Decomposing Subcubic Graphs into Claws, Paths or Triangles

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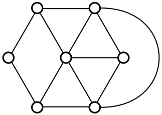

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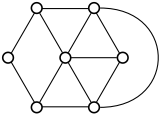

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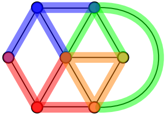

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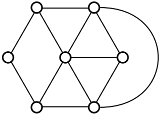

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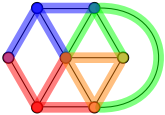

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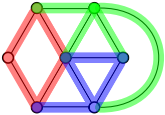

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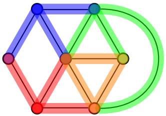

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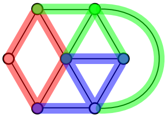

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S-DECOMPOSITION
Input: a graph $G = (V, E)$, a set S of graphs.
Question: does G admit an S-decomposition?

Motivations

Edge-partition problems appear in surprisingly diverse areas:

- database anonymisation [1];
- traffic grooming [7];
- graph drawing [4];
- . . .

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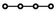
- Are there easy cases when we restrict ourselves to connected subgraphs with three edges? (i.e. $S = \{$, , $\}$)

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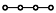
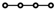
- Are there easy cases when we restrict ourselves to connected subgraphs with three edges? (i.e. $S = \{$, , $\}$)
- It turns out that the answer is yes if the input graph is subcubic (all degrees ≤ 3);

Our contributions

Here is a summary of what is known about decomposing graphs using subsets of { , , }:

Allowed subgraphs			Complexity according to graph class		
			strictly subcubic	cubic	arbitrary
✓					NP-complete [3, Theorem 3.5]
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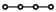
Here is a summary of what is known about decomposing graphs using subsets of { , , }:

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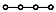
  - otherwise (i.e. as soon as we allow 's) it is NP-complete.

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  - otherwise (i.e. as soon as we allow ∘–∘–∘–∘'s) it is NP-complete.
- The △ case is trivial: *G* admits a △ -decomposition if and only if it is a disjoint union of △'s.

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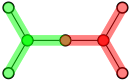

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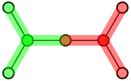

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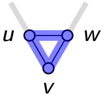

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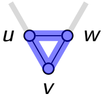

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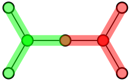

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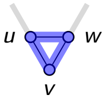

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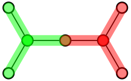

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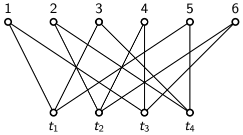

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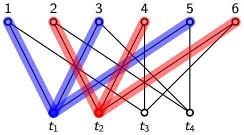

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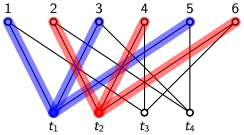

- X3C remains NP-complete if the bipartite instance graph G is planar and if $deg(w) \in \{2, 3\}\ \forall\ w \in W$;

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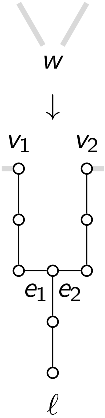

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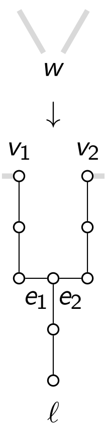

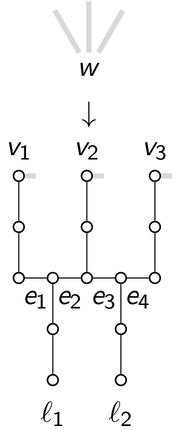

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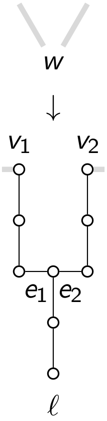

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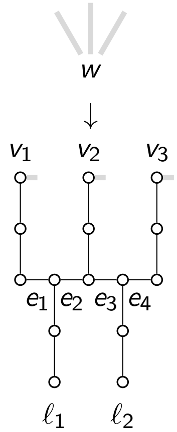
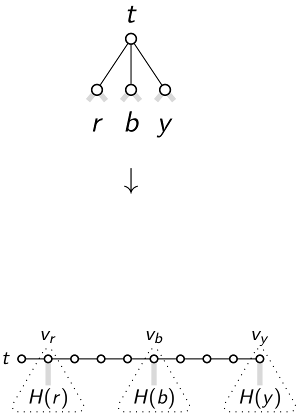


triplets



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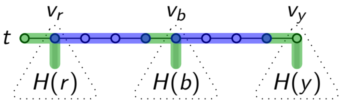

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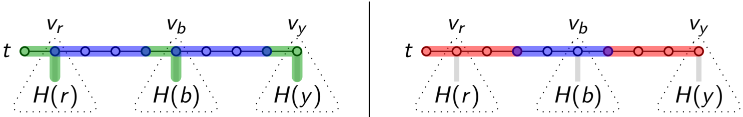

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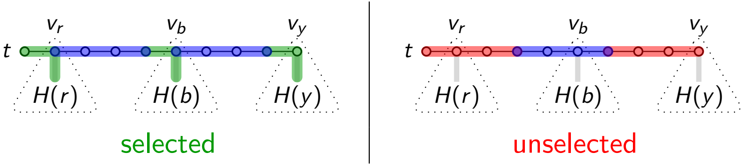

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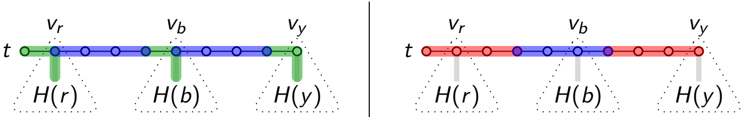

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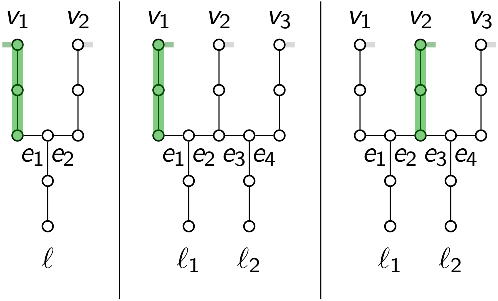

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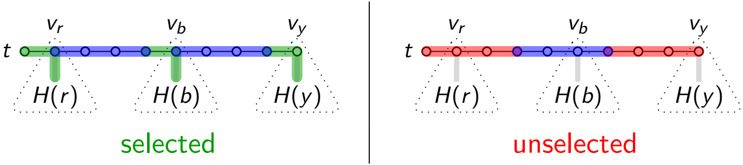

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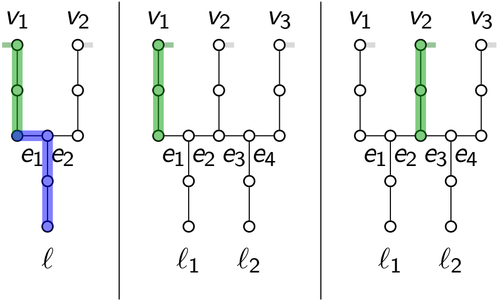

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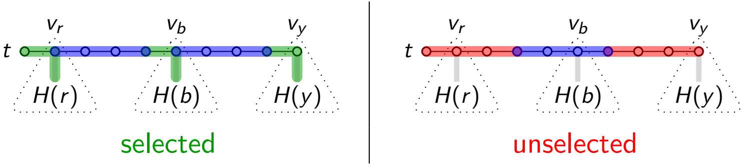

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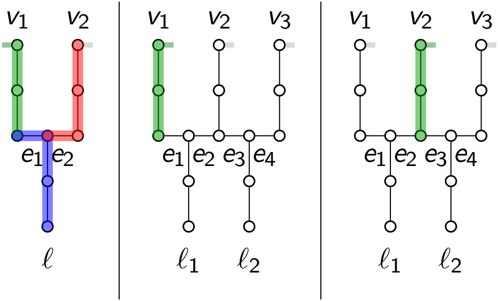

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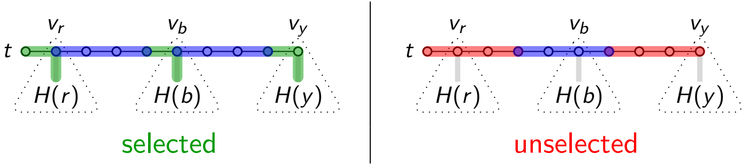

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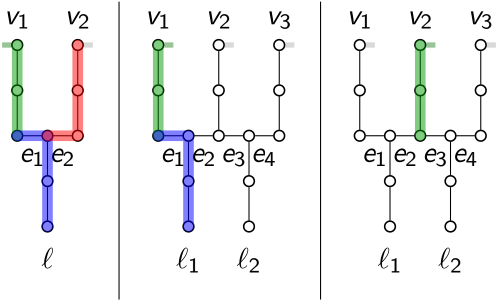

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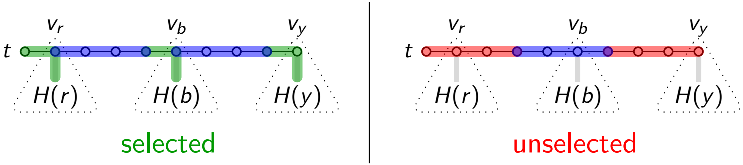

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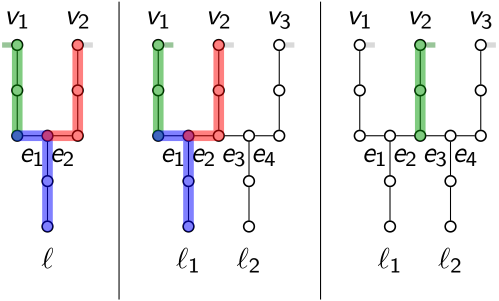

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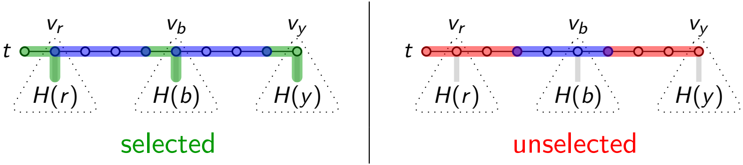

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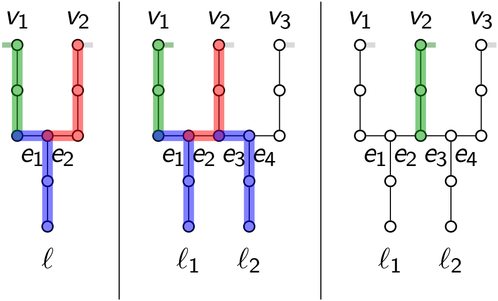

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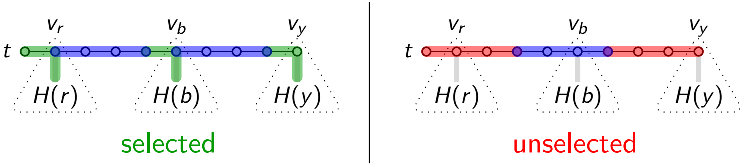

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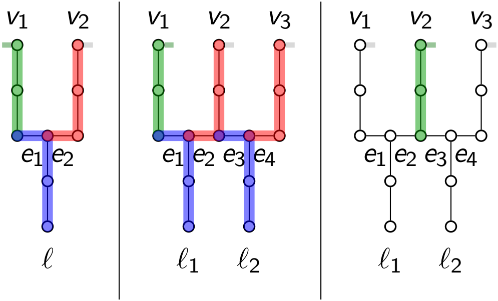

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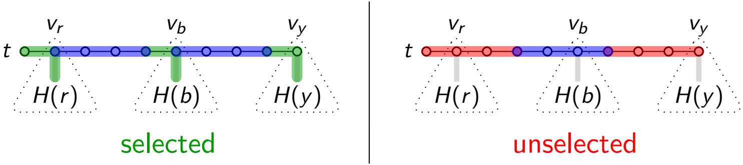

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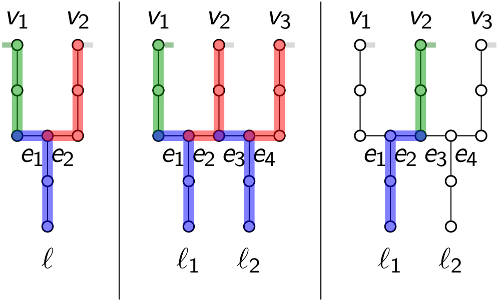

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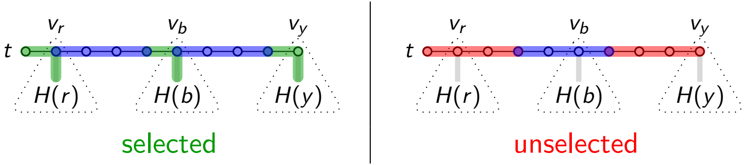

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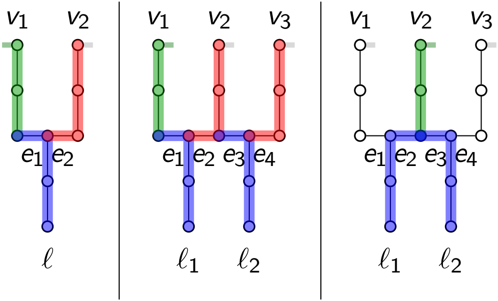

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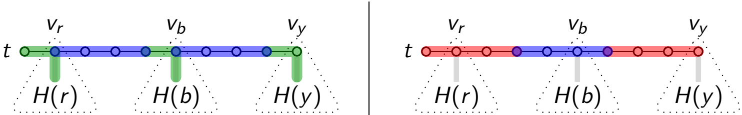

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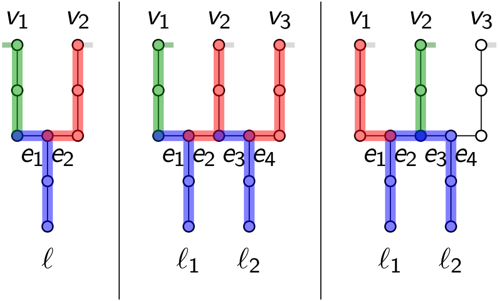

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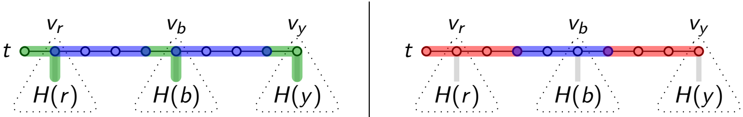

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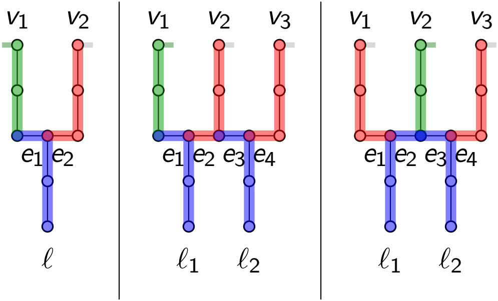

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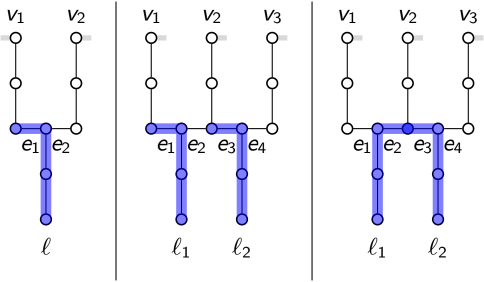

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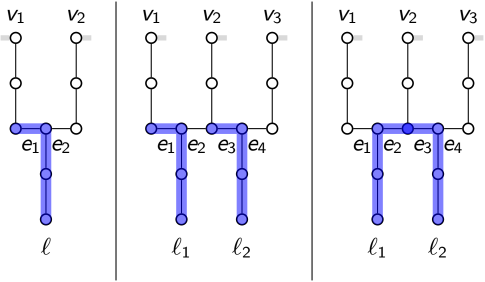

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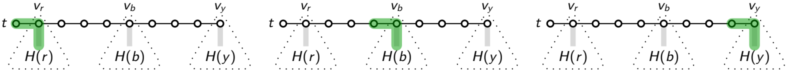

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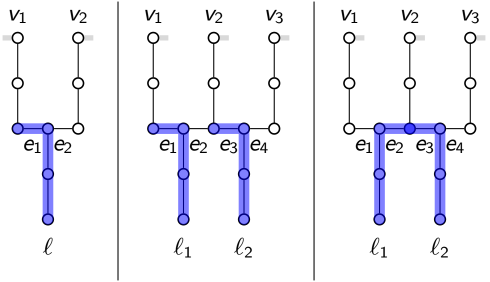

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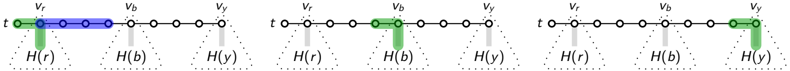

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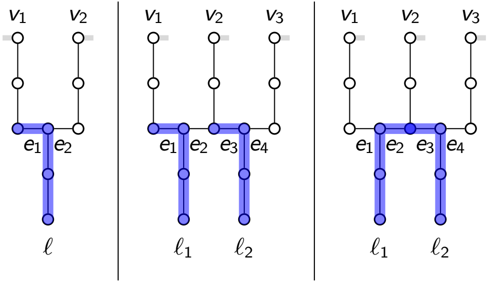

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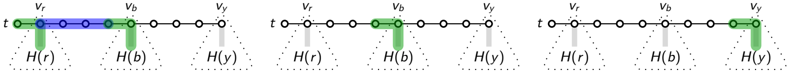

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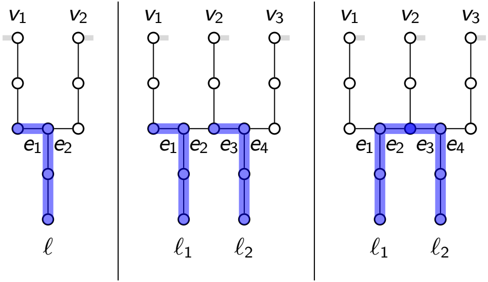

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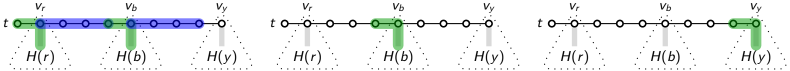

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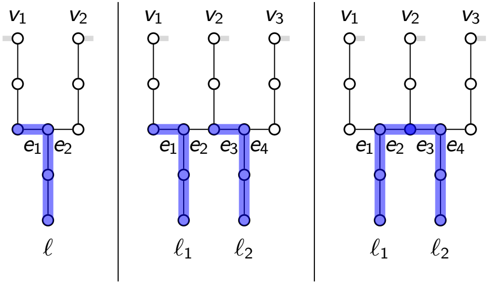

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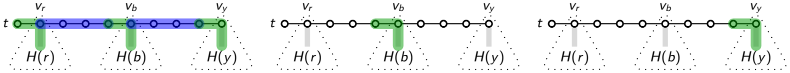

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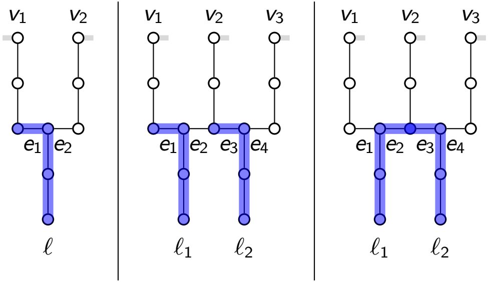

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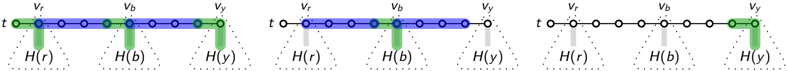

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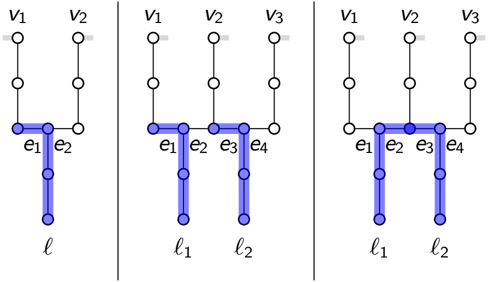

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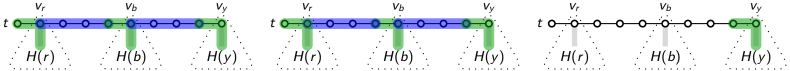

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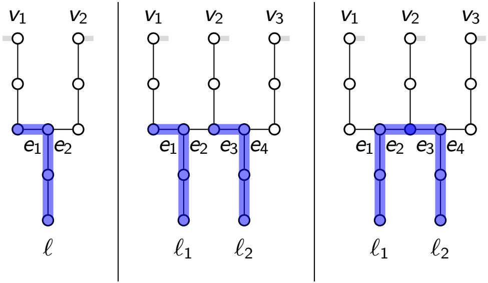

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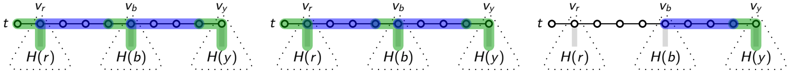

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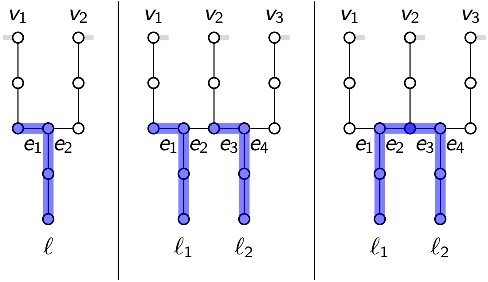

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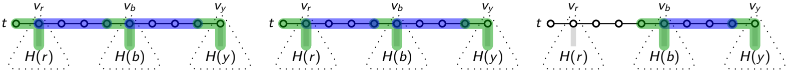

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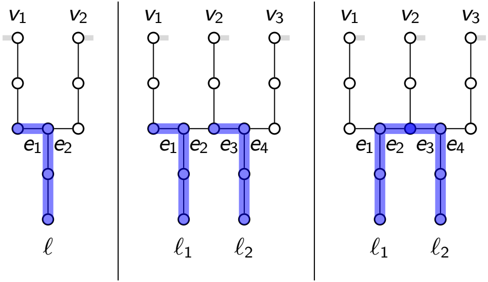

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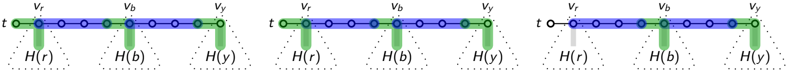

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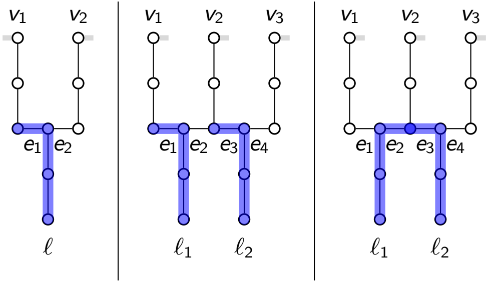

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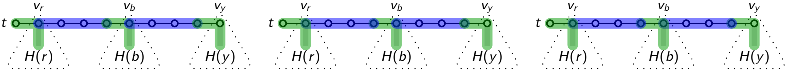

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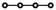
  1.  -DECOMPOSITION (just shown);

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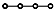
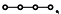
  1.  -DECOMPOSITION (just shown);
  2. { , }-DECOMPOSITION (skipped but similar);

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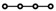
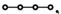
  1.  -DECOMPOSITION (just shown);
  2. { , }-DECOMPOSITION (skipped but similar);

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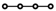
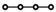
  1.  -DECOMPOSITION (just shown);
  2. { , }-DECOMPOSITION (skipped but similar);

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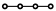
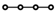
  1.  -DECOMPOSITION (just shown);
  2. {, }-DECOMPOSITION (skipped but similar);

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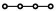
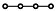
  1.  -DECOMPOSITION (just shown);
  2. { , }-DECOMPOSITION (skipped but similar);

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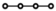
  1. { , }-DECOMPOSITION;

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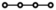
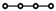
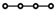
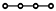
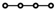
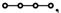
  1.  -DECOMPOSITION (just shown);
  2. $\{$ , $\}$-DECOMPOSITION (skipped but similar);

- The transformed graphs are planar and bipartite ⇒ no odd cycles ⇒ no . So we immediately obtain the hardness of:
  1. $\{$ , $\}$-DECOMPOSITION;
  2. $\{$ , , $\}$-DECOMPOSITION.

The cubic case

We now move on to the cubic case, i.e. every vertex of G has degree 3.

-  -DECOMPOSITION and { , }-DECOMPOSITION become easy!

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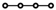
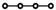
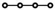
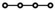
-  -DECOMPOSITION and { , }-DECOMPOSITION become easy!
- { , }-DECOMPOSITION and { , , }-DECOMPOSITION remain hard, but we need new reductions;

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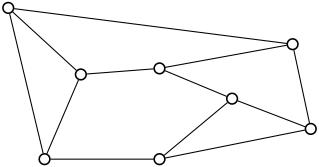
- ∘–∘–∘–∘ -DECOMPOSITION and {∘–∘–∘–∘, △}-DECOMPOSITION become easy!
- {∘–∘–∘–∘, ⅄}-DECOMPOSITION and {∘–∘–∘–∘, ⅄, △}-DECOMPOSITION remain hard, but we need new reductions;
- △ -DECOMPOSITION remains trivial (never possible in the cubic case);

∘–∘–∘–∘ -DECOMPOSITION, cubic

We need the following result:

Proposition ([6])

A cubic graph admits a ∘–∘–∘–∘ -decomposition if and only if it has a perfect matching.



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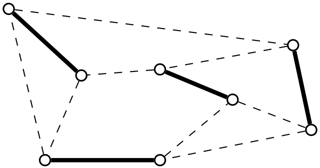

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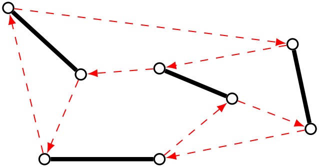

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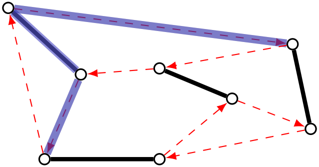

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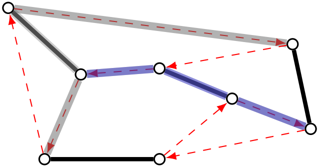

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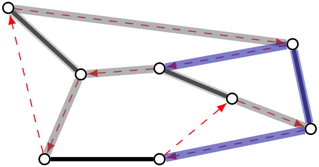

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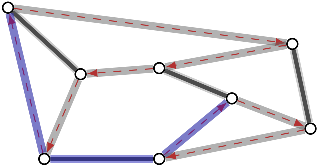

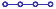 -DECOMPOSITION, cubic

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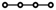
A cubic graph admits a { , }*-decomposition if and only if it has a perfect matching.*

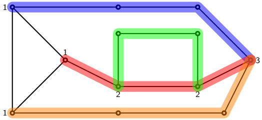
∘-∘-∘-∘ -DECOMPOSITION, cubic

Proposition

A cubic graph admits a $\{$△, ∘-∘-∘-∘$\}$*-decomposition if and only if it has a perfect matching.*

Proof.

Each vertex in V is covered by k ∘-∘-∘-∘ 's ($k \in \{1, 2, 3\}$). Example:



$\Rightarrow V = V_1 \cup V_2 \cup V_3.$

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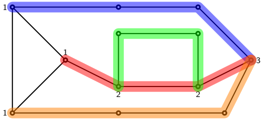

⇒ $V = V_1 \cup V_2 \cup V_3$. Let's compute the number p of ∘-∘-∘-∘ 's in a decomposition; we have (details omitted):

$$(3|V_3| + |V_2| + |V_1|)/2 = p = |V_2|/2$$

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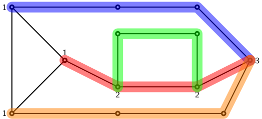

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So $V_1 = V_3 = \emptyset$; and since V_1 is the set of vertices that belong to a △ , no decomposition with a △ exists. □

⅄ -DECOMPOSITION, cubic

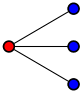
We obtain a simple characterisation of ⅄ -decomposable cubic graphs:

Proposition

A cubic graph admits a ⅄ -decomposition if and only if it is bipartite.

Proof.

$\boxed{\Rightarrow}$



A center (red) belongs to only one subgraph

⇒ Bipartition: centers – leaves

(each edge connects a center and a leaf)

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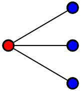

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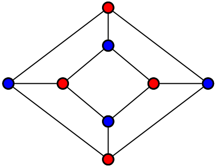

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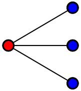

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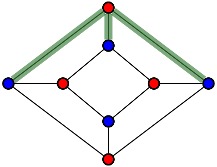

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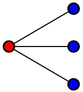

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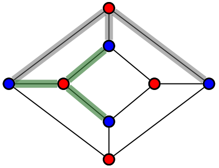

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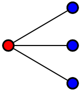

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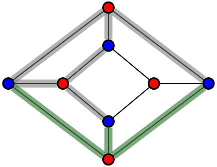

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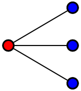

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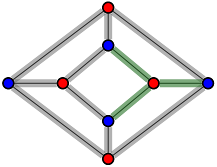

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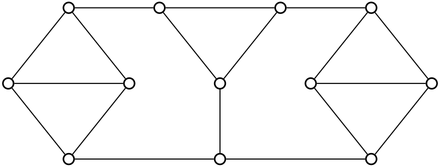
{⅄, △}-DECOMPOSITION, cubic

What if we also allow △'s?



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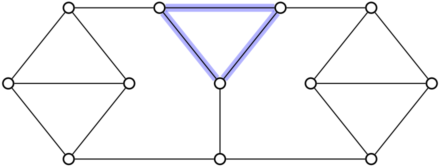

We distinguish between *isolated* and *nonisolated* triangles:

{⅄, △}-DECOMPOSITION, cubic

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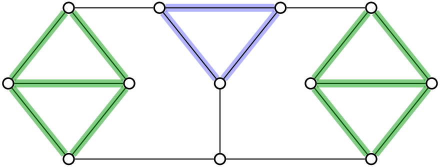

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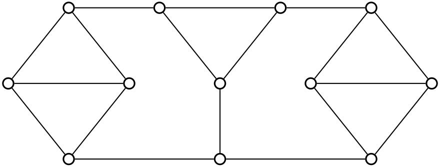

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Lemma

If a cubic graph G admits a {⅄, △}-decomposition D, then every isolated △ in G belongs to D.

{⅄, △}-DECOMPOSITION, cubic

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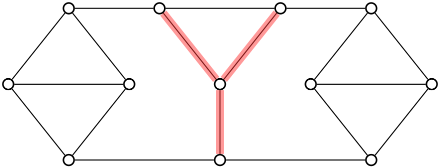

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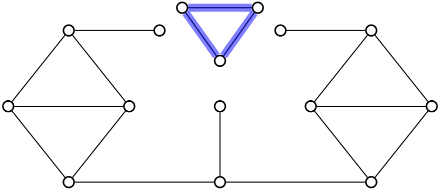

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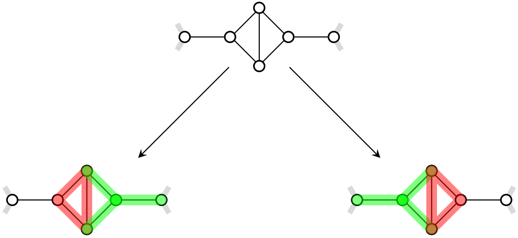
{⅄, △}-DECOMPOSITION, cubic

If G also contains nonisolated △'s, then we only have two choices to try:



{⅄, △}-DECOMPOSITION, cubic

The algorithm proceeds as follows:

1. extract all isolated triangles and add them to the decomposition;

{⅄, △}-DECOMPOSITION, cubic

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2. if there's a diamond, try either option for the decomposition;

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1. extract all isolated triangles and add them to the decomposition;
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3. if the resulting graph is still cubic, find a ⅄ -decomposition using the previous algorithm;

{⅄, △}-DECOMPOSITION, cubic

The algorithm proceeds as follows:

1. extract all isolated triangles and add them to the decomposition;
2. if there's a diamond, try either option for the decomposition;
3. if the resulting graph is still cubic, find a ⅄ -decomposition using the previous algorithm;
4. otherwise, run the {⅄, △}-decomposition algorithm for strictly subcubic graphs;

{⅄, ∘-∘-∘-∘}-DECOMPOSITION, cubic

We now show that {⅄, ∘-∘-∘-∘}-DECOMPOSITION is NP-complete, using three reductions (I'll skip tons of details and just explain the gist of the first one):

CUBIC MONOTONE 1-IN-3 SATISFIABILITY

$\leq_P$ DEGREE-2,3 {⅄, △, ∘-∘-∘-∘}-DECOMPOSITION WITH MARKED EDGES

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$\leq_P$ {⅄, ∘-∘-∘-∘}-DECOMPOSITION

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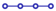
A similar approach can be used to show the NP-completeness of {⅄, △, ∘-∘-∘-∘}-DECOMPOSITION.

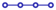
{ , }-DECOMPOSITION, cubic

We reduce from the following NP-complete problem:

SAT(ISFIABILITY)

Input: a Boolean formula $\phi = C_1 \wedge C_2 \wedge \cdots$;

Question: is there an assignment $f : \Sigma \to \{\text{TRUE}, \text{FALSE}\}$ such that each clause C_i contains one TRUE literal?

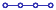
{ , }-DECOMPOSITION, cubic

We reduce from the following NP-complete problem:

MONOTONE SAT(ISFIABILITY)

Input: a Boolean formula $\phi = C_1 \wedge C_2 \wedge \cdots$ without negations;

Question: is there an assignment $f : \Sigma \to \{\text{TRUE}, \text{FALSE}\}$ such that each clause C_i contains one TRUE literal?

{ , }-DECOMPOSITION, cubic

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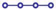
MONOTONE 1-IN-3 SAT(ISFIABILITY)

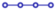
Input: a Boolean formula $\phi = C_1 \wedge C_2 \wedge \cdots$ without negations; $|C_i| = 3$ for each i

Question: is there an assignment $f : \Sigma \rightarrow \{\text{TRUE}, \text{FALSE}\}$ such that each clause C_i contains exactly one TRUE literal?

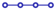
{ , }-DECOMPOSITION, cubic

We reduce from the following NP-complete problem:

CUBIC MONOTONE 1-IN-3 SAT(ISFIABILITY)
Input: a Boolean formula $\phi = C_1 \wedge C_2 \wedge \cdots$ without negations; $|C_i| = 3$ for each i and each literal appears in exactly three clauses;
Question: is there an assignment $f : \Sigma \rightarrow \{\text{TRUE}, \text{FALSE}\}$ such that each clause C_i contains exactly one TRUE literal?

{ , }-DECOMPOSITION, cubic

Echoing the steps of the previous reduction, we assume the instance to . . . SAT is a bipartite cubic graph G; so, we must:

{ , }-DECOMPOSITION, cubic

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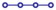
1. transform G into a graph G' to decompose;

{ , }-DECOMPOSITION, cubic

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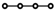
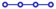
1. transform G into a graph G' to decompose;
2. convert truth assignments for G into { , }-decompositions for G';

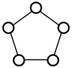
{ , }-DECOMPOSITION, cubic

Echoing the steps of the previous reduction, we assume the instance to . . . SAT is a bipartite cubic graph G; so, we must:

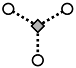
1. transform G into a graph G' to decompose;
2. convert truth assignments for G into { , }-decompositions for G';
3. convert { , }-decompositions for G' into truth assignments for G;

The reduction from CUBIC MONO-1-IN-3-SAT

Clause



Variable



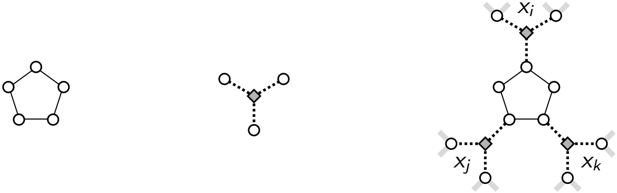
1. Map clauses onto C_5's and variables onto “marked” 's.

The reduction from CUBIC MONO-1-IN-3-SAT

Clause

Variable

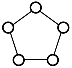
$C = x_i \vee x_j \vee x_k$



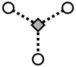
1. Map clauses onto C_5’s and variables onto “marked” ⅄’s.

The reduction from CUBIC MONO-1-IN-3-SAT

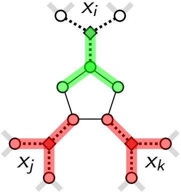
Clause



Variable

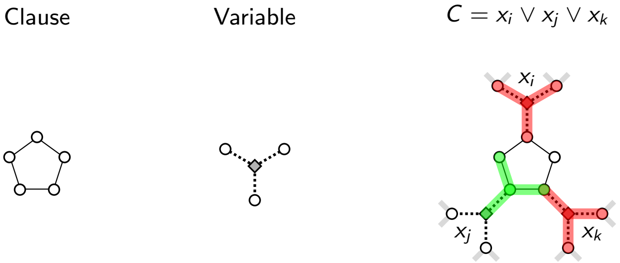


$C = x_i \vee x_j \vee x_k$



1. Map clauses onto C_5's and variables onto "marked" ⅄'s.
2. From assignments to decompositions: variables set to FALSE yield red ⅄'s, those set to TRUE yield green ⅄'s

The reduction from CUBIC MONO-1-IN-3-SAT


Clause

Variable

$C = x_i \vee x_j \vee x_k$

1. Map clauses onto C_5's and variables onto "marked" ⅄'s.
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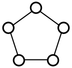
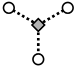
The reduction from CUBIC MONO-1-IN-3-SAT

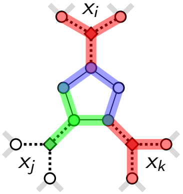
Clause



Variable



$C = x_i \vee x_j \vee x_k$



1. Map clauses onto C_5's and variables onto "marked" ⅄'s.
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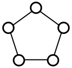
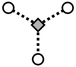
The reduction from CUBIC MONO-1-IN-3-SAT ⅄ ∘–∘–∘–∘

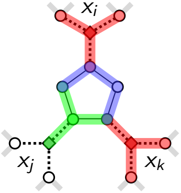
Clause



Variable



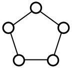
$C = x_i \vee x_j \vee x_k$



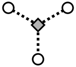
1. Map clauses onto C_5's and variables onto "marked" ⅄'s.
2. From assignments to decompositions: variables set to FALSE yield red ⅄'s, those set to TRUE yield green ⅄'s
3. From decompositions to assignments: show that a decomposable graph **must** conform to the above configuration

The reduction from CUBIC MONO-1-IN-3-SAT

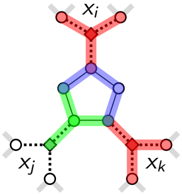
Clause



Variable



$C = x_i \lor x_j \lor x_k$

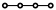
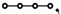


1. Map clauses onto C_5's and variables onto "marked" ⅄'s.
2. From assignments to decompositions: variables set to FALSE yield red ⅄'s, those set to TRUE yield green ⅄'s
3. From decompositions to assignments: show that a decomposable graph **must** conform to the above configuration

Marked edges are annoying and must undergo further modifications (hence the other reductions).

Encores

With (a lot) more work, we can show that

- { , }-DECOMPOSITION and
- { , , }-DECOMPOSITION

remain hard if the cubic graph is planar and -free. Ingredients:

- another variant of SAT (namely, CUBIC PLANAR MONOTONE 1-IN-3 SAT)
- another intermediate problem;
- . . . and a few more pages of reduction;

Conclusions

- We now know everything regarding S-DECOMPOSITION if G is subcubic and S is any combination of connected graphs on 3 edges.
- Possible future work:
  - what G is k-regular and $S =$ all connected subgraphs of size k for any $k > 3$?
  - do easy problems remain easy under natural generalisations? i.e.
    - P_{k+1}-DECOMPOSITION for k-regular graphs;
    - $K_{1,k}$-DECOMPOSITION for k-regular graphs;
    - ...

Thank you!

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