ON TWO-PERSON SYMMETRIC MULTI-SUIT WHIST

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ABSTRACT. The game of two-person multi-suit whist is played with a deck of cards, where each card belongs to a *suit*, and has a *rank* within its suit. The two players receive the same number of cards, and both players have complete information about the deal. Play proceeds in tricks, with the obligation to follow suit, as in many real-world card games. The objective is to win as many tricks as possible.

We study the *symmetric* case of this game, in which we assume that in each suit, the two players have the same number of cards. We show how to assign a value from a certain semigroup to each single-suit card distribution in such a way that the outcome of a multi-suit deal under optimal play is determined by the sum of the values of the individual suits. This allows us to play a deal perfectly, provided that we can compute the values of its single-suit components.

It also allows us to establish certain general properties of the game, for instance the nontrivial fact that a higher card is always at least as good as a smaller one in the same suit.

RÉSUMÉ. Le jeu de whist pour deux personnes est joué avec un jeu de cartes dans lequel chaque carte appartient à une couleur, et dans chaque couleur les cartes sont ordonnées par valeur. Les deux joueurs reçoivent le même nombre de cartes et la donne est ouverte. Le jeu se joue par levées, avec l'obligation de suivre la couleur jouée. Le but du jeu est de remporter autant de levées que possible.

Nous étudions le cas symétrique de ce jeu, c'est à dire que nous supposons que dans chaque couleur, les deux joueurs ont le même nombre de cartes. Nous montrons comment donner une valeur d'un certain monoïde à chaque distribution d'une couleur, de telle façon que le résultat d'une donne de plusieurs couleurs après un jeu optimal soit déterminé par la somme des valeurs des couleurs individuelles. Cela nous permet de jouer une donne parfaitement, à condition que l'on puisse calculer les valeurs de chaque couleur individuelle.

Cela nous permet aussi d'établir certaines propriétés générales de ce jeu, notamment le fait qu'une carte de valeur supérieure est toujours au moins aussi bonne qu'une carte inférieure de la même couleur.

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1. The game of two-person whist

1.1. **Rules of the game.** The game of two-person whist is played with a deck of cards. Each card belongs to a *suit*, and within each suit, the cards are ordered by *rank*. Real-world card packs sometimes have 4 suits with 13 cards in each, but we allow for any number of suits, and any number of cards in each suit.

The cards are distributed between the two players, so that both players receive the same number of cards. We assume that both players have complete information about the situation. One of the players is said to have the *lead*. The player with the lead plays, or *leads*, one of his cards. The other player, in response to this, plays one of his cards. If possible, the second player has to *follow suit*, that is, he has to play a card of the same suit as the one that was led. The player who played the highest card in the suit that was led wins the *trick*, and obtains the lead. The cards that were played are removed, and play continues until all cards have been played. Each player tries to win as many tricks as possible.

This game is a pure form of a common type of card game, *trick taking games*. Trick taking games exist in many different forms, and their history goes back to the early fifteenth century [6]. Here we assume that the game is played between two people, and further that it is played with perfect information. Under these assumptions, an optimal strategy exists and can in principle be computed. The outcome of the game under optimal play is determined by the distribution of the cards. The assumption of perfect information is often not realistic in actual play, but a general understanding of the game should probably start from knowledge of the playing technique in its perfect information counterpart.

In this paper, our approach is based on evaluating each suit separately, and then adding the values of the individual suits to obtain a value for the complete card distribution. We focus on a special case where this idea works well.

1.2. **The symmetric case.** Throughout the paper, we will assume that *in each suit* the two players have the same number of cards. Such a card distribution is called *symmetric*. If this condition is satisfied initially, then the player not on lead will always be able to follow suit, so the symmetry will persist throughout the game. In a symmetric deal, the number of tricks where the lead is in a particular suit is determined by the number of cards in that suit, and does not depend on how the cards are played. An advantage of studying symmetric deals is that the effect of a particular suit on the game as a whole can be measured and evaluated by comparing play and outcome with the deal obtained by removing the suit from both hands.

It is notable that symmetric positions arise "naturally" in a certain type of endgame in bridge called *strip squeeze*. This endgame arises when a player, usually a defender, has to discard on a long suit while retaining a guard in certain other suits. He can then sometimes be forced, *stripped*, down to the same card distribution as one of his opponents.

Because of the limited number of cards on each hand, such endgames involving only two players can in practice always be handled by following a few well-known rules of thumb. Therefore the problems considered in this paper generally do not present themselves in actual play, which is why the theory developed here has not been discovered by card players. The results presented here can be regarded as an answer to the question: *"How hard can a position in bridge or whist be, if there are only two players involved, and the card distribution is symmetric and known to both players?"*

1.3. **Conventions.** We assume that the game is played between two players called East and West. Our sympathies are usually with West. This convention is customary in combinatorial game theory, where the players are usually called Left and Right. The author thinks it is more in the spirit of card games to use the labels West and East. When we speak of the *outcome* of a deal, we mean the number of tricks that West will take with optimal play from both sides. When possible, we use the standard ranks from 2 to 10, Jack, Queen, King, and Ace.

1.4. **Aim of the paper.** From the point of view of computational complexity, a game such as whist can be regarded as solved when a polynomial time algorithm is found that computes the game-theoretical value of any given deal, as well as an optimal move in any given situation. A polynomial time (in fact almost linear time) algorithm for computing the outcome of single-suit whist was given in [7].

We do not solve the game of symmetric multi-suit whist in this sense. Instead, our aim is to show how to assign values (from a certain semigroup) to individual suits in such a way that the sum of the values of the suits in a multi-suit game determines the outcome of the game under optimal play. This includes assigning a rational number to each single-suit deal reflecting the average value of this suit in a multi-suit game. The theory developed for symmetric multi-suit whist includes the technique known to bridge players as *elimination and throw-in*.

Whist is not a combinatorial game in the strict sense, since the move-order is not alternating, and the objective is to win as many tricks as possible, rather than to make the last move. Therefore the theory developed in [1] and [2] does not directly apply to whist. However, as readers familiar with combinatorial game theory will necessarily notice, we make use of many of the ideas and methods of this theory. Some of the concepts that we introduce, like mean value, simplicity, numbers, and infinitesimals, have direct counterparts in the theory of combinatorial games as developed in [2].

2. From single- to multi-suit whist

2.1. **Background.** Two-person whist played with a single suit was solved by the author in [7]. We do not make use of this solution, but in principle, the outcome of a single-suit card distribution under optimal play can be regarded as known. A reasonable approach to the symmetric multi-suit game would be to calculate the number of tricks we can take in each suit, and then add these numbers together. This approach, although too naive in general, obviously works well in many cases. Consider the following deal:

(1)

West :	*East* :
♠ A K	♠ Q J
♡ A J	♡ K Q
♢ K 10 9	♢ A Q J

West can count two tricks in spades and one trick in each of hearts and diamonds. To evaluate the trick-taking potential in spades and hearts, we do not even have to take into account how the lead will pass between the players during the game, since they will produce the same number of tricks regardless of how the cards are played. In diamonds, West clearly cannot get more than one trick. On the other hand, as soon as East leads a diamond, whether high or low, West will be certain to win a trick with the king. West can therefore refuse to play diamonds as long as possible. If at the end he is on lead with only the three diamond tricks left to play, he can lead a small diamond, and then score his king in one of the last two tricks. He can therefore consider the diamond king to be worth one trick. On the deal as a whole, West will be able to take $2 + 1 + 1 = 4$ tricks.

In other cases, the outcome of a single-suit game depends on the initial location of the lead. An elementary fact about the single-suit game, proved in [3], is that having the lead is never an advantage, but on the other hand may cost at most one trick. The solution in [7] is based on assigning to each deal a number, which is half of an integer. This number is a measure of the number of tricks that West can take, and if not an integer, it should be rounded to an integer in favor of the player not on lead. Hence this number represents the mean value of the number of tricks that West can take with and without the lead. The simplest case of a non-integral value is the following:

(2)

West :	*East* :
A Q	K J

The value of this card distribution for West can be described by the number 3/2, which means that West will get 1 trick if he has the lead, but 2 tricks if East has the lead.

Interestingly, from a multi-suit perspective, the number 3/2 also happens to represent the mean value of this card distribution in another sense, analogous to the concept of mean value of a combinatorial game. Consider for example:

(3)

West :	*East* :
♠ A Q	♠ K J
♡ A Q	♡ K J
♢ A Q	♢ K J
♣ A Q	♣ K J

Whether or not West will be able to score the queen in a particular suit depends only on who makes the first lead in the suit. If East leads a certain suit, West will immediately be able to cash two tricks in that suit. If West leads the suit, then East will win a trick with the king, immediately or later. With correct play, whenever East is on lead, West will cash two tricks in the suit led. Then West on lead will cash the ace of another suit and continue with the queen. East gets a trick for his king, and the lead is back with East. Hence in this case, West will get 6 of the 8 tricks, regardless of the initial position of the lead, and in general, with any number of suits with this distribution, West will score 3/2 times the number of suits, rounded to an integer in favor of the player not on lead.

2.2. **Assigning numbers to suits.** It is natural to conjecture that the outcome of a deal in which every suit has a non-integral value in this sense can be determined by adding the values and rounding to the nearest integer. The following theorem is proved later in a more general form.

Theorem 2.1. *Suppose that we assign the number $n + 1/2$ to any single-suit deal in which West will take n tricks with the lead and $n+1$ tricks with East on lead. Then in a multi-suit deal where every single-suit component is of this type, the outcome under optimal play is obtained by summing the numbers assigned to each suit, and if the sum is not an integer, rounding in favor of the player not on lead.*

For example, in the deal

(4)

West :	*East* :
♠ A Q	♠ K J
♡ A K J	♡ Q 10 9
♢ A J 9	♢ K Q 10

we count $1 + 1/2$ for the spades, $2 + 1/2$ for the hearts, and $1 + 1/2$ for the diamonds. Note that whenever East leads the diamond king, West should play low. This adds up to $5 + 1/2$. Consequently, West can take five tricks with the lead, and six tricks if East is on lead.

If we add a club suit to make it

(5)

West:	*East*:
♠ A Q	♠ K J
♡ A K J	♡ Q 10 9
♢ A J 9	♢ K Q 10
♣ K 10 9 8	♣ A Q J 7

the sum will be $(1 + 1/2) + (2 + 1/2) + (1 + 1/2) + (1 + 1/2) = 7$, indicating that West will get 7 of the tricks regardless of the initial position of the lead.

It becomes clear from a few examples that the situation can be more complicated if the deal contains suits which played separately would yield the same number of tricks regardless of the position of the lead. We can try to evaluate the deal

(6)

West:	*East*:
♠ A Q	♠ K J
♡ A	♡ K

to $(1 + 1/2) + 1 = 2 + 1/2$, but in fact, West will not get more than two tricks even if East has the lead, since East will simply transfer the lead to West by playing hearts. With

(7)

West:	*East*:
♠ A Q	♠ K J
♡ A	♡ K
♢ K	♡ A

it is an advantage to have the lead. Apparently the number $2 + 1/2$ should be rounded in favor of the player on lead in this case.

As the following example shows, it cannot be consistent to assign the value n to every single-suit deal that, played by itself, produces n tricks for West. Hence there is no analogue of Theorem 2.1 for suits of this type.

(8)

West:	*East*:
♠ A K 10	♠ Q J 9
♡ A K 10	♡ Q J 9

Here each suit would be worth two tricks for West, if played separately, since even if East is on lead, he can secure one trick by leading a high card. However, on the deal as a whole, West can take five of the six tricks, provided East has the initial lead. In order not to give West a cheap trick immediately, East will lead one of his honors, say the spade queen. West wins the trick and plays ace, king, and ten of hearts. This way East gets the lead (unless he surrenders by playing the queen and jack of hearts under West's ace and king), and is forced to lead spades a second time. This gives West a trick for the spade ten.

In view of Theorem 2.1, we can conjecture that it is consistent to assign the value $n+1/2$ to any deal, single or multi-suit, that produces n tricks for West on lead, and $n+1$ tricks for West with East on lead. The following two theorems, as well as Theorem 2.1, are derived as corollaries of Theorem 9.1.

Theorem 2.2. *It is consistent to assign the value $n+1/2$ to any deal where West gets n tricks with the lead and $n+1$ tricks without the lead, in the sense that whenever a deal can be split into components of this type, the outcome of the deal as a whole will be the sum of the values of the components, rounded in favor of the player not on lead.*

Theorem 2.3. *It is consistent to assign the value $n+1/2$ to any deal where West gets n tricks with the lead and $n+1$ tricks without the lead, in the sense that whenever a deal has this property, we can assign values to its single suit components that add up to the value of the whole deal, and so that the value of a suit still depends only on the distribution of the cards in that suit.*

If this is correct, then the deal (8) should have value $5+1/2$, and consequently, the individual suits should have value $2+1/4$. Indeed, under reasonable assumptions on correct play, we can see that the mean value of A K 10 vs. Q J 9 ought to be $2+1/4$. Whenever East is on lead, he will lead a queen or a jack. West takes the trick and continues with ace, king, and ten of a different suit, putting East back on lead. This continues until half of the suits have been played out completely, and the remaining suits are distributed A 10 vs. J 9 (or equivalently). This combination is equivalent to A Q vs. K J discussed earlier. West will now be able to take a trick with half of his remaining tens. This means that he gets an extra trick for every four suits. Careful analysis shows that the number of tricks that West gets with n suits distributed this way is indeed $(2+1/4)n$ rounded to the nearest integer, and if $n \equiv 2 \pmod 4$, rounded in favor of the player not on lead.

3. The numerical value of a card distribution

3.1. **The working hypothesis.** In Section 2, we assigned a numerical value to certain single- and multi-suit deals, namely those that occur as components of deals where having the lead costs a trick. At the same time, it seems even more natural to evaluate a suit distributed for example A vs. K, A K vs. Q J, or A J vs K Q, to the number of tricks that the suit is bound to produce. As we prove later, these card combinations cannot occur in a deal where having the lead is a disadvantage.

The following statement has served as a working hypothesis that has motivated the approach taken in the paper. It combines the two ways of assigning numbers to individual suits. This statement too is a consequence of the main theorem (Theorem 9.1). It defines what we will refer to as the *numerical value* of a card distribution.

Theorem 3.1. *To every symmetric multi-suit deal D, we can assign a number $N(D)$ called the* numerical value *of D, satisfying the following:*

(A) *The numerical value of a multi-suit deal is the sum of the numerical values of its single suit components.*

(B) *Regardless of the location of the lead, the outcome of a deal differs by at most $1/2$ from its numerical value.*

We do not prove at this point that (A) and (B) are consistent. For the moment, we will assume the existence of a function N satisfying Theorem 3.1, and derive some of its properties. In the following, when a statement is labeled *Consequence*, we mean that it will follow from the consistency of (A) and (B). The following two statements follow from (B) since the intervals $[m-1/2, m+1/2]$ and $[n-1/2, n+1/2]$ have nonempty intersection if $|m-n| \geq 2$, and intersect only in the point $n+1/2$ if $|m-n| = 1$.

Consequence 3.2. *The difference in outcome of a deal with East and West on lead respectively is at most one trick.*

Consequence 3.3. *If D is a deal in which West gets n tricks with one of the players on lead, and $n+1$ tricks with the other player on lead, then $N(D) = n + 1/2$.*

3.2. **The mean value of a deal.** Let D be a deal, and let m be a positive integer. We let $m \cdot D$ denote a deal which consists of m copies of D. That is, for each single-suit component of D, the deal $m \cdot D$ has m corresponding suits with the same card distribution. By (A), $N(m \cdot D) = m \cdot N(D)$. If we let a_m be the outcome of $m \cdot D$ with, say, West on lead, then by (B),

$$|a_m - m \cdot N(D)| \leq 1/2.$$

If we divide by m and let $m \to \infty$, we obtain

$$\frac{a_m}{m} \to N(D).$$

From this, it follows that $N(D)$ is uniquely determined by (A) and (B). $N(D)$ is the mean value of the number of tricks that West will get per copy of D when a large number of copies of D are played simultaneously. The numerical value of a deal is therefore analogous to the mean value of a combinatorial game.

Theorem 3.4. *There is at most one function N satisfying (A) and (B).*

Consequence 3.5. *A deal which always gives West n tricks regardless of how the cards are played must have numerical value n.*

This follows since the deal must have mean value n.

Consequence 3.6. *If D is a deal with n cards on each hand, and $\overline{D}$ is the deal obtained by switching the East and West hands, then $N(D) + N(\overline{D}) = n$.*

This follows from the corresponding statement for the mean value.

3.3. **Numerical values of some card distributions.** We now show how to compute the numerical values of some card distributions using (A) and (B). The combination A vs. K always gives West one trick. By Consequence 3.5, the numerical value must be 1. Similarly, $N\,(\mathrm{K}\,,\ \mathrm{A}) = 0$. For two-card distributions, Consequence 3.5 gives

$$N\,(\mathrm{A\ K}\,,\ \mathrm{Q\ J}) = 2$$

and

$$N\,(\mathrm{A\ J}\,,\ \mathrm{K\ Q}) = 1.$$

By (B),

$$N\,(\mathrm{A\ Q}\,,\ \mathrm{K\ J}) = 1 + 1/2.$$

The numerical values of the remaining two-card deals follow from Consequence 3.6.

3.3.1. *Three-card deals.* The values

$$N\,(\mathrm{A\ K\ Q}\,,\ \mathrm{J\ 10\ 9}) = 3,$$

$$N\,(\mathrm{A\ K\ 9}\,,\ \mathrm{Q\ J\ 10}) = 2$$

and

$$N\,(\mathrm{A\ 10\ 9}\,,\ \mathrm{K\ Q\ J}) = 1$$

follow from Consequence 3.5. The values

$$N\,(\mathrm{A\ K\ J}\,,\ \mathrm{Q\ 10\ 9}) = 2 + 1/2$$

and

$$N\,(\mathrm{A\ J\ 9}\,,\ \mathrm{K\ Q\ 10}) = 1 + 1/2$$

follow immediately from (B). Notice that in the case of A J 9 versus K Q 10, if East has the lead and starts with the king or the queen, West will get two tricks by playing low in the first trick.

In the case of A Q J versus K 10 9, West will get two tricks regardless of the location of the lead. This does not prove that the numerical value of this deal is 2. However, we can prove this by considering the following two-suit deal:

	West :	*East* :
(9)	♠ A Q J	♠ K 10 9
	♡ K J	♡ A Q

Here both players will try to avoid leading hearts. If West has the lead, and leads spades, he can either cash the ace and continue with another spade, or lead one of the smaller spades immediately. In any case, East will cash his king of spades in one of the two first tricks, and then lead another spade. West will be forced to lead hearts, which restricts him to two tricks.

If on the other hand East has the lead, and starts with a spade, then West will cash two spade tricks and lead his third spade. Either East has played his king of spades under West's ace, or he is now forced to lead hearts. In any case West gets three tricks.

This shows that the numerical value of the deal as a whole is $2+1/2$. Since the value of the heart suit is already known to be $1/2$, it follows that

$$N\,(\text{A Q J}\,,\ \text{K 10 9}) = 2.$$

The situation would have been similar if the distribution of the spades had been A Q 10 versus K J 9 or A Q 9 versus K J 10. Hence

$$N\,(\text{A Q 10}\,,\ \text{K J 9}) = N\,(\text{A Q 9}\,,\ \text{K J 10}) = 2.$$

From deal (8), we know that the numerical value of

(10)

	West :	*East* :
	♠ A K 10	♠ Q J 9
	♡ A K 10	♡ Q J 9

must be $4 + 1/2$, since West will get four tricks with the lead and five tricks with East on lead. Hence

$$N\,(\text{A K 10}\,,\ \text{Q J 9}) = 2 + 1/4.$$

Similarly, with

(11)

	West :	*East* :
	♠ K Q 9	♠ A J 10
	♡ K Q 9	♡ A J 10

West will get two tricks with the lead, but three tricks if East has the lead. The reader may wish to verify this. The strategy is similar to that of (8). When East attacks one of the suits, West will use the other suit to transfer the lead back to East and force him to lead a second time from the first suit. Hence

$$N\,(\text{K Q 9}\,,\ \text{A J 10}\ ) = (2 + 1/2)/2 = 1 + 1/4.$$

The remaining three-card deals are obtained from the deals above by switching the East and West hands.

3.3.2. *Deals with more than three cards.* The example

$$N\,(\text{A K 10}\,,\ \text{Q J 9}) = 2 + 1/4$$

can be generalized in an obvious way. Consider the following four-suit deal:

	West :	*East* :
	♠ A K Q 8	♠ J 10 9 7
(12)	♡ A K Q 8	♡ J 10 9 7
	♢ A K Q 8	♢ J 10 9 7
	♣ A K Q 8	♣ J 10 9 7

West has twelve easy tricks. We claim that if East has the lead, West will be able to score a thirteenth trick with one of his eights. If East leads the spade jack say, then West will take this trick, cash the ace, king, queen of hearts and lead his fourth heart. East gets the lead, and he can do no better than lead from a new suit, say the jack of diamonds. West takes the trick, and plays four rounds of clubs, putting East on lead with the last one. The situation is now equivalent to (8) with East on lead.

We will not go through all possible lines of play, but the reader can convince himself that there is no way West can get thirteen tricks if he has the lead in (12). It follows that

$$N(\text{A K Q 8}, \text{ J 10 9 7 }) = (12 + 1/2)/4 = 3 + 1/8.$$

Similarly, we have

$$N(\text{A K Q J 6}, \text{ 10 9 8 7 5}) = 4 + 1/16,$$

$$N(\text{A K Q J 10 4}, \text{ 9 8 7 6 5 3}) = 5 + 1/32,$$

and so on.

4. Exits and stoppers

4.1. **The ambiguity of rounding.** Theorem 3.1 specifies the outcome of a deal in terms of its numerical value except when this value is half way between two integers. In this section we consider some examples of deals whose numerical value is half an odd integer. We wish to find the factors that determine which way to round the numerical value in order to obtain the outcome.

4.2. **Examples.** We already know that the rounding may depend on the location of the lead. In the example

	West :	*East* :
(13)	♠ K J	♠ A Q

the numerical value is 1/2, and this should be rounded in favor of the player not on lead. The numerical value of the deal

	West :	*East* :
(14)	♠ K J	♠ A Q
	♡ K	♡ A

is still 1/2, but here West gets a spade trick whether or not he has the lead. This is because he has what bridge players call an *exit card.* The king of hearts does not win a trick, but it provides West with a possibility to transfer the lead to East. The deal

(15)

West :	*East* :
♠ K J	♠ A Q
♡ A	♡ K
♢ K	♢ A

shows an example of a so called *elimination and throw-in.* West on lead can eliminate East's exit card, the king of hearts, by cashing the ace. Then he exits with the king of diamonds. East is "thrown in" and has to lead spades. If East has the lead, he will do the same thing to West: cash the ace of diamonds before leading hearts.

4.3. **Counting stoppers.** If both players have exits, it may apparently be an advantage to have the lead. Some exit suits are better than others though, as the following examples show. The distribution A J versus K Q provides West with an exit, since the numerical value is 1, and in

(16)

West :	*East* :
♠ K J	♠ A Q
♡ A J	♡ K Q

West gets a spade trick whether or not he has the lead. If we give East too an exit,

(17)

West :	*East* :
♠ K J	♠ A Q
♡ A J	♡ K Q
♢ A	♢ K

we would expect a situation where the lead is an advantage. However, we discover that West always gets a spade trick. West on lead can cash the red aces before putting East on lead with a second heart. Suppose now that East has the lead. He can attack West's exit by leading a heart, but West takes with the ace, and now West has time to cash the ace of diamonds, eliminating East's exit, before playing the jack of hearts.

Here the ace of hearts acts by defending West's exit in hearts. It temporarily stops East from cashing his heart trick, giving West time to eliminate East's exit in diamonds before playing his own exit card. Such a card will be called a *stopper.*

Note that the number of exits does not matter:

	West :	*East* :
	♠ K J	♠ A Q
(18)	♡ K	♡ A
	♢ A	♢ K
	♣ A	♣ K

The fact that East has two exits while West has only one is irrelevant, since West on lead can eliminate both the diamonds and the clubs before exiting in hearts.

However, the number of stoppers does matter:

	West :	*East* :
	♠ K J	♠ A Q
(19)	♡ A J	♡ K Q
	♢ K Q	♢ A J
	♣ K Q	♣ A J

Here East has two stoppers, and West has only one. This gives East an edge in the fight for the second spade trick. If West leads a club or a diamond, say a diamond, then East takes with the ace and returns a heart. His ace of clubs now guarantees that he will have time to cash his heart trick before playing the jack of clubs. This ensures him two spade tricks.

5. The semigroup of states

5.1. **Definitions.** We introduce an additive notation for card distributions. A single-suit deal is a partition of a finite totally ordered set into two sets E and W of the same cardinality, the East and West hands. This is denoted by $[W, E]$, where W and E are the West and East hands, respectively.

A multi-suit deal is a formal sum of single-suit deals. We denote the set of multi-suit deals by $\mathbf{D}$. Hence $\mathbf{D}$ is the free abelian monoid over the set of single-suit deals.

Addition is commutative, that is, we do not distinguish between the individual suits. The deal

$$[\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K\ J}\,,\ \mathrm{A\ Q}]\,,$$

for instance, may represent either of

West :	*East* :
♠ K J	♠ A Q
♡ K	♡ A

and

West :	*East* :
♠ K	♠ A
♡ K J	♡ A Q

In order to represent a state in the game in such a way that the outcome under optimal play from a game-state is a function of the state, we need to include not only the remaining cards on the two hands, but also the number of tricks that West has already taken. Therefore we let a *state* be a formal sum of an integer and a multi-suit deal. Hence the set S of states is the direct sum

$$\mathbf{Z} \oplus \mathbf{D}$$

of the integers with the set of multi-suit card distributions.

If D is a state, then the *outcome* $\chi(D)$ of D is the pair (m, n), where m is the number of tricks that West takes under optimal play if he has the lead initially, and n is the number of tricks he takes if East has the lead. Hence χ is a mapping $S \to \mathbf{Z} \times \mathbf{Z}$. Obviously χ is not a semigroup homomorphism. Our approach is to describe χ *using* a semigroup homomorphism.

When we speak of a *deal*, we mean just a card distribution. Technically, this is a state where West has not already taken any tricks, that is, the integer part of the state is zero. If D is a deal, then we let $|D|$ denote the number of cards on each of the hands, that is, the number of tricks to be played. We let $\overline{D}$ be the deal obtained by switching the East and West hands. Obviously, if $\chi(D) = (m, n)$, then $\chi(\overline{D}) = (|D| - n, |D| - m)$.

5.2. **Equivalence and order of states.** If D and E are two states, then we say that D is equivalent to E, and write $D \equiv E$, if for every state F, $\chi(D+F) = \chi(E+F)$. In other words, two states are equivalent if they behave in the same way under addition.

If (m, n) and (m', n') are two pairs of integers, we say that $(m, n) \leq (m', n')$ if $m \leq m'$ and $n \leq n'$. If D and E are states, then we say that $D \leq E$ if for every state F, $\chi(D + F) \leq \chi(E + F)$. Clearly $D \equiv E$ if and only if $D \leq E$ and $E \leq D$. Hence this gives a partial ordering on the quotient $S/\equiv$.

Addition is well-defined on the equivalence classes under $\equiv$, and $S/\equiv$ is an abelian semigroup. If D, E and F are states, and $D \leq E$, then $D + F \leq E + F$.

5.3. **Examples established by strategy-stealing.** Some properties of the ordering of states can be established by simple strategy-stealing arguments.

Example 5.1.

$$[\mathrm{K}\,,\ \mathrm{A}] \geq 0.$$

Proof. We have to show that if D is a state, then

$$\chi(D + [\mathrm{K}\,,\ \mathrm{A}]) \geq \chi(D),$$

in other words, West will get at least as many tricks in $D + [\mathrm{K}\,,\ \mathrm{A}]$ as in D, both when he has the lead and when East has the lead.

West can steal an optimal strategy for D when playing $D + [\mathrm{K}\,,\ \mathrm{A}]$ by pretending that the extra suit is not there. If at any point East leads his ace in the extra suit, West plays his king, and since the lead stays with East, West can continue to play as in D, by pretending that the extra suit was never there. If East does not lead the extra suit, then neither does West until possibly in the last trick. This strategy will give West at least as many tricks in $D + [\mathrm{K}\,,\ \mathrm{A}]$ as he can take in D. □

One would perhaps think that $[\mathrm{K}\,,\ \mathrm{A}] \equiv 0$, but this is not true. The deal $[\mathrm{K}\,,\ \mathrm{A}]$, although it does not have any trick-taking potential in itself, may give West the opportunity to put East on lead. This in turn may produce an extra trick in another suit. We have:

$$\chi([\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K\ J}\,,\ \mathrm{A\ Q}]) = (2, 2),$$

while

$$\chi([\mathrm{K\ J}\,,\ \mathrm{A\ Q}]) = (1, 2).$$

This shows that

$$[\mathrm{K}\,,\ \mathrm{A}] > 0.$$

Example 5.2.

$$[\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K}\,,\ \mathrm{A}] \equiv [\mathrm{K}\,,\ \mathrm{A}]\,.$$

Proof. It follows from Example 5.1 that

$$[\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K}\,,\ \mathrm{A}] \geq [\mathrm{K}\,,\ \mathrm{A}]\,.$$

We need to establish the opposite inequality. We do this by showing that if D is any deal, then when playing $D + [\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K}\,,\ \mathrm{A}]$, East can steal an optimal strategy for $D + [\mathrm{K}\,,\ \mathrm{A}]$. Whenever East on lead is required to cash the ace in an optimal strategy for $D + [\mathrm{K}\,,\ \mathrm{A}]$, he will cash both aces in $D + [\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K}\,,\ \mathrm{A}]$. Whenever West leads one of the two kings in $D + [\mathrm{K}\,,\ \mathrm{A}] + [\mathrm{K}\,,\ \mathrm{A}]$, East will take with the corresponding ace, and immediately cash the other. Then he will steal the strategy that he would use in $D + [\mathrm{K}\,,\ \mathrm{A}]$ if West leads the king. This will hold West to the same number of tricks in both cases. □

These two examples show that the quotient semigroup $S/\equiv$ cannot be embedded into a group.

5.4. **Further consequences of Theorem 3.1.** It seems obvious that

$$[\mathrm{A}\,,\,\mathrm{K}] > [\mathrm{K}\,,\,\mathrm{A}]\,,$$

but it is surprisingly difficult to prove this, or even to prove that

$$[\mathrm{A}\,,\,\mathrm{K}] \geq 0. \tag{20}$$

To prove the inequality (20), we need to consider $D + [\mathrm{A}\,,\,\mathrm{K}]$, where D is an arbitrary state, and show that West always gets at least as many tricks as when playing D. If West has the lead, this is obvious by strategy-stealing: West can cash the ace and then continue with an optimal strategy for D. However, if East has the lead, the problem is that East can transfer the lead to West by playing the king. Although this gives West an extra trick compared to playing D, it is not clear how West on lead can copy a strategy for D with East on lead, even if he can afford to give back one trick.

However, by Consequence 3.2, having the lead may cost at most one trick. Hence the inequality (20) follows from the consistency of (A) and (B). More generally, we have:

Consequence 5.3. *Let $D = [W, E]$, and $D' = [W\backslash\{x\} \cup \{y\}, E]$, where $x < y$. That is, D' is obtained from D by replacing one card on the West hand by a higher card. Then $D \leq D'$.*

In other words, a higher card is always at least as good as a smaller one.

Proof. We have to consider playing the two sums $D + E$ and $D' + E$, for an arbitrary state E. When playing $D' + E$, we let West steal an optimal strategy for $D + E$. West can pretend that the card y is the card x, until the optimal strategy for $D + E$ requires him to play the card x. Then instead he will play the card y. If East's card in that trick is not between x and y, West can go on pretending that the card he played was the card x. If East's card is between x and y, then West has taken an extra trick compared to playing $D + E$. West has now obtained the lead, so he cannot go on stealing the strategy for $D + E$, but by Consequence 3.2, having the lead will cost him at most one trick compared to not having the lead, so West's total number of tricks will be at least the same as when playing $D + E$. □

We can also establish that

$$[\mathrm{A\ J}\,,\,\mathrm{K\ Q}] > 1$$

by strategy-stealing. When playing $[\mathrm{A\ J}\,,\,\mathrm{K\ Q}] + D$, West can avoid leading from the suit until possibly when D is empty, and whenever East leads the suit, West takes the first trick with the ace and returns the jack. This way the lead stays with East, and West can continue stealing the strategy for D.

By Consequence 5.3, we can establish the ordering of all two-card single-suit deals, and their positions relative to the integers:

Consequence 5.4.

$$(21)\quad 0 < [Q\ J,\ A\ K] < [K\ J,\ A\ Q] < [K\ Q,\ A\ J] < 1 < [A\ J,\ K\ Q] < [A\ Q,\ K\ J] < [A\ K,\ Q\ J] < 2.$$

6. Values

We here introduce a certain semigroup whose elements will be referred to as *values*. Our aim is to prove that this semigroup is isomorphic to the quotient semigroup $S/\equiv$.

For reasons that are discussed in Section 12.3, we are constructing the values, and the mapping from states to values "by hand", before proving any of their properties. In this section, we just define the set of values, and its structure of addition, negation, order and simplicity, without proving anything. The discussion should therefore be taken as an attempt to informally motivate these definitions, based on the examples given earlier.

6.1. The semigroup $\mathcal{E}$ of infinitesimals.

6.1.1. *Unprotected exits.* We denote the value of an unprotected exit by ε. An exit for East is denoted by $-\varepsilon$. As indicated by Example 5.2, we must have $\varepsilon + \varepsilon = \varepsilon$. Hence the sum of any number of exits of the same sign equals a single exit of that sign. However, signs do not cancel. Instead, the sum of a positive and a negative exit, or any number of such, has the "fuzzy" value $\pm\varepsilon$. The values of unprotected exits form a semigroup with the four elements 0, ε, $-\varepsilon$ and $\pm\varepsilon$.

6.1.2. *Exits protected by stoppers.* Stoppers add like integers. However, the full semigroup of infinitesimals is not isomorphic to a direct sum of the integers with the semigroup of unprotected exits described above. The reason for this is that on the one hand, one cannot have a stopper without having an exit, and on the other hand, there are some equivalences to take into account. If the total number of stoppers in a deal (added with signs) is positive, so that West has more stoppers than East, then it does not matter whether East has an exit or not. For instance,

$$(22)\qquad [\text{A J}, \text{K Q}] + [\text{A}, \text{K}] \equiv 1 + [\text{A J}, \text{K Q}].$$

We indicate the number of stoppers of an exit with an index. Hence ε_k is the value of an exit for West, protected by k stoppers. An exit for East with k stoppers is denoted ε_{-k}, but can also be regarded as the negative of ε_k, that is, $-\varepsilon_k$. For consistency, the unprotected exits can be written with an index of zero.

The deal (22) shows that we must have the identity

$$\varepsilon_1 - \varepsilon_0 = \varepsilon_1.$$

6.1.3. *The set of infinitesimals.* The semigroup $\mathcal{E}$ of infinitesimals consists of the elements 0, ε_0, $-\varepsilon_0$, $\pm\varepsilon_0$, and ε_k, for nonzero integers k.

6.1.4. *Addition of infinitesimals.* The infinitesimals are added as follows: 0 is the additive identity. Moreover, $\varepsilon_0 + \varepsilon_0 = \varepsilon_0$, and similarly $(-\varepsilon_0) + (-\varepsilon_0) = -\varepsilon_0$. All other sums are evaluated by summing the indices. If the sum of the indices is a nonzero integer k, then the sum equals ε_k. If the indices sum to zero, the sum is $\pm\varepsilon_0$.

6.1.5. *The negative of an infinitesimal.* As is indicated by the notation, there is a notion of negative of an infinitesimal. We let $-(\varepsilon_0) = -\varepsilon_0$, $-(\pm\varepsilon_0) = \pm\varepsilon_0$, and for nonzero integers k, $-\varepsilon_k = \varepsilon_{-k}$. The negative of an infinitesimal is not in general an additive inverse. Negation only has the weaker property of being an automorphism with respect to addition, that is, $-(\alpha+\beta) = (-\alpha) + (-\beta)$. We still use the shorthand $\alpha - \beta$ for $\alpha + (-\beta)$.

6.1.6. *Order of infinitesimals.* The values of unprotected exits are ordered according to

$$-\varepsilon_0 < 0 < \varepsilon_0$$

and

$$-\varepsilon_0 < \pm\varepsilon_0 < \varepsilon_0,$$

with 0 and $\pm\varepsilon_0$ incompatible. If k is a positive integer, then the values of unprotected exits are greater than $-\varepsilon_k$, but smaller than ε_k. If p and q are two nonzero integers, and $p < q$, then $\varepsilon_p < \varepsilon_q$.

6.2. **The group $\mathcal{Q}$ of numbers.** We let $\mathcal{Q}$ denote the group of rational numbers that can be written with a power of 2 in the denominator. In other words, $\mathcal{Q}$ is the localization of the ring of integers to the set of 2-powers. The elements of $\mathcal{Q}$ are called *numbers*.

6.3. **The semigroup V of values.** A *value* is a sum of a number and an infinitesimal, that is, we define the set V of values by

$$V = \mathcal{Q} \oplus \mathcal{E}.$$

Values are added and negated in the obvious way: If q and r are numbers, and x and y are infinitesimals, then $(q+x) + (r+y) = (q+r) + (x+y)$, and $-(q+x) = (-q) + (-x)$. This implies that if α and β are arbitrary values, then $-(\alpha+\beta) = (-\alpha) + (-\beta)$. Negation is therefore a semigroup automorphism.

6.3.1. *Order of values.* We define order of values as follows: If q and r are numbers, and x and y are infinitesimals, then $q + x \leq r + y$ if and only if either $q < r$, or $q = r$ and $x \leq y$. This ordering has the property that $\alpha \leq \beta$ if and only if $-\beta \leq -\alpha$, for all α and β.

6.4. **Simplicity.** There is a notion of *simplicity* of values, similar to the corresponding concept for combinatorial games discussed in [2] and [1]. We classify values from simple to more complex in the following way:

(i) The simplest values are the *half-integers*, that is, the numbers of the form $k/2$ for integers k.
(ii) For numbers other than half-integers, a number with smaller denominator is simpler than a number with greater denominator.
(iii) Numbers are simpler than other values.
(iv) Values of the form $q + \varepsilon_k$ and $q - \varepsilon_k$, where q is a number, are simpler when k has smaller absolute value.
(v) Values of the form $q \pm \varepsilon_0$ are more complex than other values.

We can describe this structure by arranging values in complexity classes, labeled by ordinals, as follows:

(i) Half-integers have complexity 0.
(ii) Numbers with minimal denominator 2^k, for $k \geq 2$, have complexity $k - 1$.
(iii) Values of the form $q + \varepsilon_k$ and $q - \varepsilon_k$ have complexity $\omega + k$.
(iv) Values of the form $q \pm \varepsilon_0$ have complexity $\omega + \omega$.

This is in analogy with the classification of combinatorial games according to birthday. Since the complexity classes are well-ordered, every nonempty set of values has an element of minimal complexity.

6.5. **Rounding a value to an integer.** The value of a deal should determine its outcome. In Section 7.1, we construct a function val : $S \to V$ assigning values to states. We now define a function ρ mapping a value to the corresponding outcome, that is, mapping values to ordered pairs of integers. The idea is then to prove that $\chi = \rho \circ \mathrm{val}$.

The function $\rho : V \to \mathbf{Z} \times \mathbf{Z}$ is called the *rounding function* since first of all it rounds the numerical value to the nearest integer. Let α be a value. Then

$$\rho(\alpha) = \begin{cases} (n, n), & \text{if } n - 1/2 < \alpha < n + 1/2, \\ (n, n+1), & \text{if } \alpha = n + 1/2, \\ (n+1, n), & \text{if } \alpha = n + 1/2 \pm \varepsilon_0. \end{cases}$$

Note that the ordering of values can be characterized by the requirement that $\alpha \leq \beta$ if and only if for every value γ, $\rho(\alpha + \gamma) \leq \rho(\beta + \gamma)$. This is what we should expect in view of the definition in Section 6 of the ordering of states.

If A is any nonempty discrete set of numbers, then we can define an analogous function rounding values to pairs of elements of A. Let q be a number and x an infinitesimal. Then $\rho_A(q+x) = (a,a)$ if a is the unique element of A closest to q, while if there is a tie between two elements a and b of A, and $a < q < b$, then

$$\rho_A(q+x) = \begin{cases} (a,a), & \text{if } x \text{ is negative,} \\ (b,b), & \text{if } x \text{ is positive,} \\ (a,b), & \text{if } x = 0, \\ (b,a), & \text{if } x = \pm\varepsilon_0. \end{cases}$$

7. Mapping states to values

7.1. **The mapping** val : $S \to V$**.** In this section, we define the function val : $S \to V$. This function has to be a semigroup homomorphism which fixes the integers. Hence to define it, we need only specify its values on single-suit states. This is done inductively.

We let a *labeled value* be a pair (P, x), where P is one of the symbols E or W (for East and West), and x is a value. A labeled value represents a situation where the player P has the lead in a deal with value x.

In our analysis, an implicit hypothesis is that it is advantageous to have the lead, except if the value of the deal is a number. We therefore introduce the following ordering of labeled values:

- $(P,x) < (Q,y)$ if $x < y$,
- $(E,x) < (W,x)$ if x is not a number,
- $(E,x) > (W,x)$ if x is a number.

7.2. **Reductions.** By a *single-suit state* we mean a state which is the sum of an integer and a single-suit deal. A single-suit state $D = m + [W,E]$ is said to be an n-card state if the hands W and E have n cards each. If $W = \{W_1,\dots,W_n\}$ and $E = \{E_1,\dots,E_n\}$, where $W_1 < \cdots < W_n$ and $E_1 < \cdots < E_n$, then we define the *reduction* $D_{i,j}$ of D to be the state into which D will be transformed if in the first trick West plays the card W_i and East plays the card E_j. That is,

$$D_{i,j} = m + [W\backslash\{W_i\}, E\backslash\{E_j\}] + \begin{cases} 1, \text{ if } W_i > E_j \\ 0, \text{ if } W_i < E_j \end{cases}$$

We now let D be an n-card single-suit deal, and suppose that $\mathrm{val}(D')$ has been defined for every deal D' with fewer than n cards, and in particular, on all the reductions $D_{i,j}$ of D. For $1 \le i,j \le n$, we let

$$\alpha_{i,j} = (P, \mathrm{val}(D_{i,j})),$$

where P is the player who gets the lead if West plays his ith card and East plays his jth card. In other words, $P = W$ if $W_i > E_j$, and $P = E$

if $W_i < E_j$. We let

$$\operatorname{maxmin}(D) = \max_i \min_j \alpha_{i,j},$$

and

$$\operatorname{minmax}(D) = \min_j \max_i \alpha_{i,j}.$$

Obviously $\operatorname{maxmin}(D) \le \operatorname{minmax}(D)$.

7.3. **The left and right bounds on** $\operatorname{val}(D)$**.** For a nonempty single-suit deal D, we define two sets $L(D)$ and $R(D)$ of values, which in a certain sense correspond to the left and right options of a combinatorial game.

Definition 7.1. We let D be as above, so that the labeled values $\operatorname{maxmin}(D)$ and $\operatorname{minmax}(D)$ have been defined. Then $L(D)$ is defined as follows:

(i) If q is a number, then $q \in L(D)$ if $\operatorname{maxmin}(D) \ge (E, q)$.
(ii) Moreover, if x is a nonzero infinitesimal, then $q + x \in L(D)$ if $\operatorname{maxmin}(D) \ge (E, q)$ and $\operatorname{minmax}(D) \ge (W, q + x)$.

The set $R(D)$ is defined similarly:

(i) $q \in R(D)$ if $\operatorname{minmax}(D) \le (W, q)$, and
(ii) $q - x \in R(D)$ if $\operatorname{minmax}(D) \le (W, q)$ and $\operatorname{maxmin}(D) \le (E, q - x)$.

Notice that a value cannot belong to $L(D)$ or $R(D)$, unless its numerical part belongs to $L(D)$ or $R(D)$ respectively. Notice also that if $x \in L(D)$ and $y \le x$, then $y \in L(D)$. Similarly, if $x \in R(D)$ and $y \ge x$, then $y \in R(D)$.

7.4. **The interval** $I(D)$**.** Let D be as above. We let $I(D)$ be the set of values that lie between $L(D)$ and $R(D)$, in other words, that do not belong to $L(D)$ or to $R(D)$. The set $I(D)$ is an *interval* in the sense that if x, y, and z are values such that $x \le y \le z$, and x and z belong to $I(D)$, then so does y.

We prove that under certain conditions, an interval has a unique simplest element.

Theorem 7.2. *If a nonempty interval of values contains at most one half-integer, then it has a unique simplest element.*

Proof. Let I be a nonempty interval, and suppose that I contains at most one half-integer. Since the complexity classes are well-ordered, there is an element of I with minimal complexity. To prove uniqueness, it therefore suffices to show that if x and y are two distinct values of the same complexity, then unless they are half-integers, there is a simpler value between them.

Suppose first that x and y are numbers. Then x and y can be written $p/2^k$ and $q/2^k$ respectively, where p and q are distinct odd integers, and $k \ge 2$. Between two distinct odd integers there is always an even integer. Hence between x and y there is a number of the form $2r/2^k = r/2^{k-1}$. This number has smaller denominator, and is therefore simpler than x and y.

Suppose now that x and y are not numbers. Then we can assume that they have the same numerical part q, since otherwise there is a number between them. We must therefore have $x = q - \varepsilon_k$ and $y = q + \varepsilon_k$ for some nonzero integer k. Hence the number q is between x and y. □

Lemma 7.3. *Let D be a nonempty single-suit deal as above, so that val(D') has been defined for every reduction D' of D. Then $I(D)$ is nonempty. Moreover, $I(D)$ always contains a value whose infinitesimal part is distinct from $\pm\varepsilon$.*

Proof. If the numerical part of maxmin(D) is strictly smaller than the numerical part of minmax(D), then there is a number strictly between them, and this number must belong to $I(D)$. Suppose therefore that maxmin(D) and minmax(D) have the same numerical part q. If q does not belong to $I(D)$, then it must belong either to $L(D)$ or to $R(D)$. By symmetry it suffices to consider the case that $q \in L(D)$, that is, maxmin$(D) \geq (E, q)$. Then minmax$(D) \geq (E, q) > (W, q)$. Hence $q \notin R(D)$. It follows that no value with numerical part q belongs to $R(D)$.

For some $P = E$ or W and some nonnegative infinitesimal y, we have

$$\text{minmax}(D) = (P, q + y).$$

Hence if x is an infinitesimal greater than y, then minmax$(D) < (W, q + x)$. It follows that $q + x \notin L(D)$, and hence that $q + x \in I(D)$. □

7.5. **Definition of** val(D).

Definition 7.4. We let val(D) be an element of $I(D)$ of minimal complexity.

We prove in Theorem 8.8 that there is no ambiguity in this definition, that is, there is always a unique simplest element of $I(D)$. At this point we know that $I(D)$ contains at least one element of minimal complexity, so we can think of the function val as being defined, possibly with some arbitrary choices. This completes the definition of val(D) for every state D. We also notice that for single-suit deals, the value has infinitesimal part distinct from $\pm\varepsilon$.

8. Last trick doesn't count

8.1. **Reduced whist.** For technical reasons, we introduce two auxiliary games which have the same form as whist, but with slightly different objectives. We first briefly discuss a game which we call *reduced whist.* This game is similar to the game of whist, except that the last trick does not count. Hence in an n-card deal, the objective is to take as many as possible of the first $n - 1$ tricks.

Lemma 8.1. *In single-suit reduced whist, there is an optimal strategy that always saves the smallest card on the hand for the last trick. Hence when playing single-suit reduced whist, the players can start by removing the smallest card from their hands, and then play as in ordinary whist with the remaining cards.*

Proof. We prove this by induction on the number of cards on the hand. Consider an optimal strategy for playing a certain single-suit deal of reduced whist. We modify this strategy so that it never uses the smallest card before the last trick.

If in the first trick, the strategy requires us to play a card higher than the smallest one, then after the first trick we can, by induction, use a strategy that saves the smallest card for the last trick. Suppose therefore that the strategy requires us to play the smallest card in the first trick. Then instead, we play the next to smallest card. If our opponent plays a card which is not between our smallest and next to smallest card, then this will not make any difference. The same player will win the trick, and by induction, we can after the first trick use a strategy that makes no use of our smallest card. Hence it does not matter whether our smallest remaining card is the originally smallest card or another one.

Suppose now that in the first trick, our opponent plays a card which is between our smallest and next to smallest card. Then we have won a trick that we would not have won with the given strategy, and we have obtained the lead. By induction, we can assume that from the second trick on, both players will use a strategy that saves the smallest card for the last trick. Hence we can assume that after the first trick, both players remove their smallest remaining cards from their hands, and continue as in ordinary single-suit whist. Hence compared to the given strategy, it will do us no harm to have wasted a higher card in the first trick. The only difference in the situation is that we have obtained the lead, whereas with the original strategy we would not have had the lead. On the other hand, we have won the first trick, so in order to prove the lemma, we only have to prove that in ordinary single-suit whist, having the lead cannot cost more than one trick compared to not having the lead. This is what the following theorem tells us. ☐

To complete the proof, we cite an already mentioned theorem from [3]:

Theorem 8.2 (Kahn, Lagarias & Witsenhausen)**.** *In single-suit whist, having the lead is never an advantage, but may cost at most one trick compared to not having the lead.*

8.2. **Refined whist, a simultaneously optimal strategy.** The following theorem holds also for multi-suit deals:

Theorem 8.3. *There is a strategy which is at the same time optimal for whist and for reduced whist.*

We prove this theorem by introducing yet another form of whist, called *refined* whist. This game has the property that its optimal strategies are precisely the common optimal strategies of whist and reduced whist. Since there is an optimal strategy for refined whist, this proves Theorem 8.3.

The game is defined by the following minor adjustment of the scoring: For every trick except the last one, West scores one point for winning, and no points for losing. In the last trick, West gets 3/4 of a point for winning the trick, and 1/4 for losing it. Alternatively, we can regard this as a bonus of 1/4 for not having the lead when the game is over, and a punishment of $-1/4$ for having the lead. To make things consistent, we should therefore regard the zero state as having refined outcome $(-1/4, 1/4)$.

It turns out that the outcome of refined whist is better approximated by the value of the deal, than is the outcome of whist.

Theorem 8.4. *An optimal strategy for refined whist is optimal for both whist and reduced whist.*

Proof. Taking at least n tricks in whist is equivalent to scoring at least $n - 1/4$ in refined whist. Taking at least n tricks in reduced whist is equivalent to scoring at least $n + 1/4$ in refined whist. □

This motivates the name "refined". Note that scoring at least $n+1/4$ in refined whist cannot be expressed in terms of the outcome of whist. For example, if we have the lead with A Q versus K J, we can score $1 + 1/4$ in refined whist by starting with the ace, but in whist, it is still optimal to lead the queen, which scores only 3/4 in refined whist.

8.3. **Refined results for single-suit games.** If we know how to play refined whist, we also know how to play whist. It is therefore sufficient to study the game of refined whist. Every result about this game will yield as a corollary the corresponding result for whist.

We show that, at least for single suit hands, the converse also holds: If we know how to play single-suit whist (and by [7] we do), then we also know how to play refined single-suit whist.

We cite another theorem about single-suit whist that was proved in [3]. We refer to this theorem as the *monotonicity principle.*

Theorem 8.5 (Kahn, Lagarias & Witsenhausen). *In single-suit whist, a high card is always at least as good as a small one. In other words, if D and D' are single-suit deals, and D' is obtained from D by replacing a card on the West hand by a higher one, then $\chi(D) \leq \chi(D')$.*

This leads to the following result for refined whist.

Lemma 8.6. *In single-suit refined whist, having the lead is never an advantage, but may cost at most half a point.*

For the proof of this lemma, we introduce the following notation: If D is a nonempty single-suit deal, we let D_0 denote the deal obtained by deleting the smallest card on each of the two hands. Notice that the first statement of the lemma is obvious by strategy-stealing. If our opponent has the lead, we can still play any card we want to.

Proof. Let D be a single-suit deal with n cards on each hand. Suppose that West can score at least $k + 1/4$ when East has the lead. Then he must be able to take at least k of the first $n - 1$ tricks. By Lemma 8.1, West's smallest card cannot help him in doing this. Hence West must be able to take at least k tricks in D_0 if East has the lead. Now suppose that West has the lead in D. He can then lead his smallest card in the first trick. If East wins this trick, then by monotonicity, the situation is at least as good for West as when playing D_0 with East on lead. Hence West can take at least k tricks. If on the other hand West wins the first trick, then by Theorem 8.2, he can take at least $k - 1$ more tricks. In any case, West will get a total of at least k tricks, and thereby a score of at least $k - 1/4$.

By switching the East and West hands and applying the same argument, we see that if West can score at least $k + 3/4$ when East has the lead, then he must be able to score at least $k + 1/4$ when he has the lead himself. ☐

8.4. **Uniqueness of the simplest element of $I(D)$.** Next we show that the outcome of single-suit refined whist is determined by the value of the deal. We need this result not to solve the single-suit game, but in order to remove the potential ambiguity in the definition of the mapping val.

If D is a deal, we let $\chi'(D)$ be the outcome of D in refined whist. We let $\rho' = \rho_A$, where A is the set of numbers with fractional part $1/4$ or $3/4$. In other words, ρ' is the function that rounds a value to the nearest rational number with a minimal denominator of exactly 4, with the same tie-break rules as ρ.

Theorem 8.7. *If D is a single-suit deal, then*

$$\chi'(D) = \rho'(\mathrm{val}(D)).$$

We prove this theorem by induction on the number of cards in D, simultaneously with the following theorem:

Theorem 8.8. *If D is a single-suit deal, then $I(D)$ contains at most one number of the form $k/2$ for $k \in \mathbf{Z}$.*

Here we introduce a notational convention that will also be convenient in the following. If D is a state, and we consider the possible lines of play from D, we let D' denote the state into which D reduces after the first trick. This means that D' is obtained from D by deleting the two cards played in the first trick, and adding 1 if West won the trick. Hence D' depends not only on D, but also on the choices made by East and West in the first trick.

Proof of Theorems 8.7 and 8.8. Let D be a single-suit deal. Suppose that the two statements hold for every single-suit deal with fewer cards than D, and in particular for every reduction of D.

We first show that there cannot be two half-integers in $I(D)$. Suppose for a contradiction that k is an integer such that both $k/2$ and $(k+1)/2$ belong to $I(D)$. Then $\text{maxmin}(D) \leq (W, k/2)$. Hence if West has the lead, East can always make sure that after the first trick, either $\text{val}(D') < k/2$, or $\text{val}(D') = k/2$ with West still on lead. Hence West can score at most $k/2 - 1/4$ with the lead. On the other hand, if East has the lead, then since $\text{minmax}(D) \geq (E, (k+1)/2)$, West can make sure that on any lead from East, either $\text{val}(D') > (k+1)/2$, or $\text{val}(D') = (k+1)/2$ with East still on lead. In any case, West will score at least $(k+1)/2 + 1/4 = k/2 + 3/4$ with East on lead. This contradicts Lemma 8.6. Hence there is at most one half-integer in $I(D)$.

We now turn to the statement of Theorem 8.7. The statement clearly holds when $D = 0$, since $\chi'(0) = (-1/4, 1/4)$. Let k be an integer, and suppose that $\text{val}(D) > k/2$. We have to show that West on lead can score at least $k/2 + 1/4$. We must have $k/2 \in L(D)$. Hence $\text{maxmin}(D) \geq (E, k/2)$. This means that there is a card that West can lead, so that no matter what East does, either $\text{val}(D') > k/2$, or $\text{val}(D') = k/2$ with East on lead. By induction, West can score at least $k/2 + 1/4$.

Suppose now that East is on lead and that $\text{val}(D) \geq k/2$. Then $k/2$ cannot belong to $R(D)$. Hence $\text{minmax}(D) > (W, k/2)$. This means that on any lead from East, West has a reply such that either $\text{val}(D') > k/2$, or $\text{val}(D') = k/2$ with East still on lead. By induction, West can score at least $k/2 + 1/4$.

By interchanging the roles of East and West, we obtain the corresponding inequalities in the other direction. □

Hence there is no ambiguity in the definition of $\text{val}(D)$. We remark that the non-refined version of Theorem 8.7 is an immediate corollary:

Corollary 8.9. If D is a single-suit deal, then

$$\chi(D) = \rho(\text{val}(D)).$$

8.5. **An observation.** In this context, we make another observation which can be useful for computing the value of a deal.

Theorem 8.10. *If D is a single-suit deal, and k is an integer, then* $\text{val}(D) = k$ *if and only if* $\text{val}(D_0) = k - 1/2$.

Proof. Let D be an n-card single-suit deal. Suppose that $\text{val}(D_0) = k - 1/2$. Then $\chi(D_0) = (k-1, k)$. If West has the lead in D, then he can lead his smallest card. By the same argument as in the proof of

Lemma 8.6, West will be able to take at least k tricks. Hence he can score at least $k - 1/4$ with the lead in refined whist in D. If East has the lead in D, then West can use the same strategy as in D_0 to get at least k of the $n-1$ first tricks. Hence he can score at least $k+1/4$ in refined whist in D. This shows that $\chi'(D) \geq (k-1/4, k+1/4)$, which implies that $\mathrm{val}(D) \geq k$. By interchanging the roles of East and West, we obtain the reverse inequality. Hence we have shown that $\mathrm{val}(D_0) = k - 1/2$ implies $\mathrm{val}(D) = k$.

Suppose now that $\mathrm{val}(D) = k$. Then $\chi'(D) = (k-1/4, k+1/4)$. If West has the lead, he must therefore be able to take at least k tricks. This implies that he must be able to take at least $k-1$ tricks in D_0. If East is on lead, then West can take k of the first $n-1$ tricks. Hence he can take at least k tricks in D_0. This shows that $\chi(D_0) \geq (k-1, k)$, which implies that $\mathrm{val}(D_0) \geq k - 1/2$. Again we obtain the reverse inequality by swapping the roles of East and West. □

9. The main theorem

9.1. **The single-suit lemmas.** Our main theorem states that the function $\mathrm{val} : S \to V$ determines the outcome of symmetric multi-suit whist under optimal play. We prove this theorem simultaneously with its refined counterpart.

Theorem 9.1. *If D is a state, then*

$$\chi(D) = \rho(\mathrm{val}(D)).$$

Moreover,

$$\chi'(D) = \rho'(\mathrm{val}(D)).$$

We prove the theorem by induction on the total number of cards. Notice that it suffices to prove the second statement. The proof is divided into eight lemmas, each corresponding to a particular situation in the game. Again we use the notation D' for the state into which D is transformed after the first trick.

The first four of these lemmas deal with the situation when West has the lead.

Lemma 9.2. *Suppose that D is a single-suit deal, and that*

$$\mathrm{val}(D) = q - x,$$

where q is a number and x is a positive infinitesimal. Then there is a card that West can lead such that regardless of East's reply, either $\mathrm{val}(D') > q - x$, or $\mathrm{val}(D') = q - x$ with West still on lead.

Proof. We have $q \in R(D)$. Hence $\mathrm{minmax}(D) \leq (W, q)$. Since $q - x \notin R(D)$, we have

$$\mathrm{maxmin}(D) > (E, q - x).$$

□

Lemma 9.3. *Suppose that D is a single-suit deal, and that* $\mathrm{val}(D) = q+x$*, where q is a number and x is a positive infinitesimal. Then West has a lead such that on every reply from East, either* $\mathrm{val}(D') > q$*, or* $\mathrm{val}(D') = q$ *and East gets the lead.*

Proof. We have $q \in L(D)$. Hence

$$\mathrm{maxmin}(D) \geq (E, q).$$

□

Lemma 9.4. *Suppose that D is a single-suit deal such that*

$$\mathrm{val}(D) = \frac{a}{2^k},$$

where a is an odd integer and k is an integer greater than 1. Then West has a lead such that regardless of East's reply, either $\mathrm{val}(D') > (a-1)/2^k$*, or* $\mathrm{val}(D') = (a-1)/2^k$ *and East gets the lead.*

Proof. Since a is odd, the number $(a-1)/2^k$ can be written with a denominator of at most 2^{k-1}. This number is therefore simpler, and smaller, than $\mathrm{val}(D)$. It follows that $(a-1)/2^k \in L(D)$. Hence $\mathrm{maxmin}(D) \geq (E, (a-1)/2^k)$. □

Lemma 9.5. *Suppose that D is a nonempty single-suit deal, and that* $\mathrm{val}(D) = k/2$*, where k is an integer. Then West has a lead such that for every reply from East, either* $\mathrm{val}(D') > (k-1)/2$*, or* $\mathrm{val}(D') = (k-1)/2$ *with East on lead.*

Proof. Since there can be at most one half-integer in $I(D)$, $(k-1)/2$ must belong to $L(D)$. Hence $\mathrm{maxmin}(D) \geq (E, (k-1)/2)$. □

The following four lemmas concern the situation when East has the lead.

Lemma 9.6. *Suppose that D is a single-suit deal, and that* $\mathrm{val}(D) = q + \varepsilon_0$*. Then on any lead from East, West can make sure that either* $\mathrm{val}(D') > q$*, or* $\mathrm{val}(D') = q$ *with East still on lead.*

Proof. We have $q \notin R(D)$. This means that $\mathrm{minmax}(D) > (W, q)$. □

Lemma 9.7. *Suppose that D is a single-suit deal with value $q + \varepsilon_k$, where q is a number and k is a positive integer. Then on any lead from East, West can make sure that either* $\mathrm{val}(D') \geq q + \varepsilon_k$*, or* $\mathrm{val}(D') = q + \varepsilon_{k-1}$ *with West on lead.*

Proof. Since $q + \varepsilon_{k-1}$ is smaller and simpler than $\mathrm{val}(D)$, it belongs to $L(D)$. In particular, $\mathrm{minmax}(D) \geq (W, q + \varepsilon_{k-1})$. □

Lemma 9.8. *Suppose that D is a single-suit deal with value $q - x$, where q is a number and x is a positive infinitesimal. Then on any lead from East, West can make sure that either* $\mathrm{val}(D') > q - x$*, or* $\mathrm{val}(D') = q - x$ *with West on lead.*

Proof. Since the number q is greater and simpler than $q-x$, it belongs to $R(D)$. On the other hand $q - x \notin R(D)$. Hence $\operatorname{maxmin}(D) > (E, q-x)$. □

Lemma 9.9. *Suppose that D is a single-suit deal whose value is a number q. Then on any lead from East, West can make sure that either $\operatorname{val}(D') > q$, or $\operatorname{val}(D') = q$ with East still on lead.*

Proof. We have $q \notin R(D)$. Hence $\operatorname{minmax}(D) > (W, q)$. □

9.2. **Proof of the main theorem.** We now put together the results of the eight single-suit lemmas, to obtain a proof of Theorem 9.1. It suffices to prove the second (refined) assertion. The proof uses induction on the total number of cards. With the convention $\chi'(n) = (n-1/4, n+1/4)$ for every integer n, the statement of Theorem 9.1 is true for the integers, that is, for states with no cards. Suppose that E is a nonempty multi-suit deal, and suppose that the statement of Theorem 9.1 has been established for every deal with fewer cards. Suppose first that West has the lead. Then we have to prove that if m is an integer, and $\operatorname{val}(E) \geq m/2 \pm \varepsilon_0$, then West can score at least $m/2 + 1/4$ in refined whist.

Suppose first that there is a single-suit component D of E whose value has negative infinitesimal part. Then by Lemma 9.2, West can lead a card of this suit such that either $\operatorname{val}(E') > \operatorname{val}(E)$, or $\operatorname{val}(E') = \operatorname{val}(E)$ with West still on lead. By induction, West will get at least $m/2 + 1/4$ points.

Next suppose that there is no suit whose value has negative infinitesimal part, but that there is a suit D whose value has positive infinitesimal part, say $q + x$, where q is a number and x is a positive infinitesimal. Then the value of E, being the sum of the values of the single-suit components, must also have positive infinitesimal part. Hence $\operatorname{val}(E) \geq m/2 + \varepsilon_0$. By Lemma 9.3, West can lead a card from the suit D such that either $\operatorname{val}(D') > q$, or $\operatorname{val}(D') = q$ with East on lead after the first trick. For the compound deal E this means that either $\operatorname{val}(E') > m/2$, or $\operatorname{val}(E') = m/2$ with East on lead. By induction, West can score at least $m/2 + 1/4$.

It remains to consider the case that every single-suit component of E has a value which is a number. In this case the value of E is also a number. Suppose first that there is a single-suit component whose value has a denominator of at least 4. Let D be a suit whose value has maximal denominator among all the single-suit components of E, say 2^k. Then the value of E can also be written as a fraction with a denominator of 2^k. In order for the hypothesis to be satisfied, the value of E must therefore be at least $m/2 + 1/2^k$. By Lemma 9.4, West can lead a card from the suit D such that the value of D decreases by at most $1/2^k$, with East getting the lead if the decrease is as much as

$1/2^k$. Hence either $\text{val}(E') > m/2$, or $\text{val}(E') = m/2$ with East on lead after the first trick. By induction, West scores at least $m/2 + 1/4$.

Now consider the case that every suit in E has a value which is half an integer. Then for the hypothesis to hold, the value of E must be at least $m/2 + 1/2$. It therefore suffices to show that West can lead a card such that the value of the suit led decreases by at most $1/2$, with East getting the lead if the decrease is as large as $1/2$. This is exactly what Lemma 9.5 tells us.

We have completed the case that West is on lead, and we turn to the case that East is on lead. We have to show that if the value of E is at least $m/2$, then West can score at least $m/2 + 1/4$.

Suppose first that East leads a card from a suit D with value $q + \varepsilon_0$. Then the value of E cannot be $m/2$, so it must be at least $m/2 + \varepsilon_0$. By Lemma 9.6, West has a reply such that either $\text{val}(D') \geq q + \varepsilon_0$, or $\text{val}(D') = q$ with East still on lead. This implies that either $\text{val}(E') \geq m/2 + \varepsilon_0$, or $\text{val}(E') = m/2$ with East on lead after the first trick. By induction, West will score at least $m/2 + 1/4$.

Suppose now that East leads a card from a suit D with value $q + \varepsilon_k$, for some positive integer k. Since there is a suit with such a value, the value of E cannot be $m/2$ or $m/2 + \varepsilon_0$, and must therefore be at least $m/2 + \varepsilon_1$. By Lemma 9.7, West has a reply such that either $\text{val}(D') \geq q + \varepsilon_k$, or $\text{val}(D') = q + \varepsilon_{k-1}$ with West on lead. In the first case, $\text{val}(E') \geq \text{val}(E)$, and in the second case, the value of E' will be at least $m/2 \pm \varepsilon_0$. In both cases it follows by induction that West will score at least $m/2 + 1/4$.

Suppose now that East leads a card from a suit D with negative infinitesimal part. Again the value of E cannot be a number. By Lemma 9.8, West can make sure that either the value of D increases, or is stays the same with West getting the lead. By induction, West will get at least $m/2 + 1/4$ points.

Finally suppose East leads a card from a suit whose value is a number. By Lemma 9.9, West can then make sure that either the value of this suit increases, or it stays the same with East still on lead. This means that either $\text{val}(E') > m/2$, or $\text{val}(E') = m/2$ with East on lead after the first trick. By induction, West will score at least $m/2 + 1/4$.

Since again the corresponding inequalities in the opposite direction follow from interchanging the roles of East and West, the proof of Theorem 9.1 is complete.

9.3. **Some remarks.**

Corollary 9.10. Theorem 3.1 is true. Hence all statements labeled *Consequence* are correct.

Proof. For every state D, we define $N(D)$ to be the numerical part of $\text{val}(D)$. Since val is a semigroup homomorphism, so is N. In particular, (A) is satisfied. By Theorem 9.1, so is (B). ☐

We remark that the operation of rounding a number to the nearest integer occurs naturally also in classical combinatorial game theory. If a partizan game is played out, until one player runs out of moves, then the final position is an ordinal, or the negative of an ordinal. In particular, the final position of a short game is always an integer. Therefore, it is possible to introduce *stopping conditions* different from the so called normal playing convention.

We can introduce a stopping rule by which the game ends as soon as it reaches a position equal to an integer. This integer is then considered to be the outcome of the game. Left is trying to maximize the outcome, while Right is trying to minimize it. It can be shown that the outcome of a game under optimal play depends only on the equality-class of the game, and not on its form.

The outcome of a number with this playing convention is computed in the same way as the outcome of symmetric whist, by rounding to the nearest integer. If the number is half an odd integer, then it should be rounded in favor of the player who is not making the first move.

This playing convention does not take into account whether the last move of the game was made by Left or by Right. To incorporate this into the game, we can give a bonus of $1/4$ for making the last move. This way, the outcome will be positive if Left makes the last move, and negative if it is Right who makes the last move, thus reflecting the outcome under the normal playing convention. In this “refined” game, we compute the outcome by using the rounding function ρ', rounding to the nearest number with a denominator of exactly 4.

10. Algebraic structure and strategy of the game

10.1. **Values of some single-suit deals.** We are now in a position to rigorously establish the values of some special single-suit deals.

Lemma 10.1.

$$\mathrm{val}([K,\ A]) = \varepsilon_0.$$

Proof. The only reduction of this deal is to zero. Hence minmax([K, A]) = maxmin([K, A]) = $(E,\ 0)$. It follows that $L([\mathrm{K,\ A}]) = \{\alpha \in V : \alpha \leq 0\}$, while $R([\mathrm{K,\ A}])$ is the set of all values with positive numerical part. Hence

$$I([\mathrm{K,\ A}]) = \{\pm\varepsilon_0, \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots\},$$

the set of all nonnegative nonzero infinitesimals, of which ε_0 is the simplest. □

This can be generalized to:

Theorem 10.2. *If D is a nonempty single-suit deal, and every card on East's hand is higher than every card on West's hand, then* $\mathrm{val}(D) = \varepsilon_0$.

Proof. There is essentially only one reduction of D, and supposing that $|D| \geq 2$, this reduction is to a deal of the same type. By induction, we can assume that the value of the reduction is ε_0. Hence minmax(D) = maxmin$(D) = (E, \varepsilon_0)$. It follows that the sets $L(D)$ and $R(D)$ are equal to $L([\text{K, A}])$ and $R([\text{K, A}])$ respectively, and therefore that

$$\text{val}(D) = \text{val}([\text{K, A}]) = \varepsilon_0.$$

□

Theorem 10.3. *Let D be a single-suit deal in which West holds the k highest cards, and where every other card on West's hand is smaller than every card on East's hand. Provided that West holds the smallest card, we have*

$$\text{val}(D) = k + \varepsilon_k.$$

Proof. We prove this by induction on k. The case $k = 0$ is equivalent to Theorem 10.2. If $k \geq 1$, there are essentially two different reductions of D. If West plays a high card, the deal reduces to 1 plus a deal of the same type with k replaced by $k - 1$. By induction we can assume that the value of such a deal is $1 + (k - 1) + \varepsilon_{k-1} = k + \varepsilon_{k-1}$. If West plays a small card, then either the hand reduces to a hand of the same type with one fewer small card on West's hand, or to a deal where all West's cards are high. In the former case, we can assume, by making a simultaneous induction on the total number of cards, that the value of the reduction has already been evaluated to $k + \varepsilon_k$. In the latter case, we can use Theorem 10.2 with the East and West hands switched. Since only k cards remain, and the deal has value ε_0 from East's perspective, the value must be $k - \varepsilon$. We can conclude that maxmin(D) = minmax$(D) = (W, k + \varepsilon_{k-1})$ or $(E, k + \varepsilon_k)$ depending on whether West has one or more small cards. In any case,

$$L(D) = \{\alpha \in V : \alpha < k + \varepsilon_k\},$$

while $R(D)$ is the set of all values with numerical part greater than k. Hence

$$I(D) = \{k + \varepsilon_k, k + \varepsilon_{k+1}, k + \varepsilon_{k+2}, \dots\}.$$

The simplest value in $I(D)$ is $k + \varepsilon_k$. □

Theorem 10.4. *Let*

$$D_1 = [\text{AQ}\,,\ \text{KJ}]\,, D_2 = [\text{AK10}\,,\ \text{QJ9}]\,, D_3 = [\text{AKQ8}\,,\ \text{J1097}]\,,$$

etc. With the standard numbering, D_k is the $k+1$-card deal in which West holds $2, k+3, k+4, \dots, 2k+2$, and East holds $1, 3, 4, 5, \dots, k+2$. Then

$$\text{val}(D_k) = k + 1/2^k.$$

Proof. We have $\chi(D_1) = (1, 2)$. By Theorem 9.1, it follows that val$(D_1) = 1 + 1/2$. Consider D_k, for $k \geq 2$. There are essentially four different reductions. If both players play their highest cards (or any equivalent cards), then the deal reduces to 1 plus a deal equivalent to D_{k-1}. By induction on k, we can assume that this reduction has value $1+(k-1)+1/2^{k-1}=k+1/2^{k-1}$. If West plays a high card and East plays his smallest card, then by Theorem 10.3, the value of the reduction is $1+(k-1)+\varepsilon_{k-1}=k+\varepsilon_{k-1}$. If West plays his smallest card, then the deal reduces to a deal where all West's cards are high. The value of this deal is $k-\varepsilon_0$ or $k+1-\varepsilon_0$ depending on whether East played one of his higher cards or not. It follows that

$$\operatorname{maxmin}(D_k)=\max\left((E,k-\varepsilon_0),(W,k+\varepsilon_{k-1})\right)=(W,k+\varepsilon_{k-1}),$$

and that

$$\begin{aligned}(23)\quad \operatorname{minmax}(D_k)&=\min\big((W,k+1-\varepsilon_0),(W,k+1/2^{k-1})\big)\\ &=(W,k+1/2^{k-1}).\end{aligned}$$

Hence $I(D)$ is the set of values whose numerical part is strictly between k and $k+1/2^{k-1}$. The simplest of these values is the number $k+1/2^k$. □

Similarly one can prove that

$$\begin{aligned}\operatorname{val}([\text{K Q 9}, \text{A J 10}])&=1+1/4\\ \operatorname{val}([\text{K Q J 7}, \text{A 10 9 8}])&=2+1/8\\ \operatorname{val}([\text{K Q J 10 5}, \text{A 9 8 7 6}])&=3+1/16\end{aligned}$$

and so on.

10.2. **$S/\!\equiv$ is isomorphic to V.** The introduction of values in Section 6 was motivated only informally by examples of playing technique. We now show that the semigroup of values is isomorphic to the semigroup of equivalence classes of states. This justifies our construction of values, since it shows that it is the simplest structure for which a statement like Theorem 9.1 can hold. Our construction is just as complicated as it needs to be to characterize the game.

Theorem 10.5. *The function* val : $S\to V$ *is surjective.*

Proof. Theorem 10.4 shows that the values $1/2^k$, for positive integers k, all occur as values of states. It follows that all numbers are values of states. By Theorems 10.2 and 10.3, we have

$$\begin{aligned}\operatorname{val}([\text{K}, \text{A}])&=\varepsilon,\\ \operatorname{val}([\text{A}, \text{K}])&=1-\varepsilon,\\ \operatorname{val}([\text{A J}, \text{K Q}])&=1+\varepsilon_1,\\ \operatorname{val}([\text{K Q}, \text{A J}])&=1-\varepsilon_1.\end{aligned}$$

By combining integers and suits with these distributions, all infinitesimals can be obtained. It follows that all values occur as values of states. □

Here we have allowed states with negative integral part. The problem of characterizing the values that occur as values of states with integer part zero seems much more complicated. Obviously no negative values can then occur. Moreover, there are many positive values that can easily be seen not to occur. For instance, it is clear that no value between ε_0 and $1/2$ occurs: Either there is a card on West's hand which is higher than some card on East's hand, and then West will be able to take at least one trick if East has the lead, making the value of the deal at least $1/2$, or all East's cards are high, in which case the value of the deal is ε_0.

Theorem 10.6. *The function* val $: S \to V$ *factors through the equivalence relation* $\equiv$*. In other words, equivalent states have the same value.*

In view of Theorem 10.5, it suffices to prove the following statement:

Lemma 10.7. *If x and y are distinct values, then there is a value z such that $\rho(x+z) \neq \rho(y+z)$.*

We prove this for general rounding functions, assuming only that there are at least two distinct numbers to round to.

Lemma 10.8. *Let A be a discrete set of numbers containing at least two elements. If x and y are distinct values, then there is a value z such that $\rho_A(x+z) \neq \rho_A(y+z)$.*

Proof. Let a and b be distinct elements of A with $a < b$, and let $m = (a+b)/2$. If x and y have distinct numerical parts, then there is a number q between them. If we take $z = m - q$, then $x + z$ and $y + z$ will be rounded in different directions. Suppose therefore that x and y have the same numerical part, say r. If $x - (m - r)$ and $y - (m - r)$ are rounded to the same number, then the infinitesimal parts of x and y must either both be positive, or both negative. Suppose they are both positive. Then there are distinct nonnegative integers k and l such that $x = r + \varepsilon_k$ and $y = r + \varepsilon_l$. If we let $z = m - r - \varepsilon_k$, then $\rho_A(x+z) = (b, a)$, while $\rho_A(y+z) = (a, a)$ or (b, b) depending on whether k is greater or smaller than l. □

Theorem 10.9. *The mapping* val $: S/\!\equiv\, \to V$ *is injective. In other words, if two states have the same value, then they are equivalent.*

Proof. Let D, E and F be states, and suppose D and E have the same value. Then $D + F$ and $E + F$ have the same value. By Theorem 9.1, $D + F$ and $E + F$ have the same outcome. Since this holds for every F, we have $D \equiv E$. □

Theorem 10.10. *The function* val *is well-defined on equivalence classes of states, and provides an isomorphism between $S/\!\equiv$ and V.*

Proof. This follows from Theorems 10.5, 10.6 and 10.9. □

The fact that the function val : $S \to V$ provides an isomorphism between $S/\equiv$ and V whether we take whist or refined whist as the basis for equivalence between states shows that these two games are equivalent in a very precise and natural sense: Two states are equivalent, or one is smaller than the other, with respect to one of the games precisely when the same relation holds with respect to the other game.

10.3. **Normalization of scores and values.** We can introduce yet another rule for scoring which is symmetric around the origin, along with a new mapping of states to values, such that the integers are the unique simplest values. This is done by scoring $+1$ for each trick that West wins, and -1 for each trick that East wins, and then adding a bonus of $+1/2$ if East wins the last trick, and $-1/2$ if West wins the last trick. The effect of this is that the scores of refined whist are scaled up by a factor 2, and centered at the origin, but the game is essentially the same. The outcome of *normalized whist* is always half an odd integer. We can introduce another rounding function $\rho_{(\mathbf{Z}+1/2)}$ that rounds to the nearest number of this kind, with the usual tiebreak rules. Then for every n card deal D with $\mathrm{val}(D) = q + x$, we let

$$f(D) = 2q - n + x.$$

The outcome of normalized whist is $\rho_{(\mathbf{Z}+1/2)}(f(D))$. In this setting, there is a natural way of defining the negative of a state. If D is any state, $-D$ is obtained by interchanging the East and West hands, and changing the sign of the integer part of D. The function f now has the rather nice property that

$$f(-D) = -f(D).$$

10.4. **The structure of an optimal strategy.** From the single-suit lemmas, we can make a few observations. The optimal strategy which is implicit in the proof of Theorem 9.1 uses the following method for choosing a lead:

First, choose the suit to lead from. The suit is chosen as follows:

(i) If there is a suit whose value has negative infinitesimal part, choose any such suit.
(ii) If there is no suit whose value has negative infinitesimal part, but at least one suit whose value has positive infinitesimal part, choose any suit with positive infinitesimal part.
(iii) If the value of every suit is a number, then choose any suit whose value is at least as complex (that is, has at least the same denominator) as the value of the whole deal.

Second, choose a card to lead from that suit. When choosing a card to lead, only the local information of the card distribution in that suit is taken into account. Hence from a given single-suit card distribution, it is possible always to make the same lead. For instance, we can decide that when leading from [A Q, K J], we always lead the ace, while from [A J 10 7 , K Q 9 8], we always lead the jack.

From the way we choose a suit to lead from, we see that the so called Number Avoidance Theorem of combinatorial game theory [1] is valid also for whist. In [1], this theorem is stated as:

Never move in a Number, unless there is Nothing else to do.

This should be interpreted in the weak sense that it is possible to play optimally according to this rule. In the case of whist, we have the following theorem:

Theorem 10.11. *If D and E are states such that* $\mathrm{val}(D)$ *is a number, but* $\mathrm{val}(E)$ *is not, then there is a card in E which is an optimal lead from $D + E$.*

Hence the rule:

Never lead from a suit whose value is a Number, unless there is Nothing else to do.

The following examples show that the best reply to a lead may depend not only on the distribution in the suit that was led, but also on the distribution of the cards in the other suits. If East leads the spade king from

(24)

West:	*East*:
♠ A J 9	♠ K Q 10

then West should hold off in order to take the next two tricks. Suppose instead that East leads the spade king from

(25)

West:	*East*:
♠ A J 9	♠ K Q 10
♡ A	♡ K
♢ K	♢ A

If West plays low, East can cash the diamond ace and exit in hearts. This holds West down to two tricks. On the other hand if West takes the first trick with the spade ace, he can cash his heart trick and exit in diamonds, for a total of three tricks.

The flaw of this example is that East's spade lead is incorrect. East should immediately eliminate diamonds and exit in hearts. This will always hold West down to two tricks. If West decides to duck whenever East leads spades, he will still get as many tricks as he was entitled to in the first place. The author does not know whether there is an example where, after an optimal lead from East, West may have to look at the other suits before deciding on his reply.

11. Example deals

11.1. **Values in actual play.** We give some examples which are a little more complicated than those given elsewhere in the paper, and where our theory can be used to find an optimal strategy, and to prove its optimality. We do not prove our assertions about the values of the suits involved, but in most cases, these can quickly be verified using Theorem 8.10 and the theorems of Section 10.1. We also refer to the tables of the appendix.

11.2. **A numerical deal.** We first consider a deal whose value is a number. This deal shows that it can sometimes be necessary to play low to a lead, even if this doesn't change the number of tricks obtained in the suit that is being led.

	West :	*East* :
(26)	♠ A J 9 8	♠ <u>K</u> Q 10 7
	♡ A K 10	♡ Q J 9
	♢ K Q 9	♢ A J 10

Consider the deal (26). East leads the spade king. We can evaluate the spade suit to 2, the heart suit to $2 + 1/4$, and the diamonds to $1 + 1/4$. This sums to $5 + 1/2$, and since East has the lead, we expect West to be able to take six tricks.

If West takes the first trick with the ace of spades, then the spade suit reduces to $1 +$ [J 9 8, Q 10 7], which has the value 2. The problem is that West gets the lead, so on the deal as a whole, West will now only be able to take five tricks. On the other hand, if West plays low in the first trick, the spade suit reduces to [A J 9, Q 10 7]. This still has the value 2, but now East is on lead, which means that West will be able to take six tricks.

Whenever East plays another spade, West can cash two spade tricks and exit with the last spade. This way, East will have to make the first move in the red suits. If East leads the queen of hearts, West takes the trick and returns the diamond king. If instead East leads a small diamond, West takes the trick and clears the hearts with ace, king, and ten. No matter what East does, West will either get three heart tricks or two diamond tricks.

Here the spade suit produces two tricks almost regardless of how the cards are played. Still it is essential that West plays low in the first trick in order to get four tricks in the other suits.

11.3. **Exiting.** Our next deal (27) is an example of "exiting" in a suit with positive infinitesimal part, when there is no suit with negative infinitesimal part, in order to force the opponent to be the first to lead a suit with numerical value.

(27)

West :	*East* :
♠ A K Q 8	♠ J 10 9 7
♡ A Q 8 7	♡ K J 10 9
♢ A J 10 7	♢ K Q 9 8
♣ K Q 9	♣ A J 10

We evaluate the spades to $3+1/8$, the hearts to $2+\varepsilon_2$, the diamonds to $2+1/8$, and the clubs to $1+1/4$. This sums to $8+1/2+\varepsilon_2$, which rounds to 9. Hence West should be able to take nine tricks regardless of the position of the lead.

If West has the lead, there is only one way to accomplish this. A lead in either of the "numerical" suits, spades, diamonds or clubs, will weaken that suit. Instead, West has to exit by playing a small heart. When East takes the trick with the nine of hearts, the value of the heart suit drops to 2, but East gets the lead.

We can then distinguish two main lines: If East leads the spade jack, West will take the trick and continue with a middle diamond (nothing else will do!), while if East leads the diamond king, West will take the trick and play four rounds of spades, putting East on lead with the last one. We leave the rest of the analysis for the reader.

11.4. **A race.** Finally, we give an example of a "hot" deal where both sides have exits, and the first phase of the game is a race to knock out the opponent's stoppers. West to play and take eleven tricks in (28).

(28)

West :	*East* :
♠ K Q 10 9 8	♠ A J 7 6 5
♡ A K 9	♡ Q J 10
♢ A J 9 7 5 3	♢ K Q 10 8 6 4
♣ K Q 10 8 6	♣ A J 9 7 5

The spades actually provide an exit for East, with two stoppers. Hence this suit is worth $3-\varepsilon_2$. The hearts provide an exit for West, with value $2+\varepsilon_2$. It can be shown, for example with the methods of [7], that the outcome of the club suit played separately is $(2,3)$. It follows that the value of the clubs is $2+1/2$. Then by Theorem 8.10, the diamonds have value 3. This gives a total of $10+1/2\pm\varepsilon_0$. West should therefore be able to score eleven tricks, provided he has the lead.

When both sides have exits, the correct strategy is to attack the opponent's exit suits, that is, the suits whose values have negative infinitesimal part. In this case the only way for West to take eleven tricks is to lead a spade. On the other hand, it doesn't matter which one. West should not be afraid of giving East a cheap trick for the spade jack, since there is no way to prevent this anyway. East's natural defense is to play hearts consistently, but West is one step ahead, and will be able to clear the spades before exiting with the nine of hearts.

Now the character of the game changes from a hot race of knocking out stoppers to a cold game of numbers. East has the lead, and may set up a trap by leading the diamond king. West is certain to get three diamond tricks whether he takes the first one or not, but if he falls for the temptation and wins the first diamond trick with the ace, East will secure three club tricks by ducking when West leads the king. Instead, West has to let East win the first diamond trick. After that he will easily get eleven tricks.

Notice that starting with three rounds of hearts will not do. When East gets the lead in the third trick, he can exit with a small spade. If West then returns the spade (or club) king, East will of course refuse to take it.

12. Miscellaneous remarks

12.1. **Computing the value of a card distribution.** By inspecting the definition of the value of a deal given in Section 7.1, we find that there is an algorithm for computing it. Clearly, with a reasonable representation of the elements of V, addition in V can be computed efficiently. Hence the problem of computing the value of a deal reduces to that of computing the values of its single-suit components. Let D be a single-suit deal. If we assume that the values of all reductions $D_{i,j}$ of D have been computed, we can determine $\text{minmax}(D)$ and $\text{maxmin}(D)$. Then we can find the set $I(D)$. Clearly $\text{val}(D)$ can now be computed. The running time of this algorithm is exponential in the size of D, since the computation of the value of an n-card deal is reduced to the computation of values of n^2 $n-1$-card deals. The branching factor n^2 can be improved to $O(n)$ by using the fact that when responding to a lead, it is optimal either to take the trick as cheaply as possible, or to play low, but the running time is still exponential.

A combinatorial game can be considered solved when a polynomial time algorithm has been found that for any position in the game computes the outcome of the game under optimal play, as well as an optimal move for the player whose turn it is to play. In general, these two computational problems can be reduced to each other in polynomial time (see for example [3]), and are therefore considered equivalent.

In this sense, two-person symmetric whist is not yet solved. However, we strongly believe that an efficient algorithm for computing values exists, and that such an algorithm will soon be found. In support of this claim, we prove the following:

Theorem 12.1. *There is an algorithm running in time $O(n \log n)$ that detects if a single-suit deal has an integral or half-integral value.*

Proof. The proof of this uses an algorithm given in [7] that computes the outcome of single-suit whist in time $O(n \log n)$. Clearly a single-suit deal D has value $n + 1/2$ if and only if its outcome is $(n, n+1)$. By Theorem 8.10, a single-suit deal D has value n if and only if the deal D_0 obtained by removing the smallest card from each hand has outcome $(n-1, n)$. □

Moreover, we know from [7] that if the cards in a single suit of size $2n$ are distributed randomly, then with the notation of [7], the probability that $H(D) = H(D_0)$ will tend to 1 as $n \to \infty$. If $H(D) = H(D_0)$, then either D or D_0 will have an outcome that depends on the position of the lead, and hence a value which is half an odd integer. It follows that the value of D is half of an integer.

Hence we have an algorithm that computes the value of a random single-suit deal quickly with very high probability, and which, if it fails, still gives bounds of the form $n/2 < \mathrm{val}(D) < (n+1)/2$.

12.2. **Values of single-suit deals.** We make a few remarks on the problem of characterizing the values that occur as values of single-suit deals. This question is probably related to the problem of effectively computing the value of a single-suit deal. As mentioned in the previous section, we do not at present have a satisfactory answer to this problem. The sequence of card distributions given in Theorem 10.4 provides, for every k, a single-suit deal whose value has fractional part $1/2^k$.

The following example, discovered through a computer search, shows that a fractional part not on the form $1/2^k$ or $1 - 1/2^k$ is also possible:

$$[13\ 12\ 9\ 8\ 7\ 2\ 1\ ,\ 14\ 11\ 10\ 6\ 5\ 4\ 3]$$

To verify that this deal has value $3 + 3/8$, we can play the sum

$$[13\ 12\ 9\ 8\ 7\ 2\ 1\ ,\ 14\ 11\ 10\ 6\ 5\ 4\ 3] + [\mathrm{A\ K\ Q}\ 8\ ,\ \mathrm{J}\ 10\ 9\ 7]\,.$$

By considering the various lines of play, we can convince ourselves that West will get 6 tricks with the lead, and 7 tricks with East on lead. Hence the sum has value $6 + 1/2$. Since [A K Q 8 , J 10 9 7] is known to have value $3 + 1/8$, it follows that the other term has value $3 + 3/8$.

We now present two conjectures, both supported by computer analysis of all single-suit deals with at most 10 cards on each hand:

Conjecture 12.2. *If the value of a single-suit deal is of the form $q + x$, where q is a number and x is a positive infinitesimal, then q must be a nonnegative integer, and $x = \varepsilon_q$.*

We could easily prove this by induction if we could prove that we cannot have $\mathrm{maxmin}(D) = \mathrm{minmax}(D) = (E, q)$ unless $q = 0$. In other words, this conjecture boils down to proving that no "new" exits are created from suits with numerical value after [K , A], so that all suits with positive infinitesimal part are descendants of the ε_0 created on day one.

If this conjecture is true, then suits with non-numerical values can indeed be described using the concept of stoppers, as is implicitly suggested in the discussion in Section 4

Conjecture 12.3. *If the value of a single-suit deal is a number which is not an integer, then the fractional part of this number must be of the form* $1/2^k$, $1 - 1/2^k$, $3/8$ *or* $5/8$.

We also present another conjecture, whose solution seems to require a deeper understanding of the values of single-suit deals. The single-suit case of this conjecture was proved in [7].

Conjecture 12.4. *If the opponent leads a card that we can beat with a card smaller than the highest remaining card in the suit, then it is always optimal to do so. Hence with the standard numbering of the cards, the only situation where it can be necessary to refuse to take a trick is when we have the ace, and the opponent leads from a sequence containing the king and queen.*

12.3. **The non-absoluteness of values in whist.** This section deals with an aspect of the theory of whist that makes it fundamentally different from the classical theory of combinatorial games.

The classical theory deals with games played under the *normal playing convention*. With this convention, the move-order is alternating, and the winner of a game is the player who makes the last move.

With the normal playing convention, every game G has an inverse $-G$ with the property that $G + (-G)$ is a second player win. The relation $H \leq G$ is then defined as meaning that $G + (-H)$ is a win for Left if Right makes the first move. This means that it is a property intrinsic to the games G and H. If we are given two games, we can determine whether one of them is greater than the other simply by computing the outcome of their difference. Similarly, the games G and H are considered equivalent (in [2] they are even said to be *equal*), if and only if $G + (-H)$ is a second player win.

It is easy to prove from these definitions that G and H are equal in this respect if and only if for every game K, $G + K$ has the same outcome under optimal play as $H + K$.

Since in general a whist deal has no additive inverse, we have taken the latter property as our definition of equivalence. Hence we have defined two card distributions D and E to be equivalent if for every card distribution F, $D + F$ has the same outcome as $E + F$. This means that in order to prove that two deals are equivalent, we potentially have to investigate an infinite set of other deals. Such a problem can therefore be difficult even if the two deals in question are simple. Indeed, we had to do a certain amount of work to prove for instance that [A Q 9 , K J 10] $\equiv$ [A Q J , K 10 9], and that [A , K] > 0.

For this reason, we have no general methods for rigorously establishing algebraic properties and order relationships in the quotient semigroup $S/\equiv$ until we have constructed the semigroup V and the mapping val.

Somewhat surprisingly, it was much easier to prove that $[\mathrm{K}\,,\ \mathrm{A}] > 0$. We make a few remarks on why some statements about the ordering of deals can be proved easily by strategy-stealing, while other statements are more difficult and seem to require the development of a general theory of the game. The former are statements that hold universally, regardless of the class of games under consideration, while the latter depend on restrictions like for example the assumption that the relative rank of two cards is determined by a total ordering of the cards within a suit.

Suppose for example that in a particular suit S, there are four cards which beat each other cyclically, in a scissors-paper-stone-like way. West holds the cards a and c, while East holds b and d, and a beats b which beats c which beats d which beats a. If this suit is played on its own, the player not on lead will take both tricks. This already shows that such a suit cannot be assigned a value and incorporated in the theory of symmetric whist. Moreover, if such a situation is admissible, then it is no longer true that a higher card is at least as good as a smaller one. Indeed, we can disprove the statement that $[\mathrm{A}\,,\ \mathrm{K}] \geq 0$ by adding S to both sides. The outcome of $[\mathrm{A}\,,\ \mathrm{K}] + S$ is $(1,1)$ while the outcome of $0 + S$ is $(0,2)$.

In the same way, we can prove that $[\mathrm{A\ Q\ 9}\,,\ \mathrm{K\ J\ 10}]$ is no longer equivalent to $[\mathrm{A\ Q\ J}\,,\ \mathrm{K\ 10\ 9}]$. If East is on lead in $[\mathrm{A\ Q\ 9}\,,\ \mathrm{K\ J\ 10}]+S$, he will get only one trick, while in $[\mathrm{A\ Q\ J}\,,\ \mathrm{K\ 10\ 9}] + S$, he can get two tricks by sacrificing the king under West's ace.

However, the assumption that cards within a suit are totally ordered is not necessary for the strategy-stealing arguments that prove for instance that

$$[\mathrm{K}\,,\ \mathrm{A}] \geq 0$$

and that

$$[\mathrm{A\ Q\ J}\,,\ \mathrm{K\ 10\ 9}] \leq 2 \leq [\mathrm{A\ Q\ 9}\,,\ \mathrm{K\ J\ 10}]\,.$$

These relations therefore seem to be intrinsic to the games in question.

Hence we can distinguish between on the one hand "absolute" statements provable by strategy-stealing arguments requiring analysis of only the card distributions involved in the actual statement, and on the other hand "non-absolute" statements that require a general theory of the game.

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Appendix: Table of values of single-suit deals

The following is a table of values of single-suit deals with up to five cards, where East has the ace. The values of the single-suit deals where West has the ace can be obtained through the identity $\mathrm{val}(D) = |D| - \mathrm{val}(\overline{D})$.

West	East	value
K	A	ε_0
Q J	A K	ε_0
K J	A Q	$1/2$
K Q	A J	$1-\varepsilon_1$
J 10 9	A K Q	ε_0
Q 10 9	A K J	$1/2$
Q J 9	A K 10	$3/4$
Q J 10	A K 9	$1-\varepsilon_2$
K 10 9	A Q J	1
K J 9	A Q 10	1
K J 10	A Q 9	1
K Q 9	A J 10	$1+1/4$
K Q 10	A J 9	$1+1/2$
K Q J	A 10 9	$2-\varepsilon_1$

Table 1. Values of deals with up to three cards.

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West	East	value		West	East	value
10 9 8 7	A K Q J	ε_0		K 10 9 8	A Q J 7	$1+1/2$
J 9 8 7	A K Q 10	$1/2$		K J 8 7	A Q 10 9	$1+1/2$
J 10 8 7	A K Q 9	$3/4$		K J 9 7	A Q 10 8	$1+1/2$
J 10 9 7	A K Q 8	$7/8$		K J 9 8	A Q 10 7	$1+1/2$
J 10 9 8	A K Q 7	$1-\varepsilon_3$		K J 10 7	A Q 9 8	$1+1/2$
Q 9 8 7	A K J 10	1		K J 10 8	A Q 9 7	$1+3/4$
Q 10 8 7	A K J 9	1		K J 10 9	A Q 8 7	$2-\varepsilon_2$
Q 10 9 7	A K J 8	1		K Q 8 7	A J 10 9	$1+1/2$
Q 10 9 8	A K J 7	1		K Q 9 7	A J 10 8	$1+3/4$
Q J 8 7	A K 10 9	$1+1/4$		K Q 9 8	A J 10 7	$1+7/8$
Q J 9 7	A K 10 8	$1+1/2$		K Q 10 7	A J 9 8	2
Q J 9 8	A K 10 7	$1+1/2$		K Q 10 8	A J 9 7	2
Q J 10 7	A K 9 8	$1+1/2$		K Q 10 9	A J 8 7	2
Q J 10 8	A K 9 7	$1+3/4$		K Q J 7	A 10 9 8	$2+1/8$
Q J 10 9	A K 8 7	$2-\varepsilon_2$		K Q J 8	A 10 9 7	$2+1/4$
K 9 8 7	A Q J 10	$1+\varepsilon_1$		K Q J 9	A 10 8 7	$2+1/2$
K 10 8 7	A Q J 9	$1+1/4$		K Q J 10	A 9 8 7	$3-\varepsilon_1$
K 10 9 7	A Q J 8	$1+1/2$				

TABLE 2. Values of four-card deals.

West	East	value	West	East	value	West	East	value
98765	AKQJ10	ε_0	QJ976	AK1085	2	KJ1085	AQ976	$2+1/4$
108765	AKQJ9	$1/2$	QJ985	AK1076	2	KJ1086	AQ975	$2+1/2$
109765	AKQJ8	$3/4$	QJ986	AK1075	2	KJ1087	AQ965	$2+1/2$
109865	AKQJ7	$7/8$	QJ987	AK1065	2	KJ1095	AQ876	$2+1/4$
109875	AKQJ6	$15/16$	QJ1065	AK987	2	KJ1096	AQ875	$2+1/2$
109876	AKQJ5	$1-\varepsilon_4$	QJ1075	AK986	2	KJ1097	AQ865	$2+3/4$
J8765	AKQ109	1	QJ1076	AK985	2	KJ1098	AQ765	$3-\varepsilon_2$
J9765	AKQ108	1	QJ1085	AK976	$2+1/4$	KQ765	AJ1098	2
J9865	AKQ107	1	QJ1086	AK975	$2+1/2$	KQ865	AJ1097	2
J9875	AKQ106	1	QJ1087	AK965	$2+1/2$	KQ875	AJ1096	2
J9876	AKQ105	1	QJ1095	AK876	$2+1/4$	KQ876	AJ1095	2
J10765	AKQ98	$1+1/4$	QJ1096	AK875	$2+1/2$	KQ965	AJ1087	$2+1/4$
J10865	AKQ97	$1+1/2$	QJ1097	AK865	$2+3/4$	KQ975	AJ1086	$2+1/4$
J10875	AKQ96	$1+1/2$	QJ1098	AK765	$3-\varepsilon_2$	KQ976	AJ1085	$2+1/4$
J10876	AKQ95	$1+1/2$	K8765	AQJ109	$1+\varepsilon_1$	KQ985	AJ1076	$2+1/4$
J10965	AKQ87	$1+1/2$	K9765	AQJ108	$1+1/4$	KQ986	AJ1075	$2+1/2$
J10975	AKQ86	$1+3/4$	K9865	AQJ107	$1+1/2$	KQ987	AJ1065	$2+1/2$
J10976	AKQ85	$1+3/4$	K9875	AQJ106	$1+3/4$	KQ1065	AJ987	$2+1/4$
J10985	AKQ76	$1+3/4$	K9876	AQJ105	$1+3/4$	KQ1075	AJ986	$2+1/2$
J10986	AKQ75	$1+7/8$	K10765	AQJ98	$1+1/2$	KQ1076	AJ985	$2+1/2$
J10987	AKQ65	$2-\varepsilon_3$	K10865	AQJ97	$1+1/2$	KQ1085	AJ976	$2+1/2$
Q8765	AKJ109	$1+\varepsilon_1$	K10875	AQJ96	$1+3/4$	KQ1086	AJ975	$2+1/2$
Q9765	AKJ108	$1+1/4$	K10876	AQJ95	$1+3/4$	KQ1087	AJ965	$2+1/2$
Q9865	AKJ107	$1+1/2$	K10965	AQJ87	2	KQ1095	AJ876	$2+1/2$
Q9875	AKJ106	$1+1/2$	K10975	AQJ86	2	KQ1096	AJ875	$2+1/2$
Q9876	AKJ105	$1+1/2$	K10976	AQJ85	2	KQ1097	AJ865	$2+3/4$
Q10765	AKJ98	$1+1/2$	K10985	AQJ76	2	KQ1098	AJ765	$3-\varepsilon_2$
Q10865	AKJ97	$1+1/2$	K10986	AQJ75	2	KQJ65	A10987	$2+1/4$
Q10875	AKJ96	$1+1/2$	K10987	AQJ65	2	KQJ75	A10986	$2+1/2$
Q10876	AKJ95	$1+1/2$	KJ765	AQ1098	2	KQJ76	A10985	$2+1/2$
Q10965	AKJ87	$1+1/2$	KJ865	AQ1097	2	KQJ85	A10976	$2+3/4$
Q10975	AKJ86	$1+3/4$	KJ875	AQ1096	2	KQJ86	A10975	$2+3/4$
Q10976	AKJ85	$1+3/4$	KJ876	AQ1095	2	KQJ87	A10965	$2+7/8$
Q10985	AKJ76	$1+3/4$	KJ965	AQ1087	2	KQJ95	A10876	3
Q10986	AKJ75	$1+7/8$	KJ975	AQ1086	2	KQJ96	A10875	3
Q10987	AKJ65	$2-\varepsilon_3$	KJ976	AQ1085	2	KQJ97	A10865	3
QJ765	AK1098	$1+1/2$	KJ985	AQ1076	2	KQJ98	A10765	3
QJ865	AK1097	$1+1/2$	KJ986	AQ1075	2	KQJ105	A9876	$3+1/16$
QJ875	AK1096	$1+3/4$	KJ987	AQ1065	2	KQJ106	A9875	$3+1/8$
QJ876	AK1095	$1+3/4$	KJ1065	AQ987	2	KQJ107	A9865	$3+1/4$
QJ965	AK1087	2	KJ1075	AQ986	2	KQJ108	A9765	$3+1/2$
QJ975	AK1086	2	KJ1076	AQ985	2	KQJ109	A8765	$4-\varepsilon_1$

TABLE 3. Values of five-card deals.