Quasi-tilings

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ABSTRACT. The aim of this paper is to unify the definition of tilings on rectangular and triangular lattices. We first define quasi-tilings which are a simple extension of tilings. These quasi-tilings are considered as words on $\{0,1\}^n$. Then, usual problems on tilings are mapped into problems on permutation on these words. The lattice structure of tilings and so the connectivity of the tiling space under the flip operation is proved by this way.

RÉSUMÉ. Nous prśentons ici une généralisation des pavages. Nous nous intéressons surtout au cas des pavages par des dominos ou des losanges. Nous définissons une bijection entre des mots sur $\{0,1\}$ et nos quasi-pavages. Muni de cette bijection, nous montrons comment les résultats classiques sur les pavages s'expliquent de manière naturelle dans ce nouveau formalisme.

1. Introduction

A traditionnal tiling problem is tiling a picture of the plane with dominoes or rhombus [1, 2, 3, 4]. Furthermore it is interesting to check the connectivity of the tiling configuration space under the flip operation. This operation which corresponds to local moves in the subsequent lattice is the basis for physical tiling issues. These proofs are based on geometric definition of the height function [4, 5, 6]. In this paper, we give a new approach of these questions from a purely algebraic and algorithmic point of view. To each tiling is associated a word and a flip could be seen as a product of two transpositions of letters on these words. With this formal description the notion on connectivity comes from the connectivity of the permutohedron.

2. Quasi-tilings

2.1. **Definitions.** In this section, we will study quasi-tilings on rectangular grids. Most of the examples are taken on even square grids but the extension to rectangular case is straightforward.

Let G_n be the square $n \times n$ grid. A diagonal D_i $(0 \leq i \leq 2n-5)$ of the grid is the set of edges beginning with a vertical step at coordinates $(n-2-i, 0)$, ending with a horizontal step at coordinates $(n-1, i+1)$ and made of up and right steps alternatively for the first $n-2$ diagonals. - see for instance figure 1.

For the last $n-2$ ones, they begin at coordinates $(0, 2n-4-i)$, end at coordinates $(i-n+3, n-1)$, begin with a horizontal step and alternate horizontal and vertical steps.

Put the following orientation on diagonals :

(1) Odd ones are oriented from top right to bottom left
(2) Even ones are oriented from bottom left to top right

A *quasi-tiling* is a map f from the set of internal edges to $\{0,1\}$. We will only consider a subclass of quasi-tilings ie those who have the following property:

$$\sum_{e \in D_i} f(e) = \alpha_i$$

with $(\alpha_0, \ldots, \alpha_{2n-5})$ a fixed vector of $\mathbb{N}^{2n-4}$.

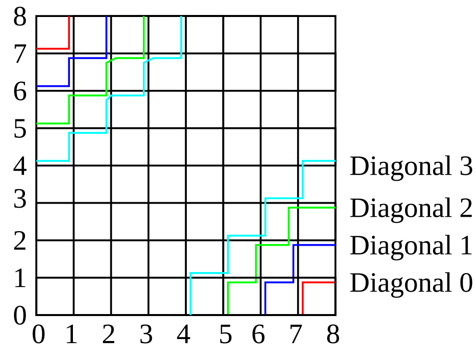

FIGURE 1. Definition of diagonals on a 9×9 grid

For the square case, we will take $\alpha_i = u_i$ for $i \in \{0, \ldots, n-2\}$ and $\alpha_i = u_{2n-5-i}$ otherwise with $(u_i)_{i \in \mathbb{N}}$ defined as follows:

$$\begin{cases} u_{2n} = 1 + 3n \\ u_{2n+1} = 3(n+1) \end{cases}$$

2.2. **Word and graphical representations.** The above definition of quasi-tilings has the drawback and the advantage to avoid the geometric point of view of traditionnal tilings. The translation of our definition into usual tiling standard is to consider a $n \times n$ square and represent the map of each diagonal as follows:

$$\begin{cases} \text{If an edge is mapped into 1 then draw a black line} \\ \text{Otherwise, draw a white line} \end{cases}$$

For example, consider the 5×5 square and the following mapping.

Diagonal	Mapping	Weigth
0	10	1
1	1101	3
2	101110	4
3	101110	4
4	1101	3
5	10	1

The graphical representation of this quasi-tiling is given in figure 2 (a dotted line is a white line).

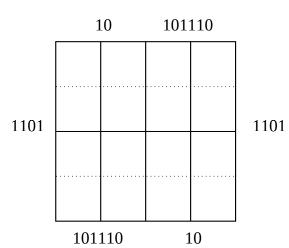

FIGURE 2. A quasi-tiling (also a tiling)

Note that in this case the quasi-tiling is in fact a tiling. But this is not always the case even with the same α vector as the following example shows:

Word representation

Diagonal	Mapping	Weigth
0	10	1
1	1101	3
2	101110	4
3	101110	4
4	1110	3
5	10	1

Graphical representation

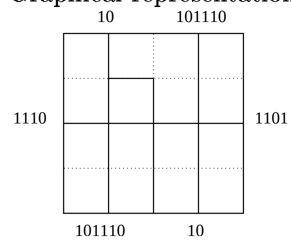

FIGURE 3. A quasi-tiling

2.3. **Quasi-flip.**

2.3.1. *Definition of quasi-flip.* To make a parallel with classical tilings and the usual flip operation, we define a quasi-flip on quasi-tiling. A flip is the exchange of two contiguous horizontal -vertical- dominoes with vertical -horizontal- ones such that they have the same shape as shown in figure 4.

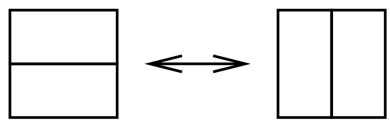

FIGURE 4. Example of flip for domino tiling

For quasi-tilings, consider the mapping of each diagonal as words on $\{0,1\}$ and the quasi-flip operation is a canonical transposition of two adjacent letters. We denote by σ_i the transposition of letters i and $i+1$.

In figure 2, the words associated to the quasi-tiling are:

$$(10, 1101, 101110, 101110, 11\underline{01}, 10)$$

Let's transpose the third and fourth letters of the fifth word. We obtain the following words

$$(10, 1101, 101110, 101110, 11\underline{10}, 10)$$

which is the quasi-tiling of figure 3.

In the graphical representation a quasi-flip can be seen as the exchange of the color of two adjacent edges belonging to the same diagonal.

Note that in this transformation, the weight (ie the number of black edges) of each diagonal is preserved.

2.3.2. *Order on quasi-tilings.* We consider each diagonal as a word in $\{0,1\}$. For each diagonal there is a natural partial order $<_d$ which is given by the transitive closure of the following one :

$$u < v \text{ iff } \begin{cases} \exists \sigma_i \text{ s.t. } v = u\sigma_i \\ \text{Inv}(v) > \text{Inv}(u) \end{cases}$$

where $Inv(u)$ is the number of inversions in u.

As diagonals does not intersect, there is no possible exchange between two diagonals under the quasi-flip action. It is then natural to take the direct product over all diagonals of the above order as partial order on quasi-tilings. We will denote it by $<_q$.

2.3.3. *Quasi-tilings lattice.* It is known that by taking a word $w = a_1^{(i_1)} a_2^{(i_2)} ... a_k^{(i_k)}$ and the canonical transpositions, the graph where vertices are all permutations of w and edges of the form (w_1, w_2) where $w_1 < w_2$ is a distributive lattice. One can see it as the permutohedron where some letters have been identified.

But we can give a proof of this fact by exhibiting the lowest upper bound (and upper lower bound) for two distinct elements x and y. Let $l_k(w)$ be the position of the k^{th} number 1 in w.If $x = x_1 x_2 \dots x_n$ and $y = y_1 y_2 \dots y_n$ with x_i and y_j in $\{0, 1\}$ then the lowest upper bound exists and it is $z = z_1 z_2 \dots z_n$ where $\begin{cases} z_i = 1 \text{ iff } \exists k s.t. \min(l_k(x), l_k(y)) = i \\ z_i = 0 \text{ otherwise} \end{cases}$

It is straightforward to check that this lattice is distributive.

Then, taking $(01, 0111, 001111, 001111, 0111, 01)$ as the first quasi-tiling, the lattice of all permutations obtained by quasi-flips is the direct product of each diagonal lattice. (see e.g. figures 5,6 for the 2×3 rectangle case) Thus, it is a distributive lattice.

FIGURE 5. Lattice of permutations on words for the word 01 and 011

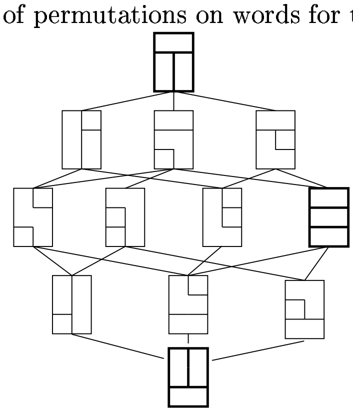

FIGURE 6. Lattice of quasi-tilings on a 2×3 rectangle

Notice that tilings seem to be a sublattice - see figure 7 of the quasi-tiling lattice.

2.3.4. *Height functions on quasi-tilings on grids.* A height function for quasi-tiling is a mapping from V the vertex set into $\mathbb{N}^2$. We first define the height function on the border of the $n \times n$ grid. Start with putting a $(0, 0)$ on lower left corner then make alternatively $(+1, +1)$ and $(-1, -1)$ for the height function at each step except in corners where the last step before the corner as the same value as the first step after it. -ie two increasing steps or decreasing steps around a corner-. See figure 8 for an example.

Odd diagonals act on the first component of the height function and even ones on the second component. To number the diagonals, start from bottom left of each even one and go to the upper right (and conversely for the odd ones). When going from a vertex to an adjacent one along an edge, we use the following rule on the coordinate of the height function:

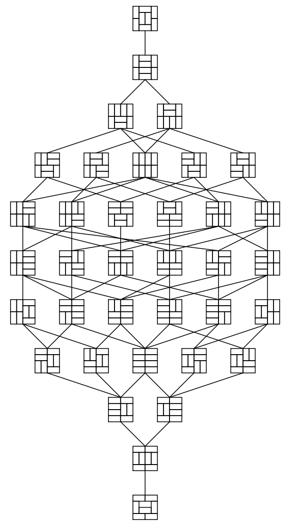

FIGURE 7. Lattice of dominoe tilings on the 4×4 grid

- If the edge is black then subtract 1 to the height
- If the edge is white then add 3 to the height

See figure 8 for an example.

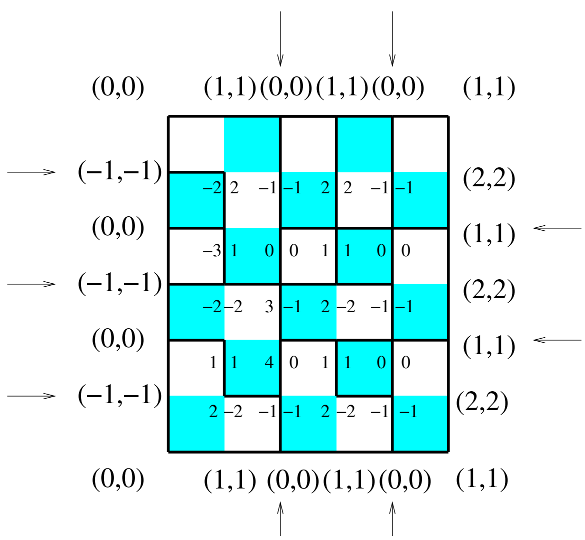

FIGURE 8. Height function on a quasi-tiling

This is the natural extension of the usual definition of height function. Usually, this function is defined in a geometrical way. Let's color the cells of the grid in black and white like a checkerboard. Then put the height on the border of the grid like in quasi-tilings. Then, if you follow a black edge from a vertex to another one, check the color of the cell on the left. If the cell is black then you add one otherwise subtract one. This function of $\mathbb{Z}$ is well defined.

Lemma 1. *A quasi-tiling is a tiling if and only if at each vertex both components of the height function are equal.*

Proof. Notice first that on the border of the grid both components of the height function are equal and equal to the usual tiling height function. Then, suppose that at every vertex of the grid those components are also equal. Consider a cell. Then turning clockwise around it, the only possible modifications of the height functions is to subtract 1 or add 3 (or the converse). But the only possible way to obtain the same height is to add (or subtract) three times one and subtract (or add) one time three to the height function. This means that there are three black edges and one white edge. And this is a characterization of dominoes. The converse is straightforward. □

3. Quasi-tiling on triangular lattice

In this part, we will show how quasi-tiling could be generalized on other lattices like triangular one. The main difference in this case is that there are three directions so that the height function should be taken in $\mathbb{Z}^3$ rather than $\mathbb{Z}^2$.

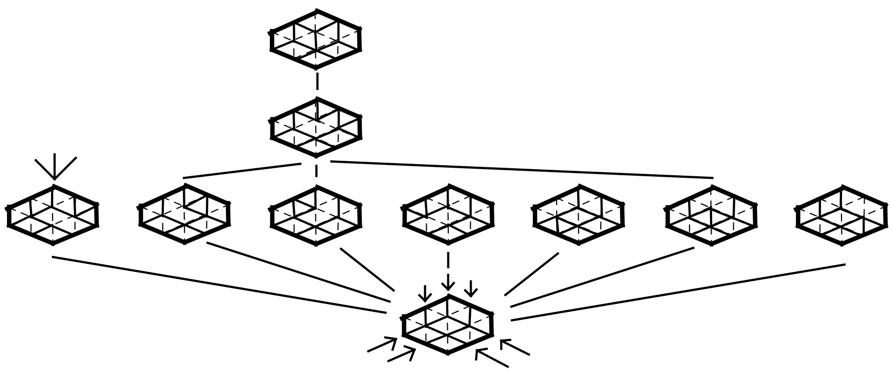

FIGURE 9. Beginning of quasi-tiling lattice on triangular lattice

The three directions are represented on figure 9 with arrows. The rules for computing the height function are the following :

(1) The oriented diagonals are the verticals -top to down-, and both other directions -from lower left to upper right and from lower right to upper left-.
   - The first coordinate is given by diagonals beginning in the lower left.
   - The second coordinate is given by the verticals.
   - The third one by the last direction of diagonals.
   - Going from one vertex to an adjacent one in a positive way, subtract one to the associated coordinate if the edge is black, add two otherwise.

(2) To number the border of the hexagon just put a number on it $(0, 0, 0)$ for example and following the standard numbering and the *border* put height on all vertices.

For a detailed example, see figure 10. A quasi-tiling on a triangular lattice is a mapping of edges of the lattice into {0,1} with constant weight on each diagonal.

The quasi-flip operation is the transposition of two adjacent edges on a diagonal. See for instance figure 11 and figure 12.

As figure 10 shows, when the three heights are equal then it is a tiling. The same proof as for dominoes goes. Turning around a triangle, the only way to be coherent is to have two black edges and one white edge. This means that it is a tiling. The converse is straightforward. Now we define two binary operators $\vee, \wedge$ on these quasi-tilings. Taking the word representation of a diagonal, we define the operator the same way as for dominoes. For example : $01\bar{1}001 \vee \bar{1}001\bar{1}0 = 101010$ where $\bar{1}$ are the letters 1 which are reported in the final word.

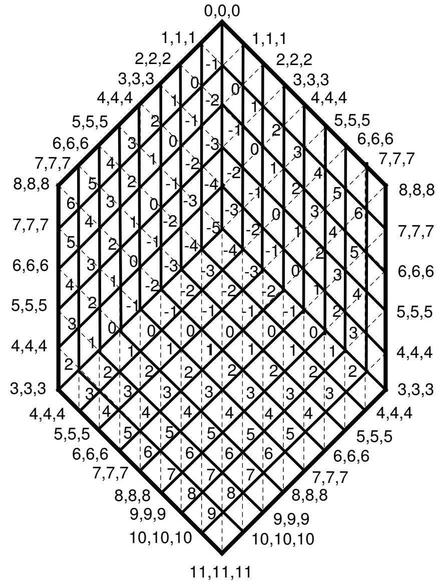

FIGURE 10. Numbering example

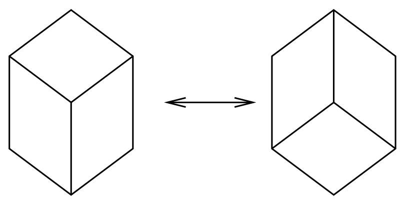

FIGURE 11. Example of flip in the triangular lattice

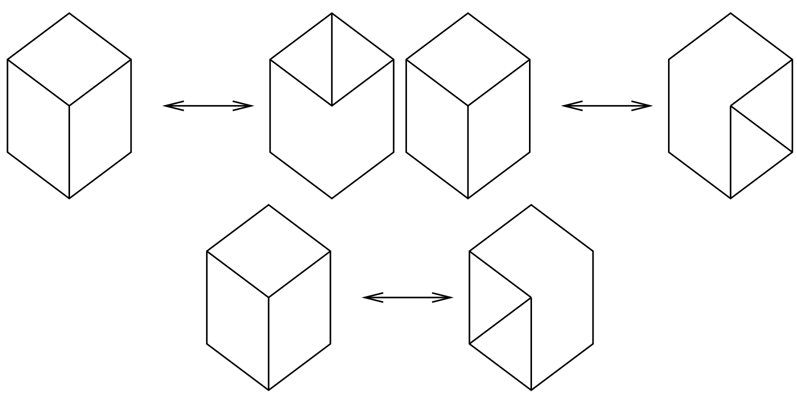

FIGURE 12. Example of quasi-flip in the triangular lattice

3.1. **Tiling lattice.** Thanks to our definition of $\vee$ and $\wedge$ on quasi-tilings we can prove that these operators are in fact the usual operators for lattices. For a introduction to lattices and ordered sets see [7].

We already proved that quasi-tilings with these operators form a lattice. We first proved that quasi-tilings with the quasi-flip operation is a lattice and then we noticed that $\vee$ and $\wedge$ are in fact the *sup* and *inf* operators of this lattice.

Now we keep our defintions of $\vee$ and $\wedge$ but we work on tiling configuration space instead of quasi-tilings.

All our proofs are based on the following remark:

Remark 1. *The action of* $\vee$ *and* $\wedge$ *on tilings or quasi-tilings could indeed be expressed in terms of height functions as taking the maximal or the minimal height for each coordinate between both tilings or quasi-tilings.*

First we have to prove that the tiling configuration space $\mathcal{P}$ is stable under the action of $\vee$ and $\wedge$. By remark 1 if P and Q are two tilings then $P \wedge Q = R$ where the height - taken either as a vector in $\mathbb{Z}^2$ for domino or $\mathbb{Z}^3$ for rhombus- on each vertex of the grid for R is the maximum between the height of P and Q. But P and Q are tilings so that the height function is in fact a function in $\mathbb{Z}$ -all coordinates are equal for tilings-. So that R is a quasi-tiling where all coordinates are equal on each vertex: R is a tiling.

Now we can prove that $(\mathcal{P}, \vee, \wedge)$ is a lattice. So we have to prove that $\vee$ and $\wedge$ are associative, commutative and idempotent. Finally we have to check the absorption law. So for every P, Q, R we have to check the following rules:

- $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$ - same with $\vee$- (associative law)
- $P \wedge Q = Q \wedge P$ -same with $\vee$- (commutative law)
- $P \wedge P = P$ - same with $\vee$- (idempotency law)
- $P \wedge (P \vee Q) = P$ -same when inverting $\vee$ and $\wedge$ (absorption law)

By remark 1, we have indeed to check these relations for the *max* and the *min* operators which is straightforward.

This proved also the connectivity of the tiling configuration space under the flip operation as the definitions of $\vee$ and $\wedge$ restricted to tilings induce the flip operation as order relation.

3.2. **Quasi-tiling lattice and tiling lattice.** We can notice that a flip is exactly two quasi-flips for dominoes and three for triangle and that the definition of $\vee$ and $\wedge$ for tilings is the restriction of the quasi-tiling ones. Thus, the lattice of tilings is a sub-lattice of the lattice of quasi-tilings.

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