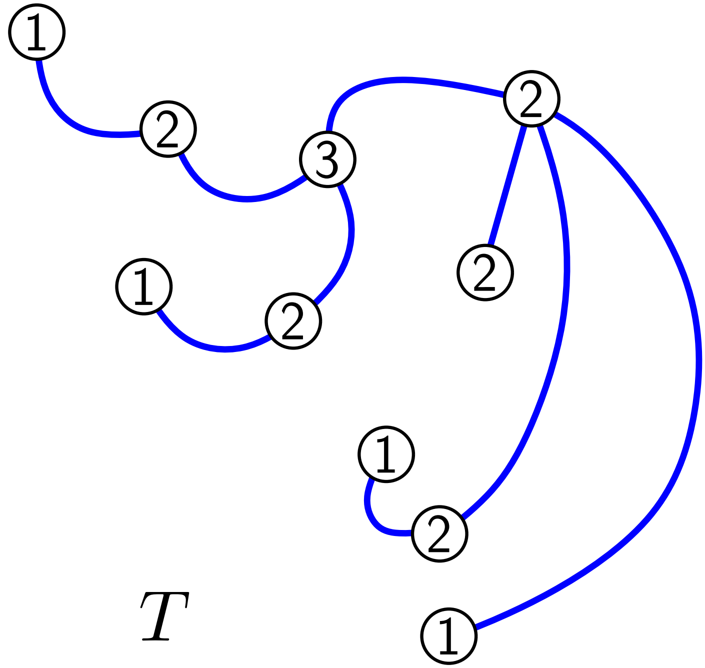

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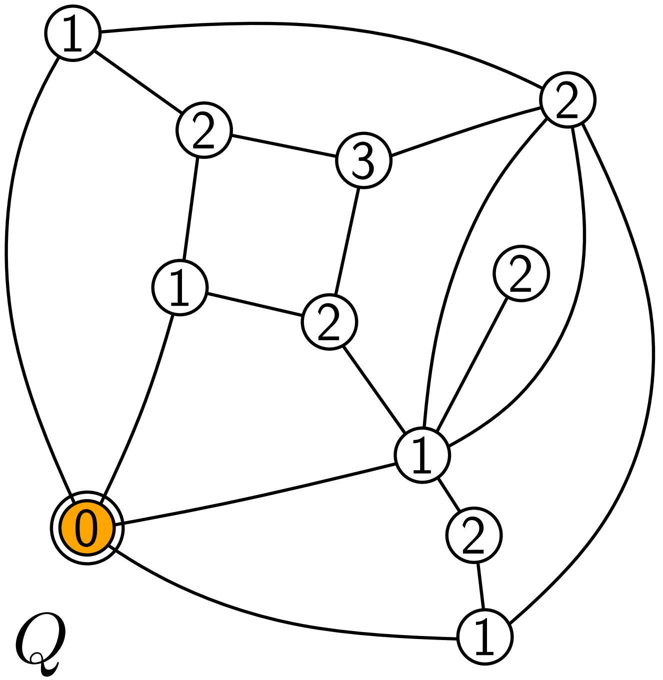

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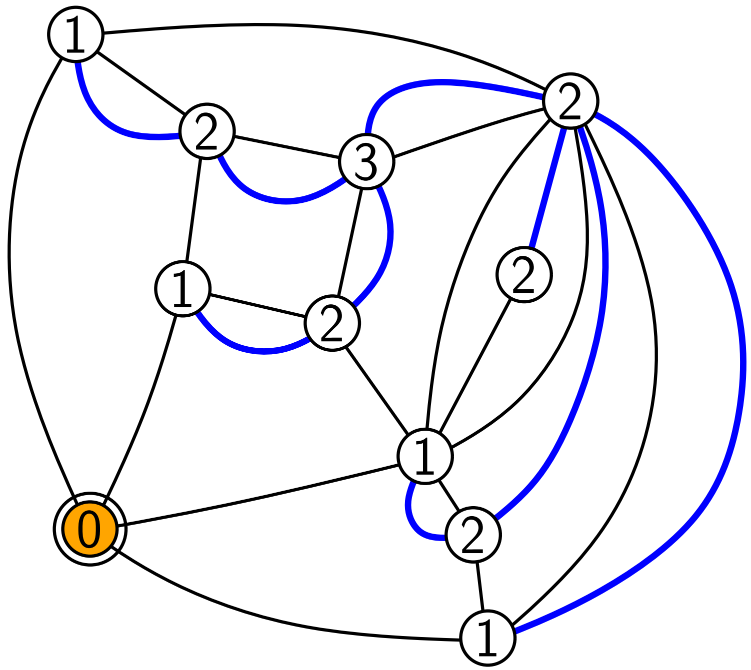

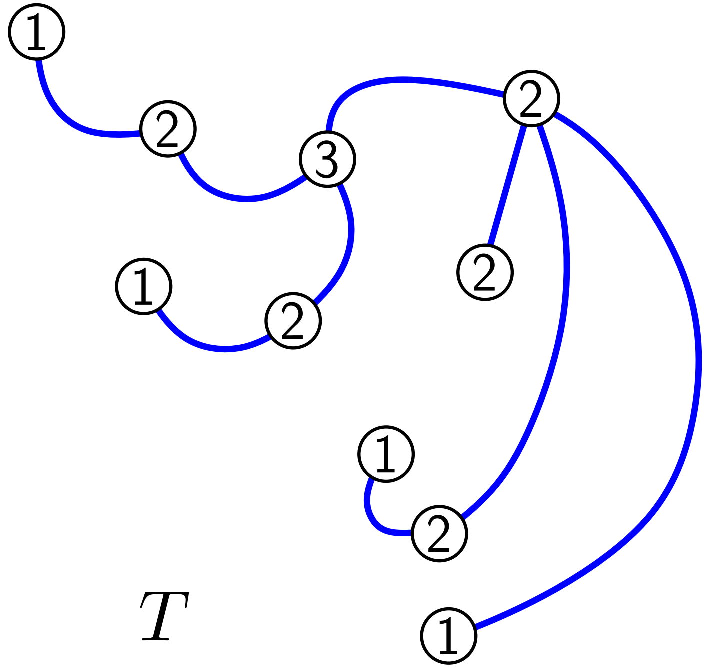

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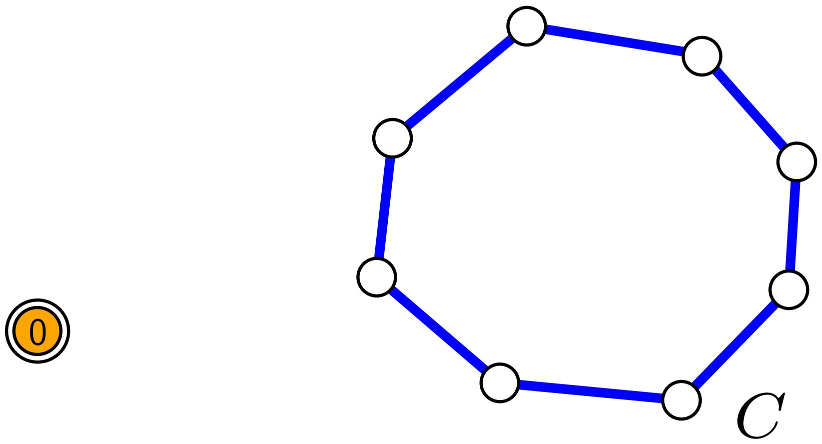
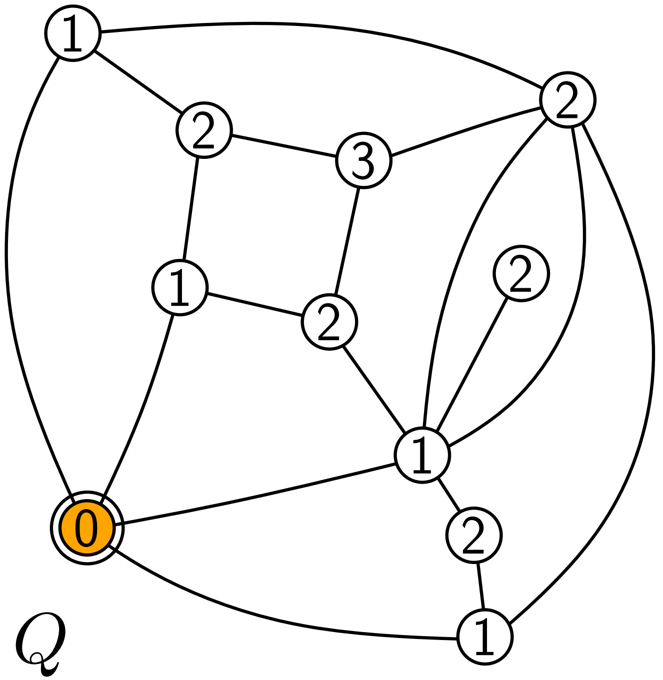


Assume that
T has a cycle C

Proof that it gives a tree



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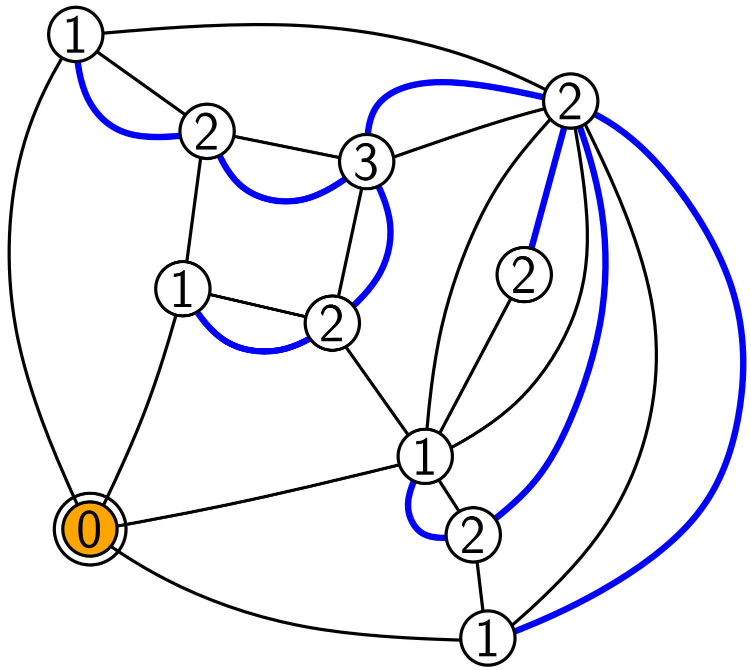

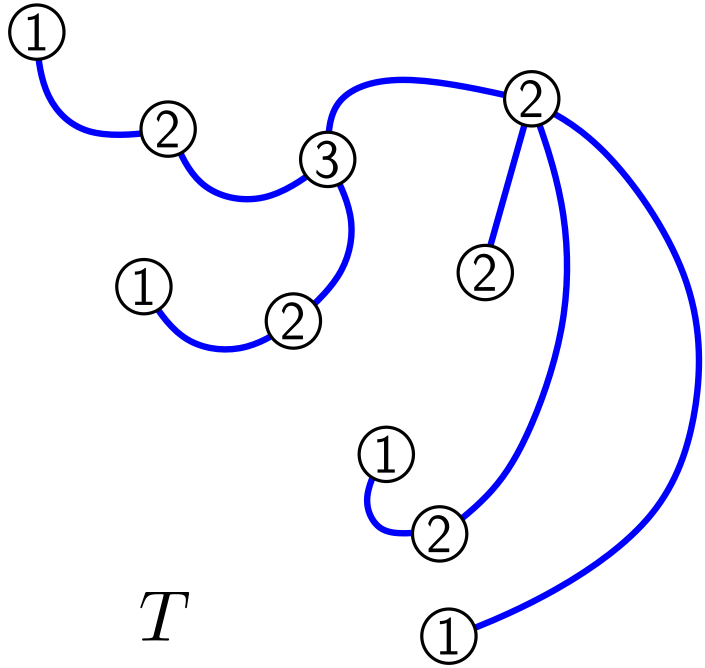

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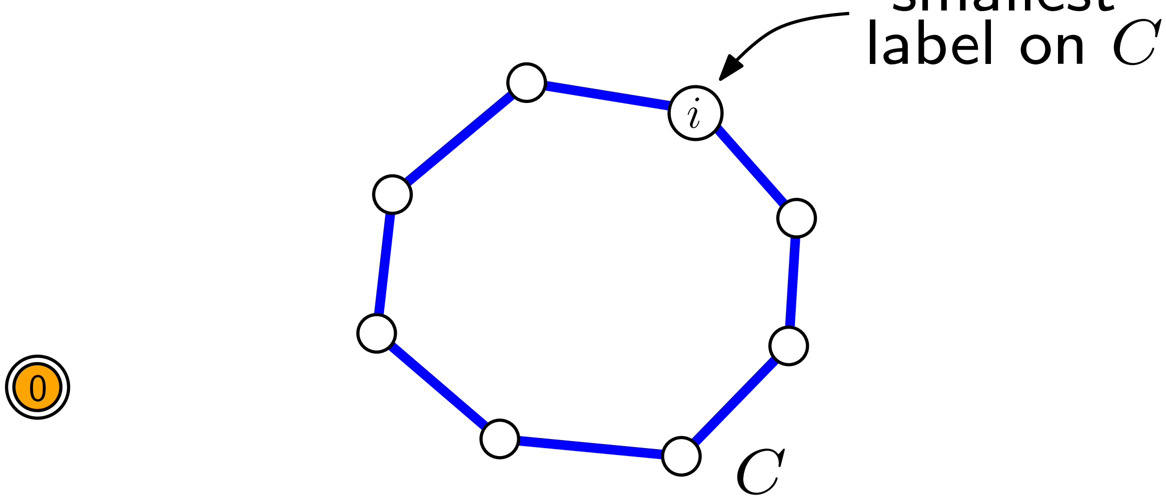

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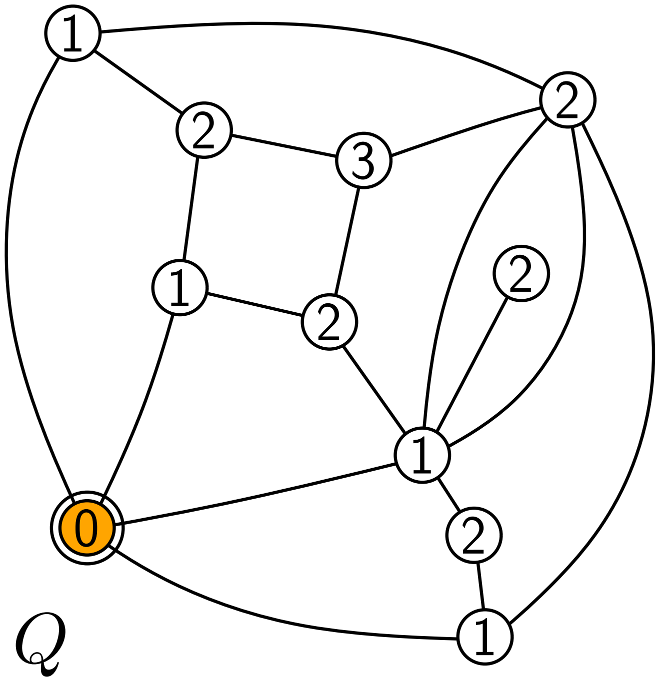

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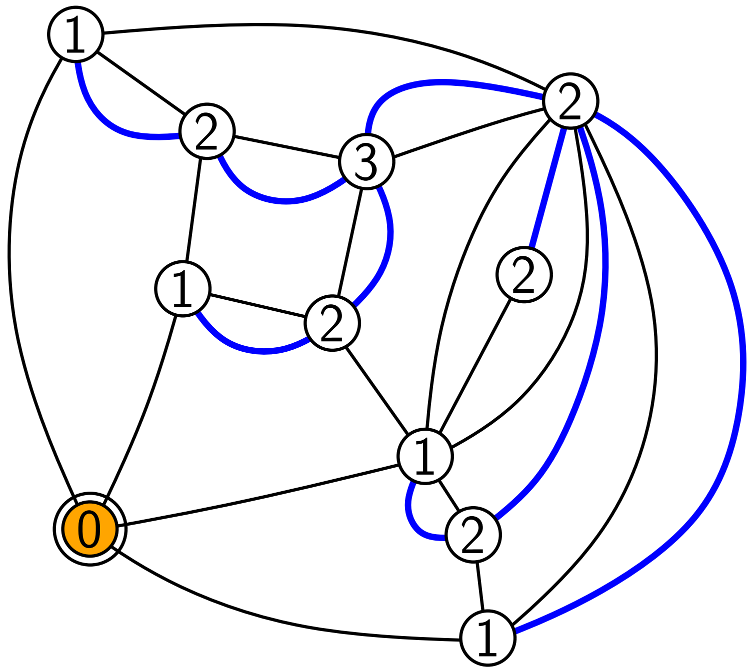

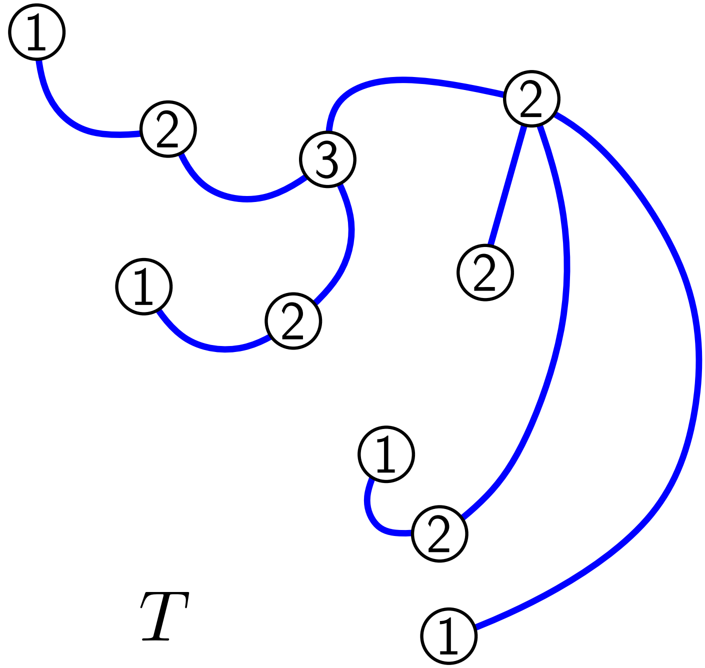

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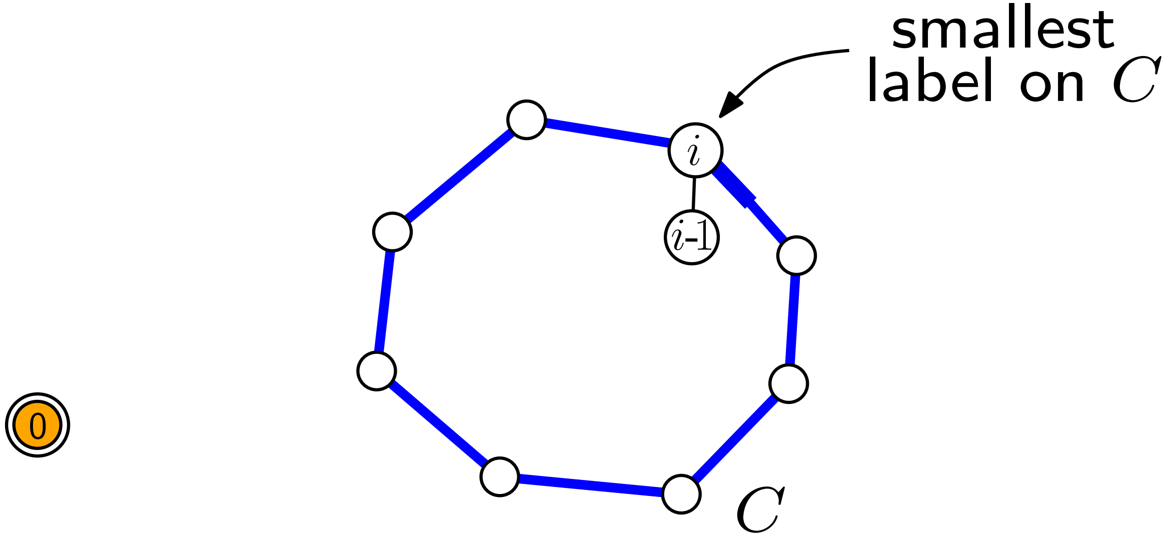

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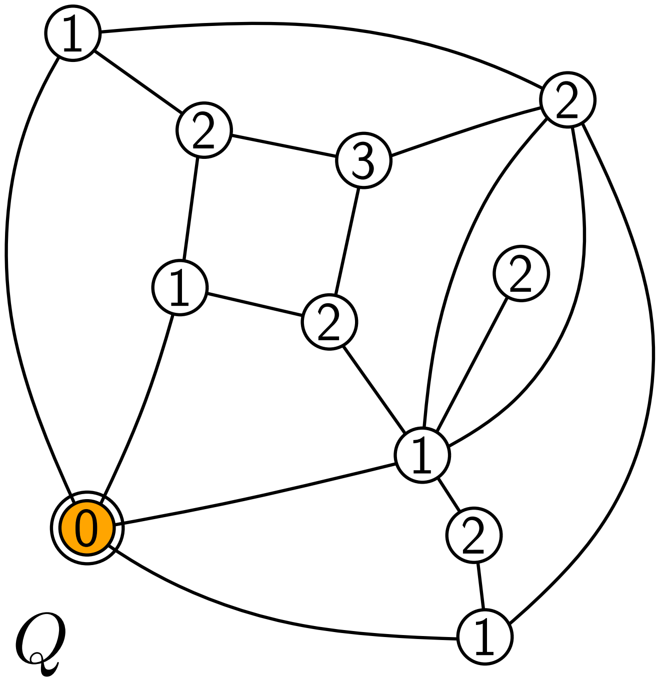

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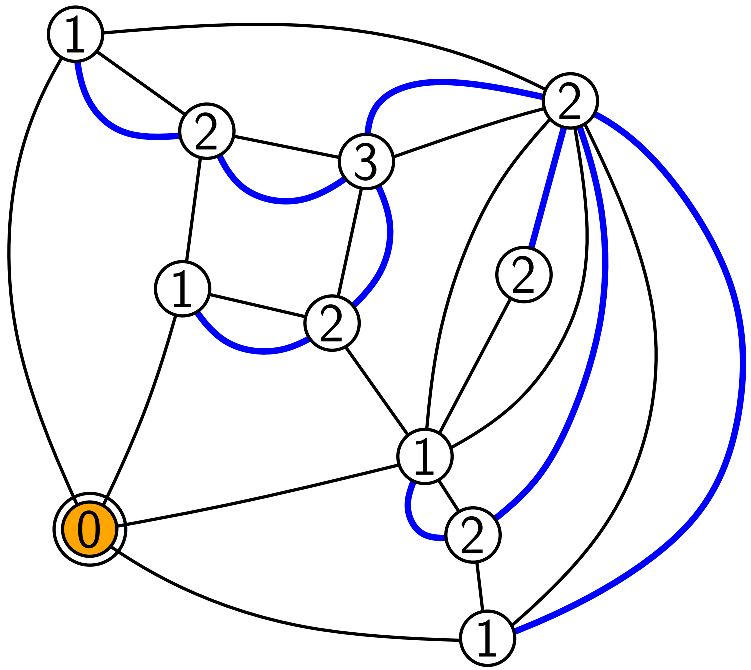

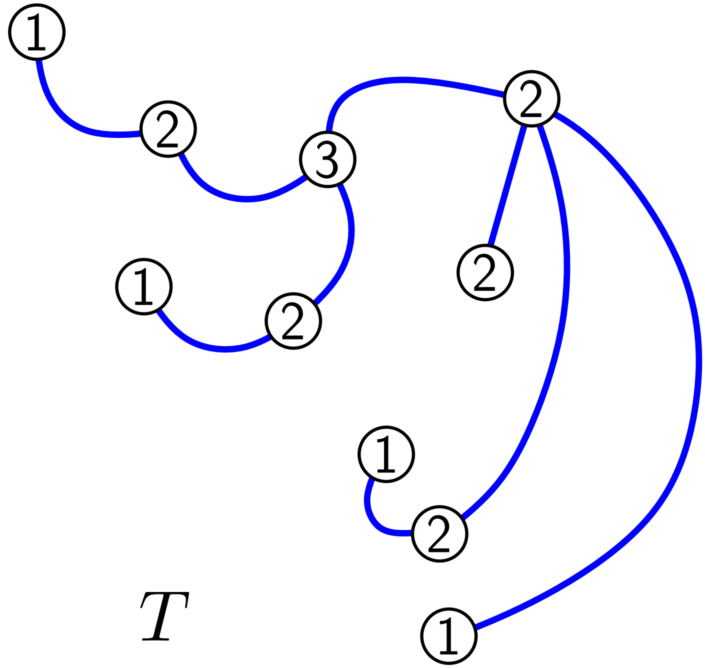

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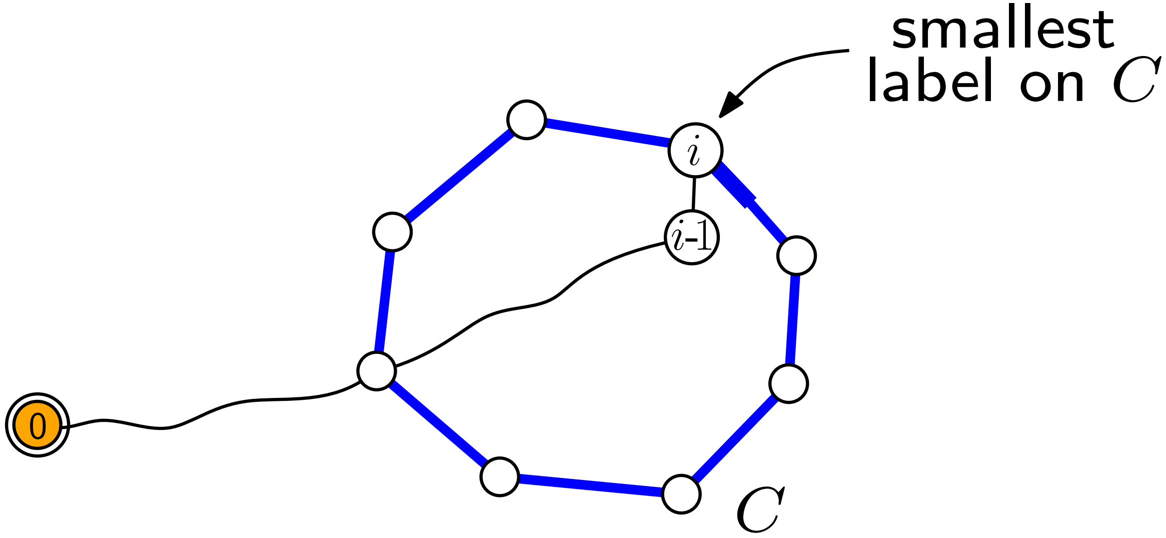

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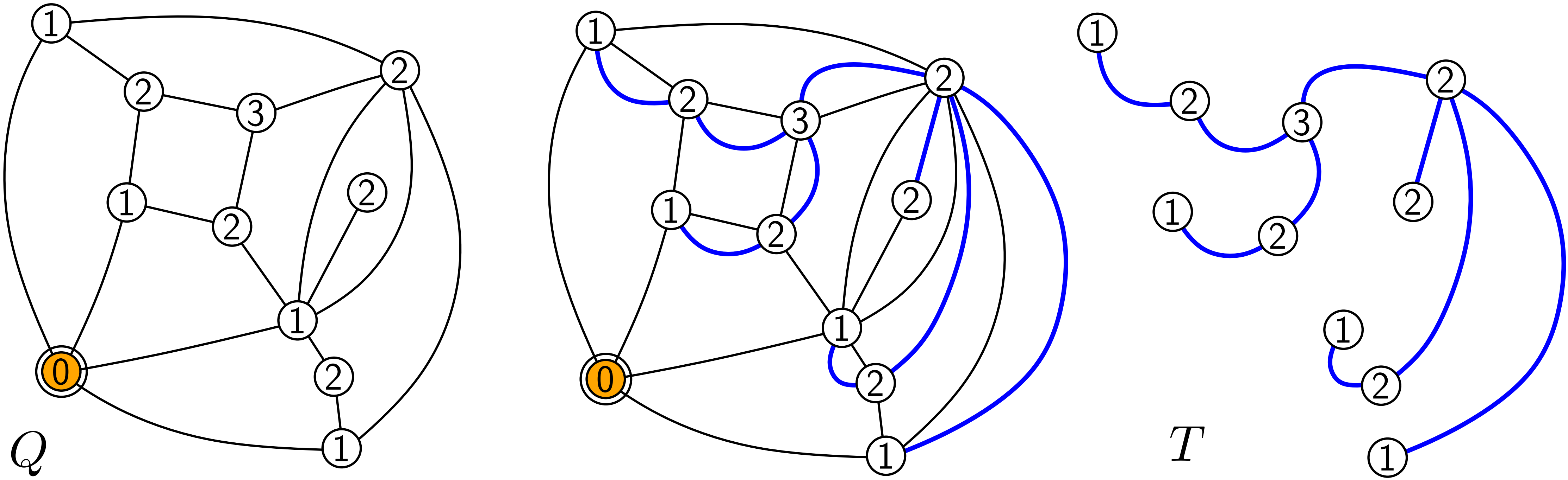

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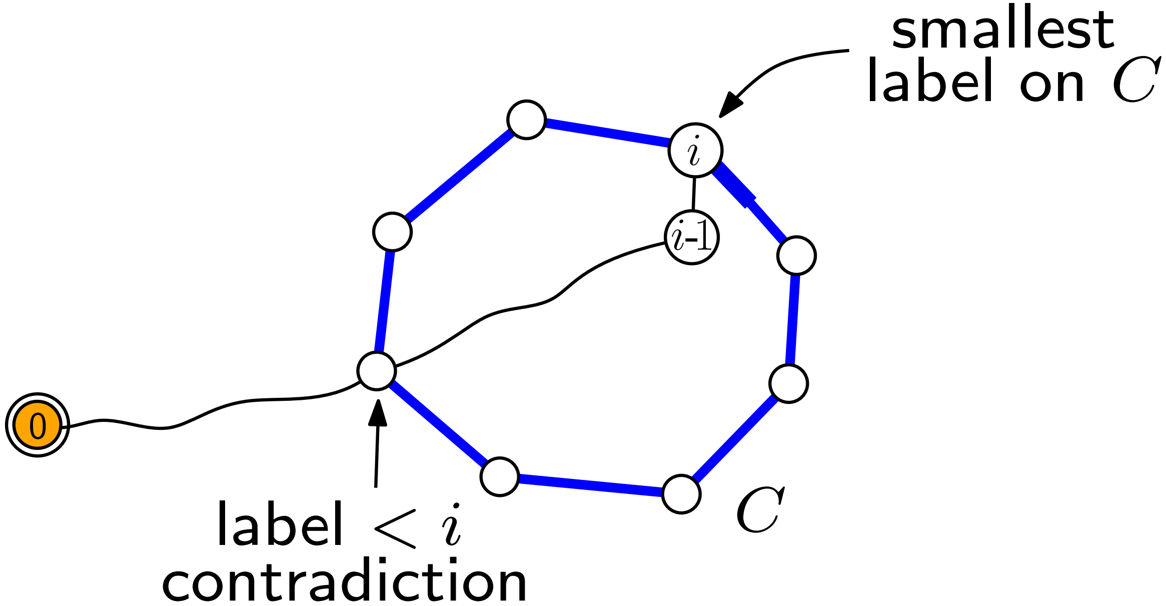

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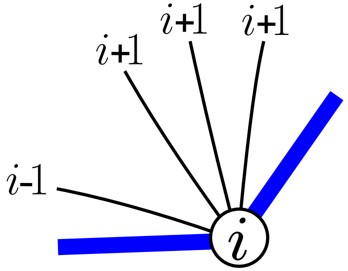
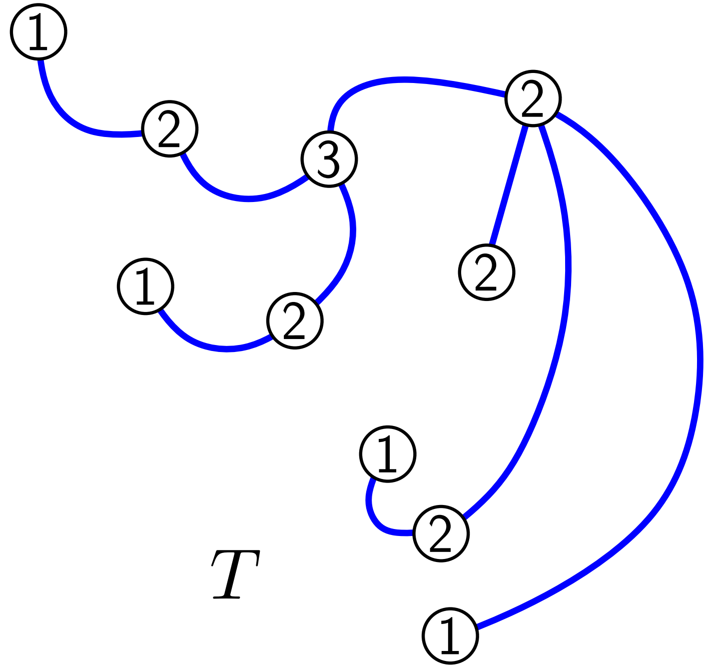
Rightmost geodesic paths

situation at a corner of the tree





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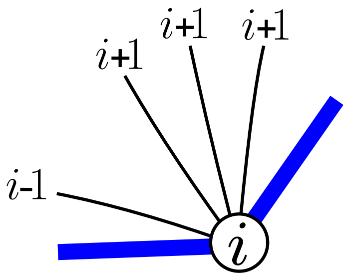

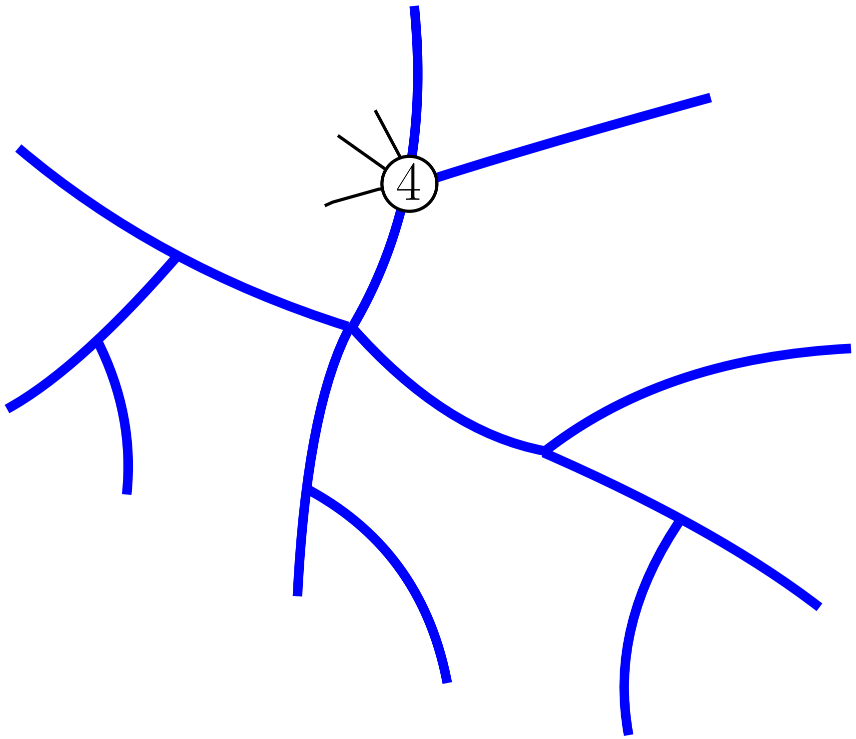

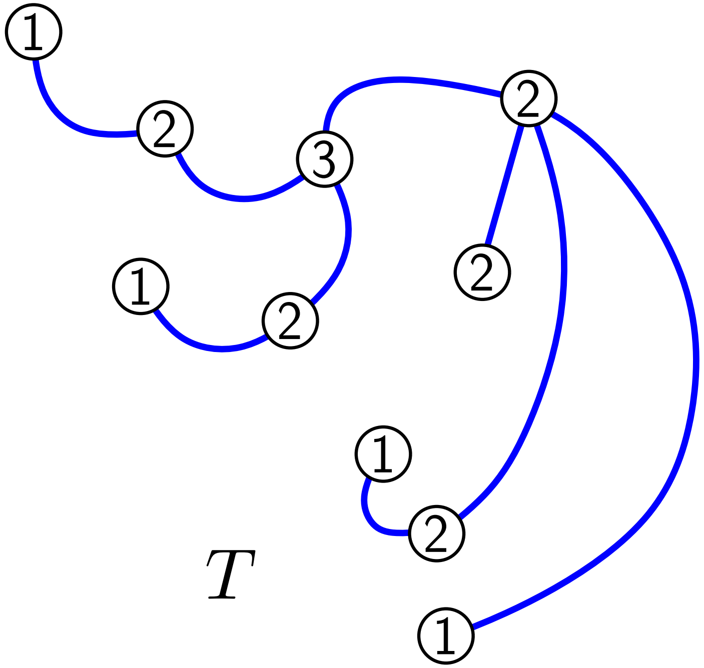

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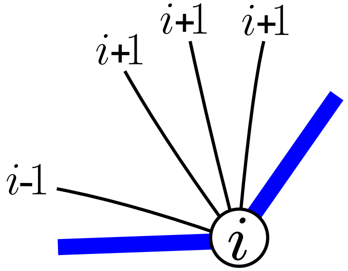

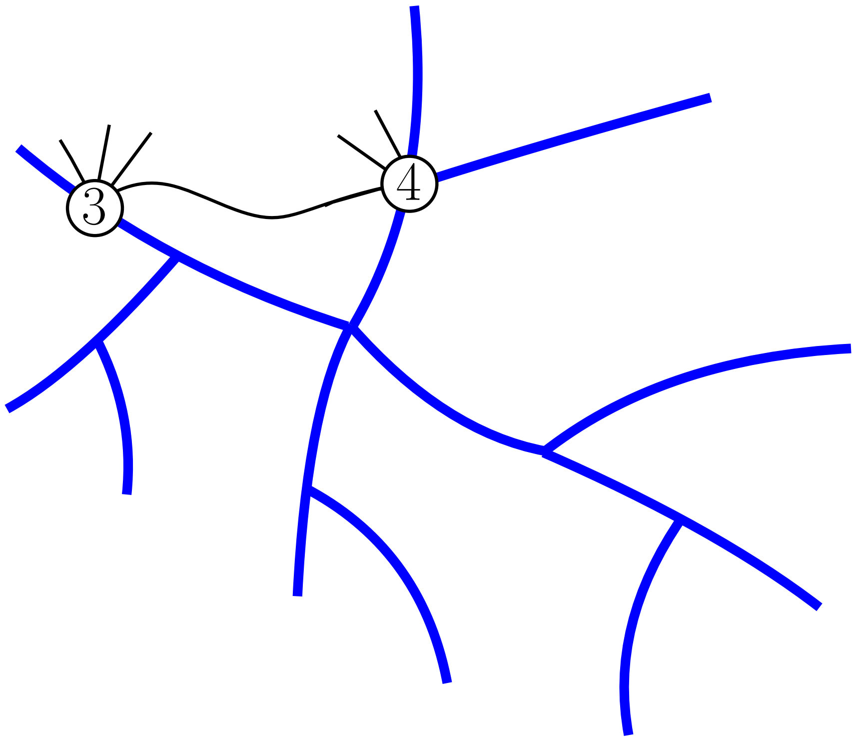

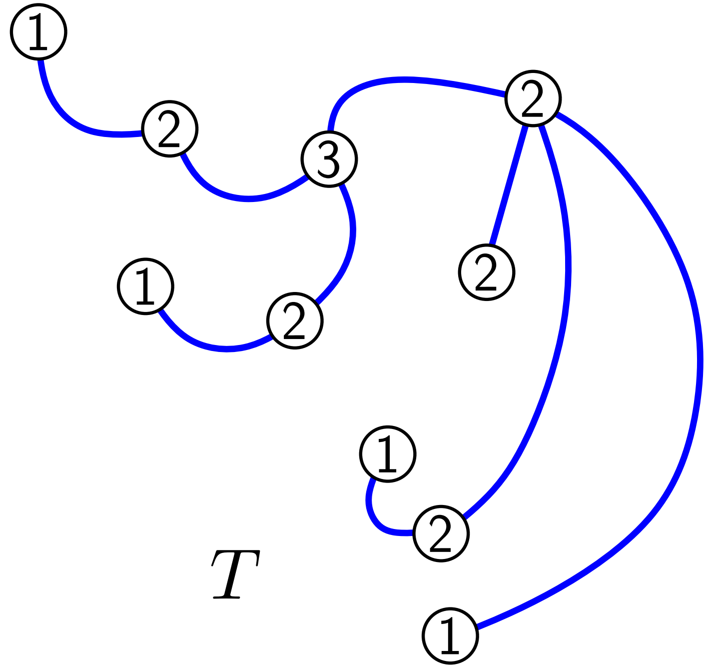

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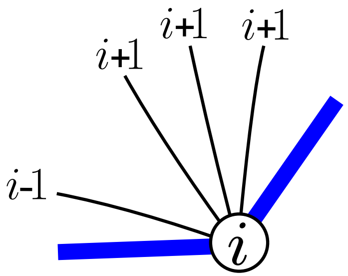

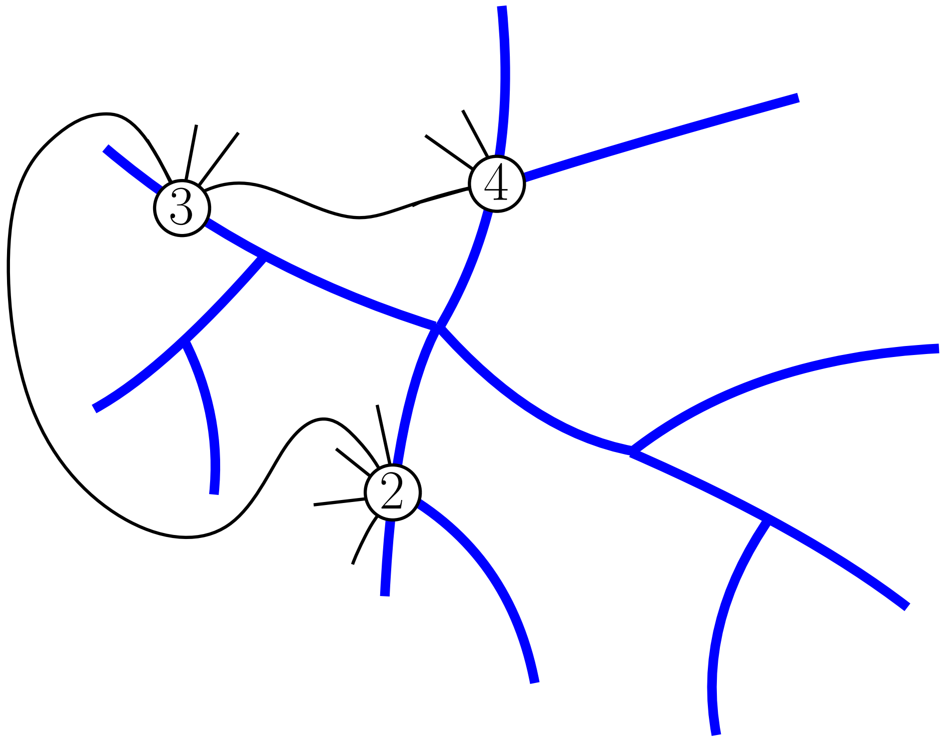

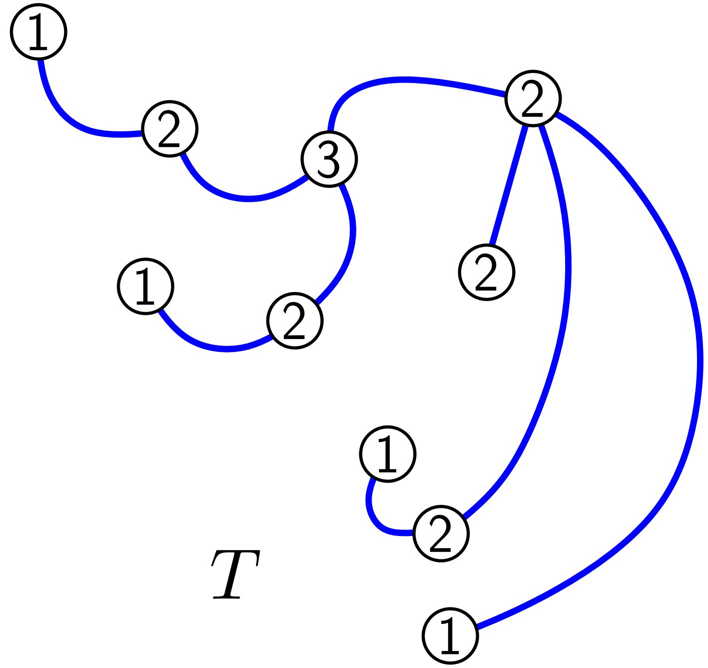

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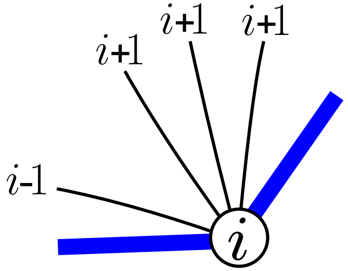

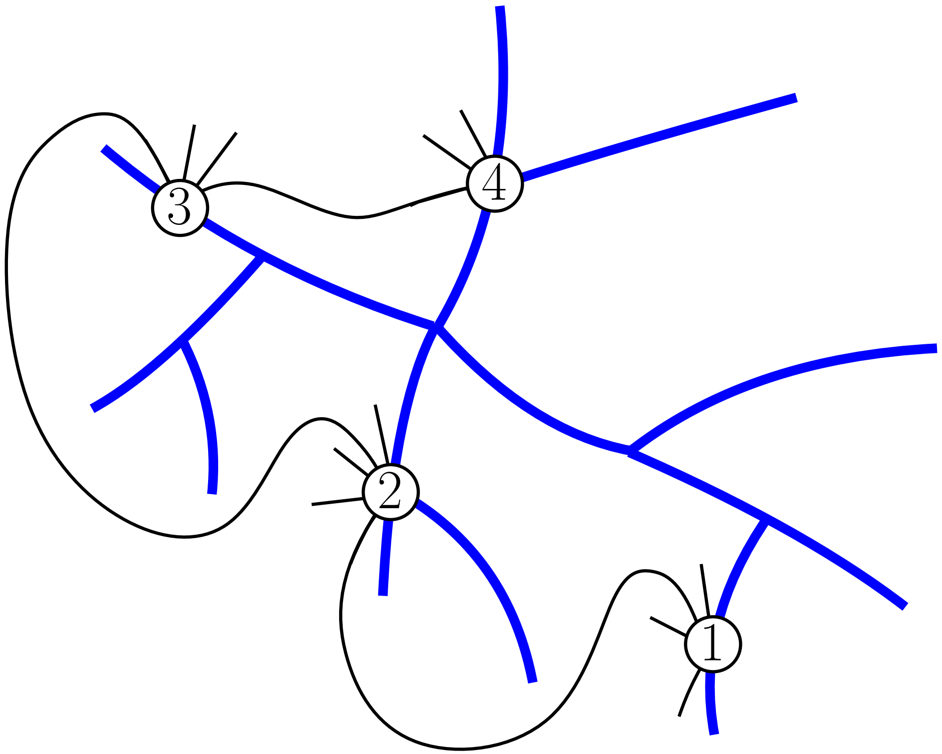

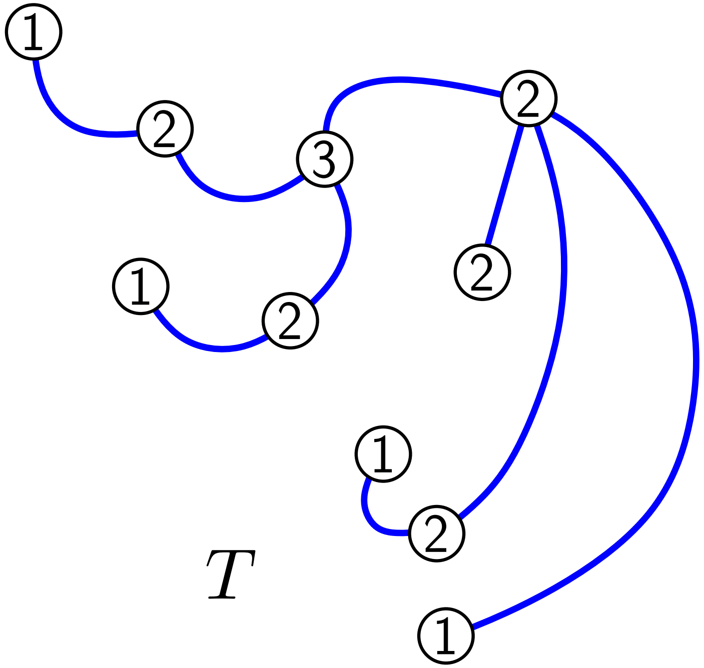

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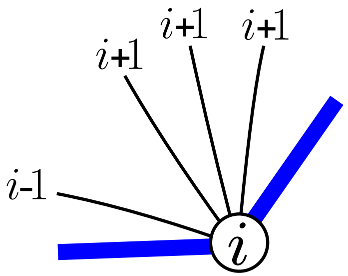

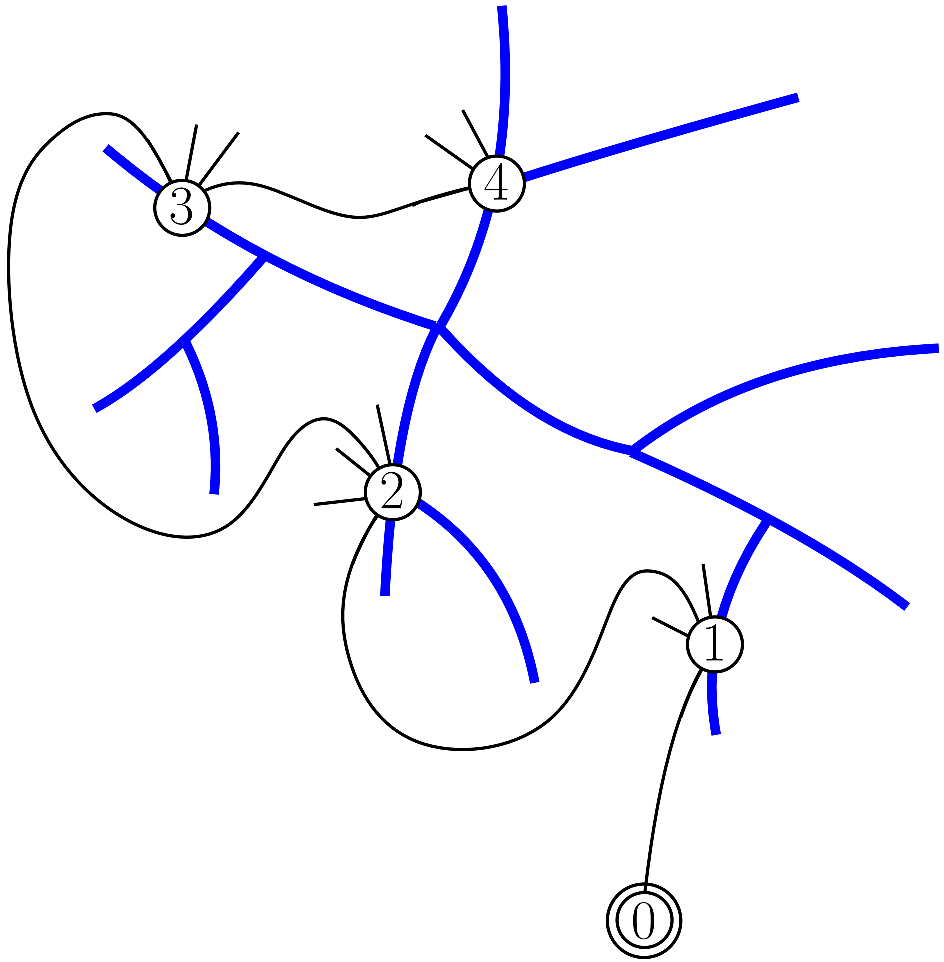

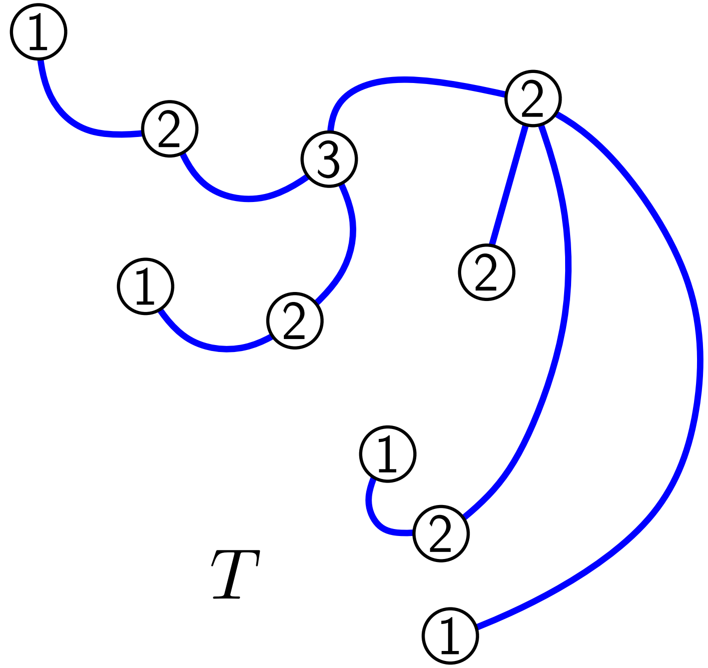

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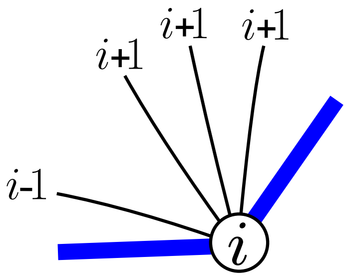
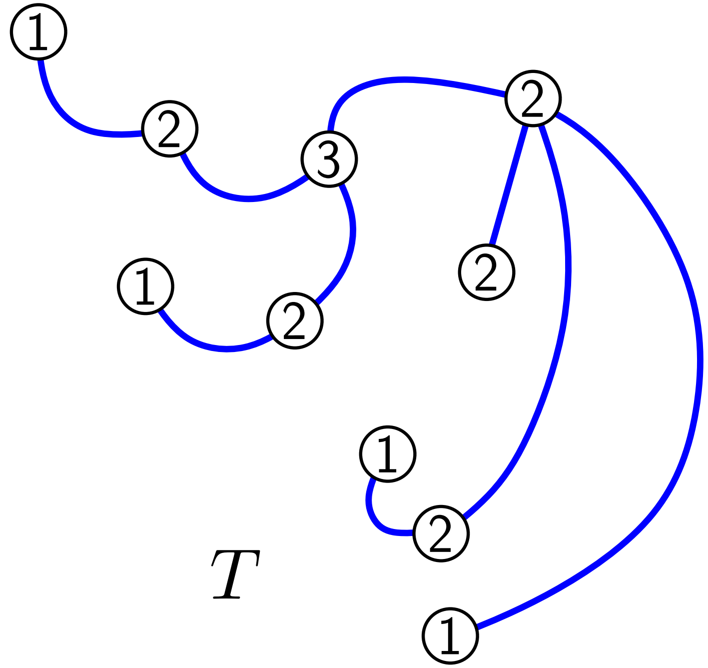
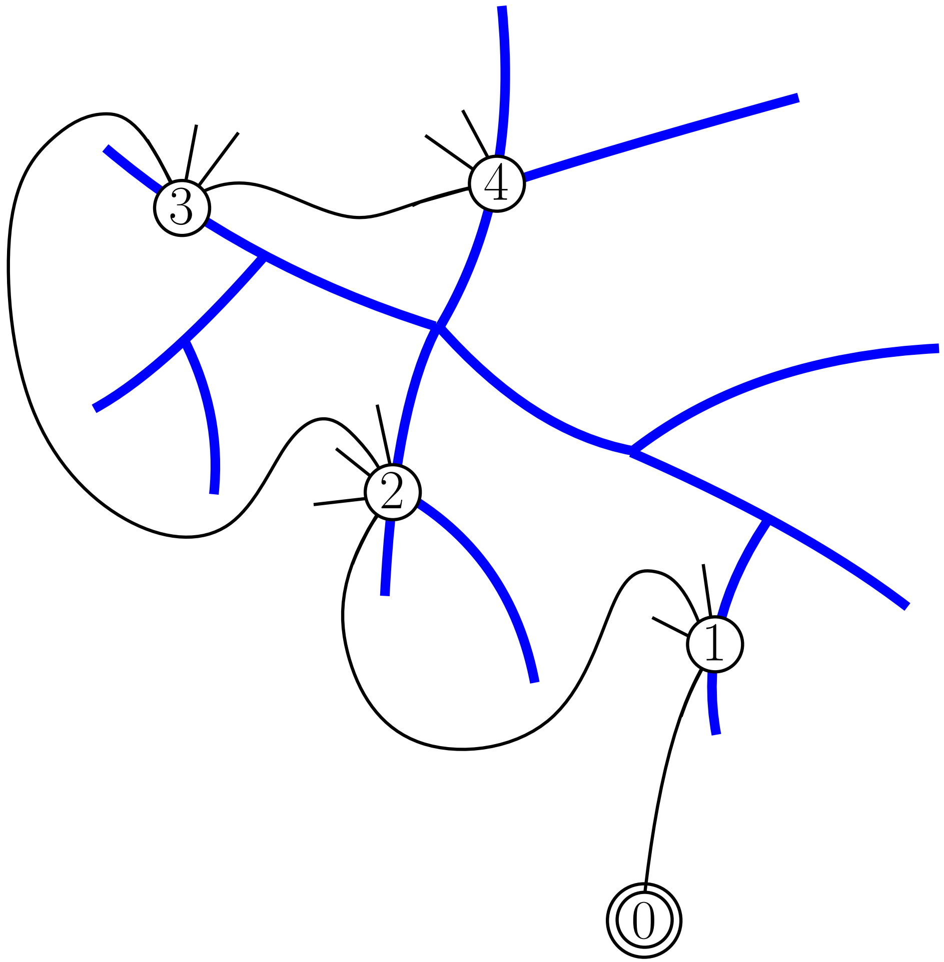
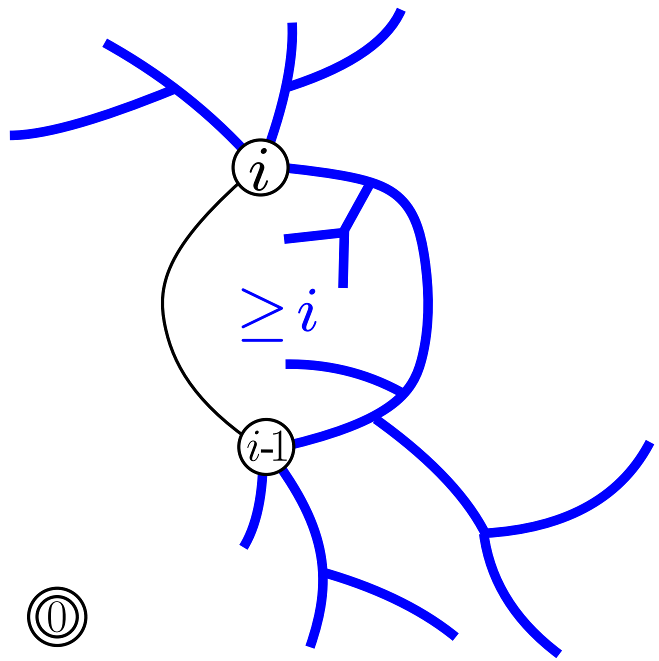




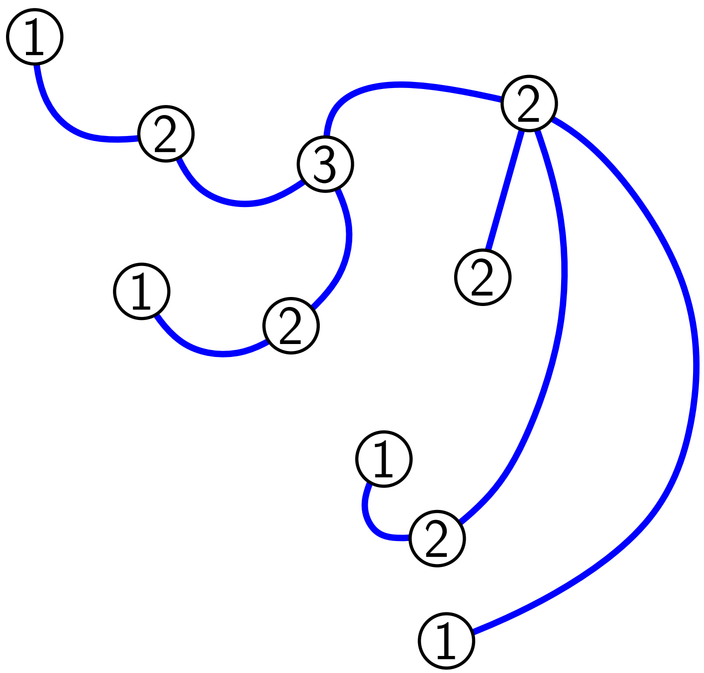


implies property



The inverse construction **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

From a well-labelled tree to a pointed quadrangulation



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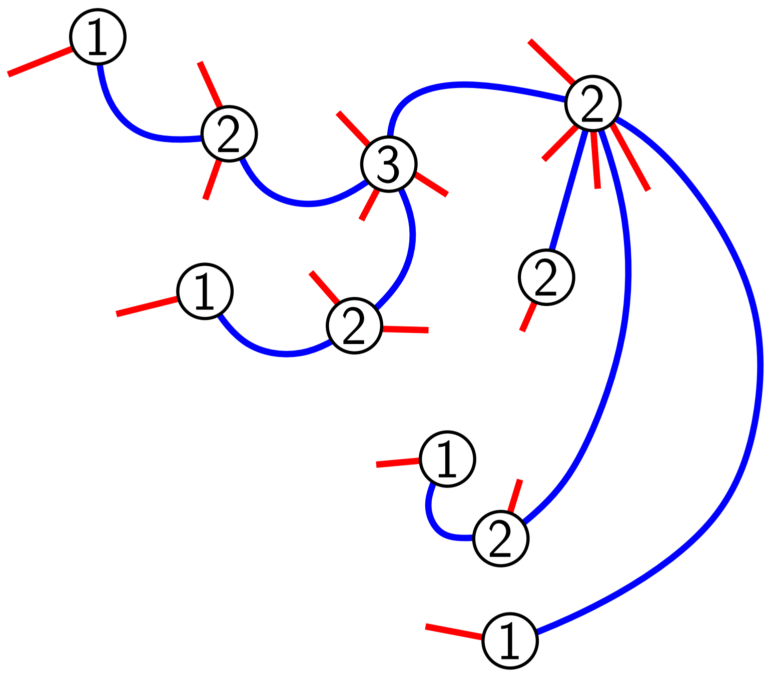

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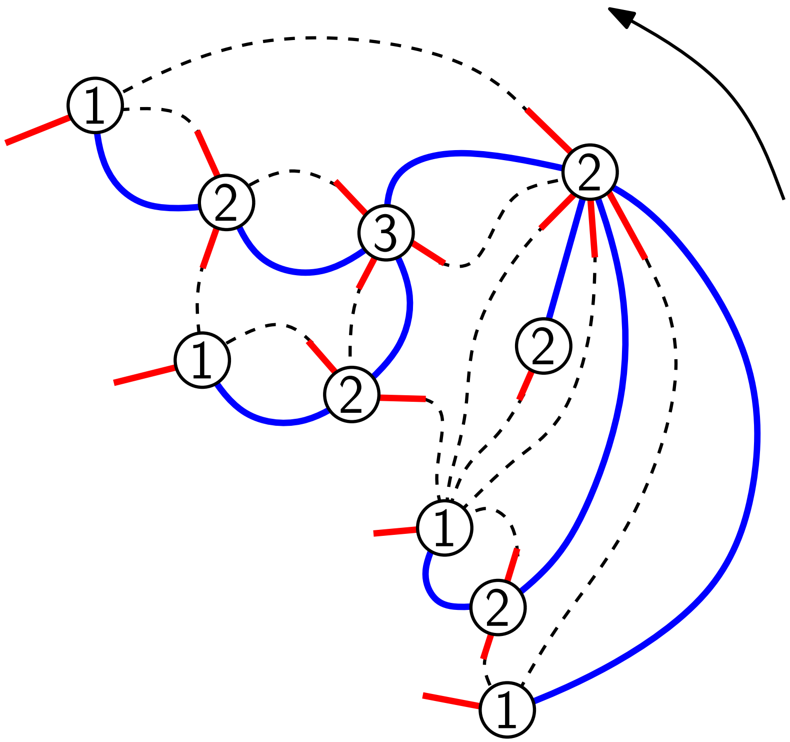

1) insert a "leg" at each corner

2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree

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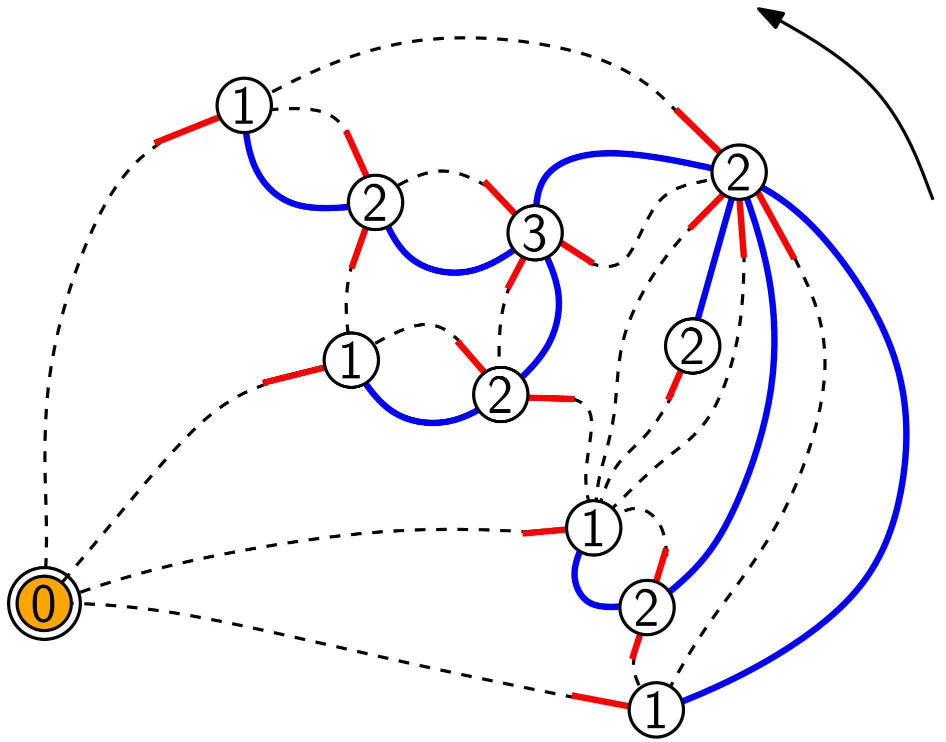

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3) create a new vertex v_0 outside and connect legs of label 1 to it

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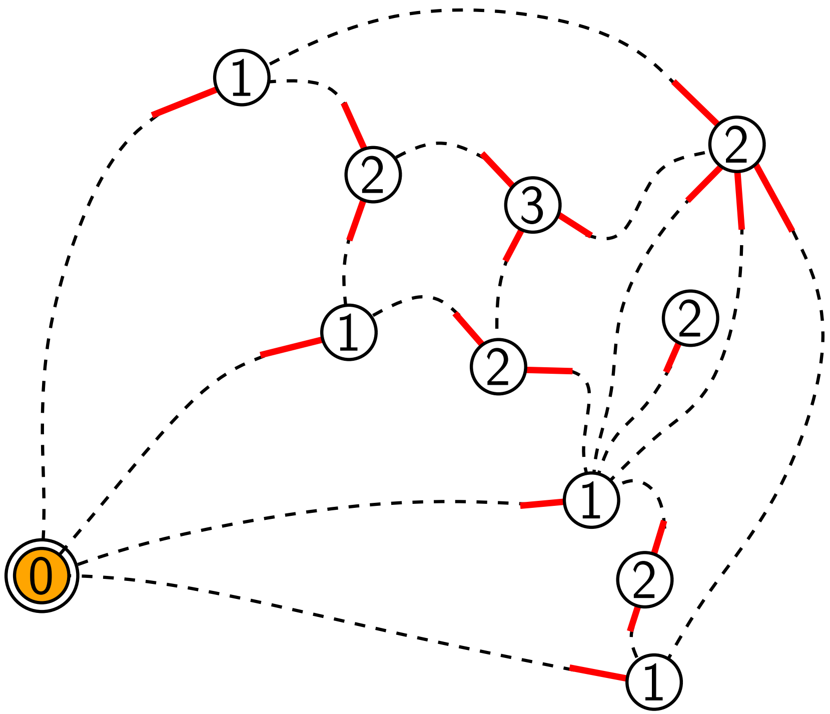

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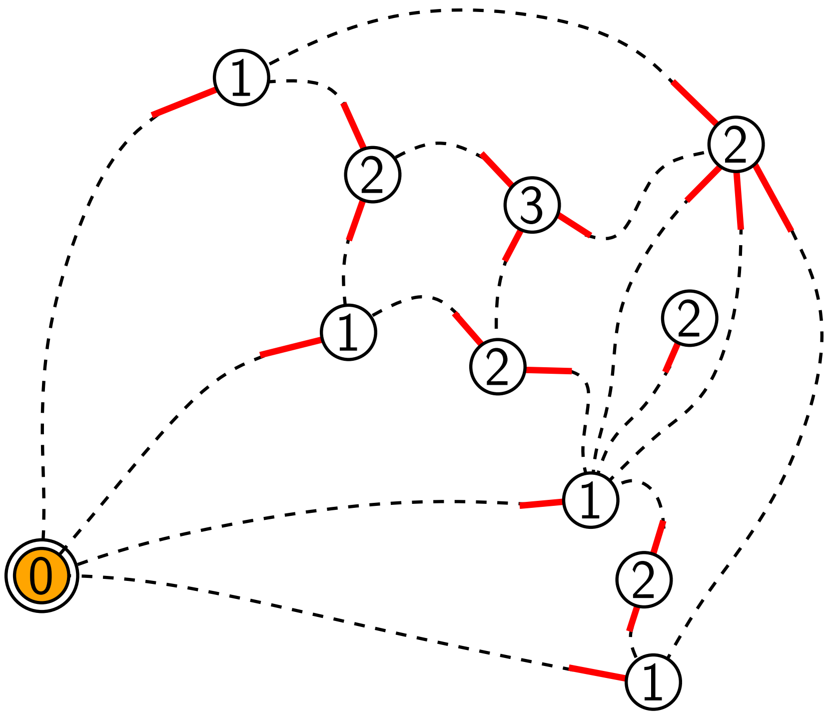

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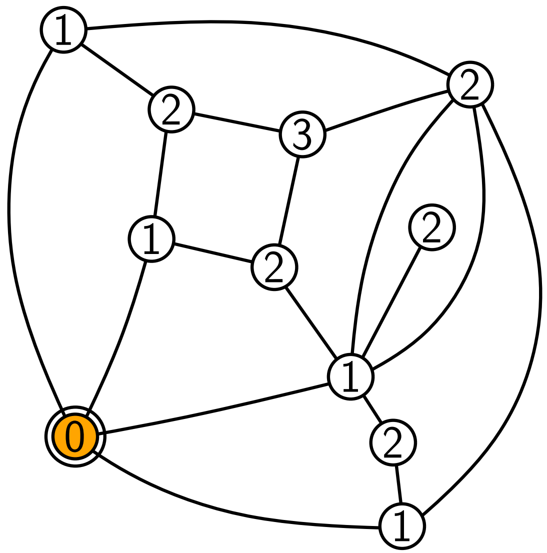
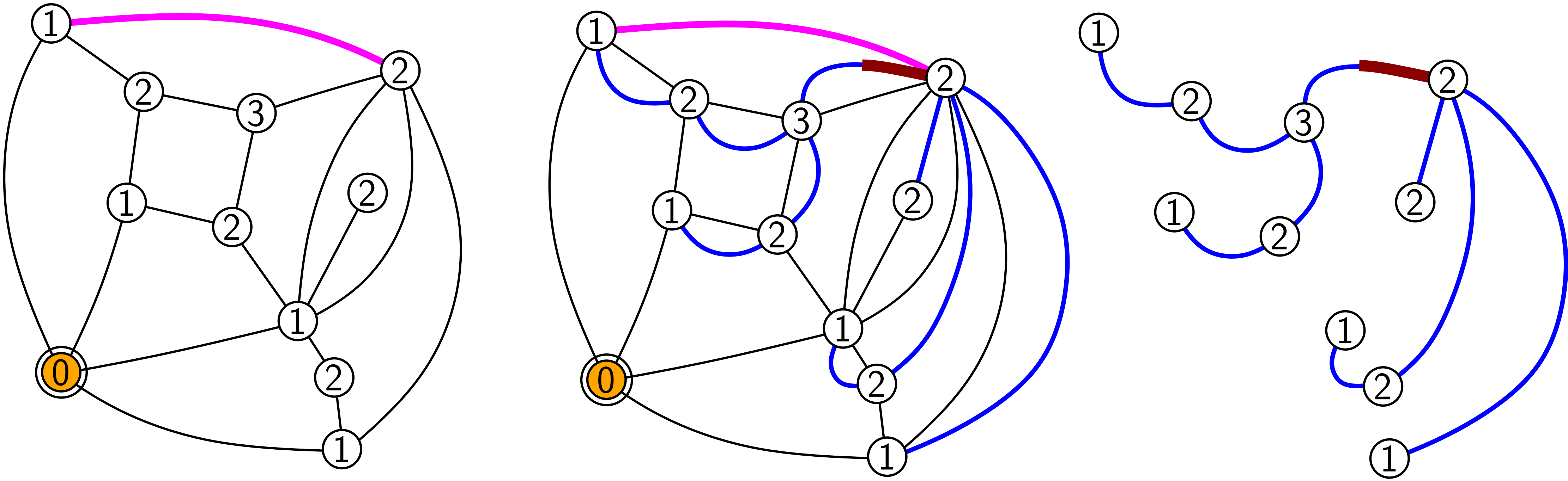
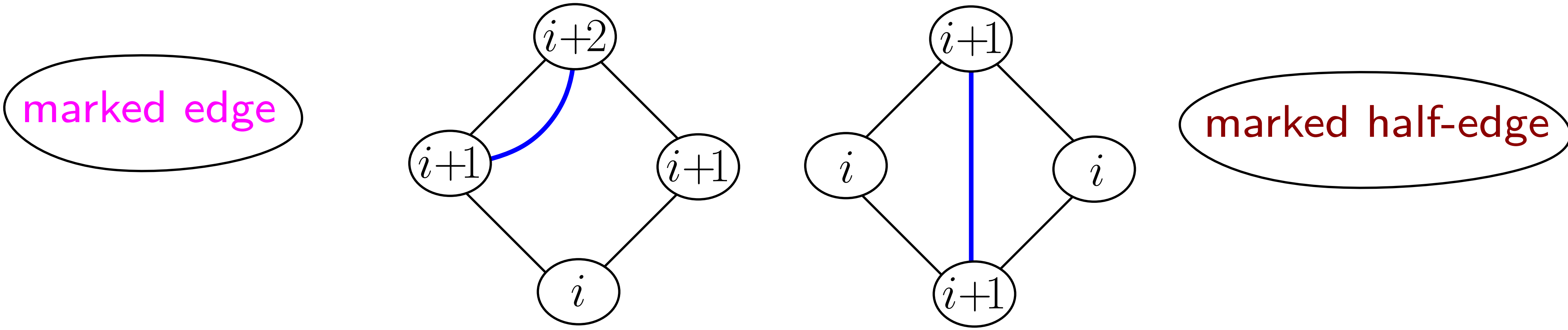
recover the original pointed quadrangulation



The effect of marking an edge



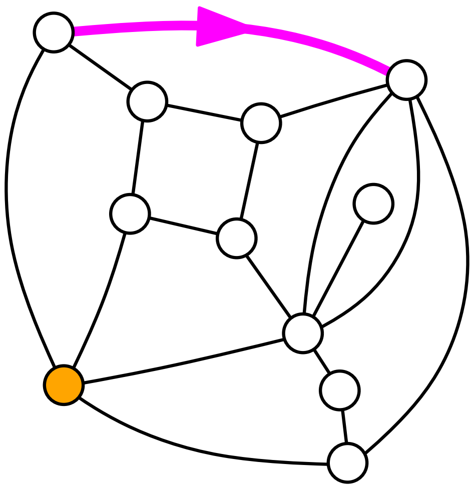
Local rule in each face:



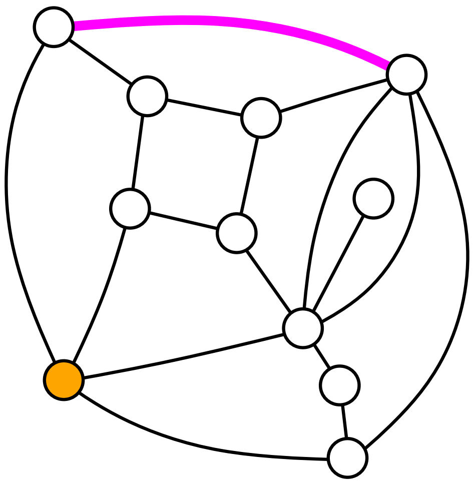
Bijective proof of counting formula

Let $q_n = \#(\text{rooted quadrangulations with } n \text{ faces})$

We want to show (bijectively) that $q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$



Rk: $q_n \times (n+2) = \#$ rooted quadrangulations with n faces
+ marked vertex



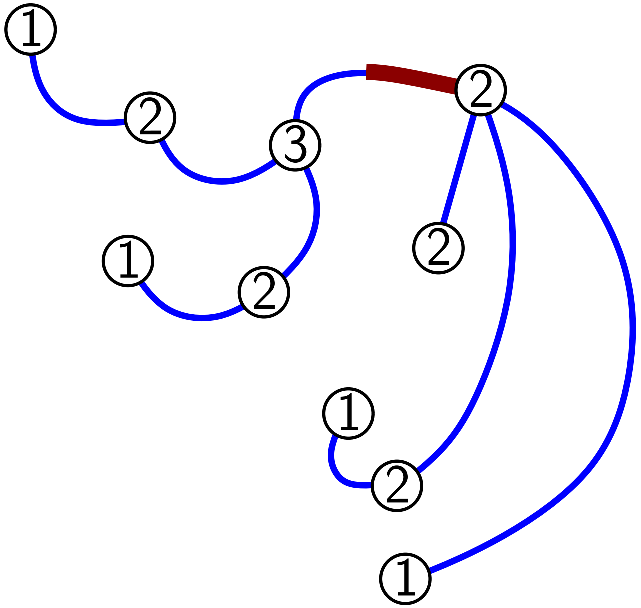
Hence if $b_n := \#$ quadrangulations with n faces
+ marked edge + marked vertex

then $b_n = \frac{n+2}{2} q_n$

Hence proving formula for q_n amounts to proving $\boxed{b_n = 3^n \mathrm{Cat}_n}$

Bijective proof of counting formula

Schaeffer's bijection $\Rightarrow b_n = \#$(rooted well-labelled trees with n edges)



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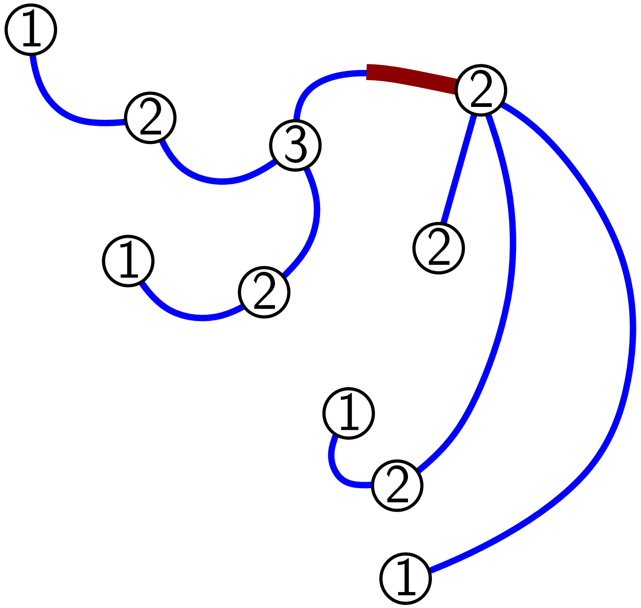
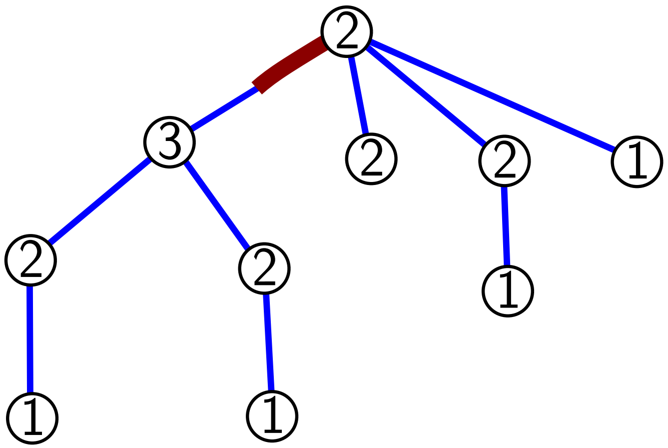
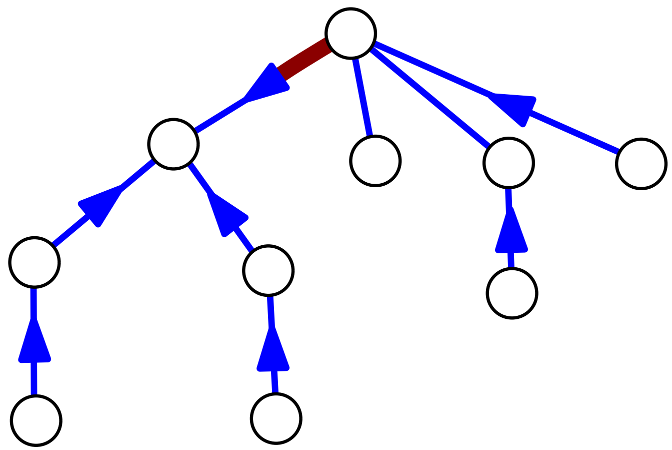


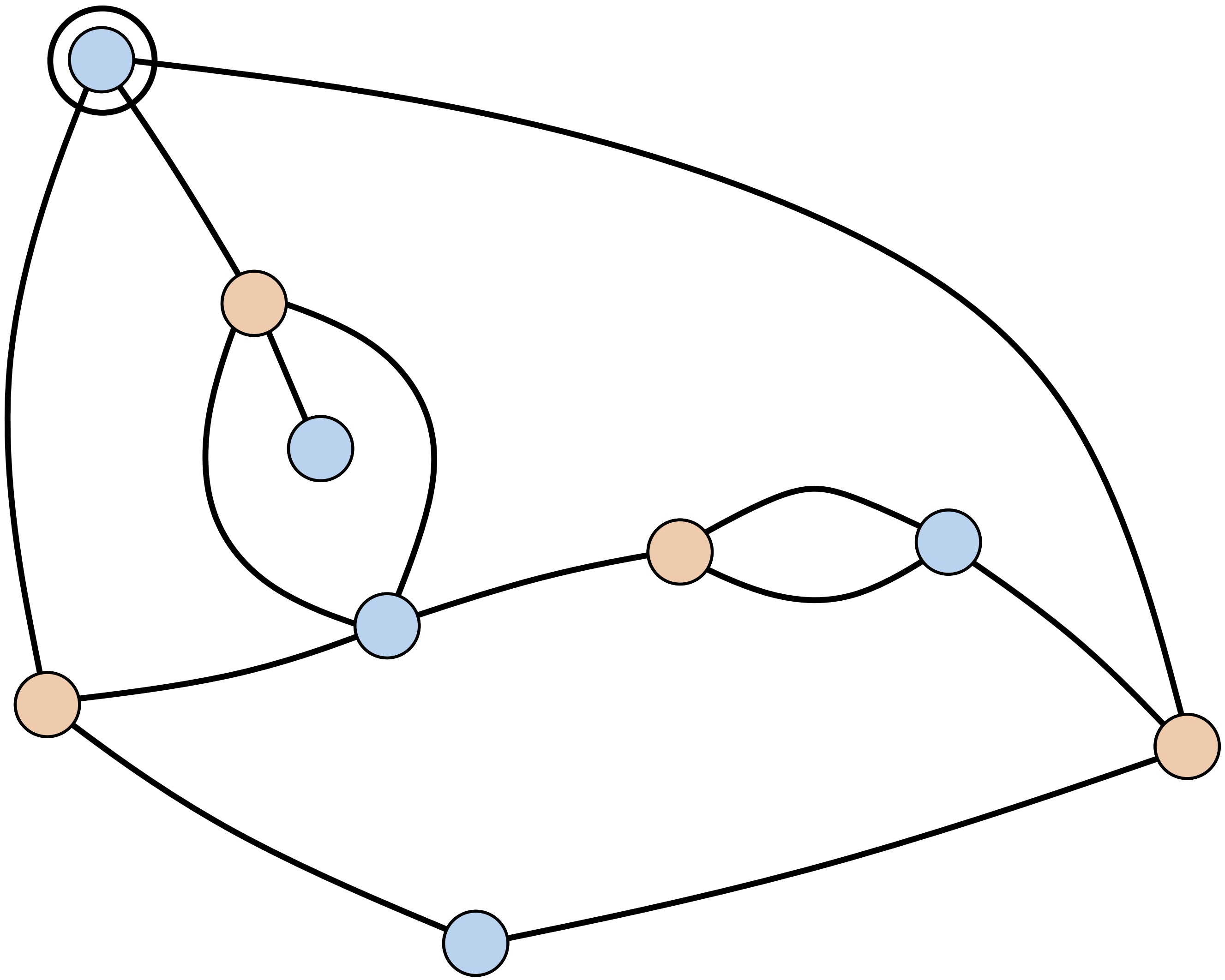




$$b_n = 3^n \mathrm{Cat}_n = 3^n \frac{(2n)!}{n!(n+1)!}$$

The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]



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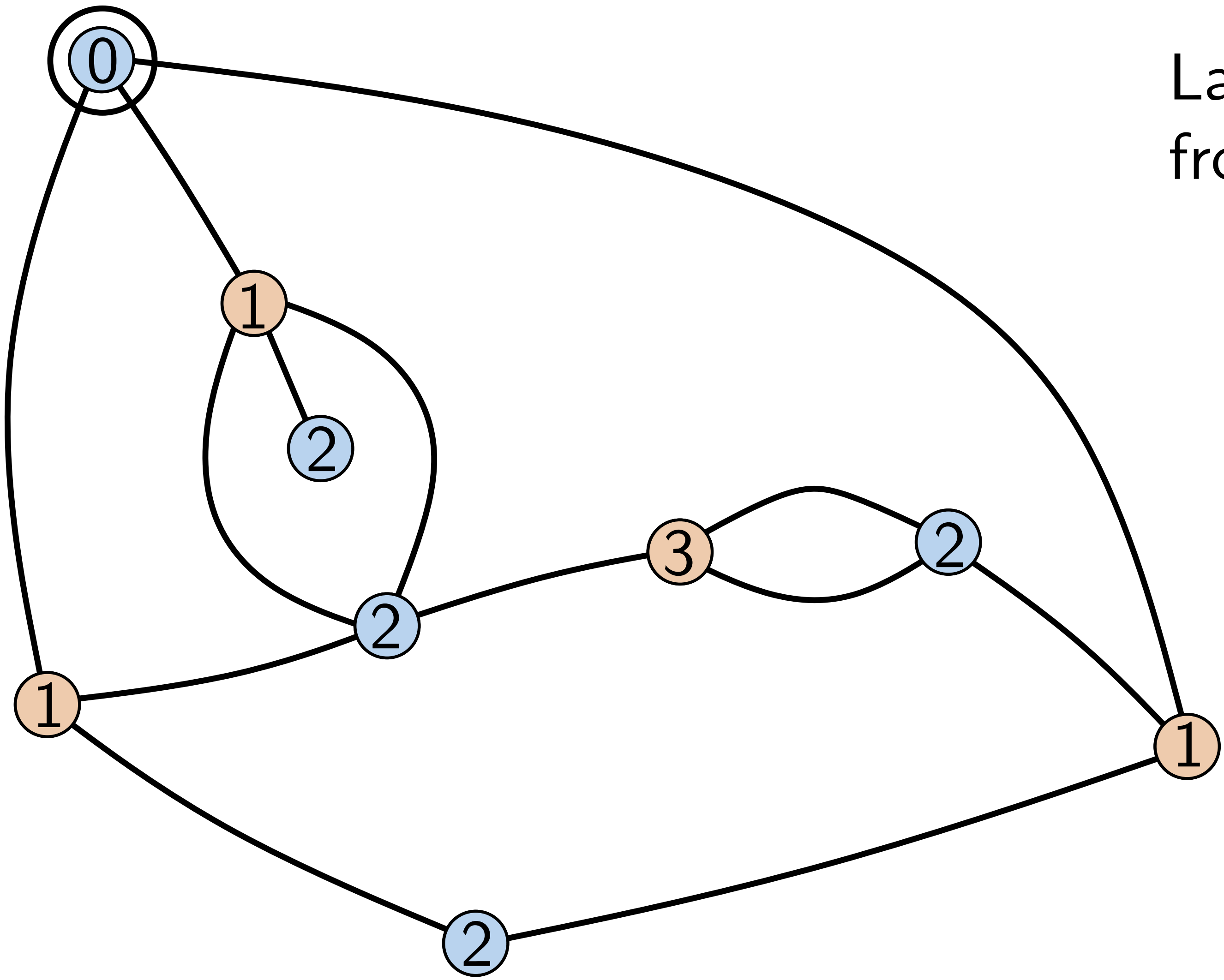
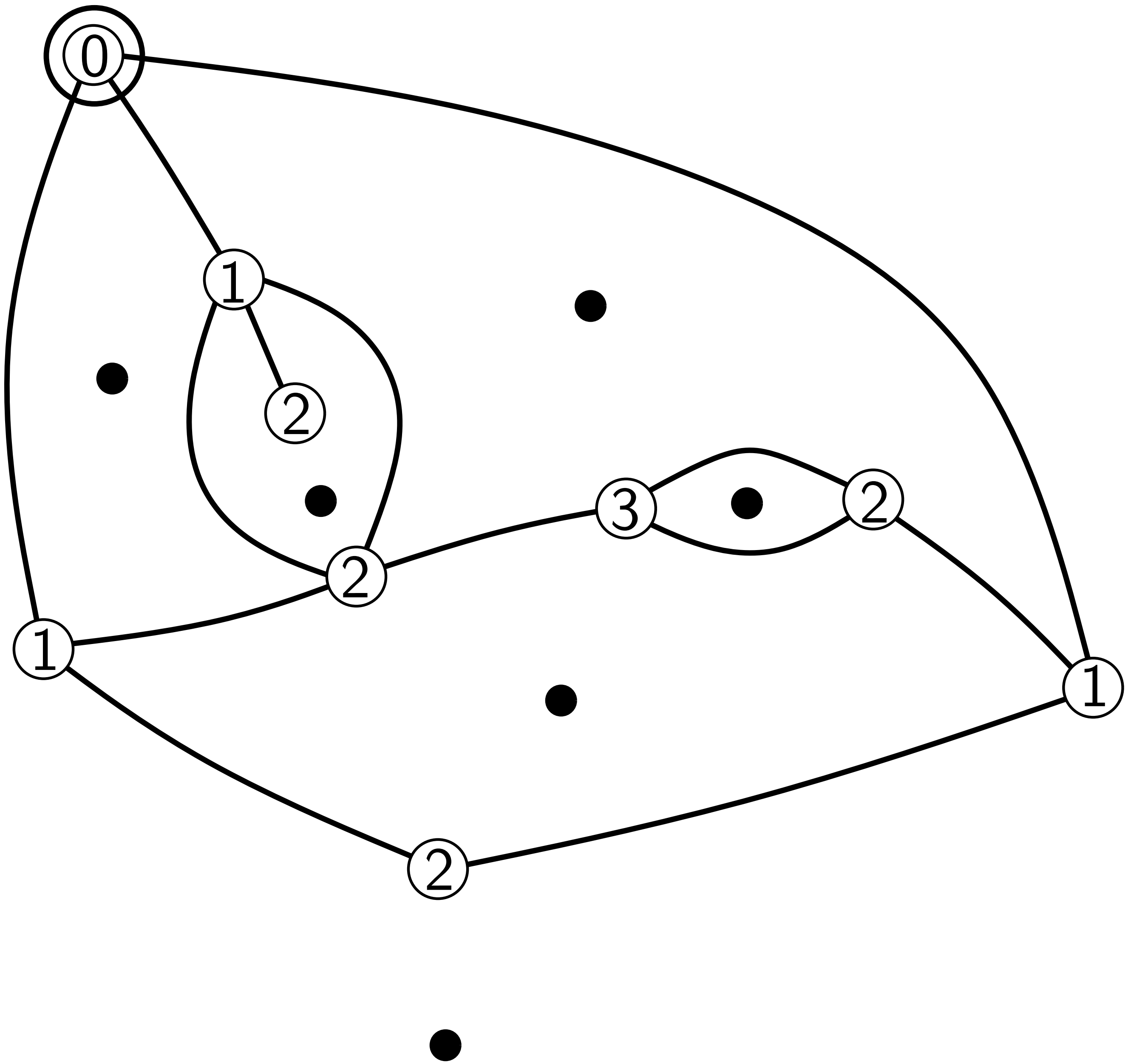


Label vertices by distance from the marked vertex

The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]

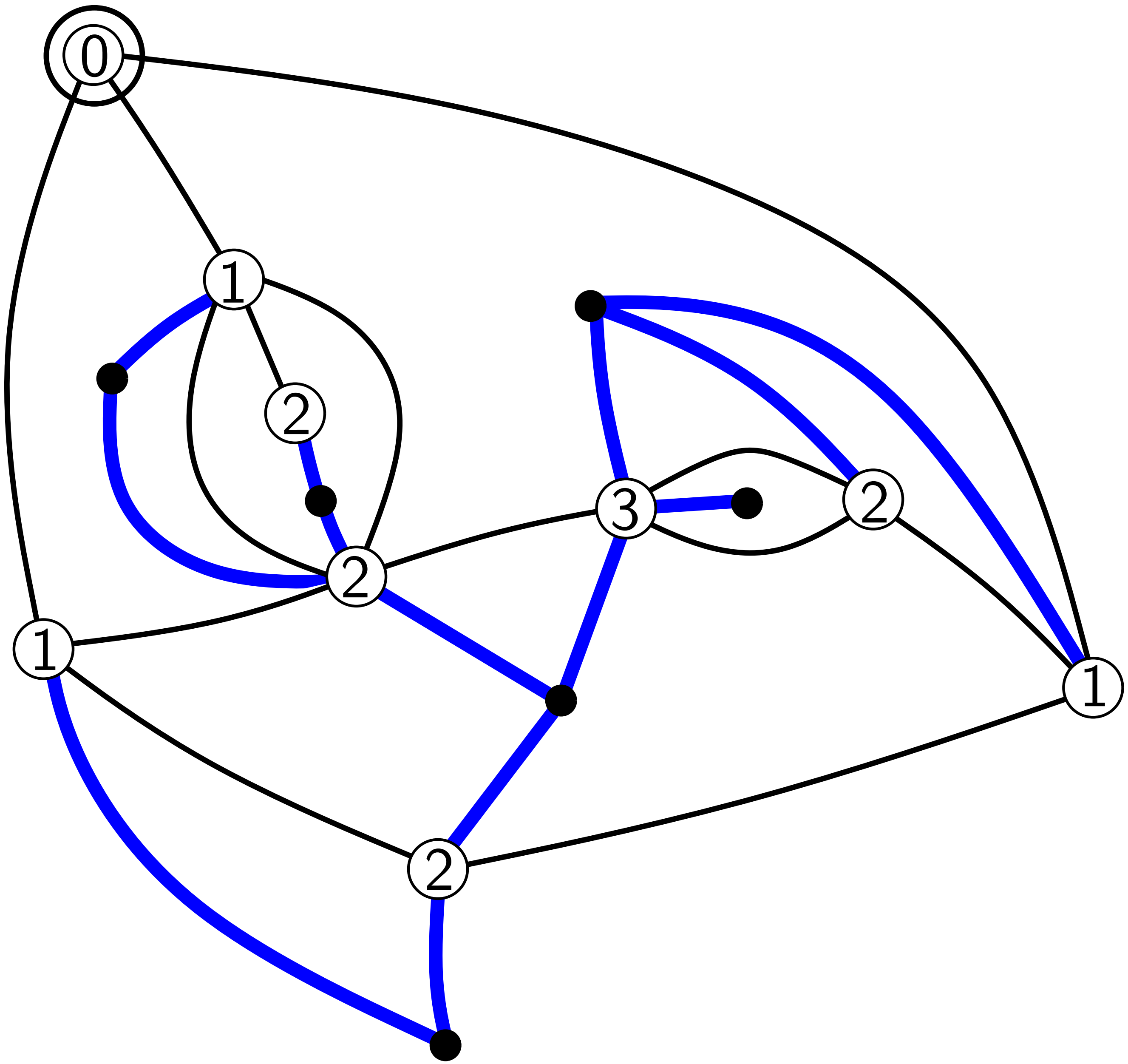


Construction of a labeled mobile

(i) Add a black vertex in each face

The BDG bijection for pointed bipartite maps

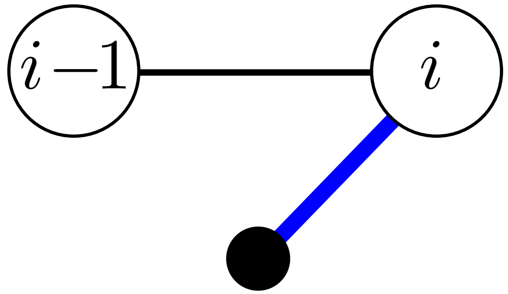
[Bouttier, Di Francesco, Guitter'04]

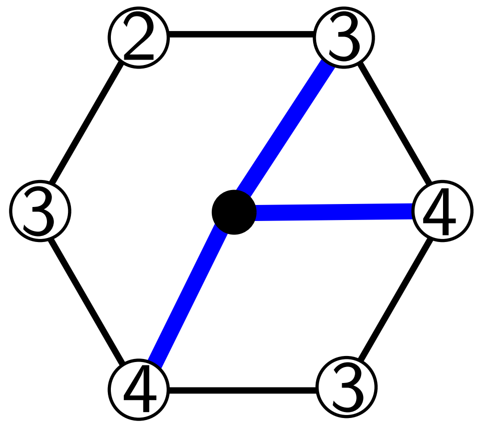


Construction of a labeled mobile

(i) Add a black vertex in each face

(ii) Each map-edge gives a mobile-edge using the local rule





The BDG bijection for pointed bipartite maps

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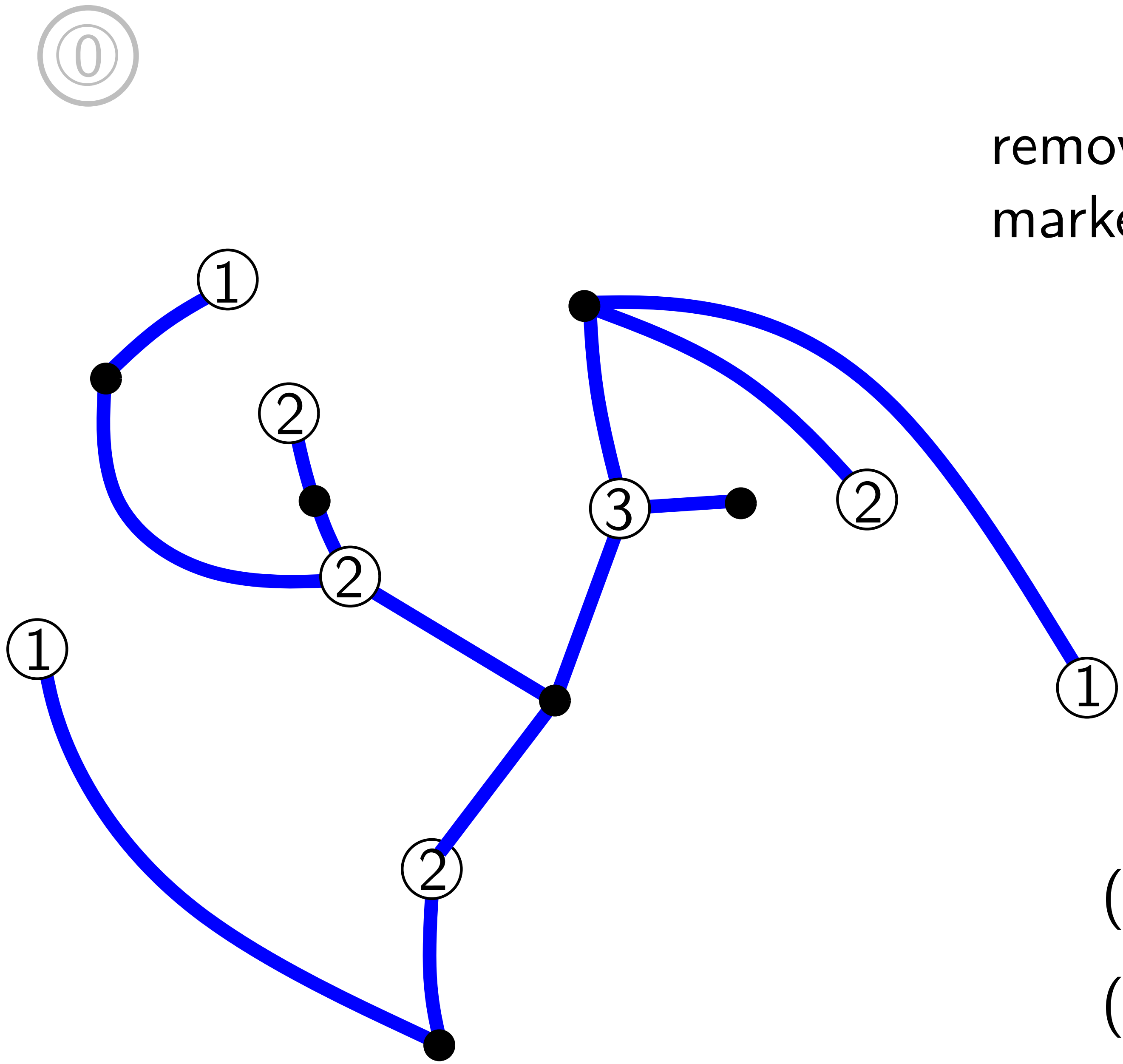
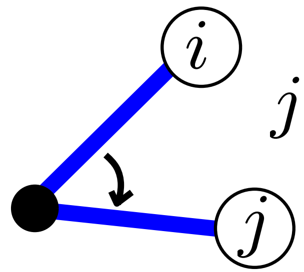


remove the map-edges and the marked vertex ⓪

Conditions:

(i) $\exists$ vertex of label 1

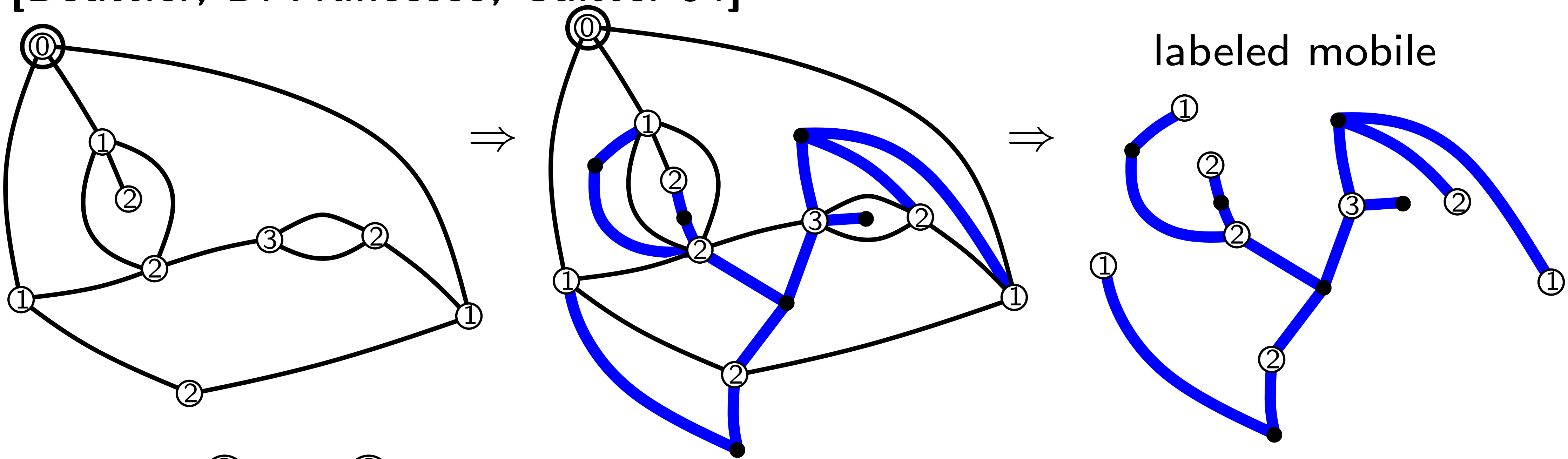
(ii)

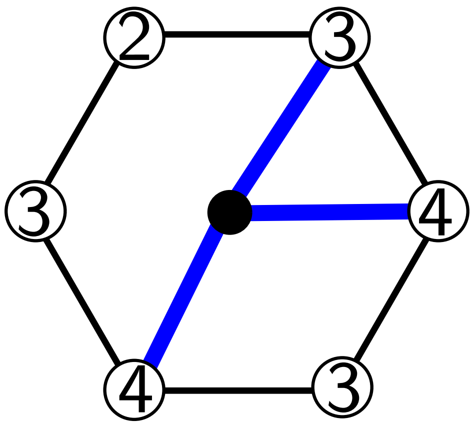


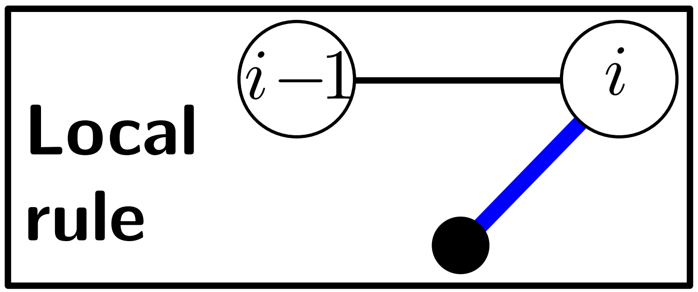
$j \leq i+1$

The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]







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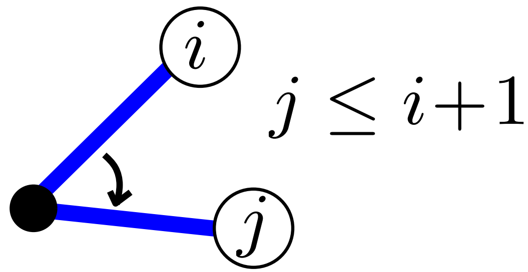
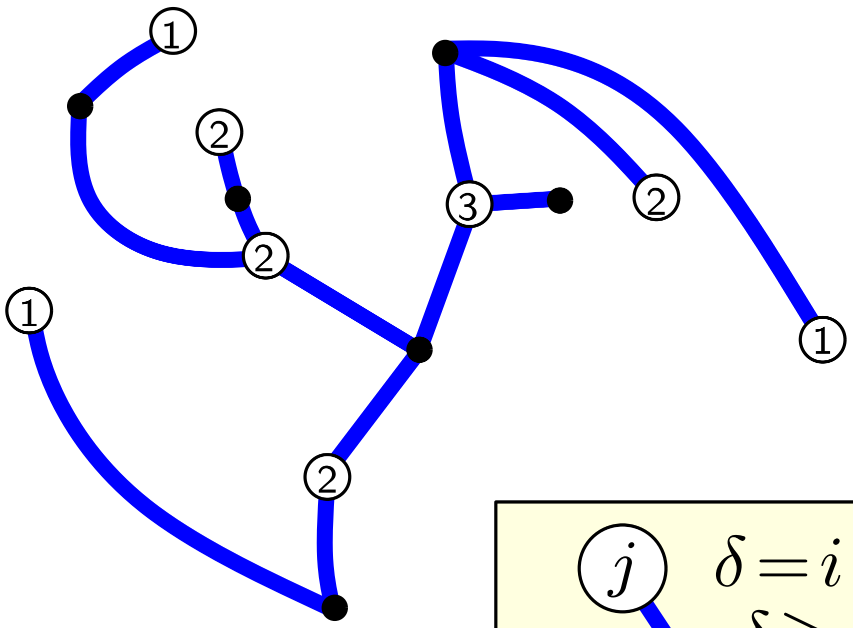


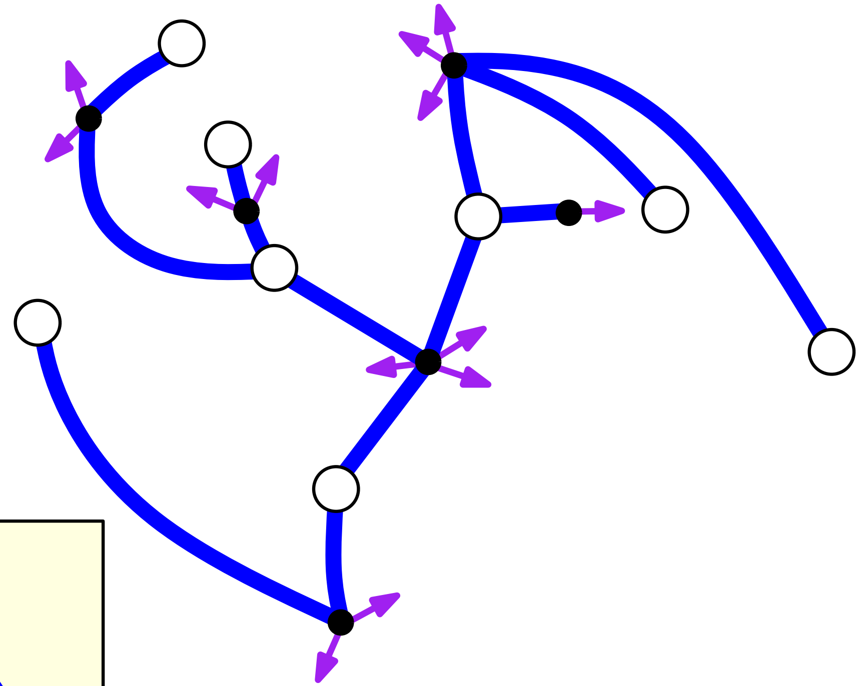
Theorem: The mapping is a **bijection**.

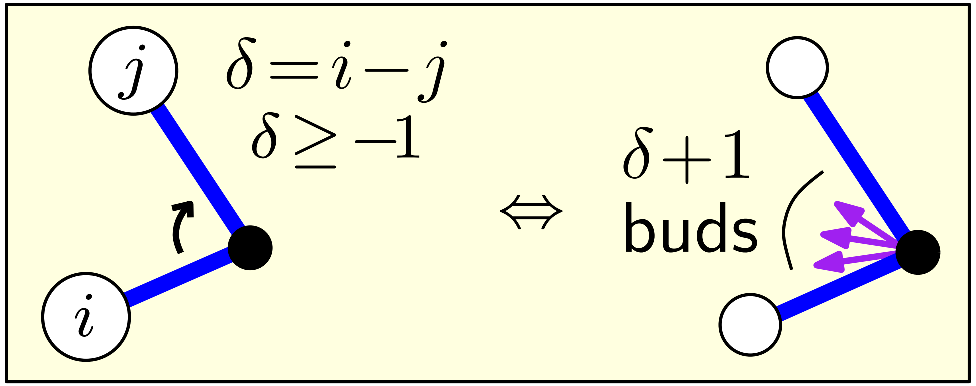
face of degree $2i$ ⟷ black vertex of degree i

Rewriting labelled mobiles as trees with arrows



$\Longleftrightarrow$





Conditions:

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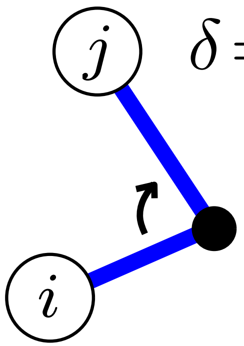
(ii) 

$\delta = i - j \geq -1$

Condition:

each black vertex has as many buds as neighbors

Enumerative consequence

Tutte's slicings formula (1962):

Let $B[n_1, n_2, \ldots, n_k]$ be the number of rooted bipartite maps with n_i faces of degree $2i$ for $i \in [1..k]$. Then

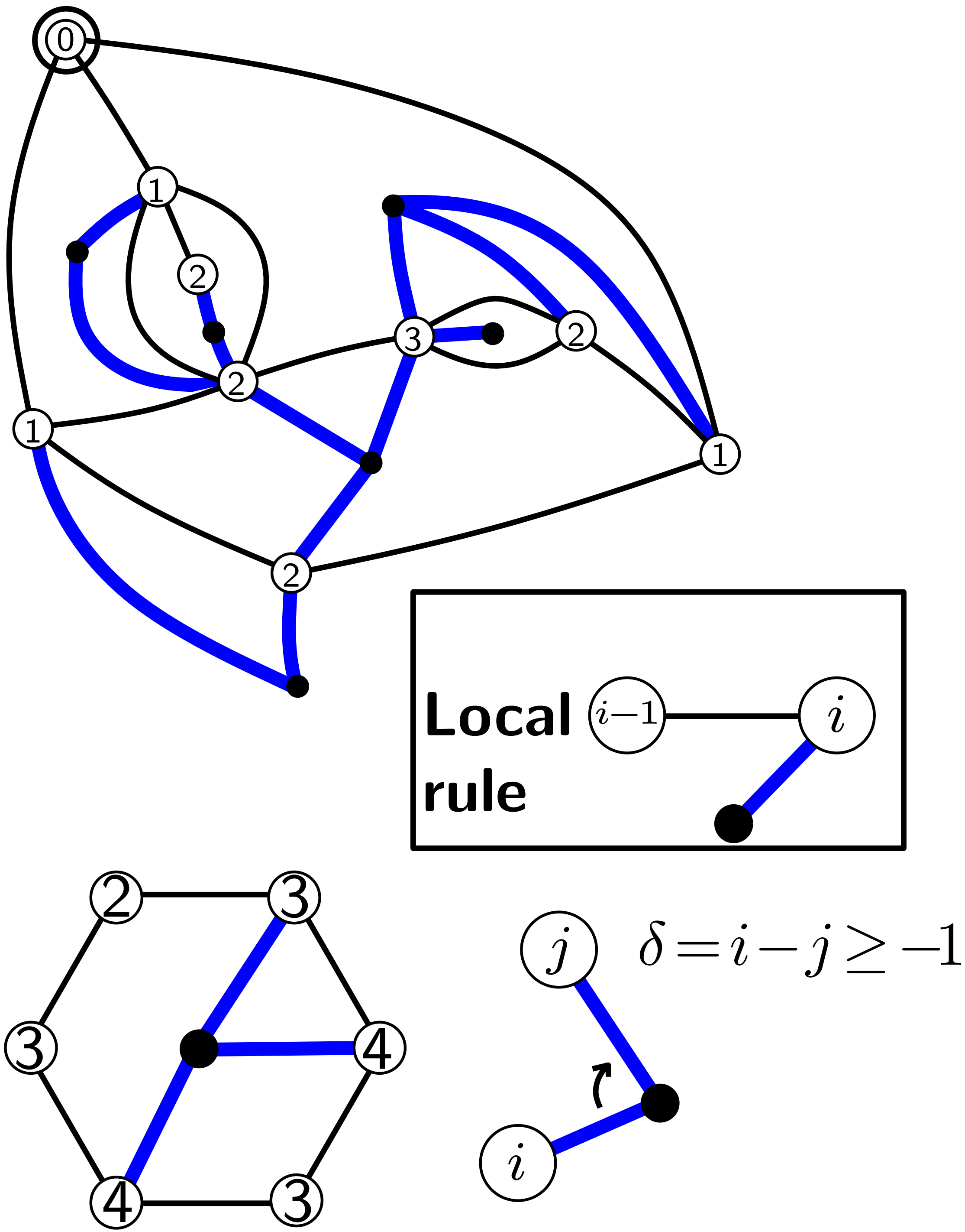
$$B[n_1, \ldots, n_k] = 2\frac{e!}{v!} \prod_{i=1}^{k} \frac{1}{n_i!} \binom{2i-1}{i-1}^{n_i}$$

where $e = \#\text{edges} = \sum_i i n_i$ and $v = \#\text{vertices} = e - k + 2$

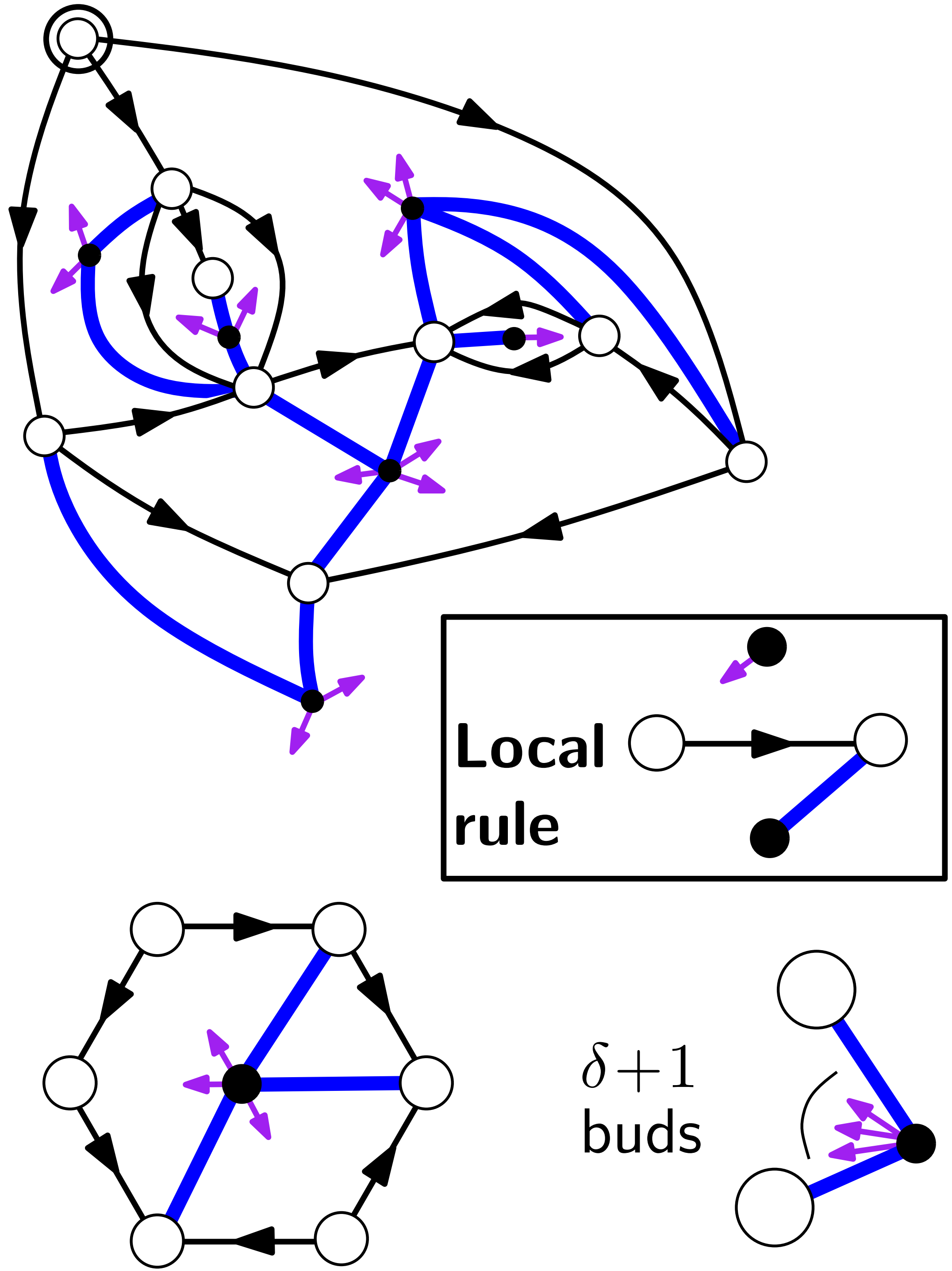
('contains' formula for rooted quadrangulations, $n_2 = n$, $n_i = 0$ for $i \neq 2$)

Reformulation of bijection using orientations

Distance-labeling



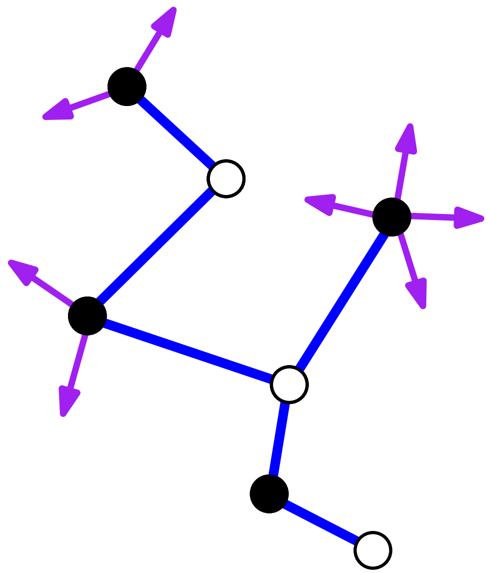
Geodesic orientation



Definition of blossoming mobiles

- **Blossoming mobile**= bipartite tree (black/white vertices) where each corner at a black vertex carries $i \geq 0$ buds

excess $=$ number of edges - number of buds



a blossoming mobile of excess -2

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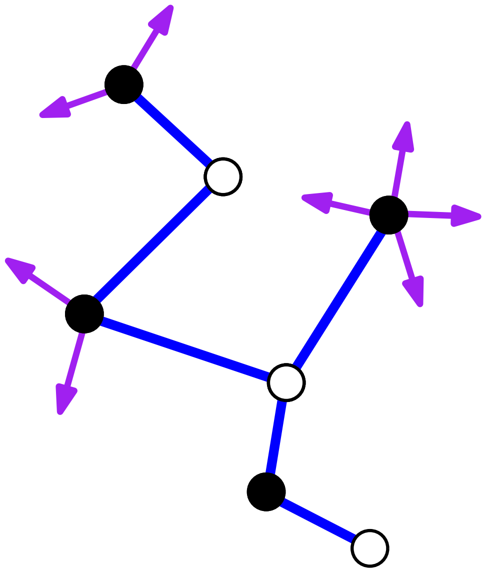
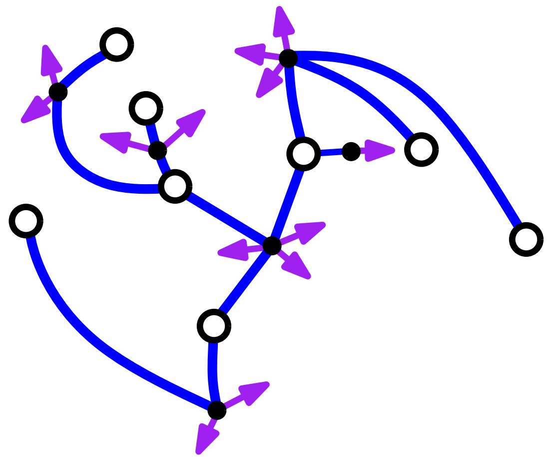


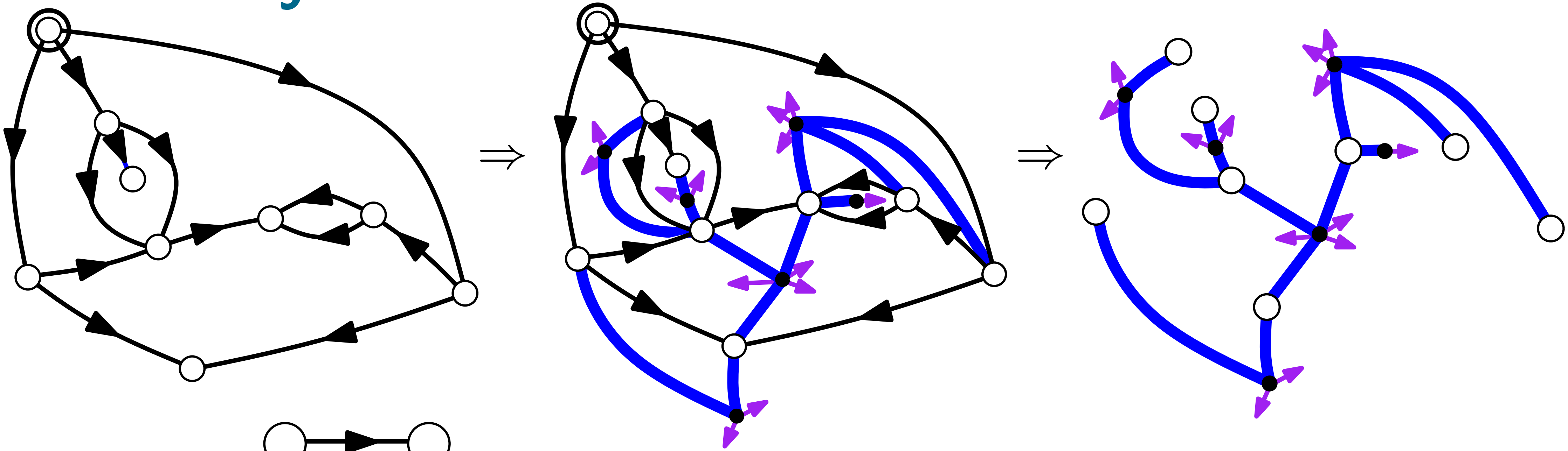
a blossoming mobile of excess -2

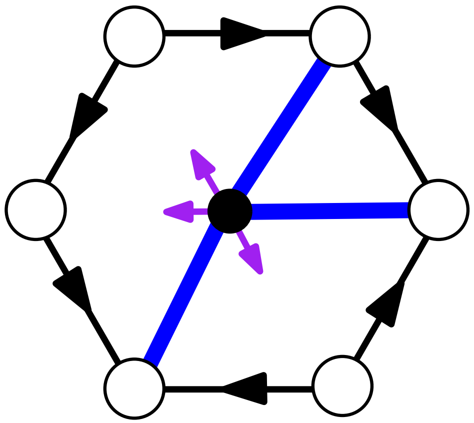
- A blossoming mobile is called **balanced** iff each black vertex has as many buds as neighbors

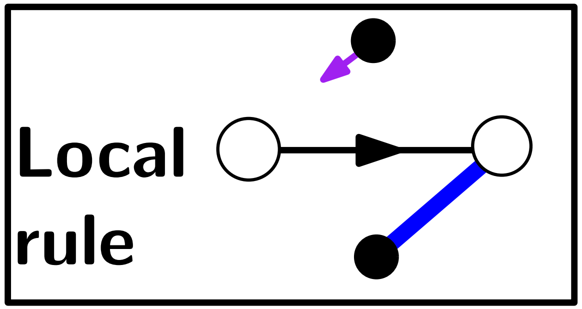
Rk: implies that the excess is 0



Summary of the reformulation







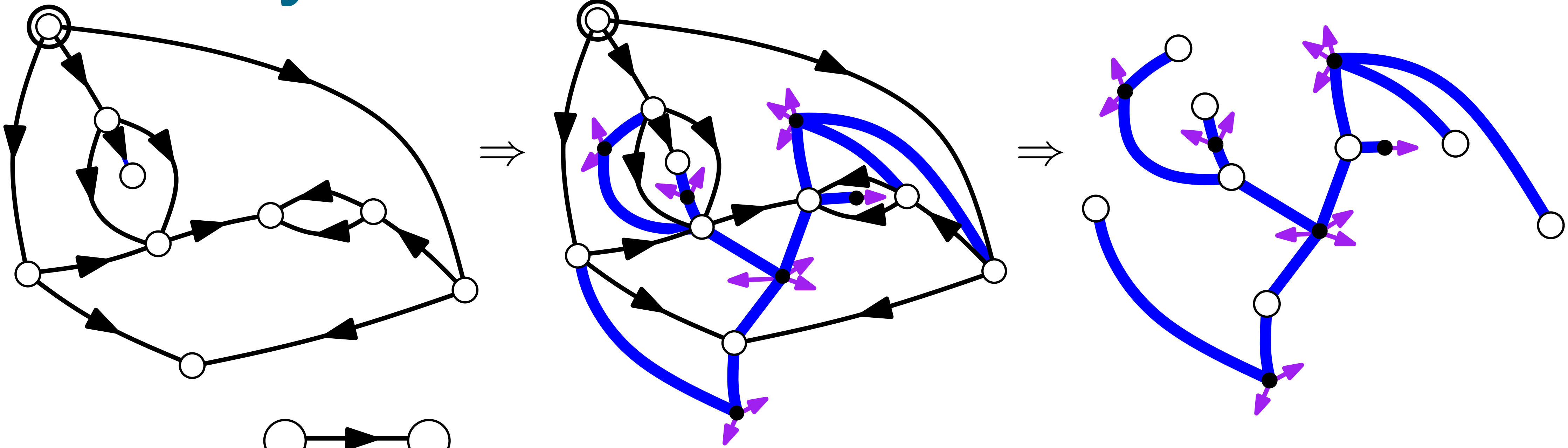
Condition:

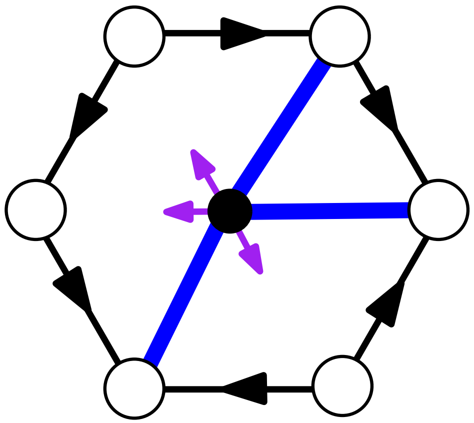
Each black vertex has as many buds as neighbors

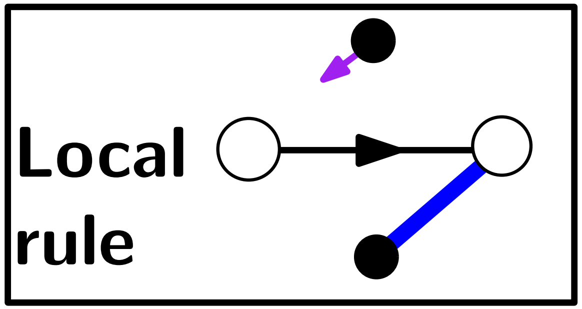
Theorem: The mapping is a bijection between pointed bipartite maps and balanced blossoming mobiles

face of degree $2i$ $\longleftrightarrow$ black vertex of degree $2i$

Summary of the reformulation







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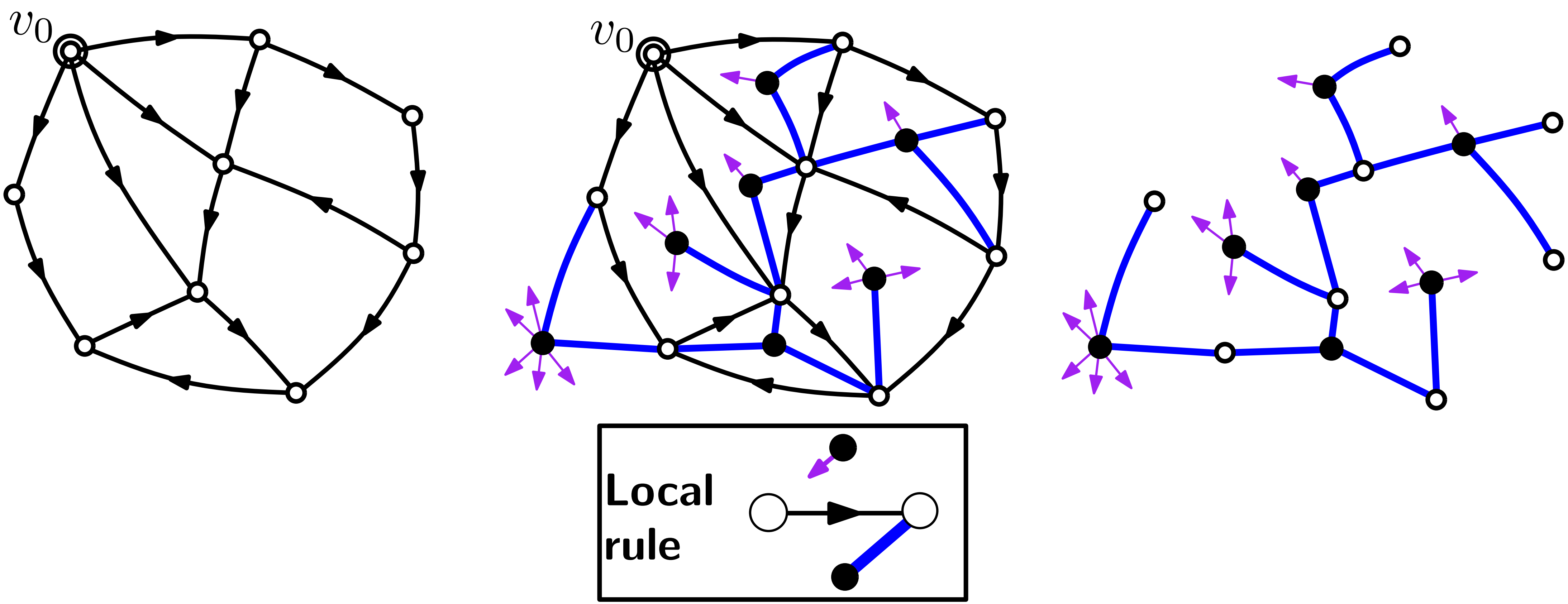
(other bijection by Schaeffer'97 in the dual setting of eulerian maps)

Extension for pointed orientations with no ccw cycle

- More generally, we **obtain a blossoming mobile** (of excess 0) if we start from a vertex-pointed orientation such that :
  - the marked vertex v_0 is a **"source"** (no incoming edge)
  - every vertex is **accessible** from v_0 by a directed path
  - **there is no ccw cycle** (with $v_0 \in$ outer face)



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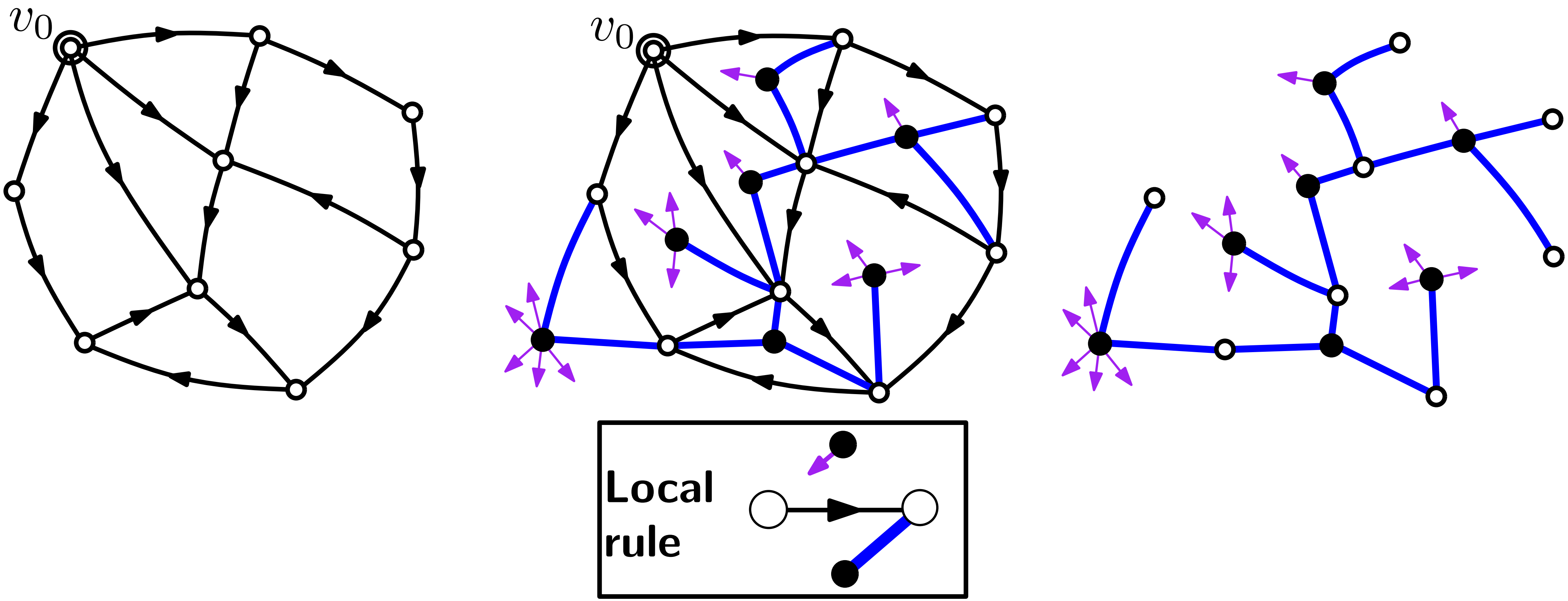
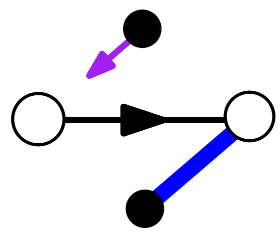


Theorem : Let $\mathcal{O}_0$ be this family of orientations, then the correspondence is a bijection with mobiles of excess 0

Proof that it gives a tree

Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

Let G be the graph of red edges and their incident vertices

Proof that it gives a tree

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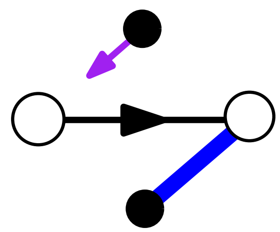

Let G be the graph of red edges and their incident vertices

G has $|V_M| - 1$, white vertices, $|F_M|$ black vertices, et $|E_M|$ edges

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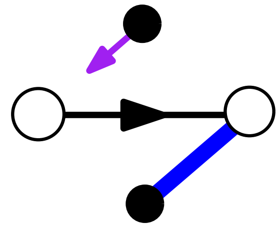

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Euler relation: $|E_M| = |V_M| + |F_M| - 2$

$\Rightarrow G$ has one more vertices than edges

hence G is a tree iff G is acyclic

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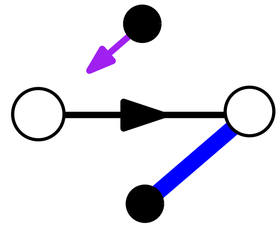
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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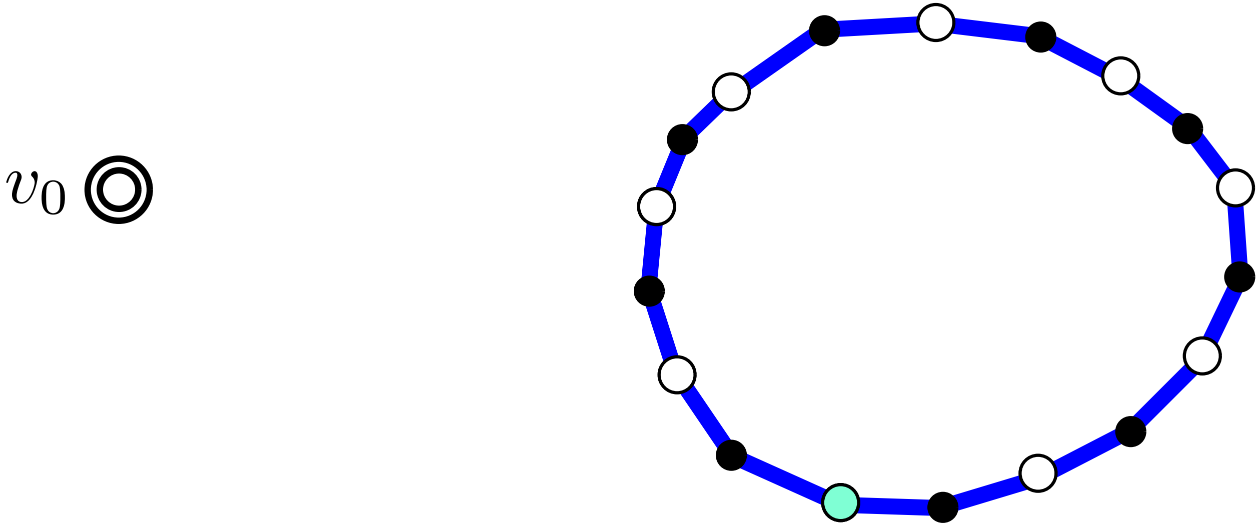

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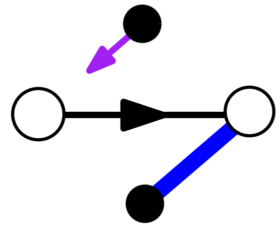
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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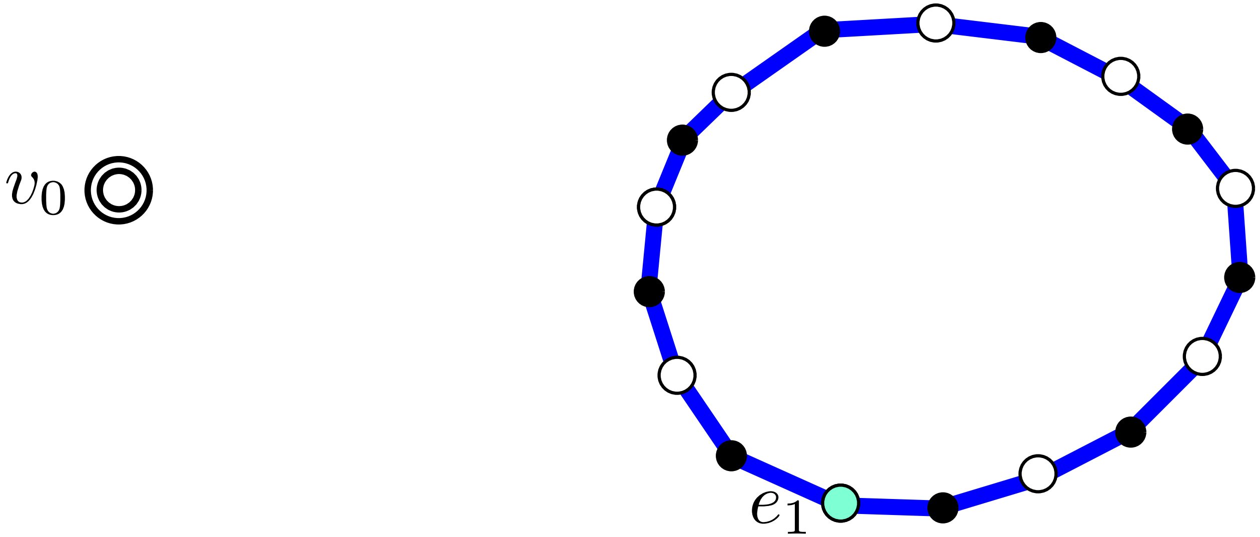

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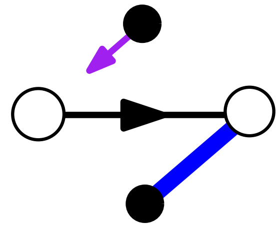
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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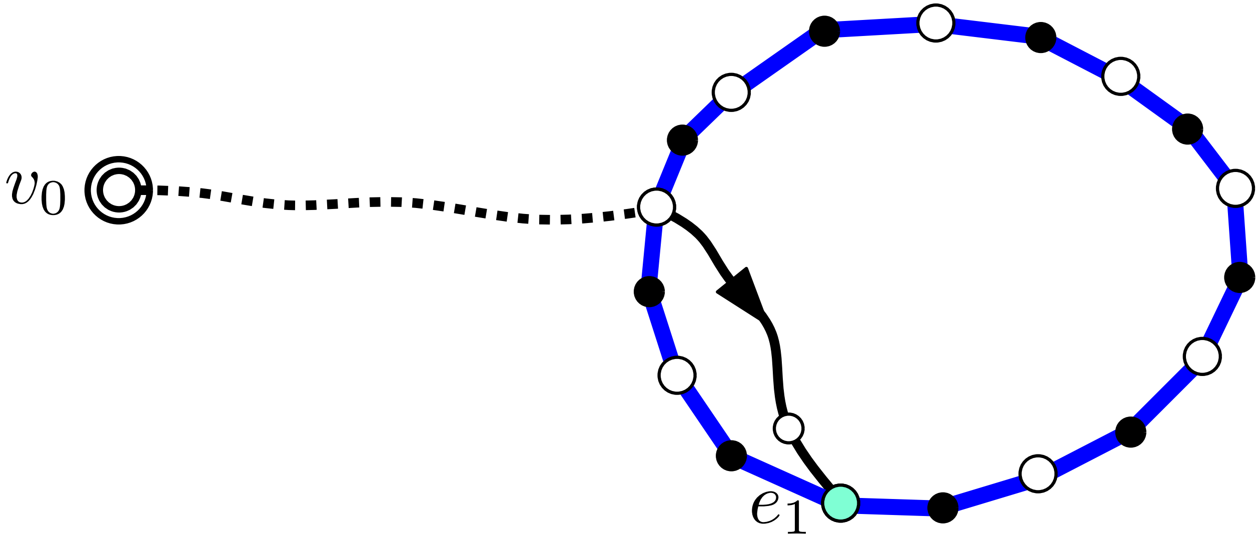

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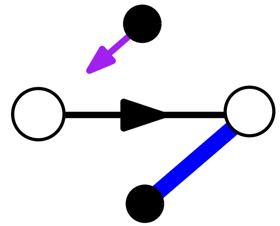
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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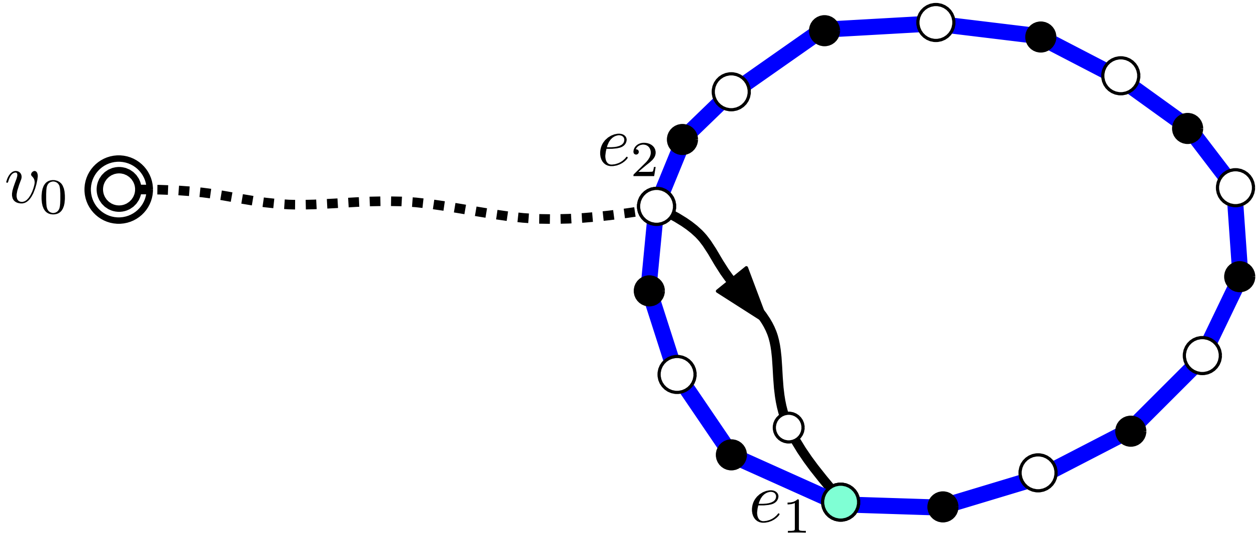

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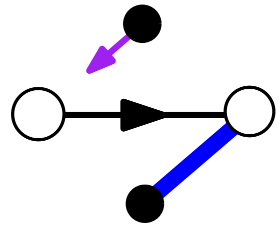
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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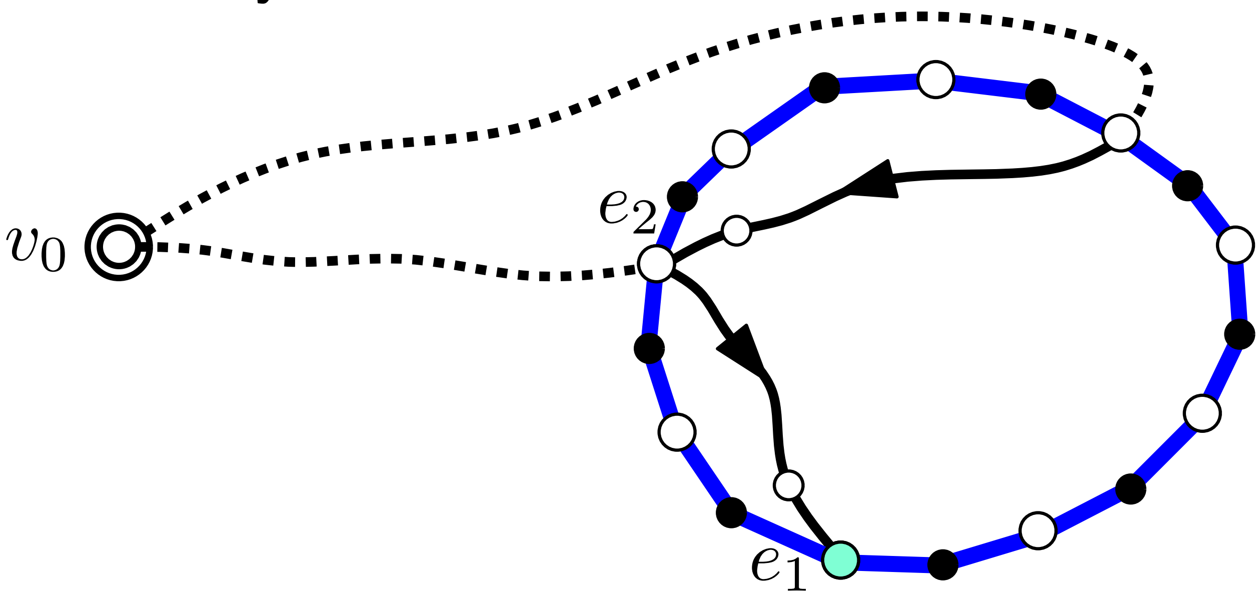

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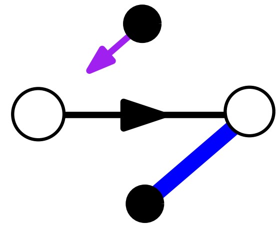
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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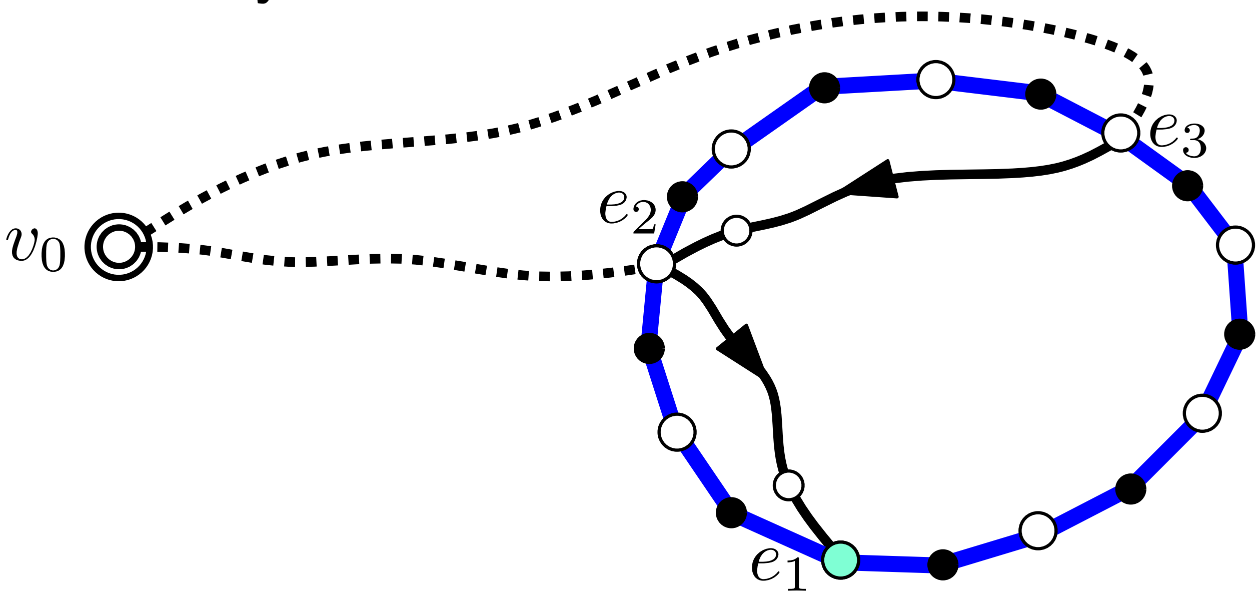

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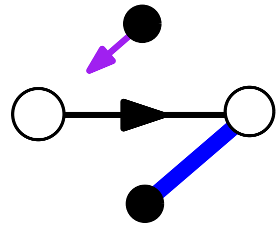
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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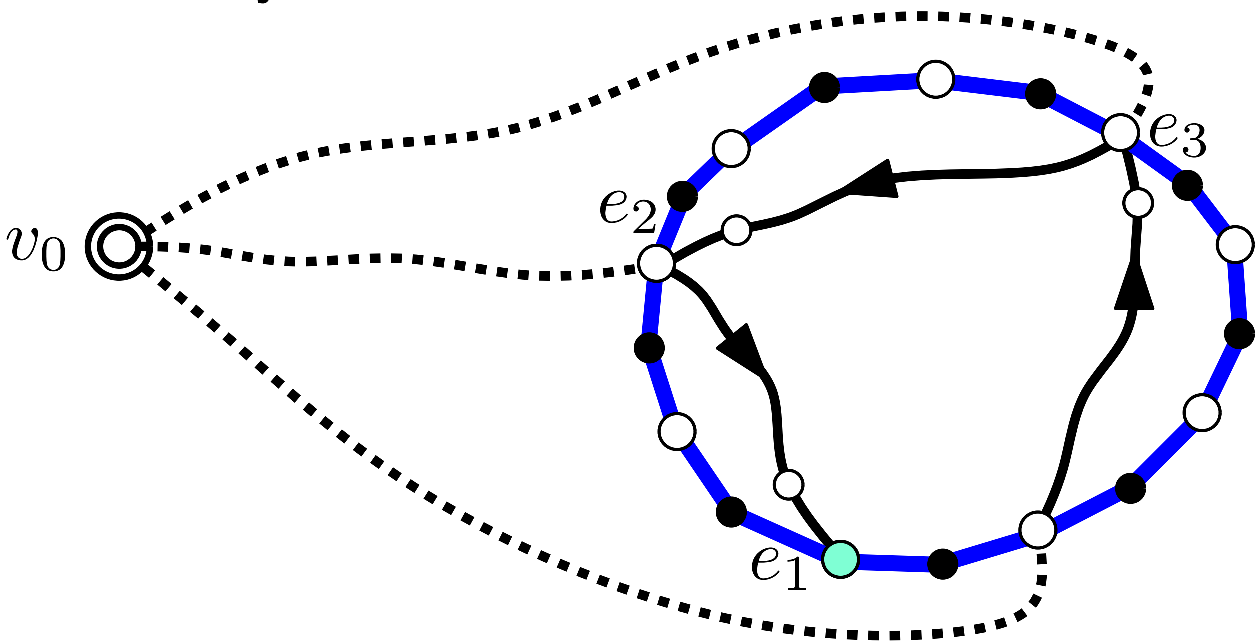
Euler relation: $|E_M| = |V_M| + |F_M| - 2$

$\Rightarrow G$ has one more vertices than edges

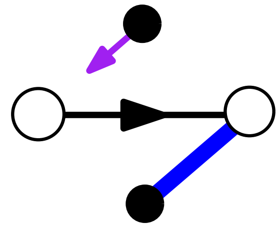
hence G is a tree iff G is acyclic

Assume G has a cycle :



Proof that it gives a tree

Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule



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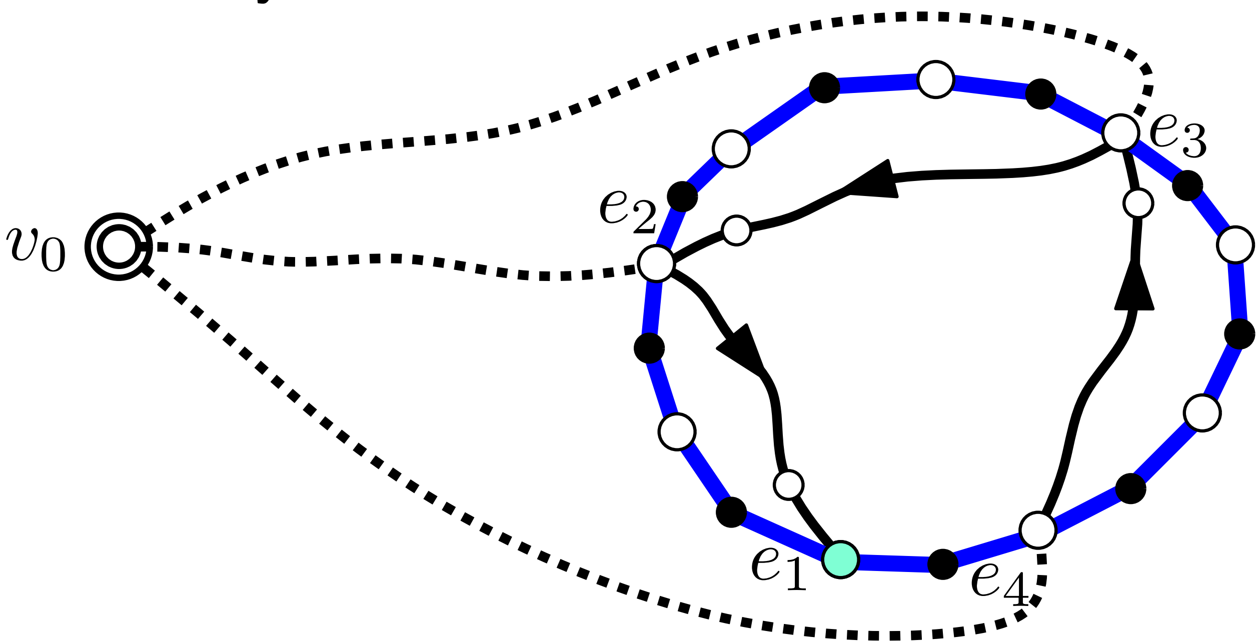

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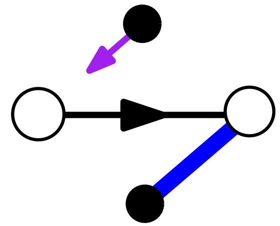
Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule 

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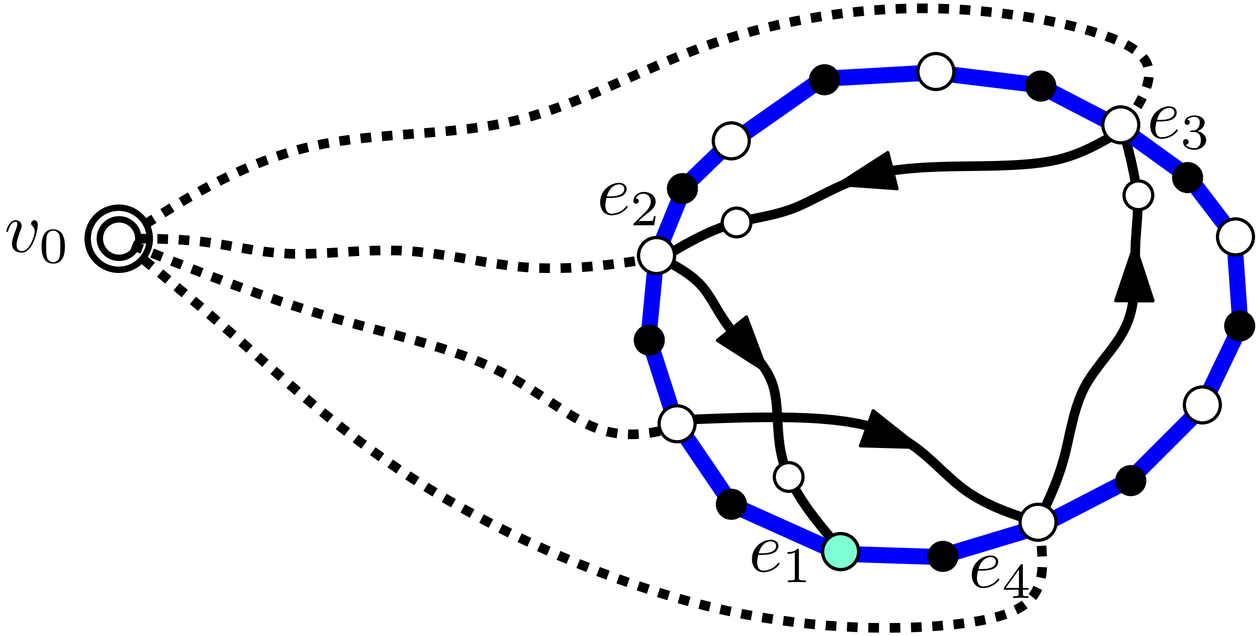

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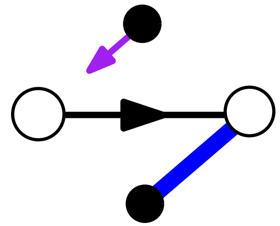

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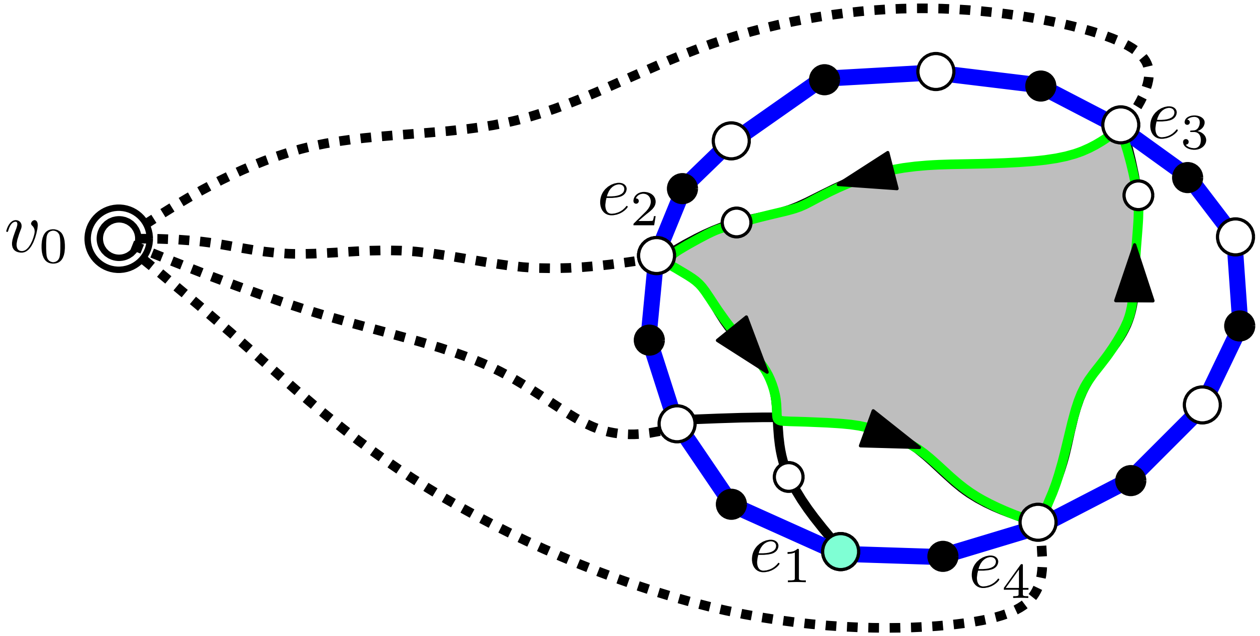


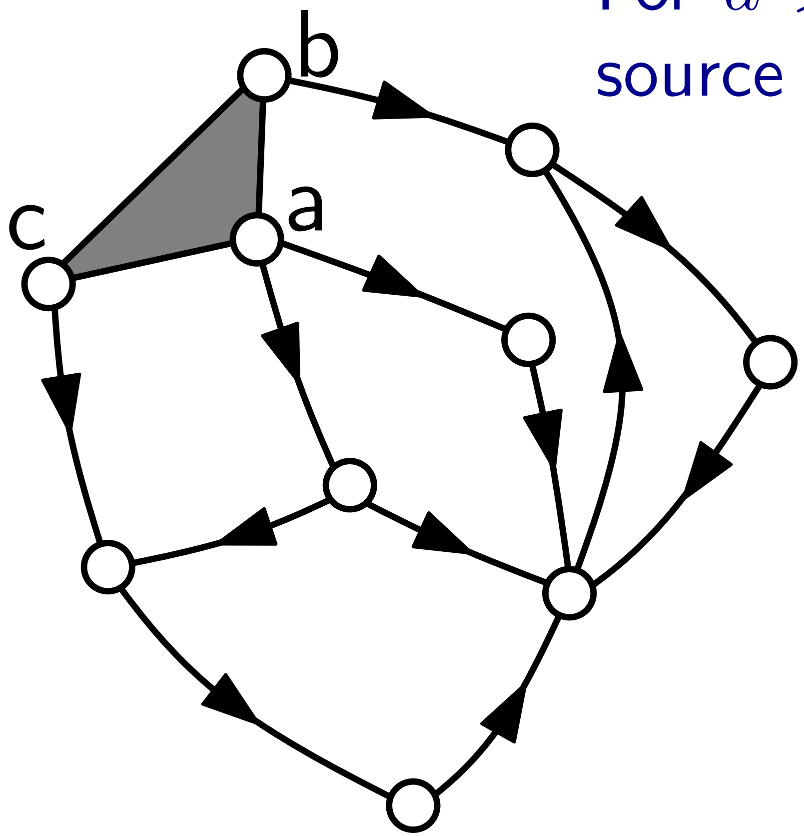
prisoner ccw cycle

$\Rightarrow$ contradiction

Extension for mobiles of excess ≤ 0

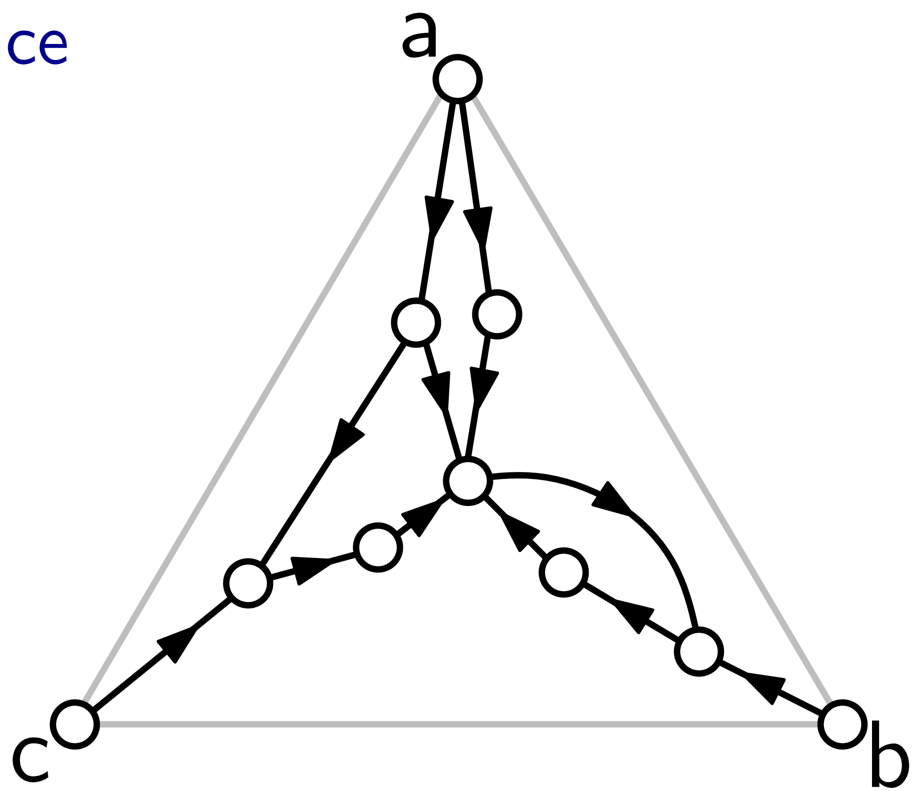
More generally the "source" can be a d-gon, for any $d \geq 0$

Example for $d = 3$



For $d > 0$, we take the d-gonal source as the outer face

$\Rightarrow$



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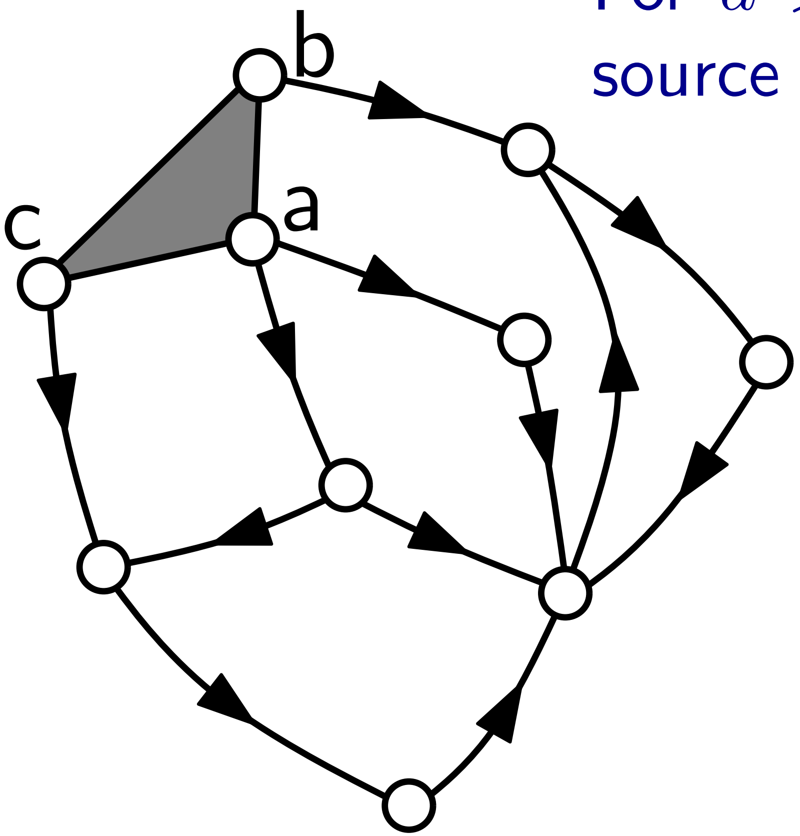

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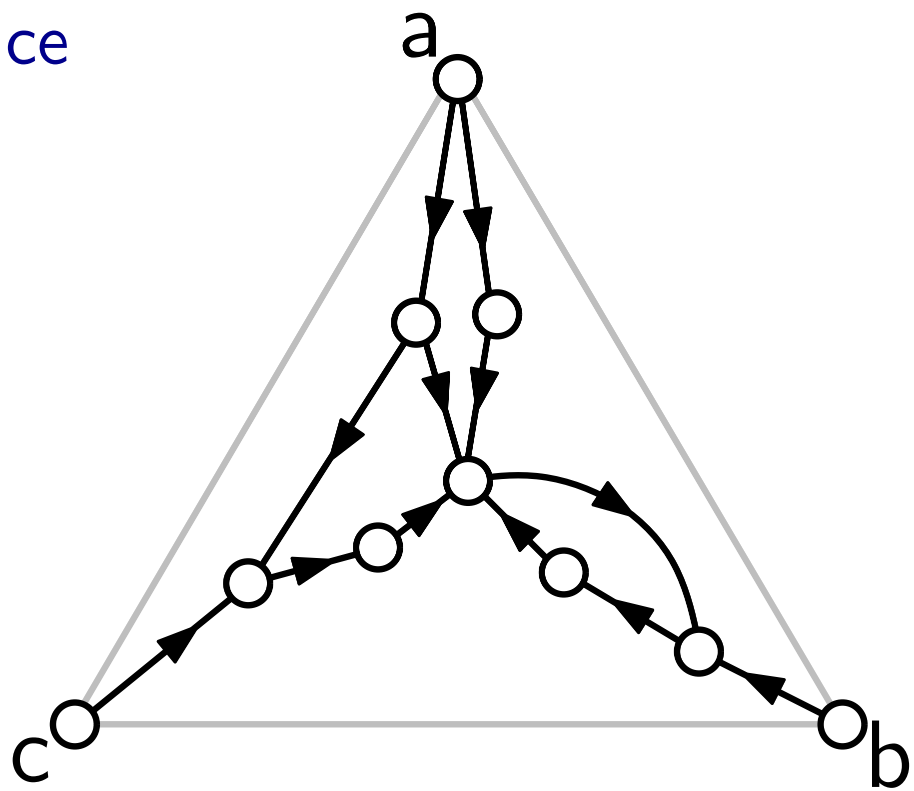
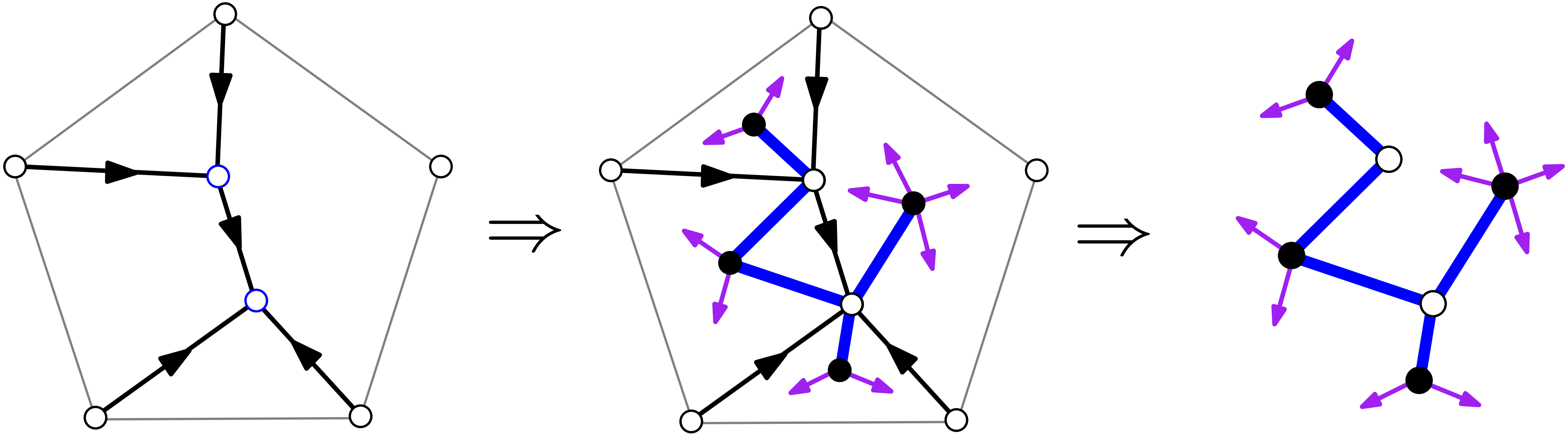
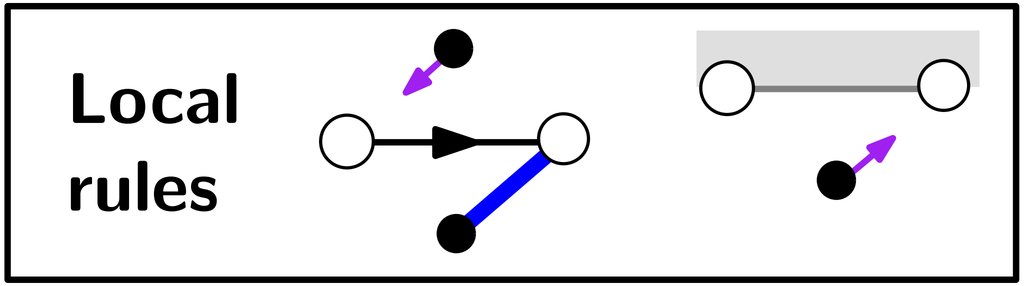


Let $\mathcal{O}$ be the family of these orientations, still with the conditions

- the d-gonal **source** has no ingoing edge
- **accessibility** of every vertex from the source
- **no ccw cycle**

Extension for mobiles of excess ≤ 0



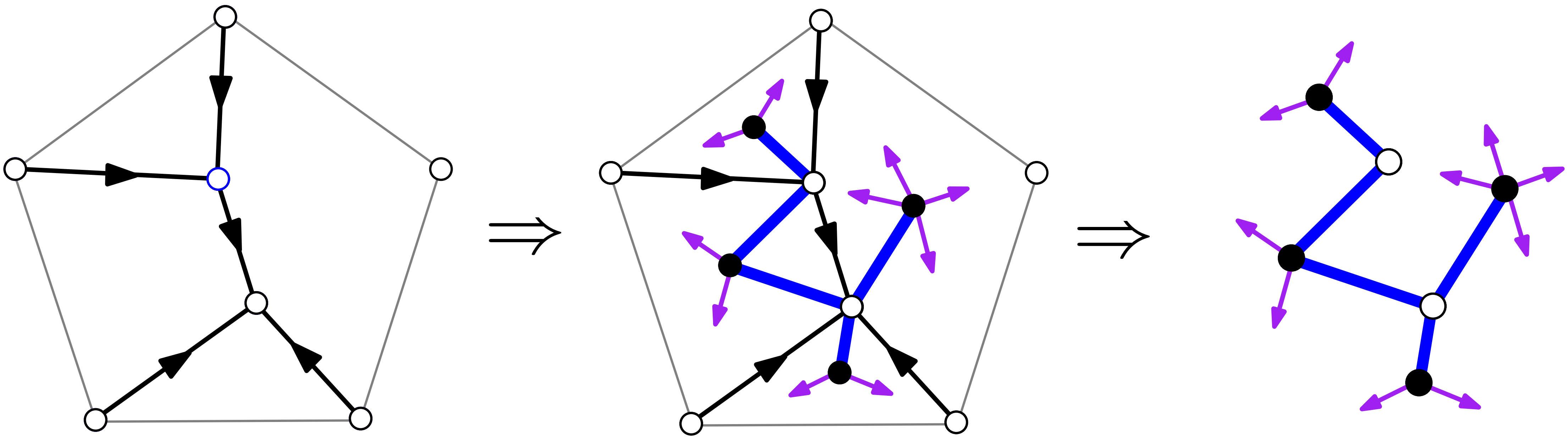


Theorem [Bernardi-F'10]: Φ is a **bijection** between $\mathcal{O}$ and blossoming mobiles of ≤ 0 excess. Moreover,

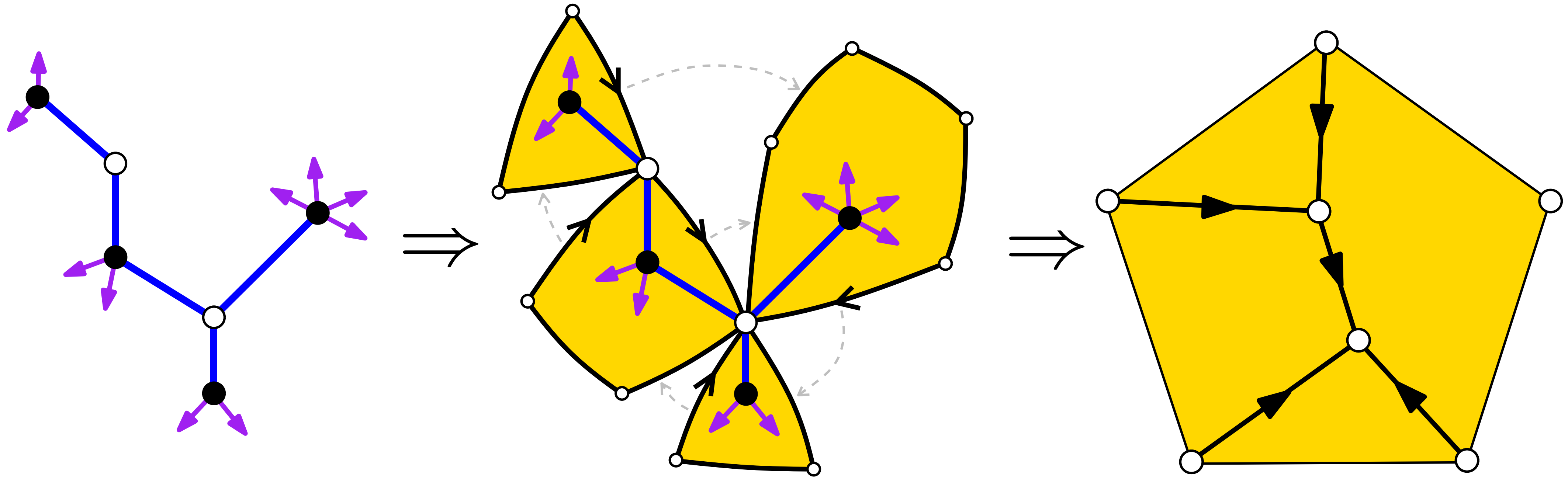
degree of external face $\longleftrightarrow$ $-$excess
degree of internal faces $\longleftrightarrow$ degree of black vertices
indegree of internal vertices $\longleftrightarrow$ degree of white vertices

cf [Bernardi'07], [Bernardi-Chapuy'10]

Extension for mobiles of excess ≤ 0



- Inverse mapping (tree $\rightarrow$ cactus $\rightarrow$ closure operations)



Scheme for a general bijective strategy

1) Map family $\mathcal{C}$ identifies with a **subfamily** $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees

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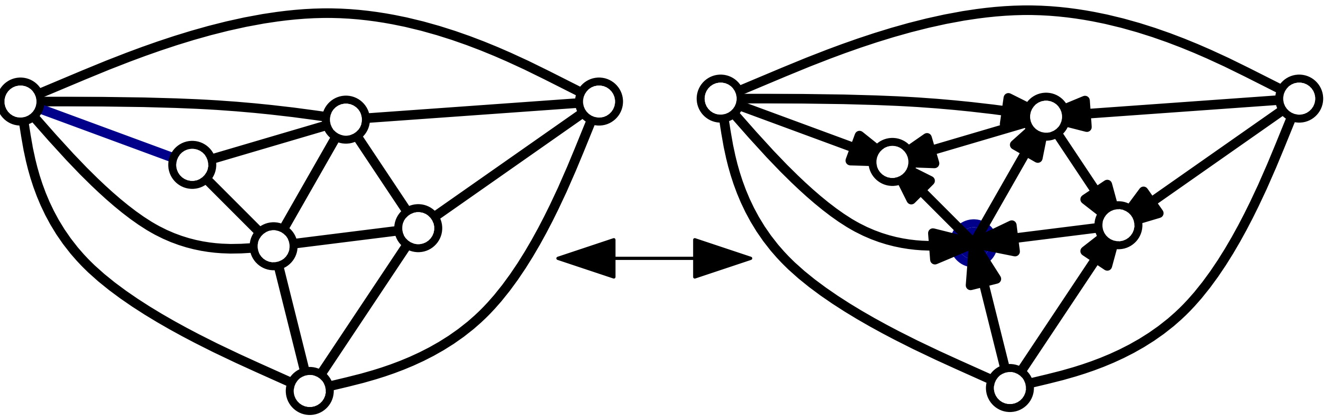
Example: $\mathcal{C}$ = Family of **simple triangulations**



$\mathcal{C} \simeq$ subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with

- Face-degree $= 3$
- Vertex-indegree $= 3$

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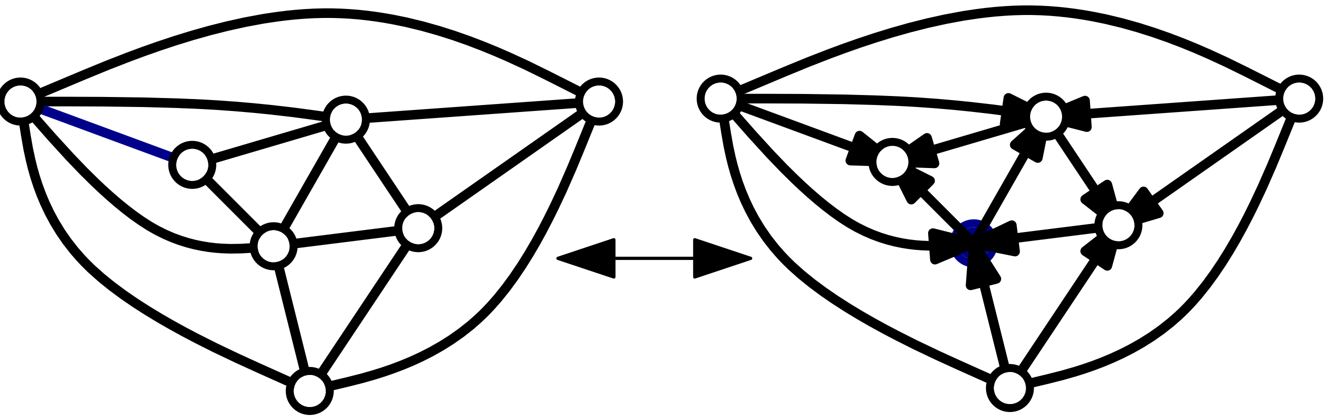

$\mathcal{C} \simeq$ subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with

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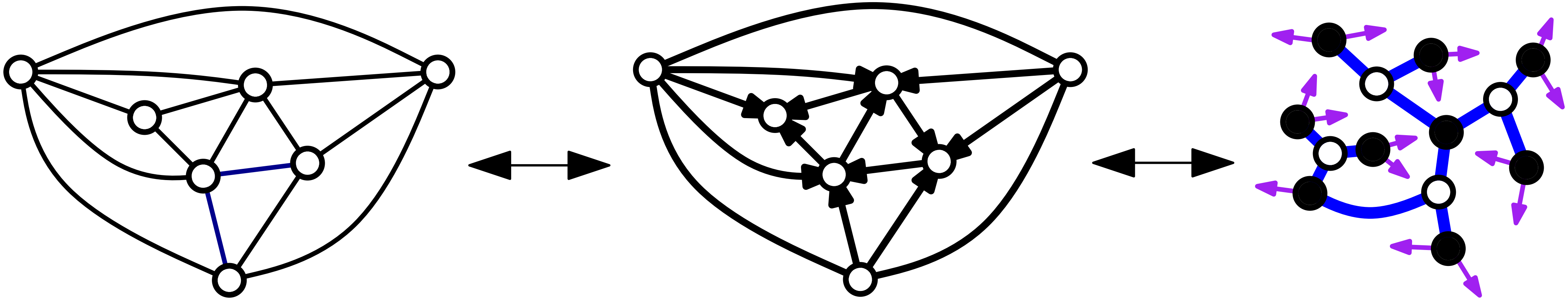
(2) **Specialize** the 'meta bijection' Φ to the subfamily $\mathcal{O}_C$



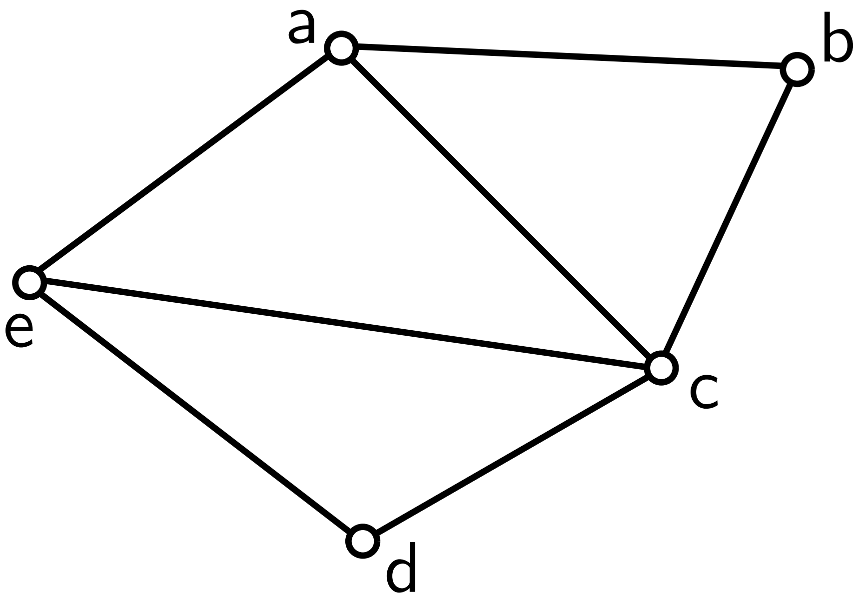
degree of internal faces $\longleftrightarrow$ degree of black vertices

indegree of internal vertices $\longleftrightarrow$ degree of white vertices

α-orientations

Let $G = (V, E)$ be a graph

Let α be a function from V to $\mathbb{N}$



$$\alpha: \begin{array}{l} a \rightarrow 2 \\ b \rightarrow 1 \\ c \rightarrow 2 \\ d \rightarrow 0 \\ e \rightarrow 2 \end{array}$$

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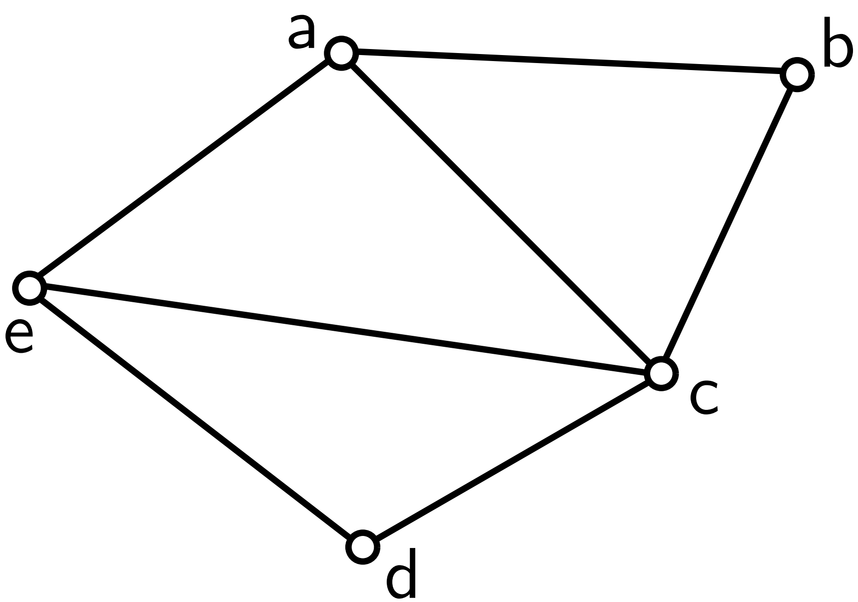

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Def: An **α-orientation** is an orientation of G where for each $v \in V$

$$\boxed{\text{indegree}(v) = \alpha(v)}$$

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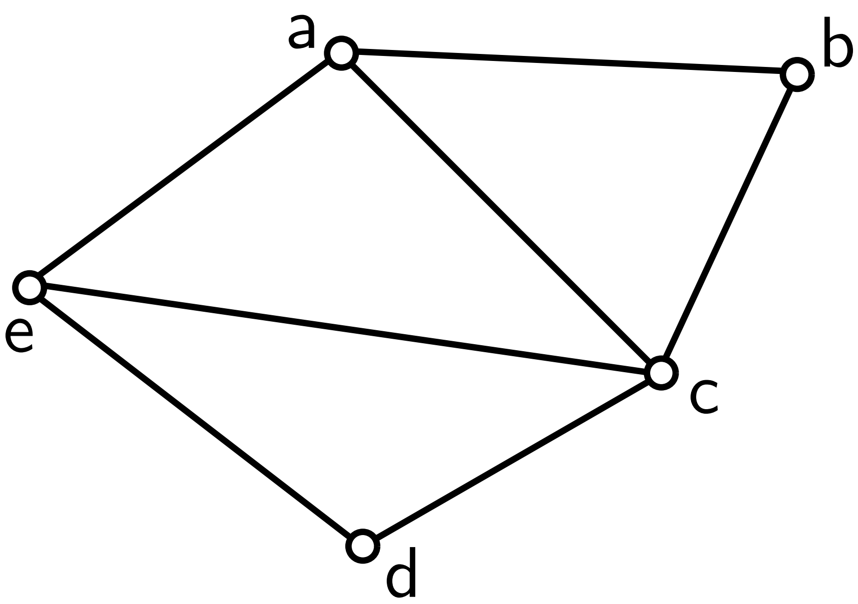

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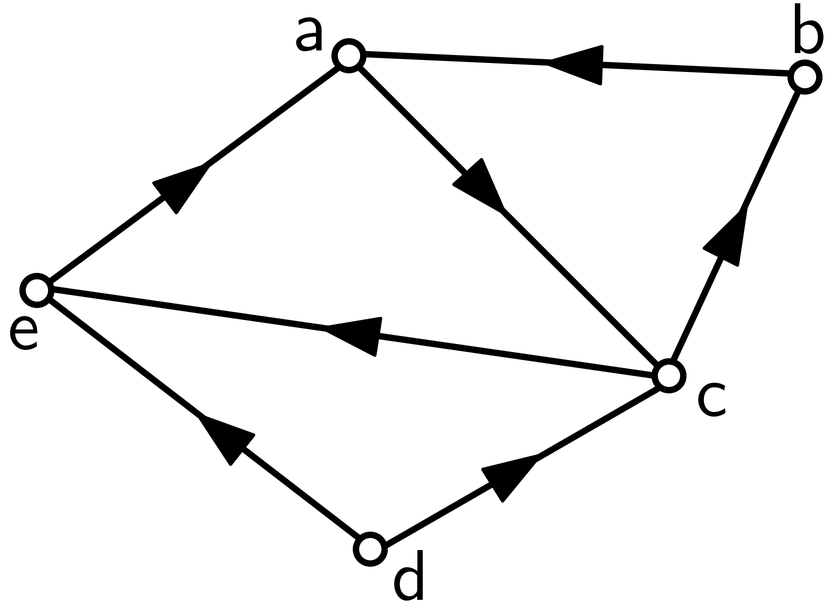

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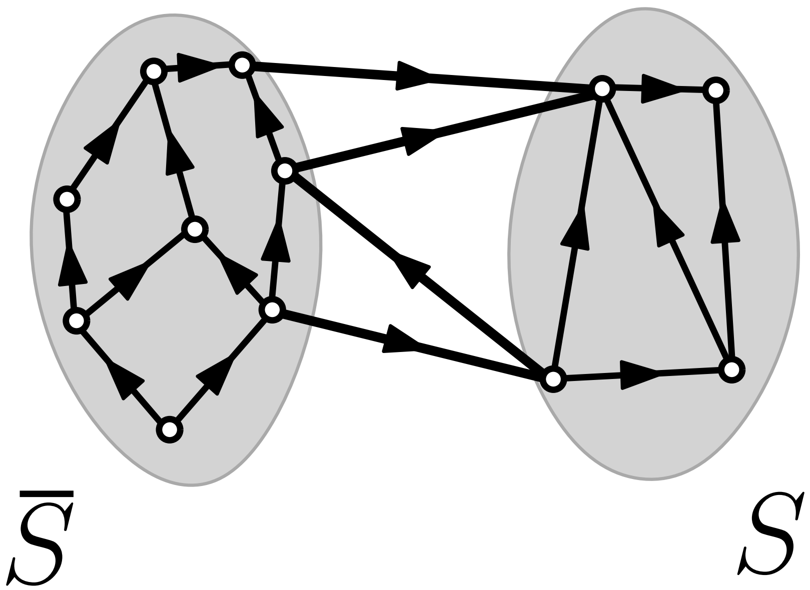
α-orientations: criteria for existence

- If an α-orientation **exists**, then



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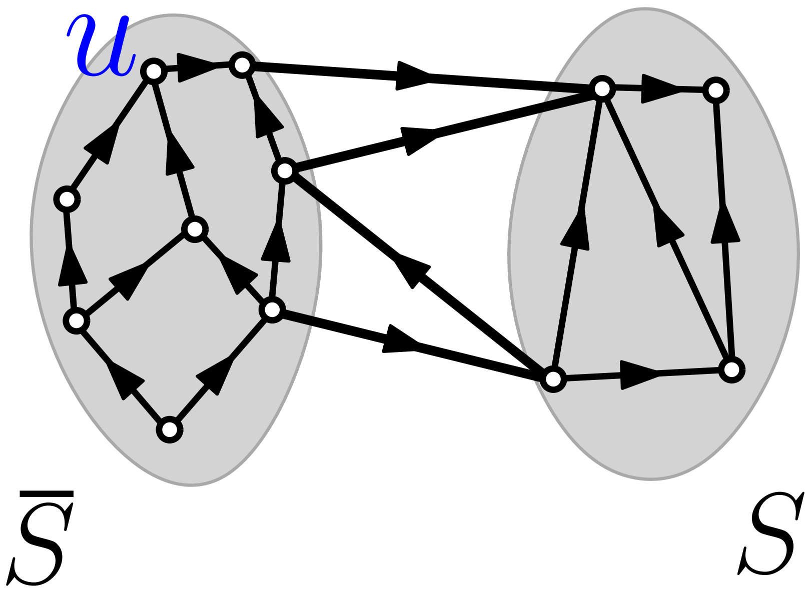

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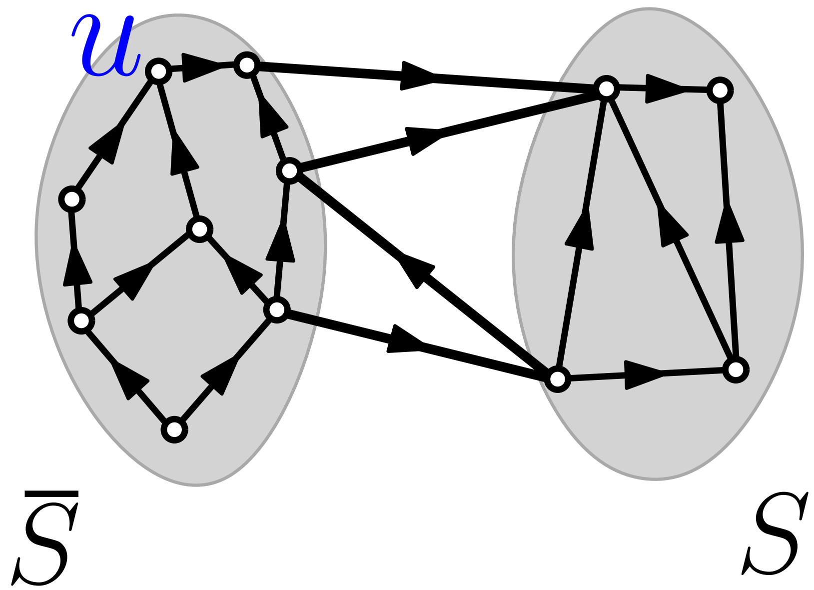

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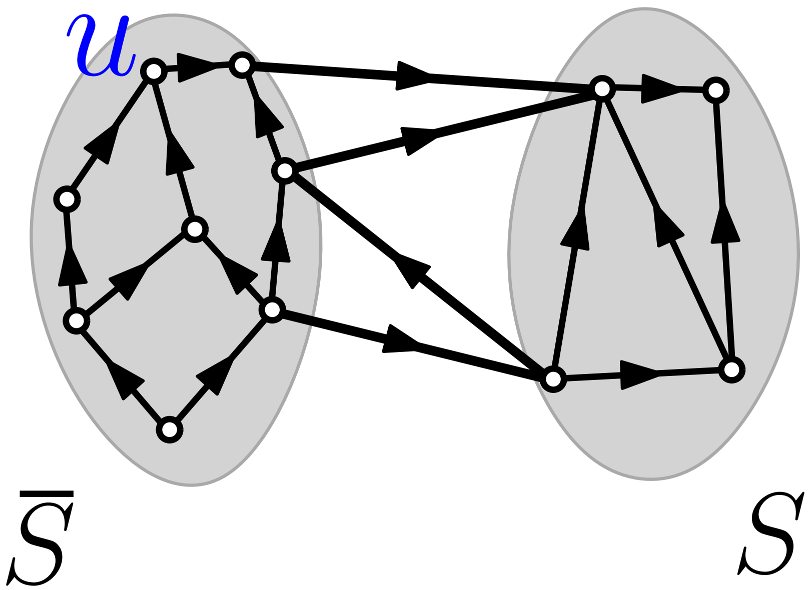

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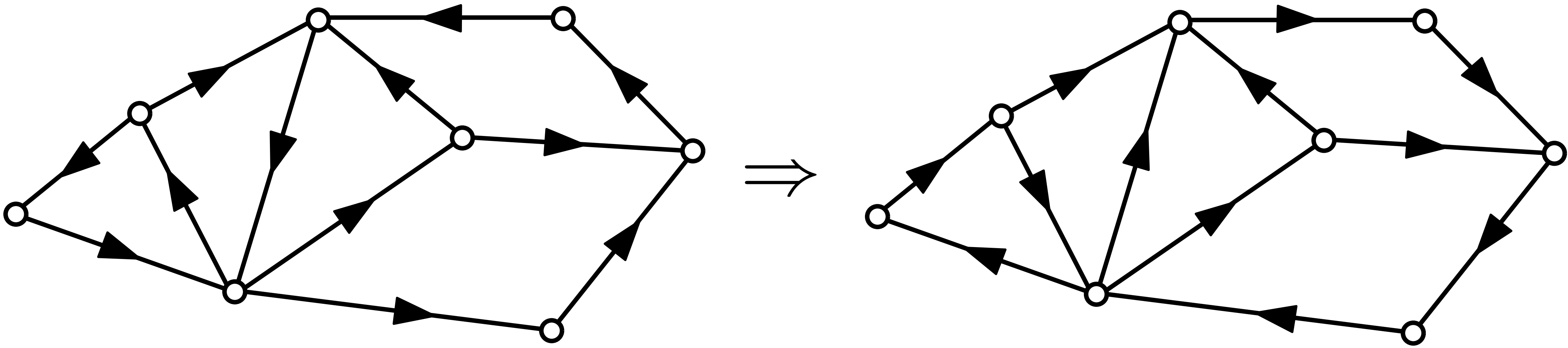
$\Rightarrow$ accessibility from $u \in V$ just depends on α (not on which α-orientation)

α-orientations for plane maps

Fundamental lemma: If a plane map admits an α-orientation, then it admits a **unique** α-orientation **without ccw circuit**, called **minimal**



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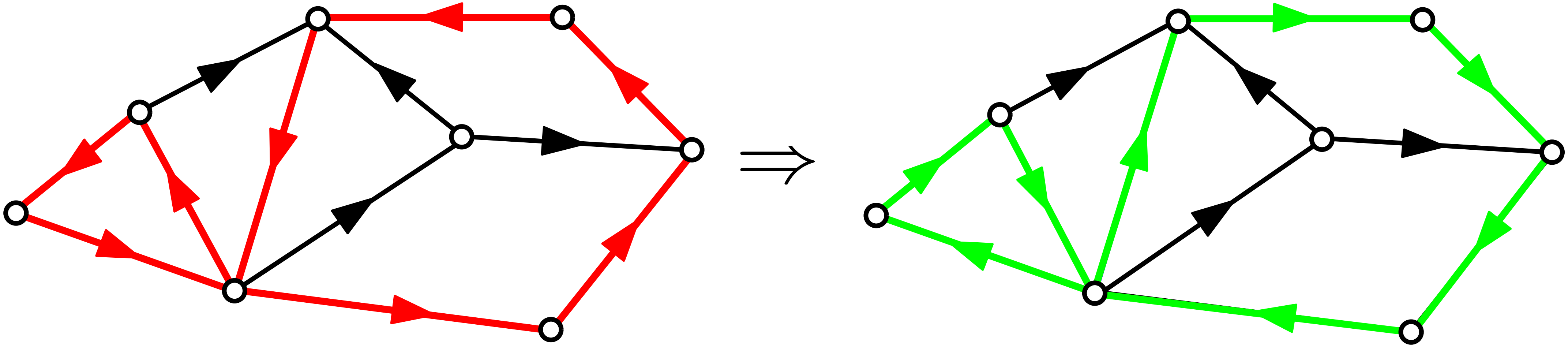

Uniqueness proof: if $O_1 \neq O_2$, edges where O_1 and O_2 **disagree** form an **eulerian suborientation** of $O_1 \Rightarrow$ contains a circuit (ccw in O_1 or O_2)

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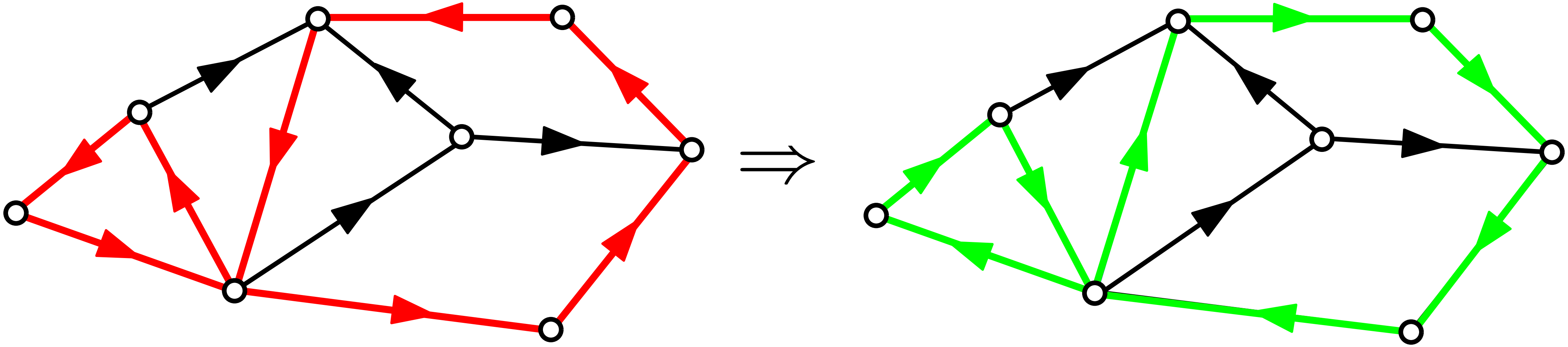

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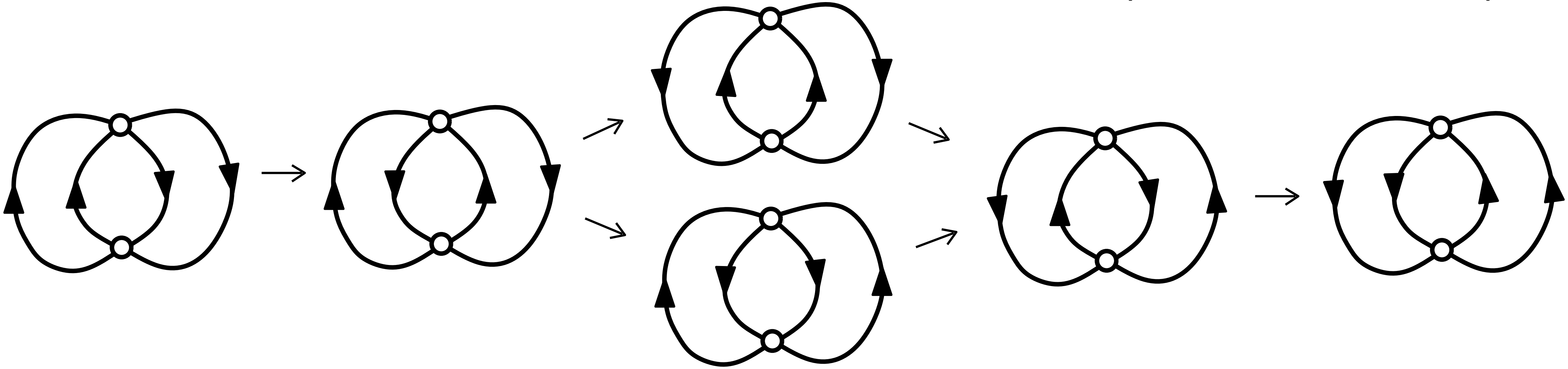

Set of α-orientations = **distributive lattice**

[Khueller et al'93], [Propp'93], [O. de Mendez'94], [Felsner'03]

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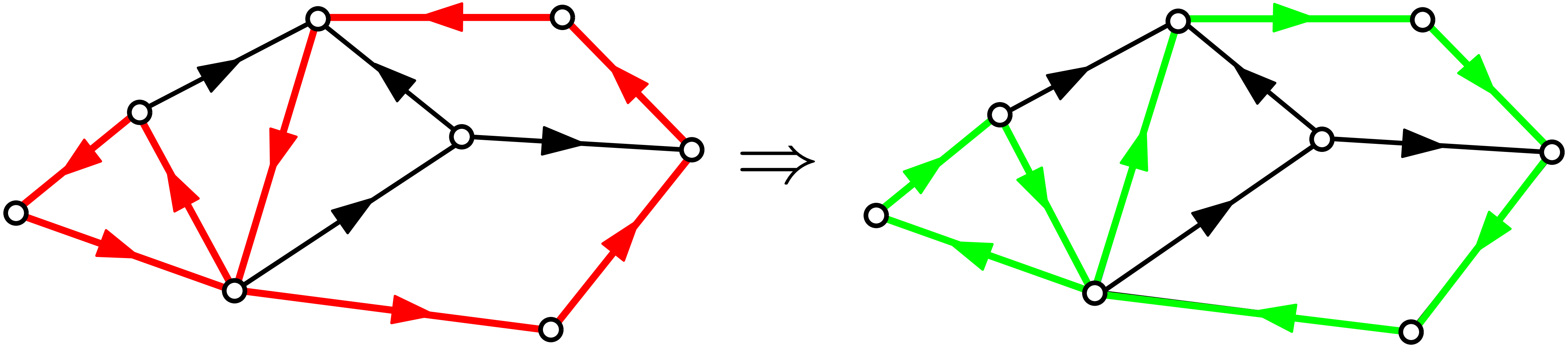

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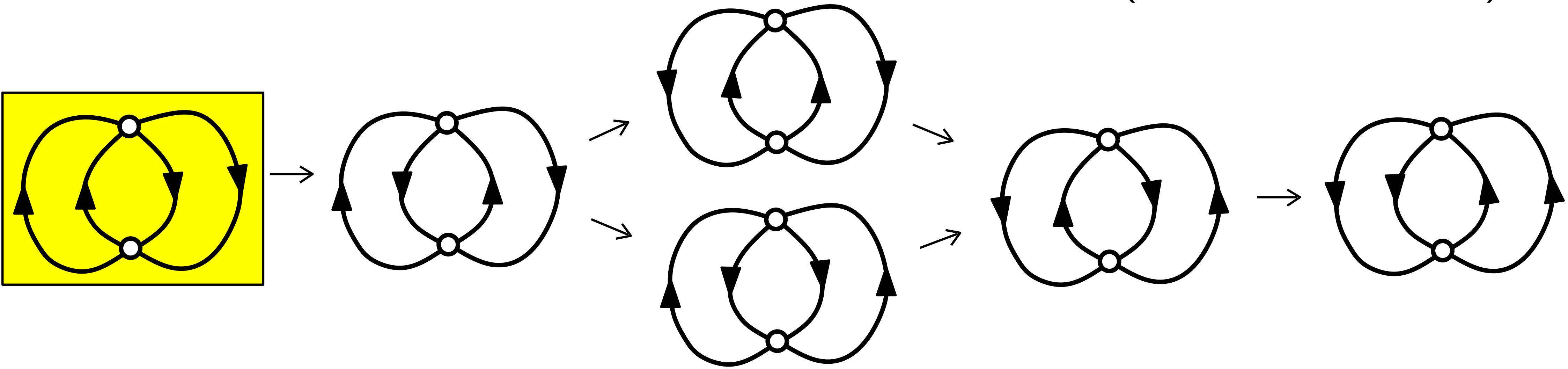

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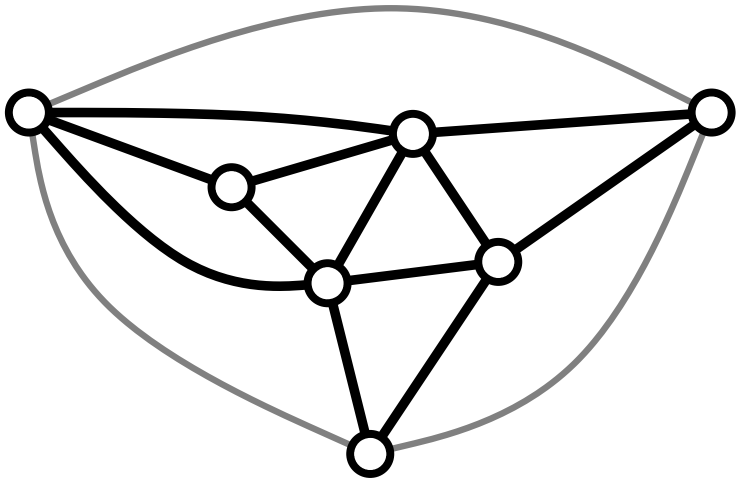
Application to simple triangulations

Fact: A triangulation with n internal vertices has $3n$ internal edges.



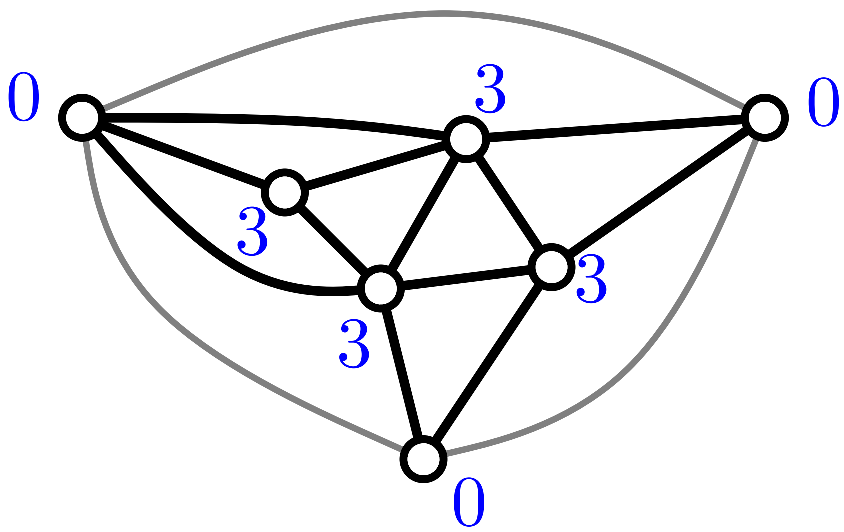
Application to simple triangulations

Fact: A triangulation with n internal vertices has $3n$ internal edges.

Natural candidate for indegree function:

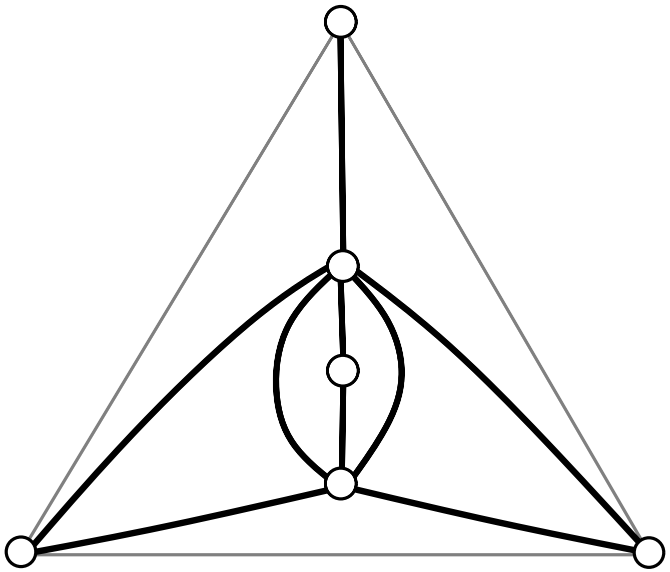
$$\alpha : \; v \mapsto 3 \text{ for each internal vertex } v.$$

call 3-**orientation** such an α-orientation

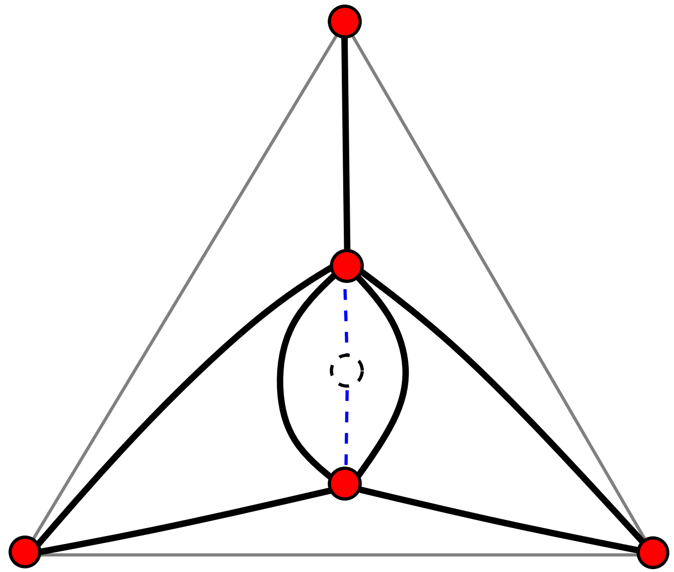


Application to simple triangulations

Fact: A triangulation admitting a 3-orientation is simple



$\Rightarrow$



k internal vertices
$3k + 1$ internal edges

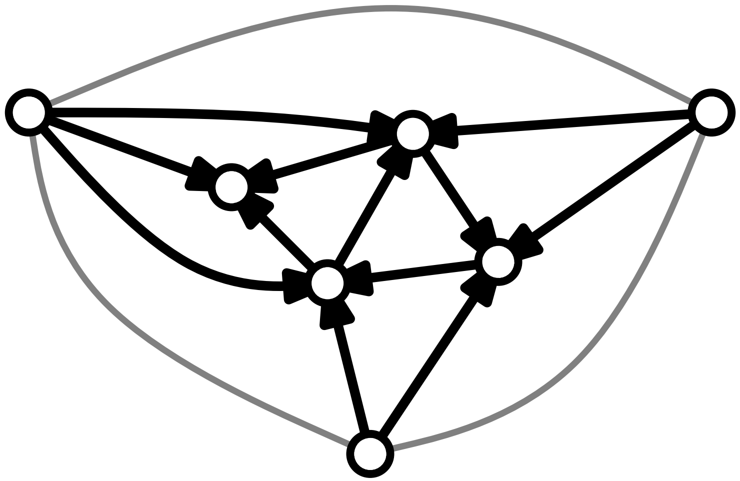
Application to simple triangulations

Thm [Schnyder 89]: A simple triangulation admits a 3-orientation.
(proof by shelling procedure)

Easier proof: Any simple planar graph $G = (V, E)$ satisfies

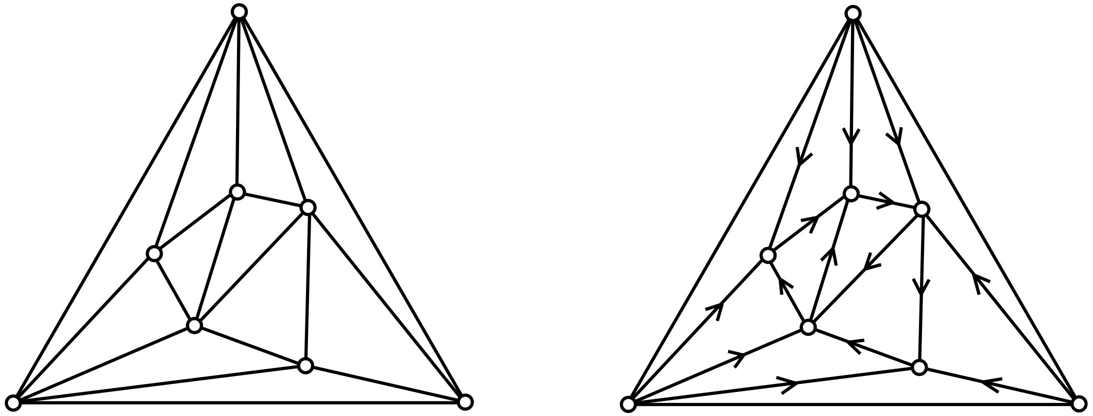
$$|E| \leq 3|V| - 6 \quad \text{(Euler relation)}$$

hence the existence/accessibility conditions are satisfied.$\square$

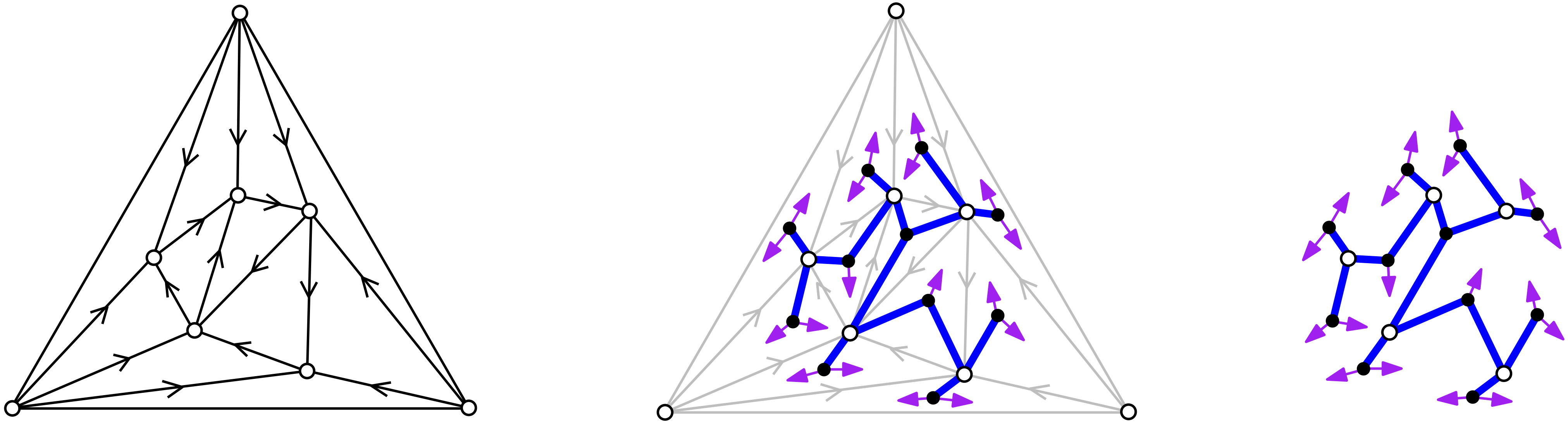


Application to simple triangulations

- From the lattice property (**taking the min**) we have
  family $\mathcal{F}$ of simple triangulations $\leftrightarrow$ **subfamily $\mathcal{O}_T$ of $\mathcal{O}$** where:
  - faces have degree 3
  - inner vertices have indegree 3



- From the **bijection Φ specialized to $\mathcal{O}_T$**, we have
  $\mathcal{F}$ $\leftrightarrow$ **mobiles** where all vertices have **degree 3**

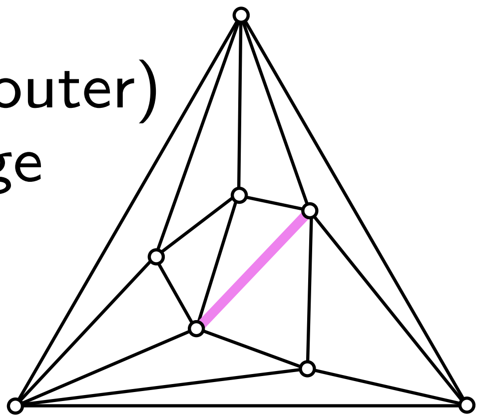


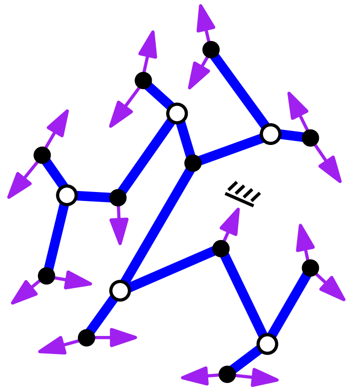
[Bernardi, F'10], other bijection in [Poulalhon, Schaeffer'03]

Counting formula for simple triangulations

Let $T_n = \#$ rooted simple triangulations with $n+3$ vertices

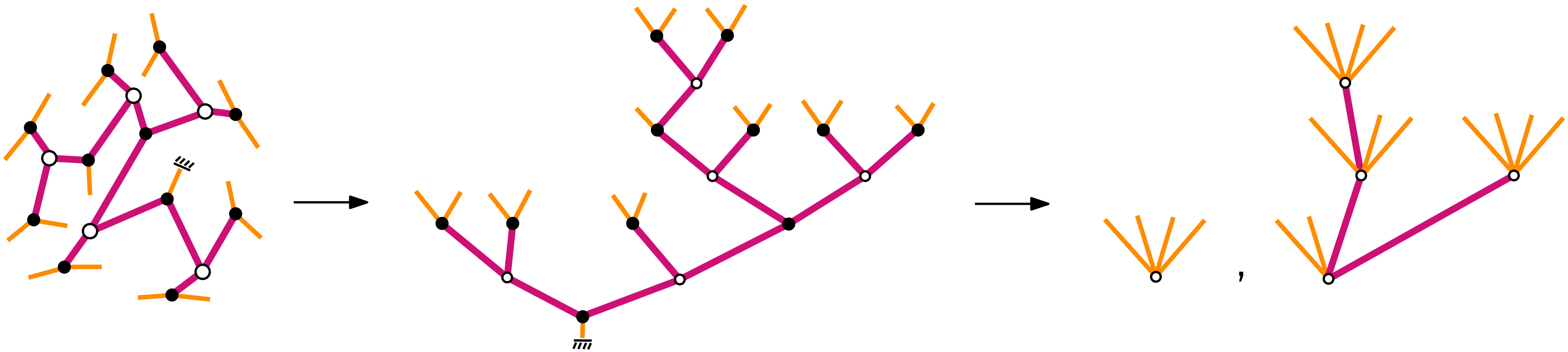
marked face (outer)
$+$ marked edge





marked bud

cardinality $= \frac{(2n+2)}{2} T_n$



$$\Rightarrow \boxed{T_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}}$$

pair of quaternary trees, n nodes

Application to simple quadrangulations

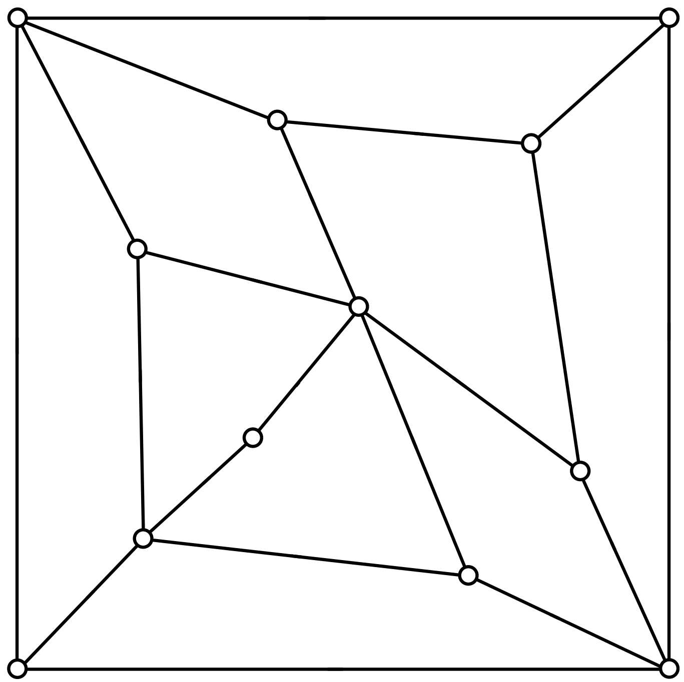
2-orientation $=$ orientation where each internal vertex has indegree 2

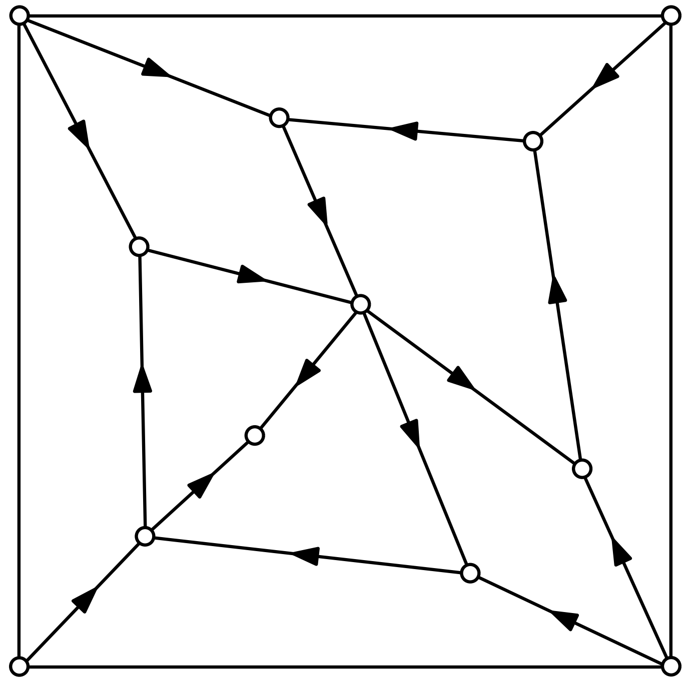
[de Fraysseix, Ossona de Mendez'01]:

A quadrangulation Q admits a 2-orientation iff Q is simple

Every 2-orientation is accessible from the outer contour

(proof by shelling algorithm)



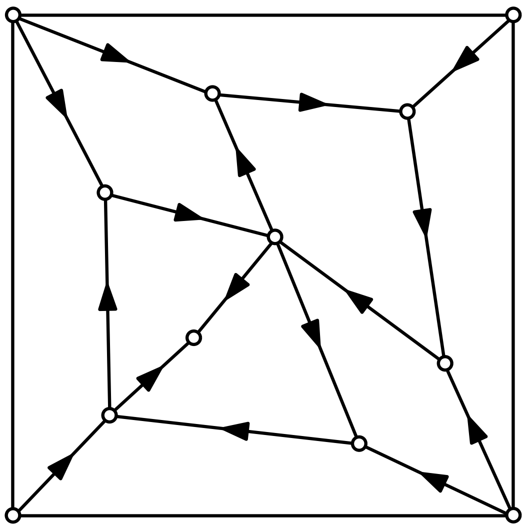


Proof from existence criterion:

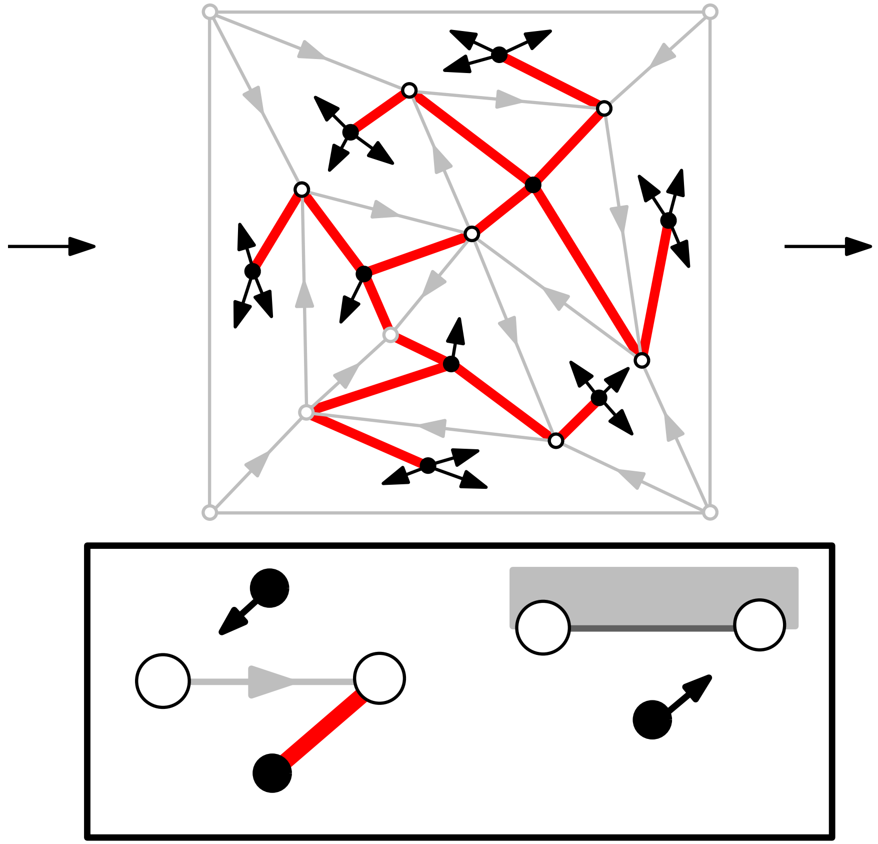
for every simple bipartite graph $G=(V,E)$, one has $|E| \leq 2|V|-4$

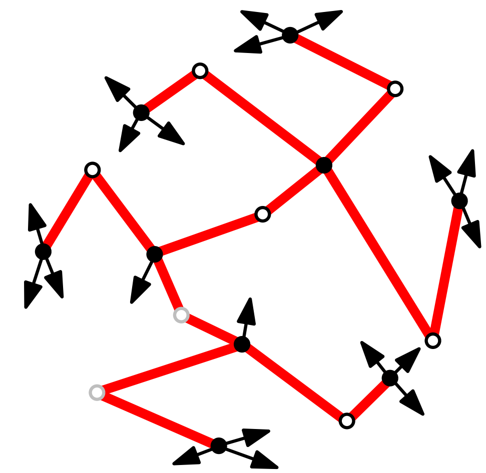
Application to simple quadrangulations

- Specializing the meta bijection Φ we get



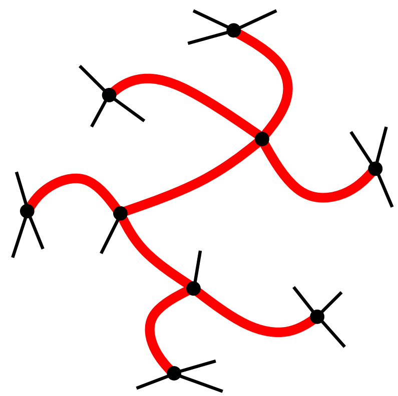
indegrees $= 2$
face-degrees $= 4$





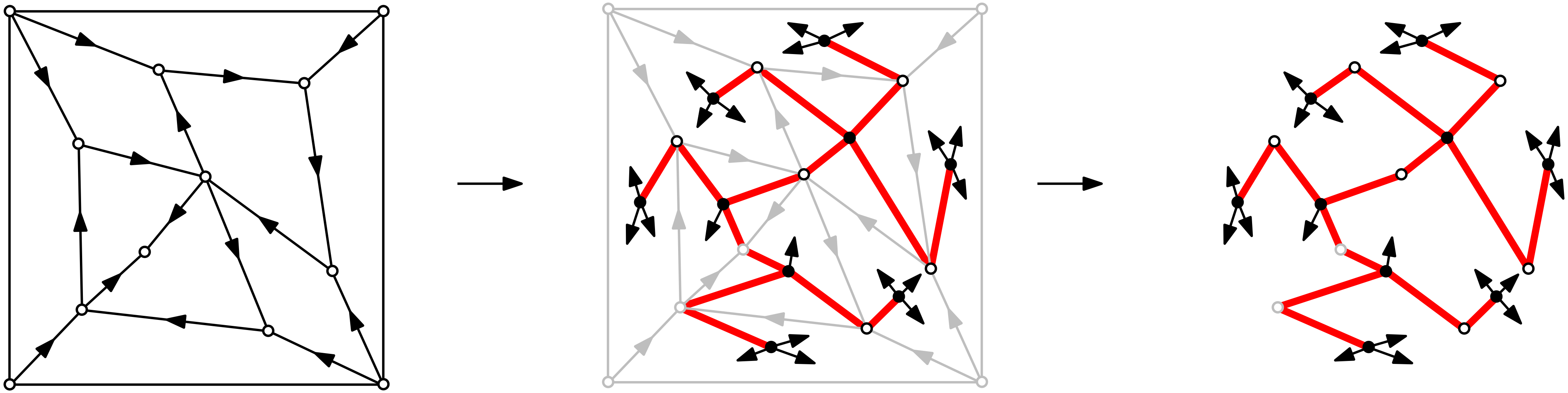
every ○ has degree 2
every ● has degree 4

($\simeq$ unrooted ternary tree)

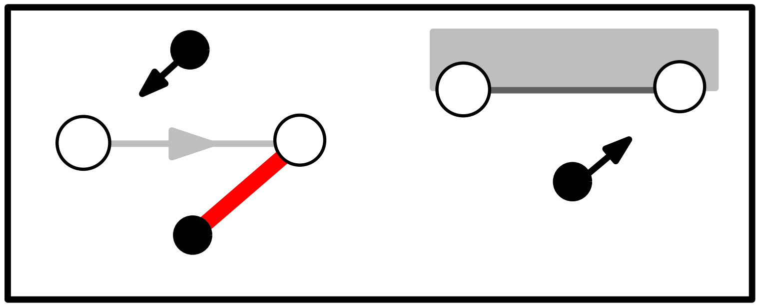


Application to simple quadrangulations

- Specializing the meta bijection Φ we get



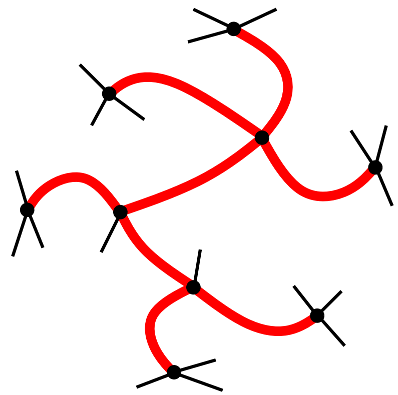
indegrees $= 2$
face-degrees $= 4$



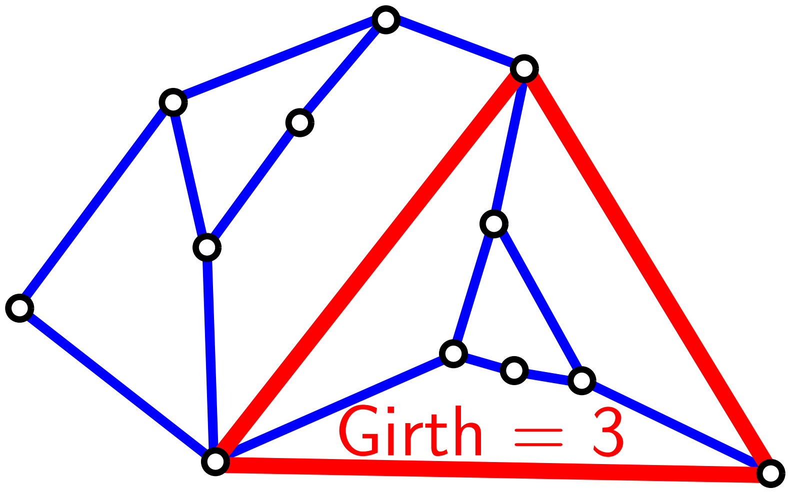
every ○ has degree 2
every ● has degree 4
($\simeq$ unrooted ternary tree)

- recover a bijection in **[Schaeffer'99]**

- bijection $\Rightarrow$ there are $\frac{4(3n)!}{n!(2n+2)!}$ rooted simple quadrangulations with n faces



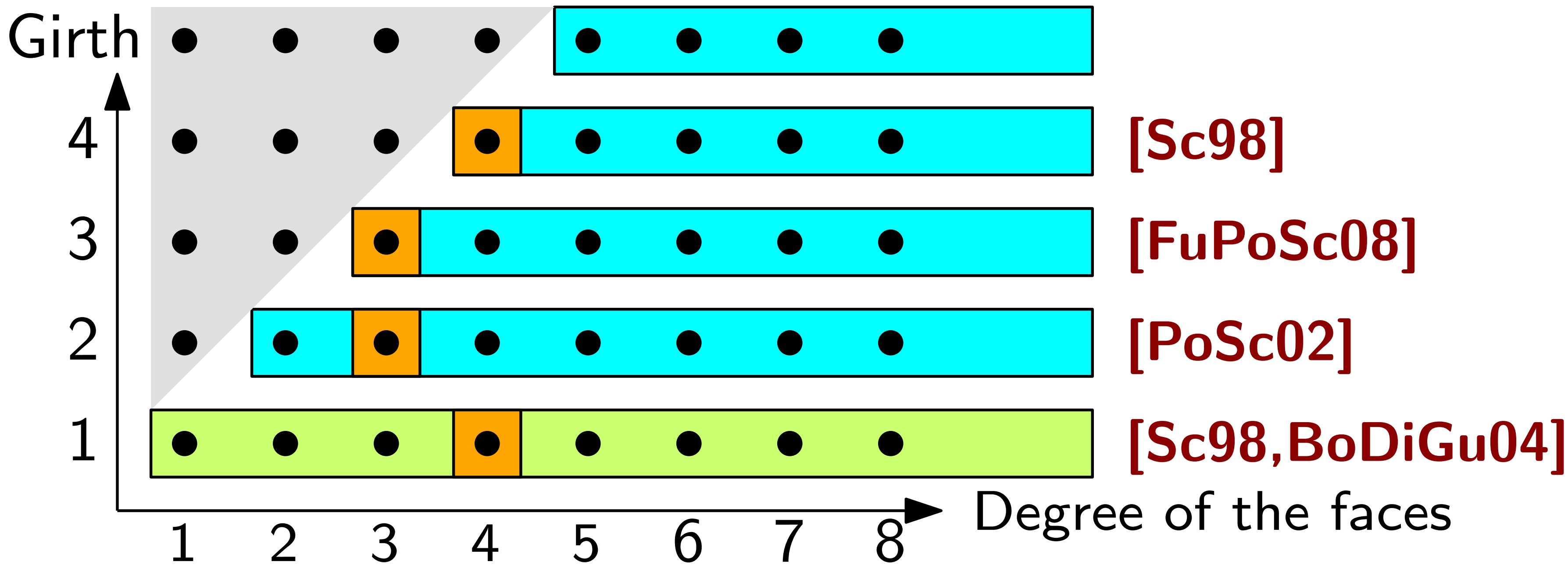
Extension to any girth and face-degrees



girth=length shortest cycle

Rk: girth ≤minimal face-degree

Our approach works in any girth d, with control on the face-degrees



Other approach using slice decompositions **[Bouttier,Guitter'15]**