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Schur Function Identities and Hook Length Posets

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Abstract. In this paper we find new classes of posets which generalize the d-complete posets. In fact the d-complete posets are classified into 15 irreducible classes in the paper "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets" (*J. Algebraic Combin.* **9** (1999), 61 – 94) by R. A. Proctor. Here we present six new classes of posets of hook-length property which generalize the 15 irreducible classes. Our method to prove the hook-length property is based on R. P. Stanley's (P,ω)-partitions and Schur function identities.

Résumé. Dans cet article nous trouvons des nouvelles classes de posets qui généralisent les posets d-complets. En fait, les posets d-completes sont classés en 15 classes irréducibles dans l'article "Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets" (*J. Algebraic Combin.* **9** (1999), 61 – 94) par R. A. Proctor. Dans cet article nous présentons six nouvelles classes de posets ayant la propriété de longueur de crochet, qui généralisent les 15 classes irréductibles. Notre méthode pour prover la propriété de longueur de crochet est basée sur les (P,ω)-partitions de R. P. Stanley et identités de fonctions de Schur.

1. Introduction

In [**18**] R. A. Proctor defined d-complete posets, which include shapes, shifted shapes and trees, by certain local structural conditions and showed that arbitrary connected d-complete poset is decomposed into a slant sum of irreducible ones. He also classified 15 exhaustive classes of irreducible d-complete components and described all of the members of each class. In this paper we define six types of posets as subposets of the posets of lattice points in the plane, and these six types generalize the 15 types of irreducible d-complete posets. First we enumerate eight product formulas involving the Schur functions in Section 2, which will be applied to obtain the hook formulas of the new posets, which we call "leaf posets". Then, in Section 3, we define certain subposets of the lattice points, which we call "pre-leaf posets". In fact, a pre-leaf posets is defined by gluing components of lattice points along the main diagonal. Although we can give the one-variable generating function of (P,ω)-partitions for any pre-leaf poset by the theory of Stanley's (P,ω)-partitions, the generating function is not always a product. Thus we specify six special cases, which we call "leaf-posets", and in these cases we can show the generating function becomes a product by the Schur function identities. We present these six types of leaf-posets and the hook formulas. Our definition is not based on local structural conditions, and our proof of the product formulas are based on determinant or Pfaffian expressions of the generating functions, but we don't have enough space to state the proofs.

Throughout this paper, let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_{>0}$ denote the set of integers, non-negative integers and positive integers, respectively. For a set S, we denote the cardinality of S by $|S|$. From now on, P is a partially ordered set (poset) and is assumed to be finite. Let P be a finite poset. If $x, y \in P$, then we say *y covers x* if $x < y$ and no $z \in P$ satisfies $x < z < y$. When y covers x, we denote $y \gtrdot x$. A tree T is a finite connected

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poset with a maximum element such that every element except the maximum element is covered by exactly one element.

Let ω be a labeling of P, i.e. ω is a bijection from P to $\{1, 2, \dots, |P|\}$. If $x \lessdot y$ is an edge in the diagram of P and $\omega(x) > \omega(y)$, we say $x \lessdot y$ is a *strict* edge. Otherwise, we say $x \lessdot y$ is a *weak* edge. In [**22**], R. P. Stanley defined the (P, ω)-partitions and obtained the several results on their generating functions.

DEFINITION 1.1. A (P, ω)-*partition* is a map φ from P to $\mathbb{N}$ satisfying the following conditions:

(i) $\varphi(x) \geq \varphi(y)$ if $x < y$ in P, i.e. φ is order reversing.
(ii) $\varphi(x) > \varphi(y)$ if $x < y$ and $\omega(x) > \omega(y)$.

When a labeling ω is natural, a (P, ω)-partition is simply called a P-partition. We can easily see that φ is a P-*partition* if and only if φ is an order-reversing map from P to $\mathbb{N}$. We denote the set of all (P, ω)-partitions by $\mathscr{A}(P, \omega)$, and the set of all P-partitions by $\mathscr{A}(P)$.

As one can easily see, $\mathscr{A}(P, \omega)$ depends only on which edges are strict and which edges are weak. Let $\mathscr{E} = \{(x, y) : x, y \in P, x \lessdot y\}$ denote the set of edges in P. We call a map $\epsilon : \mathscr{E} \to \{0, 1\}$ an *orientation*, and we regard the edges assigned 1 are the strict edges and the edges assigned 0 are the weak edges. We refer to (P, ϵ) an *oriented poset*. A labeling ω gives rise to an orientation, but not all orientation can come from a labeling (see [**13**]).

If P is a finite poset and ω is a labeling of P, then

$$F(P, \omega; q) := \sum_{\varphi \in \mathscr{A}(P, \omega)} q^{|\varphi|} \tag{1.1}$$

is said to be the *one-variable generating function* of (P, ω)-partitions. Here $|\varphi| = \sum_{x \in P} \varphi(x)$ is the *weight* of the (P, ω)-partition φ, which is the sum of its entries.

We say that P has *hook-length property* if there exists a map h_P from P to $\mathbb{Z}_{>0}$ satisfying

$$\sum_{\varphi \in \mathscr{A}(P)} q^{|\varphi|} = \prod_{x \in P} \frac{1}{1 - q^{h_P(x)}}. \tag{1.2}$$

If P has hook-length property, then $h_P(x)$ is called the *hook length* of x and h_P is called the *hook-length function* of P. A *hook-length poset* is a poset which has hook-length property.

2. Schur function identities

In this section we state eight Cauchy type identities of the Schur functions, which will be applied in the following sections. Our proof of these product formulas is based on the determinant or Pfaffian evaluations. We omit the proof and just present the identities, but the proof will be stated in [**6**].

The *Schur function* $s_\lambda(x_1, \dots, x_n)$ of variables $x_1, \dots, x_n$ with respect to a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is defined to be

$$s_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}}. \tag{2.1}$$

For detailed explanation of symmetric functions, the reader can find in [**12**]. For a positive integer m, we write $X_m = (x_1, x_2, \dots, x_m)$, $Y_m = (y_1, y_2, \dots, y_m)$ and $Z_m = (z_1, z_2, \dots, z_m)$ in short, where m is the number of variables. Let $\mathscr{P}$ denote the set of all partitions. One of the most well-known identities is the Cauchy identity which reads

$$\sum_{\lambda \in \mathscr{P}} s_\lambda(X_m) s_\lambda(Y_n) = \frac{1}{\prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)}. \tag{2.2}$$

If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition and a and b are positive integers such that $a \leq b$, then we write $\lambda[a, b]$, in short, for the partition $(\lambda_a, \lambda_{a+1}, \dots, \lambda_b)$. If $X_m = (x_1, x_2, \dots, x_m)$ is an m-tuple of variables, then we use the notation $\|X_m\| := \prod_{i=1}^m x_i$ for brevity. The aim of this section is to prove the following variants of the Cauchy identity.

THEOREM 2.1. Let m be a positive integer.

(i) If $m \geq 1$, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\mathscr{P}} w^{\lambda_m} s_\lambda(X_m)s_\lambda(Y_m) = \frac{1-\|X_m\|\|Y_m\|}{(1-w\|X_m\|\|Y_m\|)\prod_{i,j=1}^{m}(1-x_iy_j)}. \tag{2.3}$$

(ii) If $m \geq 2$, and $v = 1$ or 2, then we have

$$\begin{aligned}&\sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\mathscr{P}} z_v^{|\lambda|-\lambda_m-\lambda_{m-1}} w^{\lambda_m} s_\lambda(X_m)s_{\lambda[1,m-1]}(Y_{m-1})s_{\lambda[m-1,m]}(Z_2)\\ &= \frac{\prod_{k=1}^{m-1}(1-z_v^{m-2}y_k\|X_m\|\|Y_{m-1}\|\|Z_2\|)}{(1-wz_v^{m-2}\|X_m\|\|Y_{m-1}\|\|Z_2\|)\prod_{i=1}^{m}\prod_{j=1}^{m-1}(1-x_iy_jz_v)\prod_{k=1}^{m}(1-z_v^{m-3}x_k^{-1}\|X_m\|\|Y_{m-1}\|\|Z_2\|)}.\end{aligned} \tag{2.4}$$

(iii) If $v = 1$ or 2, then we have

$$\begin{aligned}&\sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_6)\in\mathscr{P}} z_v^{\lambda_1+\lambda_2} w^{\lambda_6} s_{\lambda[1,3]}(X_3)s_{\lambda[3,4]}(Z_2)s_{\lambda[4,6]}(X_3)s_{\lambda[1,5]}(Y_5)s_{\lambda[5,6]}(Z_2)\\ &= \frac{1}{(1-wz_v^2\|X_3\|^2\|Y_5\|\|Z_2\|^2)\prod_{i=1}^{3}\prod_{j=1}^{5}(1-x_iy_jz_v)}\\ &\quad\times\frac{\prod_{k=1}^{5}(1-z_v^2y_k\|X_3\|^2\|Y_5\|\|Z_2\|^2)}{\prod_{k=1}^{3}(1-z_vx_k^{-1}\|X_3\|^2\|Y_5\|\|Z_2\|^2)\prod_{1\leq i<j\leq 5}(1-z_vy_i^{-1}y_j^{-1}\|X_3\|\|Y_5\|\|Z_2\|)}.\end{aligned} \tag{2.5}$$

(iv) If $v = 1$ or 2, then we have

$$\begin{aligned}&\sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{2r})\in\mathscr{P}} z_v^{\lambda_1} w^{\lambda_{2r}} s_\lambda(X_{2r})\prod_{i=1}^{r}s_{\lambda[2i-1,2i]}(Y_2)\prod_{i=1}^{r-1}s_{\lambda[2i,2i+1]}(Z_2)\\ &= \frac{\prod_{i=1}^{2}(1-z_vz_i\|X_{2r}\|\|Y_2\|^r\|Z_2\|^{r-1})}{(1-wz_v\|X_{2r}\|\|Y_2\|^r\|Z_2\|^{r-1})\prod_{i=1}^{2r}\prod_{j=1}^{2}(1-x_iy_jz_v)\prod_{1\leq i<j\leq 2r}(1-x_ix_j\|Y_2\|\|Z_2\|)}.\end{aligned} \tag{2.6}$$

(v) If $v = 1$ or 2, then we have

$$\begin{aligned}&\sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{2r+1})\in\mathscr{P}} z_v^{\lambda_1} w^{\lambda_{2r+1}} s_\lambda(X_{2r+1})\prod_{i=1}^{r}s_{\lambda[2i-1,2i]}(Y_2)\prod_{i=1}^{r}s_{\lambda[2i,2i+1]}(Z_2)\\ &= \frac{\prod_{i=1}^{2}(1-z_vy_i\|X_{2r+1}\|\|Y_2\|^r\|Z_2\|^r)}{(1-wz_v\|X_{2r+1}\|\|Y_2\|^r\|Z_2\|^r)\prod_{i=1}^{2r+1}\prod_{j=1}^{2}(1-x_iz_jz_v)\prod_{1\leq i<j\leq 2r+1}(1-x_ix_j\|Y_2\|\|Z_2\|)}.\end{aligned} \tag{2.7}$$

(vi) If $r \geq 2$, $v \in \{s,t\} \subseteq \{1,2,3\}$ and $s \neq t$, then we have

$$\begin{aligned}&\sum_{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{2r})\in\mathscr{P}} x_v^{\lambda_1} w^{\lambda_{2r}} s_{\lambda[1,2r-1]}(Y_{2r-1})s_{\lambda[2r-2,2r]}(X_3)\prod_{i=1}^{r}s_{\lambda[2i-1,2i]}(Z_2)\prod_{i=1}^{r-2}s_{\lambda[2i,2i+1]}(x_s,x_t)\\ &= \frac{1}{(1-w(x_sx_t)^{r-2}x_v\|X_3\|\|Y_{2r-1}\|\|Z_2\|^r)\prod_{i=1}^{2r-1}\prod_{j=1}^{2}(1-x_vy_iz_j)\prod_{1\leq i<j\leq 2r-1}(1-x_sx_ty_iy_j\|Z_2\|)}\\ &\quad\times\frac{\prod_{k=1}^{2r-1}(1-(x_sx_t)^{r-2}x_vy_k\|X_3\|\|Y_{2r-1}\|\|Z_2\|^r)}{\prod_{k=1}^{2}(1-(x_sx_t)^{r-2}z_k\|X_3\|\|Y_{2r-1}\|\|Z_2\|^{r-1})\prod_{k=1}^{2r-1}(1-(x_sx_t)^{r-3}x_vy_k^{-1}\|X_3\|\|Y_{2r-1}\|\|Z_2\|^{r-1})}.\end{aligned} \tag{2.8}$$

(vii) If $r \geq 1$, $v \in \{1,2\}$ and $1 \leq s \neq t \leq 3$, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{2r+1})\in\mathscr{P}} z_v^{\lambda_1} w^{\lambda_{2r+1}} s_{\lambda[1,2r]}(Y_{2r}) s_{\lambda[2r-1,2r+1]}(X_3) \prod_{i=1}^{r} s_{\lambda[2i,2i+1]}(Z_2) \prod_{i=1}^{r-1} s_{\lambda[2i-1,2i]}(x_s,x_t)$$

$$= \frac{1}{(1-w(x_sx_t)^{r-1}z_v\|X_3\|\|Y_{2r}\|\|Z_2\|^r)\prod_{i=s,t}\prod_{j=1}^{2r}(1-x_iy_jz_v)\prod_{1\leq i<j\leq 2r}(1-x_sx_ty_iy_j\|Z_2\|)}$$

$$\times \frac{\prod_{k=1}^{2r}(1-(x_sx_t)^{r-1}y_kz_v\|X_3\|\|Y_{2r}\|\|Z_2\|^r)}{\prod_{k=s,t}(1-(x_sx_t)^{r-2}x_k\|X_3\|\|Y_{2r}\|\|Z_2\|^r)\prod_{k=1}^{2r}(1-(x_sx_t)^{r-2}y_k^{-1}z_v\|X_3\|\|Y_{2r}\|\|Z_2\|^{r-1})}. \tag{2.9}$$

(viii) If $v = 1, 2, 3$ or 4, then we have

$$\sum_{\lambda=(\lambda_1,\lambda_2,\dots,\lambda_6)\in\mathscr{P}} y_v^{\lambda_1} w^{\lambda_6} s_{\lambda[1,3]}(X_3) s_{\lambda[3,4]}(Z_2) s_{\lambda[4,6]}(X_3) s_{\lambda[1,2]}(Z_2) s_{\lambda[2,5]}(Y_4) s_{\lambda[5,6]}(Z_2)$$

$$= \frac{1}{(1-wy_v\|X_3\|^2\|Y_4\|\|Z_2\|^3)\prod_{\substack{1\leq k\leq 4\\ k\neq v}}\prod_{j=1}^{2}(1-y_k^{-1}z_j\|X_3\|\|Y_4\|\|Z_2\|)}$$

$$\times \frac{\prod_{1\leq k\leq 4}(1-y_ky_v\|X_3\|^2\|Y_4\|\|Z_2\|^3)}{\prod_{k=1}^{3}(1-x_k\|X_3\|\|Y_4\|\|Z_2\|^2)\prod_{i=1}^{3}\prod_{j=1}^{2}(1-x_iy_vz_j)\prod_{\substack{1\leq k\leq 4\\ k\neq v}}\prod_{j=1}^{3}(1-x_j^{-1}y_ky_v\|X_3\|\|Z_2\|)}. \tag{2.10}$$

3. Pre-leaf posets

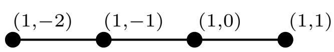
In this section we introduce a new class of posets, which we call pre-leaf posets, and give a formula of the generating function of the (P,ω)-partitions for any pre-leaf poset and its column-strict labeling. Unfortunately the generating functions does not factor out in general, but in the next section we present six classes of pre-leaf posets which has the hook-length property.

3.1. Pre-leaf posets. Let $\mathbb{Z}^2$ denote the set of lattice points in the plane, and let $D = \{(i,i) : i \in \mathbb{Z}\}$ denote the main diagonal of $\mathbb{Z}^2$. We call $x \in D$ a *diagonal point*. We can make $\mathbb{Z}^2$ into a poset by defining $(i_1,j_1) \leq (i_2,j_2)$ in $\mathbb{Z}^2$ if $i_1 \geq i_2$ and $j_1 \geq j_2$. It is easy to see that $(i_1,j_1) \lessdot (i_2,j_2)$ in $\mathbb{Z}^2$ if, and only if $(i_1-i_2, j_1-j_2) = (1,0)$ or $(0,1)$. When we draw a diagram, we use the matrix coordinates in which the first coordinate i (the row index) increases as one goes downwards, and the second coordinate j (the column index) increases as one goes from left to right. Thus we call $(1,0)$ a *vertical edge*, and $(0,1)$ a *horizontal edge*. If P is any subset of $\mathbb{Z}^2$, we regard P as a poset by the induced order of $\mathbb{Z}^2$, i.e. $x \leq y$ in P if, and only if $x \leq y$ in $\mathbb{Z}^2$. Let $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\dots,\alpha_r)$ be a strictly decreasing sequence of nonnegative integers of length r, and let d be an integer. Let $P^{(r)}(\boldsymbol{\alpha},d)$ denote the set of points $(i,j) \in \mathbb{Z}^2$ such that $d \leq i \leq d+r-1$ and $i \leq j \leq \alpha_{i-d+1}+i$. Note that the ith row of $P^{(r)}(\boldsymbol{\alpha},d)$ contains α_i+1 vertices starting from (i,i) for $d \leq i \leq d+r-1$. We introduce an order structure into $P^{(r)}(\boldsymbol{\alpha},d)$ as an induced subposet of $\mathbb{Z}^2$, which we call a *right half-leaf poset of type* $(\boldsymbol{\alpha},d)$. For example, if $\boldsymbol{\alpha} = (3,2,0)$ and $d = 2$, then the diagram of $P^{(r)}((3,2,0),2)$ is as in Figure 1. Similarly we define $P^{(l)}(\boldsymbol{\alpha},d)$ by the transpose of $P^{(r)}(\boldsymbol{\alpha},d)$, i.e. $P^{(l)}(\boldsymbol{\alpha},d) = \{(j,i) \in \mathbb{Z}^2 : (i,j) \in P^{(r)}(\boldsymbol{\alpha},d)\}$, which we call a *left half-leaf poset of type* $(\boldsymbol{\alpha},d)$. We call a poset *half-leaf* if it is a right or left half-leaf poset. For example, $P^{(l)}((3,2,0),2)$ is designated by the diagram in Figure 1. Especially, when the length of $\boldsymbol{\alpha}$ is one, i.e. $\boldsymbol{\alpha} = (a)$, then $P^{(l)}((a),d)$ is an $(a+1)$-element chain composed of vertical edges. We write $P^{(c)}(a,d)$ for $P^{(l)}((a),d)$ in brevity.

For a nonnegative integer f, let L_f denote the subset of $\mathbb{Z}^2$ defined by

$$L_f := \{(1,i) : -f+1 \leq i \leq 1, i \in \mathbb{Z}\},$$

which is $(f+1)$-element chain composed of horizontal edges. For example, if $f = 3$, then the diagram of L_3 is as follows.

$L_3 =$ .

Assume P and Q are any induced subposets of $\mathbb{Z}^2$. We build a new poset, denoted by $P \sqcup Q$, from half-leaf posets by gluing them together along the main diagonal.

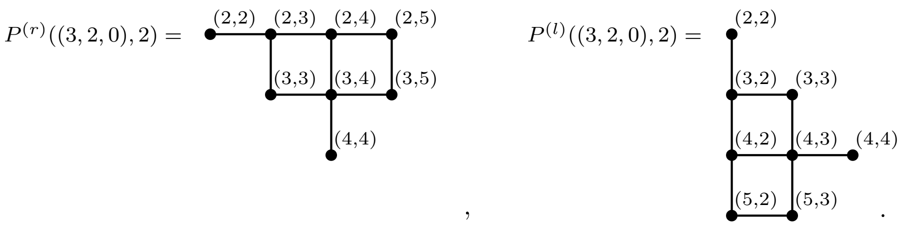


FIGURE 1. Half-leaf posets

DEFINITION 3.1. Let P and Q be induced subposets of $\mathbb{Z}^2$. Let $P \uplus Q$ denote the disjoint union of P and Q as a set, i.e. $P \uplus Q = \{(x,1) : x \in P\} \cup \{(x,2) : x \in Q\}$. We identify the points in the main diagonal of $P \uplus Q$, i.e. $((i,i),1) = ((i,i),2)$ for $i \in \mathbb{Z}$, and denote the resulting poset by $P \sqcup Q$, which we call *diagonally glued poset* of P and Q. Thus the order relation $x \leq y$ in $P \sqcup Q$ is the transitive closure of the following binary relation $x \prec y$: We say $x \prec y$ in $P \sqcup Q$ if one of the following conditions holds;

(i) x, y are both in P, and $x \leq y$ in P,
(ii) x, y are both in Q, and $x \leq y$ in Q.

For example, if we glue four posets $P^{(r)}((3,2,0),1)$, $P^{(l)}((3,2),1)$, $P^{(l)}((3,1),2)$ and L_2 along the main diagonal, then we obtain the following poset in Figure 2. Here we write (i,i) for the identified diagonal

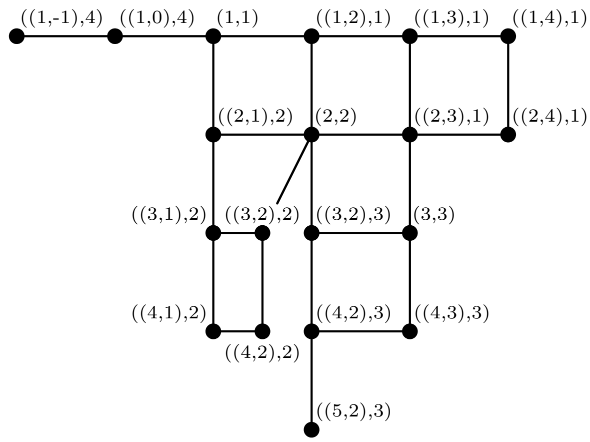


FIGURE 2. A pre-leaf poset

point $((i,i),k)$, $k = 1,2,3,4$.

As in this example, we define a new poset as a diagonally glued poset of several $P^{(r)}(\boldsymbol{\alpha}, d)$'s and $P^{(l)}(\boldsymbol{\alpha}, d)$'s. Thus, hereafter, we use the following notation.

DEFINITION 3.2. Let $\vec{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(k)})$ be a k-tuple of strictly decreasing sequences of nonnegative integers, and let $\vec{\boldsymbol{\beta}} = (\boldsymbol{\beta}^{(1)}, \dots, \boldsymbol{\beta}^{(l)})$ be an l-tuple of strictly decreasing sequences of nonnegative integers. Let r_i (resp. s_j) denote the length of $\boldsymbol{\alpha}^{(i)}$ (resp. $\boldsymbol{\beta}^{(j)}$) for $i = 1, \dots, k$ (resp. $j = 1, \dots, l$). Put $d_i = \sum_{\nu=1}^{i} r_\nu - i + 1$ (resp. $e_j = \sum_{\nu=1}^{j} s_\nu - j + 1$) for $i = 0, \dots, k$ (resp. $j = 0, \dots, l$). Assume $d_k = e_l$ which we denote by n. Put

$$R(\vec{\boldsymbol{\alpha}}) = \bigsqcup_{i=1}^{k} P^{(r)}(\boldsymbol{\alpha}^{(i)}, d_{i-1}), \quad L(\vec{\boldsymbol{\beta}}) = \bigsqcup_{j=1}^{l} P^{(l)}(\boldsymbol{\beta}^{(j)}, e_{j-1}),$$

then $R(\vec{\boldsymbol{\alpha}})$ and $L(\vec{\boldsymbol{\beta}})$ both have n elements in the main diagonal. Let f be a nonnegative integer, and let $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$ be a sequence of nonnegative integers of length n. Hereafter we are mainly concerned with the poset defined by

$$P_f(\vec{\boldsymbol{\alpha}}; \vec{\boldsymbol{\beta}}; \boldsymbol{c}) = L_f \sqcup R(\vec{\boldsymbol{\alpha}}) \sqcup L(\vec{\boldsymbol{\beta}}) \sqcup \bigsqcup_{k=1}^{n} P^{(c)}(c_k, k),$$

which we call the *pre-leaf poset* of type $(f, \vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}, \boldsymbol{c})$. We also write

$$P_f(\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(k)}; \boldsymbol{\beta}^{(1)}, \dots, \boldsymbol{\beta}^{(l)}; c_1, \dots, c_n)$$

for $P_f(\vec{\boldsymbol{\alpha}}; \vec{\boldsymbol{\beta}}; \boldsymbol{c})$.

For example, the poset in Figure 2 is denoted by $P_2\left((3,2,0);(3,2),(3,1);0,0,0\right)$.

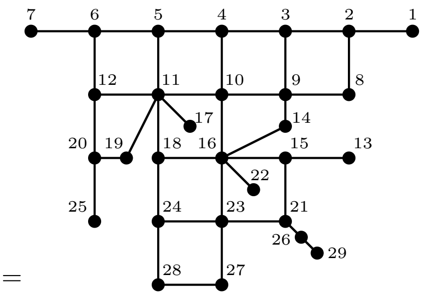
3.2. Generating functions. In this section we define a labeling, which we call the column-strict labeling, denoted by ω_c, for any pre-leaf poset $P_f(\vec{\boldsymbol{\alpha}}; \vec{\boldsymbol{\beta}}; \boldsymbol{c})$. The goal of this section is to obtain the generating function of (P, ω_c)-partitions for a pre-leaf poset $P_f(\vec{\boldsymbol{\alpha}}; \vec{\boldsymbol{\beta}}; \boldsymbol{c})$ (see Proposition 3.3).

We use the standard notation [**4**]: The *q-shifted factorial* is, by definition,

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1 - aq^{n-1})(a;q)_{n-1} & \text{if } n = 1, 2, \dots. \end{cases}$$

If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ is a strictly decreasing sequence of nonnegative integers, we write $(q;q)_{\boldsymbol{\alpha}}$ for the product $\prod_{k=1}^{r}(q;q)_{\alpha_k}$. Throughout this section we assume that $\boldsymbol{\alpha}^{(i)}$ (resp. $\boldsymbol{\beta}^{(j)}$) is a strictly decreasing sequence of nonnegative integers of length r_i (resp. s_j) for $i = 1, \dots, k$ (resp. $j = 1, \dots, l$). We also put $d_i = \sum_{\nu=1}^{i} r_\nu - i + 1$ (resp. $e_j = \sum_{\nu=1}^{j} s_\nu - j + 1$) for $i = 0, \dots, k$ (resp. $j = 0, \dots, l$) and assume $d_k = e_l = n$ as in Definition 3.2. Let $f, c_1, c_2, \dots, c_n$ be any nonnegative integers, and, hereafter, we use the notation $\vec{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(k)})$, $\vec{\boldsymbol{\beta}} = (\boldsymbol{\beta}^{(1)}, \dots, \boldsymbol{\beta}^{(l)})$ and $\boldsymbol{c} = (c_1, \dots, c_n)$. We also use the notation $(q;q)_{\vec{\boldsymbol{\alpha}}} = \prod_{i=1}^{k}(q;q)_{\boldsymbol{\alpha}^{(i)}}$ and $(q;q)_{\vec{\boldsymbol{\beta}}} = \prod_{j=1}^{l}(q;q)_{\boldsymbol{\beta}^{(j)}}$.

Let $P := P_f(\vec{\boldsymbol{\alpha}}; \vec{\boldsymbol{\beta}}; \boldsymbol{c})$ denote the pre-leaf poset of type $(f, \vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}, \boldsymbol{c})$. We define the *column-strict orientation* $\epsilon_c : P \to \{0, 1\}$ by assigning 1 to the vertical edges and 0 to the horizontal edges. One can easily see that a column-strict orientation always come from a labeling in which we linearly order the vertices of P by saying that (i_1, j_1) proceeds (i_2, j_2) if either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 > j_2$. If two different vertices from different half-leaf posets have the same coordinate (i, j), then we can order them appropriately. We write this labeling ω_c and call it a *column-strict labeling*. For example, if $P = P_1((5,3,1),(2,0);(3,1),(3,2,0);0,1,1,2)$, then Figure 3 gives a column-strict labeling. We obtain the following proposition which gives a general formula

$\omega_c =$ 

.

FIGURE 3. A column-strict labeling

to compute $F(P, \omega_c; q)$.

PROPOSITION 3.3. Let $P := P_f(\vec{\boldsymbol{\alpha}}; \vec{\boldsymbol{\beta}}; \boldsymbol{c})$ denote the pre-leaf poset of type $(f, \vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}, \boldsymbol{c})$, where $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$, $\boldsymbol{c}$ and f are as before. Let $\boldsymbol{\alpha}^{(i)} = \left(\alpha_t^{(i)}\right)_{t=1}^{r_i}$ $\left(\text{resp. } \boldsymbol{\beta}^{(j)} = \left(\beta_t^{(j)}\right)_{t=1}^{s_j}\right)$ for $i = 1, \dots, k$ (resp. $j = 1, \dots, l$). Then we have

$$F(P, \omega_c; q) = \frac{q^{\sum_{j=1}^{l}\sum_{t=1}^{s_j}\binom{\beta_t^{(j)}+1}{2} + \sum_{t=1}^{n}\binom{c_t+1}{2}}(q;q)_{|P|-f-1}}{(q;q)_{|P|}(q;q)_{\vec{\boldsymbol{\alpha}}}(q;q)_{\vec{\boldsymbol{\beta}}}(q;q)_{\boldsymbol{c}}} \times \sum_{0 \le p(2) \le p(3) \le \cdots \le p(n)} q^{\sum_{i=2}^{n}(c_i+1)p(i)}$$

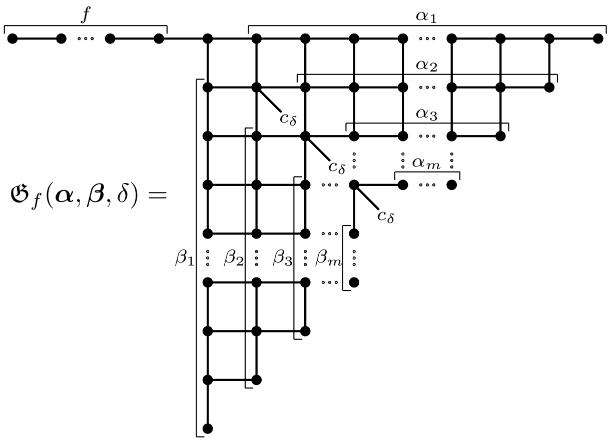
$$\times \prod_{u=1}^{k} \det\left(q^{\alpha_i^{(u)} p(d_{u-1}+j-1)}\right)_{1 \le i,j \le r_u} \prod_{v=1}^{l} \det\left(q^{\beta_i^{(v)} p(e_{v-1}+j-1)}\right)_{1 \le i,j \le s_v}.$$

Here we use the convention that $p(1) = 0$.

4. Leaf posets

4.1. Basic leaf posets. In the former subsection, we define a pre-leaf poset. But, unfortunately, a pre-leaf poset may not have hook-length property in general. In this section we define special cases of pre-leaf posets whose generating functions have product formulas. We call these classes basic leaf posets, and denote them by $\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta)$, $\mathfrak{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$, $\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$, $\mathfrak{W}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$, $\mathfrak{F}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,s,t,v)$, $\mathfrak{C}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)$. Throughout this section, for a non-negative integer n, $(n)_q$ denotes $1-q^n$.

DEFINITION 4.1. (i) Let $m\geq 2$ be an integer, and let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$ and $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_m)$ be strictly decreasing sequences of nonnegative integers of length m. Let δ and f be nonnegative integers which satisfy $f\geq\delta\geq 0$. Then, we let $\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta):=P_f\left(\boldsymbol{\alpha};\boldsymbol{\beta};0,\delta,\delta,\dots,\delta\right)$, and we call it a *ginkgo*. The diagram of $\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta)$ looks as follows.

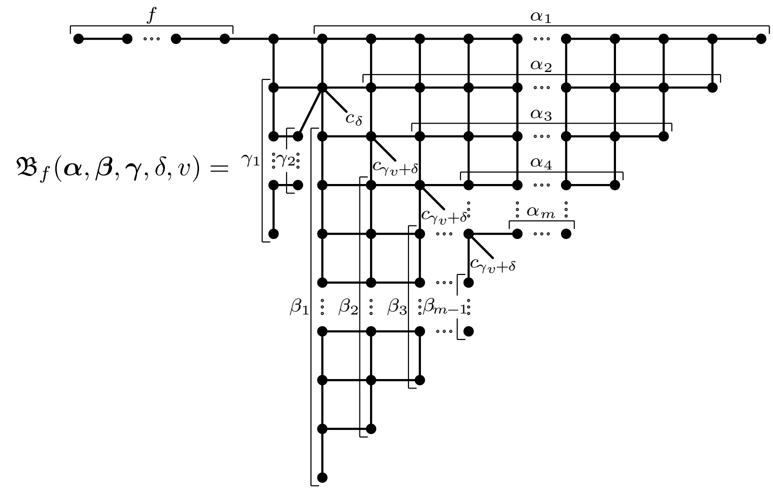


In this diagram c_δ denotes the chain of length δ. Then the generating function of $(\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta),\omega_c)$-partitions is equal to

$$
\begin{aligned}
F(\mathfrak{G}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\delta),\omega_c;q)&=\frac{q^{\sum_{i=1}^m\binom{\beta_i+1}{2}+(m-1)\binom{\delta+1}{2}+\binom{m}{2}(\delta+1)}(q;q)_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+(m-1)(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}(q;q)_{\boldsymbol{\beta}}(q;q)_\delta^{m-1}(q;q)_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+(m-1)\delta+f+m}}\\
&\quad\times\frac{(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+m(\delta+1))_q\prod_{1\leq i<j\leq m}(q^{\alpha_j}-q^{\alpha_i})(q^{\beta_j}-q^{\beta_i})}{\prod_{i,j=1}^m(\alpha_i+\beta_j+\delta+1)_q},
\end{aligned}\tag{4.1}
$$

where $|\boldsymbol{\alpha}|=\sum_{i=1}^m\alpha_i$ and $|\boldsymbol{\beta}|=\sum_{i=1}^m\beta_i$.

(ii) Let $m\geq 3$ be an integer, and let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_{m-1})$, $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be strictly decreasing sequences of nonnegative integers. Let δ and f be nonnegative integers which satisfy $f\geq\beta_1+\delta\geq 0$. For $v=1,2$, we let $\mathfrak{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=P_f(\boldsymbol{\alpha};\boldsymbol{\gamma},\boldsymbol{\beta};0,\delta,\gamma_v+\delta,\dots,\gamma_v+\delta)$, and we call it a *bamboo*. Its diagram is as follows.

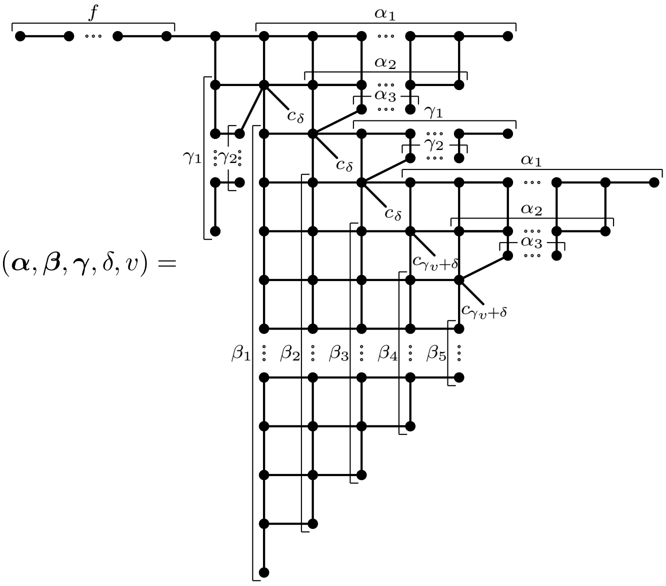


Then the generating function of $(\mathfrak{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v),\omega_c)$-partitions is equal to

$$\begin{aligned}
&F(\mathfrak{B}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v),\omega_c;q)\\
&\quad=\frac{q^{\sum_{i=1}^{m-1}\binom{\beta_i+1}{2}+\sum_{i=1}^{2}\binom{\gamma_i+1}{2}+\binom{\gamma_v+\delta+1}{2}^{m-2}+\binom{\delta+1}{2}+|\boldsymbol{\beta}|+\frac{(m+1)(m-2)}{2}\gamma_v+\binom{m}{2}(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}(q;q)_{\boldsymbol{\beta}}(q;q)_{\boldsymbol{\gamma}}(q;q)_{\gamma_v+\delta}^{m-2}(q;q)_{\delta}}\\
&\qquad\times\frac{(q;q)_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|+(m-2)\gamma_v+(m-1)(\delta+1)}(q^{\gamma_2}-q^{\gamma_1})\prod_{1\le i<j\le m}(q^{\alpha_j}-q^{\alpha_i})\prod_{1\le i<j\le m-1}(q^{\beta_j}-q^{\beta_i})}{(q;q)_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|+(m-2)\gamma_v+f+(m-1)\delta+m}\prod_{i=1}^{m}\prod_{j=1}^{m-1}(\alpha_i+\beta_j+\gamma_v+\delta+1)_q}\\
&\qquad\times\frac{\prod_{i=1}^{m-1}(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+\beta_i+|\boldsymbol{\gamma}|+(m-2)\gamma_v+m(\delta+1))_q}{\prod_{i=1}^{m}(|\boldsymbol{\alpha}|-\alpha_i+|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|+(m-3)\gamma_v+(m-1)(\delta+1))_q},
\end{aligned}\tag{4.2}$$

where $|\boldsymbol{\alpha}|=\sum_{i=1}^{m}\alpha_i$, $|\boldsymbol{\beta}|=\sum_{i=1}^{m-1}\beta_i$, and $|\boldsymbol{\gamma}|=\sum_{i=1}^{2}\gamma_i$.

(iii) Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5)$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be strictly decreasing sequences of nonnegative integers. Let δ and f be nonnegative integers which satisfy $f\ge\beta_1+\delta\ge0$. For $v=1,2$, we let $\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=P_f(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\alpha};\boldsymbol{\gamma},\boldsymbol{\beta};0,\delta,\delta,\delta,\gamma_v+\delta,\gamma_v+\delta)$ and call it an *ivy*. Its diagram looks as follows.

$\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v)=$ 

Then the generating function of $(\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v),\omega_c)$-partitions is equal to

$$\begin{aligned}
&F(\mathfrak{I}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v),\omega_c;q)\\
&\quad=\frac{q^{\sum_{i=1}^{5}\binom{\beta_i+1}{2}+\sum_{i=1}^{2}\binom{\gamma_i+1}{2}+2\binom{\gamma_v+\delta+1}{2}+3\binom{\delta+1}{2}+3|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+2|\boldsymbol{\gamma}|+9\gamma_v+15(\delta+1)}(q;q)_{2|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+2|\boldsymbol{\gamma}|+2\gamma_v+5(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}^2(q;q)_{\boldsymbol{\beta}}(q;q)_{\boldsymbol{\gamma}}^2(q;q)_{\gamma_v+\delta}^2(q;q)_{\delta}^3(q;q)_{2|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+2|\boldsymbol{\gamma}|+2\gamma_v+f+5\delta+6}}\\
&\qquad\times\frac{(q^{\gamma_2}-q^{\gamma_1})^2\prod_{1\le i<j\le 3}(q^{\alpha_j}-q^{\alpha_i})^2\prod_{1\le i<j\le 5}(q^{\beta_j}-q^{\beta_i})}{\prod_{i=1}^{3}\prod_{j=1}^{5}(\alpha_i+\beta_j+\gamma_v+\delta+1)_q\prod_{i=1}^{3}(2|\boldsymbol{\alpha}|-\alpha_i+|\boldsymbol{\beta}|+2|\boldsymbol{\gamma}|+\gamma_v+5(\delta+1))_q}\\
&\qquad\times\frac{\prod_{i=1}^{5}(2|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+\beta_i+2|\boldsymbol{\gamma}|+2\gamma_v+6(\delta+1))_q}{\prod_{1\le i<j\le 5}(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-\beta_i-\beta_j+|\boldsymbol{\gamma}|+\gamma_v+3(\delta+1))_q}.
\end{aligned}\tag{4.3}$$

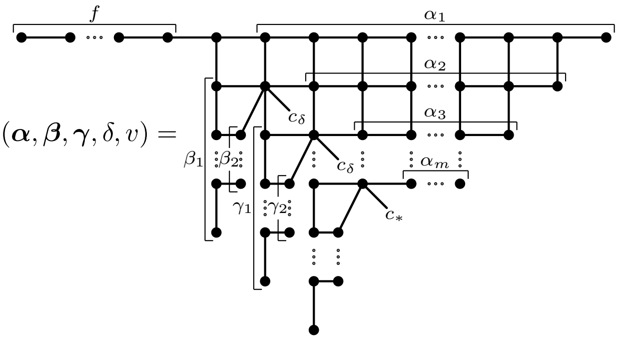
(iv) Let $m\ge4$ be a positive integer, and let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2)$ and $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)$ be strictly decreasing sequences of nonnegative integers. Let δ and f be nonnegative integers which satisfy $f\ge\gamma_1+\delta\ge0$. Assume $v=1$ or 2. If m is even, then we let

$$\mathfrak{W}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=P_f(\boldsymbol{\alpha};\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\gamma},\dots,\boldsymbol{\gamma},\boldsymbol{\beta};0,\delta,\dots,\delta,\gamma_v+\delta),$$

where $m=2r$ and we have r $\boldsymbol{\beta}$'s and $(r-1)$ $\boldsymbol{\gamma}$'s alternatively in the second index. If m is odd, then we let

$$\mathfrak{W}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v):=P_f(\boldsymbol{\alpha};\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\gamma},\dots,\boldsymbol{\beta},\boldsymbol{\gamma};0,\delta,\dots,\delta,\beta_v+\delta),$$

where $m = 2r + 1$ and we have r $\boldsymbol{\beta}$'s and r $\boldsymbol{\gamma}$'s alternatively in the second index. We call $\mathfrak{W}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v)$ a *wisteria*, and its diagram is as follows.

$\mathfrak{W}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v) =$ 

If $m = 2r$ ($r \geq 2$), then the generating function of $(\mathfrak{W}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v), \omega_c)$-partitions is equal to

$$\begin{aligned}
&F(\mathfrak{W}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v), \omega_c; q) \\
&= \frac{q^{r\sum_{i=1}^{2}\binom{\beta_i+1}{2}+(r-1)\sum_{i=1}^{2}\binom{\gamma_i+1}{2}+\binom{\gamma_v+\delta+1}{2}+(2r-2)\binom{\delta+1}{2}+r(r-1)|\boldsymbol{\beta}|+(r-1)^2|\boldsymbol{\gamma}|+(2r-1)\gamma_v+r(2r-1)(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}(q;q)_{\boldsymbol{\beta}}^{r}(q;q)_{\boldsymbol{\gamma}}^{r-1}(q;q)_{\gamma_v+\delta}(q;q)_{\delta}^{2r-2}} \\
&\quad\times \frac{(q;q)_{|\boldsymbol{\alpha}|+r|\boldsymbol{\beta}|+(r-1)|\boldsymbol{\gamma}|+\gamma_v+(2r-1)(\delta+1)}(q^{\beta_2}-q^{\beta_1})^r(q^{\gamma_2}-q^{\gamma_1})^{r-1}\prod_{1\leq i<j\leq 2r}(q^{\alpha_j}-q^{\alpha_i})}{(q;q)_{|\boldsymbol{\alpha}|+r|\boldsymbol{\beta}|+(r-1)|\boldsymbol{\gamma}|+\gamma_v+f+(2r-1)\delta+2r}\prod_{i=1}^{2r}\prod_{j=1}^{2}(\alpha_i+\beta_j+\gamma_v+\delta+1)_q} \\
&\quad\times \frac{\prod_{i=1}^{2}(|\boldsymbol{\alpha}|+r|\boldsymbol{\beta}|+(r-1)|\boldsymbol{\gamma}|+\gamma_i+\gamma_v+2r(\delta+1))_q}{\prod_{1\leq i<j\leq 2r}(\alpha_i+\alpha_j+|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|+2\delta+2)_q},
\end{aligned} \tag{4.4}$$

and, if $m = 2r + 1$ ($r \geq 2$), then we have

$$\begin{aligned}
&F(\mathfrak{W}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, v), \omega_c; q) \\
&= \frac{q^{r\sum_{i=1}^{2}\binom{\beta_i+1}{2}+r\sum_{i=1}^{2}\binom{\gamma_i+1}{2}+\binom{\beta_v+\delta+1}{2}+(2r-1)\binom{\delta+1}{2}+r(r-1)|\boldsymbol{\beta}|+r^2|\boldsymbol{\gamma}|+2r\beta_v+r(2r+1)(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}(q;q)_{\boldsymbol{\beta}}^{r}(q;q)_{\boldsymbol{\gamma}}^{r}(q;q)_{\beta_v+\delta}(q;q)_{\delta}^{2r-1}} \\
&\quad\times \frac{(q;q)_{|\boldsymbol{\alpha}|+r|\boldsymbol{\beta}|+\beta_v+r|\boldsymbol{\gamma}|+2r(\delta+1)}(q^{\beta_2}-q^{\beta_1})^r(q^{\gamma_2}-q^{\gamma_1})^{r}\prod_{1\leq i<j\leq 2r+1}(q^{\alpha_j}-q^{\alpha_i})}{(q;q)_{|\boldsymbol{\alpha}|+r|\boldsymbol{\beta}|+\beta_v+r|\boldsymbol{\gamma}|+f+2r\delta+2r+1}\prod_{i=1}^{2r+1}\prod_{j=1}^{2}(\alpha_i+\beta_v+\gamma_j+\delta+1)_q} \\
&\quad\times \frac{\prod_{i=1}^{2}(|\boldsymbol{\alpha}|+r|\boldsymbol{\beta}|+r|\boldsymbol{\gamma}|+\beta_v+\gamma_i+(2r+1)(\delta+1))_q}{\prod_{1\leq i<j\leq 2r+1}(\alpha_i+\alpha_j+|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|+2\delta+2)_q}.
\end{aligned} \tag{4.5}$$

(v) Let $m \geq 4$ be a positive integer, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_{m-1})$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be strictly decreasing sequences of nonnegative integers. Let δ and f be nonnegative integers which satisfy $f \geq \beta_1 + \delta \geq 0$. Fix positive integers s, t which satisfy $1 \leq s < t \leq 3$, and let $v \in \{s, t\}$ if m is even, and let $v \in \{1, 2\}$ if m is odd. Write $\tilde{a} := (\alpha_s, \alpha_t)$. If m is even, then we let

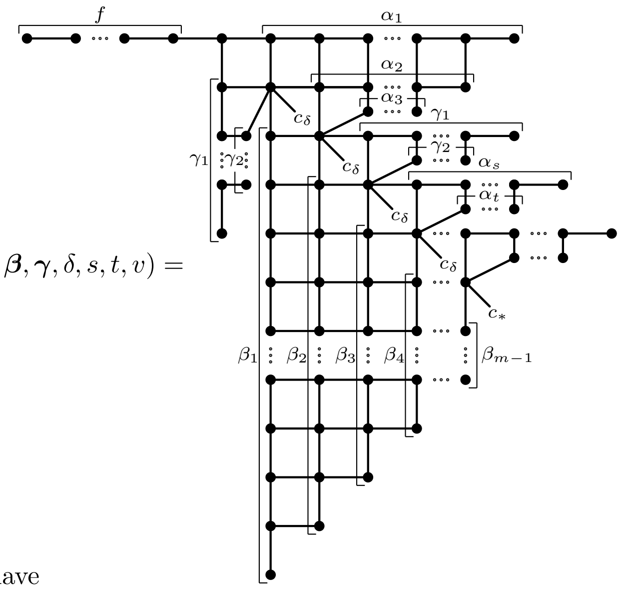
$$P = \mathfrak{F}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, s, t, v) = P_f(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \tilde{a}, \boldsymbol{\gamma}, \tilde{a}, \dots, \tilde{a}, \boldsymbol{\gamma}; \boldsymbol{\gamma}, \boldsymbol{\beta}; 0, \delta, \dots, \delta, \alpha_v + \delta)$$

where $m = 2r$ ($r \geq 2$) and we have one $\boldsymbol{\alpha}$ followed by $(r-1)$ $\boldsymbol{\gamma}$'s and $(r-2)$ $\tilde{a}$'s alternatively in the first index. If m is odd, then we let

$$P = \mathfrak{F}_f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta, s, t, v) = P_f(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \tilde{a}, \boldsymbol{\gamma}, \tilde{a}, \dots, \boldsymbol{\gamma}, \tilde{a}; \boldsymbol{\gamma}, \boldsymbol{\beta}; 0, \delta, \dots, \delta, \gamma_v + \delta)$$

where $m = 2r + 1$ ($r \geq 2$) and we have one $\boldsymbol{\alpha}$ followed by $(r-1)$ $\boldsymbol{\gamma}$'s and $(r-1)$ $\tilde{a}$'s alternatively in the first index. We call P a *fir*, and its diagram is as follows.

$$\mathfrak{F}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,s,t,v) =$$



If $m = 2r$ $(r \geq 2)$, then we have

$$\begin{aligned}
&F(\mathfrak{F}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,s,t,v),\omega_c;q)\\
&\quad= \frac{q^{\binom{\alpha_v+\delta+1}{2}+\sum_{i=1}^{2r-1}\binom{\beta_i+1}{2}+\sum_{i=1}^{2}\binom{\gamma_i+1}{2}+(2r-2)\binom{\delta+1}{2}+r(r-2)(\alpha_s+\alpha_t)+(2r-1)\alpha_v+|\boldsymbol{\beta}|+r(r-1)|\boldsymbol{\gamma}|+r(2r-1)(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}(q;q)_{\boldsymbol{\beta}}(q;q)_{\boldsymbol{\gamma}}^{r}(q;q)_{\alpha_s}^{r-2}(q;q)_{\alpha_t}^{r-2}(q;q)_{\alpha_v+\delta}(q;q)_{\delta}^{2r-2}}\\
&\qquad\times \frac{(q;q)_{|\boldsymbol{\alpha}|+(r-2)(\alpha_s+\alpha_t)+\alpha_v+|\boldsymbol{\beta}|+r|\boldsymbol{\gamma}|+(2r-1)(\delta+1)}}{(q;q)_{|\boldsymbol{\alpha}|+(r-2)(\alpha_s+\alpha_t)+\alpha_v+|\boldsymbol{\beta}|+r|\boldsymbol{\gamma}|+f+(2r-1)\delta+2r}}\\
&\qquad\times \frac{(q^{\gamma_2}-q^{\gamma_1})^r(q^{\alpha_t}-q^{\alpha_s})^{r-2}}{\prod_{i=1}^{2}\prod_{j=1}^{2r-1}(\alpha_v+\beta_j+\gamma_i+\delta+1)_q\prod_{1\leq i<j\leq 2r-1}(\alpha_s+\alpha_t+\beta_i+\beta_j+|\boldsymbol{\gamma}|+2\delta+2)_q}\\
&\qquad\times \frac{\prod_{1\leq i<j\leq 3}(q^{\alpha_j}-q^{\alpha_i})\prod_{1\leq i<j\leq 2r-1}(q^{\beta_j}-q^{\beta_i})}{\prod_{i=1}^{2}(|\boldsymbol{\alpha}|+(r-2)(\alpha_s+\alpha_t)+|\boldsymbol{\beta}|+(r-1)|\boldsymbol{\gamma}|+\gamma_i+(2r-1)(\delta+1))_q}\\
&\qquad\times \frac{\prod_{i=1}^{2r-1}(|\boldsymbol{\alpha}|+(r-2)(\alpha_s+\alpha_t)+\alpha_v+|\boldsymbol{\beta}|+\beta_i+r|\boldsymbol{\gamma}|+2r(\delta+1))_q}{\prod_{i=1}^{2r-1}(|\boldsymbol{\alpha}|+(r-3)(\alpha_s+\alpha_t)+\alpha_v+|\boldsymbol{\beta}|-\beta_i+(r-1)|\boldsymbol{\gamma}|+2(r-1)(\delta+1))_q},
\end{aligned}\tag{4.6}$$

and, if $m = 2r+1$ $(r \geq 2)$, then we have

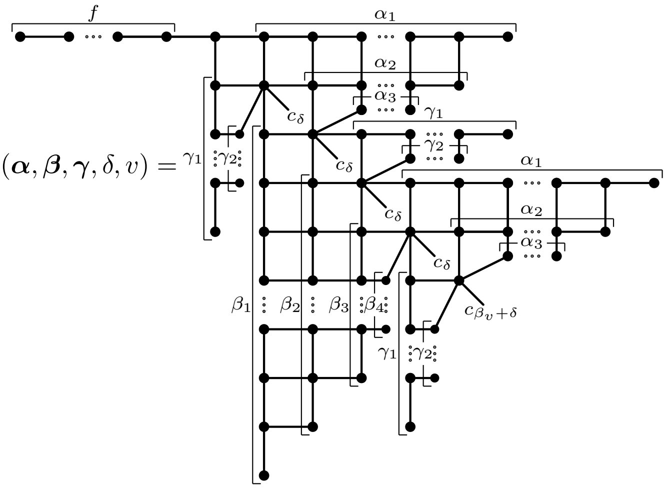
$$\begin{aligned}
&F(\mathfrak{F}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,s,t,v),\omega_c;q)\\
&\quad= \frac{q^{\sum_{i=1}^{2r}\binom{\beta_i+1}{2}+\sum_{i=1}^{2}\binom{\gamma_i+1}{2}+\binom{\gamma_v+\delta+1}{2}+(2r-1)\binom{\delta+1}{2}+(r^2-1)(\alpha_s+\alpha_t)+|\boldsymbol{\beta}|+r(r-1)|\boldsymbol{\gamma}|+2r\gamma_v+r(2r+1)(\delta+1)}}{(q;q)_{\boldsymbol{\alpha}}(q;q)_{\boldsymbol{\beta}}(q;q)_{\boldsymbol{\gamma}}^{r}(q;q)_{\alpha_s}^{r-1}(q;q)_{\alpha_t}^{r-1}(q;q)_{\gamma_v+\delta}(q;q)_{\delta}^{2r-1}}\\
&\qquad\times \frac{(q;q)_{|\boldsymbol{\alpha}|+(r-1)(\alpha_s+\alpha_t)+|\boldsymbol{\beta}|+r|\boldsymbol{\gamma}|+\gamma_v+2r(\delta+1)}}{(q;q)_{|\boldsymbol{\alpha}|+(r-1)(\alpha_s+\alpha_t)+|\boldsymbol{\beta}|+r|\boldsymbol{\gamma}|+\gamma_v+f+2r\delta+2r+1}}\\
&\qquad\times \frac{(q^{\gamma_2}-q^{\gamma_1})^r(q^{\alpha_t}-q^{\alpha_s})^{r-1}}{\prod_{i=s,t}\prod_{j=1}^{2r}(\alpha_i+\beta_j+\gamma_v+\delta+1)_q\prod_{1\leq i<j\leq 2r}(\alpha_s+\alpha_t+\beta_i+\beta_j+|\boldsymbol{\gamma}|+2\delta+2)_q}\\
&\qquad\times \frac{\prod_{1\leq i<j\leq 3}(q^{\alpha_j}-q^{\alpha_i})\prod_{1\leq i<j\leq 2r}(q^{\beta_j}-q^{\beta_i})}{\prod_{i=s,t}(|\boldsymbol{\alpha}|+(r-2)(\alpha_s+\alpha_t)+\alpha_i+|\boldsymbol{\beta}|+r|\boldsymbol{\gamma}|+2r(\delta+1))_q}\\
&\qquad\times \frac{\prod_{i=1}^{2r}(|\boldsymbol{\alpha}|+(r-1)(\alpha_s+\alpha_t)+|\boldsymbol{\beta}|+\beta_i+r|\boldsymbol{\gamma}|+\gamma_v+(2r+1)(\delta+1))_q}{\prod_{i=1}^{2r}(|\boldsymbol{\alpha}|+(r-2)(\alpha_s+\alpha_t)+|\boldsymbol{\beta}|-\beta_i+(r-1)|\boldsymbol{\gamma}|+\gamma_v+(2r-1)(\delta+1))_q}.
\end{aligned}\tag{4.7}$$

(vi) Let $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3)$, $\boldsymbol{\beta} = (\beta_1,\beta_2,\beta_3,\beta_4)$ and $\boldsymbol{\gamma} = (\gamma_1,\gamma_2)$ be strictly decreasing sequences of nonnegative integers. Let δ and f be nonnegative integers which satisfy $f \geq \beta_1+\delta \geq 0$. For $v = 1,2,3,4$, we let

$$\mathfrak{C}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v) := P_f(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\alpha};\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\gamma};0,\delta,\delta,\delta,\delta,\beta_v+\delta)$$

which we call a *chrysanthemum*. Its diagram is as follows.

$\mathfrak{C}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v) =$



Then we have

$$
\begin{aligned}
&F(\mathfrak{C}_f(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\delta,v),\omega_c;q)\\
&= \frac{q^{\sum_{i=1}^4\binom{\beta_i+1}{2}+\binom{\beta_v+\delta+1}{2}+2\sum_{i=1}^2\binom{\gamma_i+1}{2}+4\binom{\delta+1}{2}+3|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+5\beta_v+6|\boldsymbol{\gamma}|+15\delta+15}(q;q)_{2|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+\beta_v+3|\boldsymbol{\gamma}|+5\delta+5}}{(q;q)_{\boldsymbol{\alpha}}^2(q;q)_{\boldsymbol{\beta}}(q;q)_{\boldsymbol{\gamma}}^3(q;q)_{\beta_v+\delta}(q;q)_{\delta}^4(q;q)_{2|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+\beta_v+3|\boldsymbol{\gamma}|+f+5\delta+6}}\\
&\quad\times\frac{(q^{\gamma_2}-q^{\gamma_1})^3\prod_{1\le i<j\le 3}(q^{\alpha_j}-q^{\alpha_i})^2\prod_{1\le i<j\le 4}(q^{\beta_j}-q^{\beta_i})}{\prod_{i=1}^2\prod_{j=1}^3(\alpha_j+\beta_v+\gamma_i+\delta+1)_q\prod_{\substack{1\le k\le 4\\ k\ne v}}\prod_{j=1}^3(|\boldsymbol{\alpha}|-\alpha_j+\beta_v+\beta_k+|\boldsymbol{\gamma}|+2\delta+2)_q}\\
&\quad\times\frac{\prod_{1\le k\le 4}(2|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+\beta_k+\beta_v+3|\boldsymbol{\gamma}|+6\delta+6)_q}{\prod_{j=1}^3(|\boldsymbol{\alpha}|+\alpha_j+|\boldsymbol{\beta}|+2|\boldsymbol{\gamma}|+4\delta+4)_q\prod_{\substack{1\le k\le 4\\ k\ne v}}\prod_{i=1}^2(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-\beta_k+|\boldsymbol{\gamma}|+\gamma_i+3\delta+3)_q}. \qquad (4.8)
\end{aligned}
$$

4.2. General leaf posets. In this section we explain how to compose a general leaf poset from the basic ones. An operation called "slant sum" which combines two posets to generate a new one is introduced in [**18**]. Here we slightly modify the definition, and call it a "joint sum".

Definition 4.2. Let P_1 be a finite poset and let y be any element of P_1. Let P_2 be a finite poset which is non-adjacent to P_1, i.e. P_2 shares no element with P_1 and there is no order relation between elements of P_1 and P_2. Let $x_1, x_2, \cdots, x_m$ be all maximal elements of P_2. Set Q to be $P_1 \cup P_2$ as set, and make it a partially ordered set by inserting additional covering relations $x_1 \lessdot y, \cdots, x_m \lessdot y$ besides the order relations among the elements of P_1 or P_2. We use ${P_1}^y\backslash P_2$ (or more explicitly ${P_1}^y\backslash_{x_1,\cdots,x_m}P_2$) to denote this new poset Q, and call it the *joint sum* of P_1 with P_2 at y.

An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y \le x$, then $y \in I$. The order ideal $\langle x\rangle = \{y \in P : y \le x\}$ is the *principal order ideal* generated by x.

Definition 4.3. Let P be a poset, and let (x, y) be a pair of elements in P such that y covers x. We say (x, y) is an *joint pair* if $\langle x\rangle$ is a chain with the maximum element x and P is equal to $(P - \langle x\rangle)^y\backslash\langle x\rangle$, i.e. removing $\langle x\rangle$ from P and then making the joint sum of $P - \langle x\rangle$ with $\langle x\rangle$ at y recovers P. An element $y \in P$ is said to be a *joint element* if there exists $x \in P$ which is covered by y and (x, y) is a joint pair.

Now we are in position to define the notion of general leaf posets as follows.

Definition 4.4. First we inductively define k-level leaf posets for a positive integer k. A poset P is said to be a 1*-level leaf poset* if it is a basic leaf poset, a tree, or obtained as a disjoint union of several basic leaf posets and trees. Let LP_1 denote the set of 1-level leaf posets. For $k \ge 2$, let Q be a $(k-1)$-level leaf poset and (x, y) is a joint pair in Q. We construct a *k-level leaf poset* P by removing the chain $\langle x\rangle$ from Q and make the joint sum of Q with P_1 at y, where P_1 is a 1-level leaf poset which has the same number of elements as $\langle x\rangle$, i.e. $P = (Q - \langle x\rangle)^y\backslash P_1$ and $|P_1| = |\langle x\rangle|$. Let LP_k denote the set of k-level leaf posets, and put

$$\mathrm{LP} := \bigcup_{k\ge 1}\mathrm{LP}_k.$$

We call an element of LP a *leaf poset*.

Note that a poset can be k-level leaf poset for several k, i.e. there exist some i, j such that $\mathrm{LP}_i \cap \mathrm{LP}_j \neq \emptyset$.

Proposition 4.5. Let P_1 be a finite poset and let y be any element of P_1. Let P_2 be any n-element poset which is non-adjacent to P_1. Let $\langle z\rangle$ denote the n-element chain whose maximum element is z. We put $P = {P_1}^y\backslash_z\langle z\rangle$ and $P' = {P_1}^y\backslash P_2$. Let ω be a labeling on P whose restriction on $\langle z\rangle$ is a natural labeling and $\omega(y) > \omega(z)$. Let ω' be a labeling on P' such that the restriction of ω' on P_1 coincides with the restriction of ω on P_1. Let ω_2 denote the restriction of ω' on P_2. Then the generating function of (P', ω')-partitions is given by $F(P', \omega'; q) = (q; q)_n F(P, \omega; q) F(P_2, \omega_2; q)$. Especially, if P and P_2 have hook length property and $h_P(x) = |\langle x\rangle|$ for any $x \in \langle z\rangle$, then P' has hook length property, where h_P is the hook length function of P.

5. Concluding Remarks

Proposition 4.5 and Definition 4.1 immediately implies that the generating function of P-partitions is given by a product formula, where P is any leaf poset. Thus we conclude that any leaf poset has hook-length property. Our proof is based on Stanley's (P, ω)-partitions and determinant or Pfaffian computations, in which some of the proofs have elegant Pfaffian expressions and some of them are direct computations by brute forces. We think that one more interesting problem, which is still left, is to consider a bijective proof of the hook formulas.

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