Unshuffling Permutations

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Definition (shuffle product of words)

The shuffle product $\sqcup\!\sqcup$ on words of A^* is defined recursively by

$$u \sqcup\!\sqcup \epsilon := \{u\} =: \epsilon \sqcup\!\sqcup u,$$
$$(ua \sqcup\!\sqcup vb) := (ua \sqcup\!\sqcup v)b \cup (u \sqcup\!\sqcup vb)a.$$

To take into account multiplicities, we consider $\sqcup\!\sqcup$ as a linear product

$$\sqcup\!\sqcup : \mathbb{Q}[A^*] \otimes \mathbb{Q}[A^*] \to \mathbb{Q}[A^*]$$

on $\mathbb{Q}[A^*]$, the $\mathbb{Q}$-linear span of words defined by

$$u \sqcup\!\sqcup \epsilon := u =: \epsilon \sqcup\!\sqcup u,$$
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Example

$$\mathtt{ab} \sqcup\!\sqcup \mathtt{ba} = \mathtt{abba} + \mathtt{abba} + \mathtt{abab} + \mathtt{baba} + \mathtt{baab} + \mathtt{baab}$$
$$= 2\,\mathtt{abba} + \mathtt{abab} + \mathtt{baba} + 2\,\mathtt{baab}$$

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Key results

Given two words v_1 and v_2, the set $v_1 \sqcup\!\sqcup v_2$ can be computed in

$$O\left((|v_1| + |v_2|) \binom{|v_1| + |v_2|}{|v_1|}\right)$$

time [**Spehner**, 1986].

Given three words u, v_1, and v_2, deciding if u is in $v_1 \sqcup\!\sqcup v_2$ can be done in

$$O\left(\frac{|u|^2}{\log(|u|)}\right)$$

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Given a word u, deciding if there is a word v such that u is in $v \sqcup\!\sqcup v$ is **NP**-complete [**Rizzi, Vialette**, 2013] [**Buss, Soltys**, 2014].

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Square words

Definition (square words)

A word u is a square (w.r.t. the shuffle product $\shuffle$) if there is a word v such that u appears in $v \shuffle v$.

Example

The word $u :=$ `ccababbb` is a square since u can be obtained by shuffling `cabb` with itself.

The first numbers of square binary words of length $2n$ are

$$1, 2, 6, 22, 82, 320, 1268, 5102, 20632, 83972.$$

Open problem [**Henshall, Rampersad, Shallit**, 2012]

Enumeration of square (binary) words.

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Recognizing square words

Definition (perfect matchings for words)

A perfect matching of a word u is a graph (V, E) such that

- $V := \{(u_i, i) : i \in \{1, \ldots, |u|\}\}$;
- every vertex of V belongs to exactly one edge of E;
- $(u_i, i) - (u_j, j) \in E$ implies $u_i = u_j$.

Example

The word `b a b a b b c c` admits the perfect matching

.

Definition (inclusion-free perfect matchings)

A perfect matching (V, E) of a word u is inclusion-free if there are no edges $(u_i, i) - (u_j, j)$ and $(u_k, k) - (u_\ell, \ell)$ of E such that $i < k < \ell < j$.

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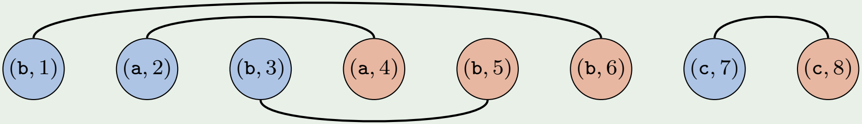

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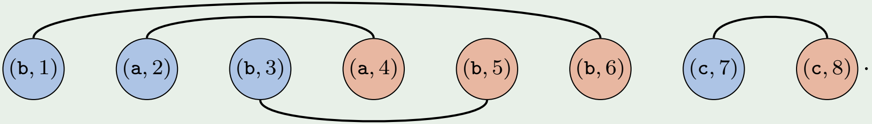

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Proposition [**Rizzi, Vialette**, 2013]

A word u is a square iff u admits an inclusion-free perfect matching.

Example

The word c c a b a b b b admits the perfect matching

which is inclusion-free and hence is a square.

Example

The word a b b a is not a square. Its only perfect matching

is not inclusion-free.

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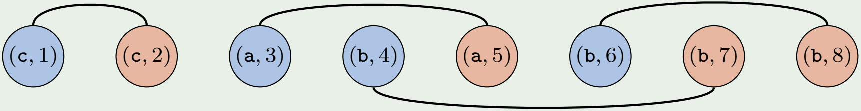

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The word `abba` is not a square. Its only perfect matching

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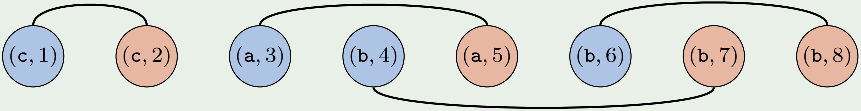

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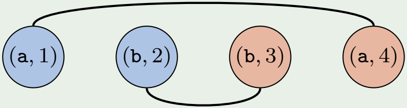

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Main motivations

Let S_n be the set of permutations of size n and S be the set of all permutations.

There are several products on permutations, analogs of the shuffle product of words. Among these, there are

- the shifted shuffle product [**Duchamp, Hivert, Thibon**, 2002];
- the convolution product [**Duchamp, Hivert, Thibon**, 2002];
- the supershuffle [**Vargas**, 2014].

Main questions

- Combinatorial properties of "square permutations" w.r.t. the supershuffle?
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Combinatorial products and coproducts

Let C be a set of combinatorial objects and $\mathbb{Q}[C]$ be the $\mathbb{Q}$-linear span of C.

Key idea

Knowing how to break combinatorial objects explains how to combine these.

To define a product $\cdot : \mathbb{Q}[C] \otimes \mathbb{Q}[C] \to \mathbb{Q}[C]$, it is in some cases more convenient to start by defining a coproduct $\Delta : \mathbb{Q}[C] \to \mathbb{Q}[C] \otimes \mathbb{Q}[C]$.

Every coproduct fits into the general form

$$\Delta(z) = \sum_{x,y \in C} \lambda_{x,y}^{z} \, (x \otimes y)$$

where the $\lambda_{x,y}^{z} \in \mathbb{Q}$ are structure coefficients.

Then, Δ leads by duality to the definition of $\cdot$ by

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$$\Delta(u) := \sum_{P \sqcup Q = \{1, \dots, |u|\}} u_{|P} \otimes u_{|Q}.$$

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$$\Delta(\mathtt{b\,a\,a}) = \epsilon \otimes \mathtt{b\,a\,a} + \mathtt{b} \otimes \mathtt{a\,a} + 2(\mathtt{a} \otimes \mathtt{b\,a}) + 2(\mathtt{b\,a} \otimes \mathtt{a}) + \mathtt{a\,a} \otimes \mathtt{b} + \mathtt{b\,a\,a} \otimes \epsilon$$

Proposition [**Reutenauer**, 1993]

The shuffle product of words is the dual product of the unshuffling coproduct of words.

Example

Since $\mathtt{a} \otimes \mathtt{b\,a}$ has multiplicity 2 in $\Delta(\mathtt{b\,a\,a})$, the coeff. of $\mathtt{b\,a\,a}$ is 2 in $\mathtt{a} ⧢ \mathtt{b\,a}$:

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Unshuffling permutations

Definition (standardization of words)

If u is a word of integers without multiple occurrence of a same letter, the standardized $\mathrm{std}(u)$ of u is the unique permutation of $S_{|u|}$ order-isomorphic to u.

Example

$$\mathrm{std}(82194) = 42153$$

Definition (unshuffling coproduct of permutations)

The unshuffling coproduct of permutations is the coproduct on $\mathbb{Q}[S]$ defined by

$$\Delta(\pi) := \sum_{P \sqcup Q = \{1, \dots, |\pi|\}} \mathrm{std}\left(\pi_{|P}\right) \otimes \mathrm{std}\left(\pi_{|Q}\right).$$

Example

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Shuffling permutations

Definition (shuffling product of permutations)

The shuffling product of permutations (supershuffle) is the product $\bullet$ on $\mathbb{Q}[S]$ defined as the dual product of the unshuffling product of permutations.

Proposition

The permutations appearing in $\sigma \bullet \nu$ are exactly the one obtained by shuffling two words u and v respectively order-isomorphic to σ and ν.

Example

$$12 \bullet 21 = 1243 + 1324 + 2(1342) + 2(1423) + 3(1432) + 2134 + 2(2314)$$
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Shuffling permutations

Definition (shuffling product of permutations)

The shuffling product of permutations (supershuffle) is the product $\bullet$ on $\mathbb{Q}[S]$ defined as the dual product of the unshuffling product of permutations.

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The permutations appearing in $\sigma \bullet \nu$ are exactly the one obtained by shuffling two words u and v respectively order-isomorphic to σ and ν.

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A permutation π is a square (w.r.t. the shuffle product $\bullet$) if there is $\sigma \in S$ such that π appears in $\sigma \bullet \sigma$. We say that σ is a square root of π.

By duality, π is a square if there is $\sigma \in S$ such that $\sigma \otimes \sigma$ appears in $\Delta(\pi)$.

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A permutation π is a square iff π can be obtained by shuffling two order-isomorphic words.

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The permutation $\pi := 25167834$ is square since π can be obtained by shuffling 1683 and 2574 and $\mathrm{std}(1683) = \mathrm{std}(2574) = 1342$.

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Some properties of square permutations

The first numbers of square permutations of size $2n$ are

$$1, 2, 20, 504, 21032, 1293418.$$

Square permutations are compatible with some involutions on permutations:

Proposition

Let π be a square permutation and σ be a square root of π. Then,

1. $\widetilde{\pi}$ is a square and $\widetilde{\sigma}$ is one of its square roots;
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The set of binary words of length n that are square w.r.t. $\sqcup\!\sqcup$ is in one-to-one correspondence with the set of square permutations of length n avoiding the patterns 213 and 231.

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A permutation π contains a permutation σ if there exists a subsequence of (not necessarily consecutive) letters of π that has the same relative order as σ, and in this case σ is said to be a pattern of π, written $\sigma \leq \pi$.

Otherwise, π avoids σ.

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The permutation $\pi = 391867452$ contains the pattern $\sigma := 51342$ since $\mathrm{std}(91674) = 51342$.

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The pattern involvement problem is the decision problem consisting in, given $\pi, \sigma \in S$, decide whether $\sigma \leq \pi$.

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The permutation $\pi := 35867241$ is a square since it admits the perfect matching

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Moreover, this shows that π appears in the shuffle of the words 3641 and 5872, both order-isomorphic to 2431.

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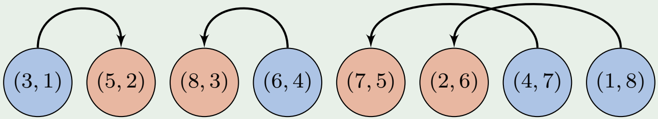

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Deciding whether a permutation is a square is **NP**-complete.

Main ideas of the proof

- We show that the pattern involvement problem is reducible in polynomial time to the problem of deciding if a permutation is a square.

- For this, given two permutations σ and π, we show that $\sigma \preceq \pi$ iff $\mu_{\sigma,\pi}$ is a square, where $\mu_{\sigma,\pi}$ is a permutation constructed in polynomial time from σ and π.

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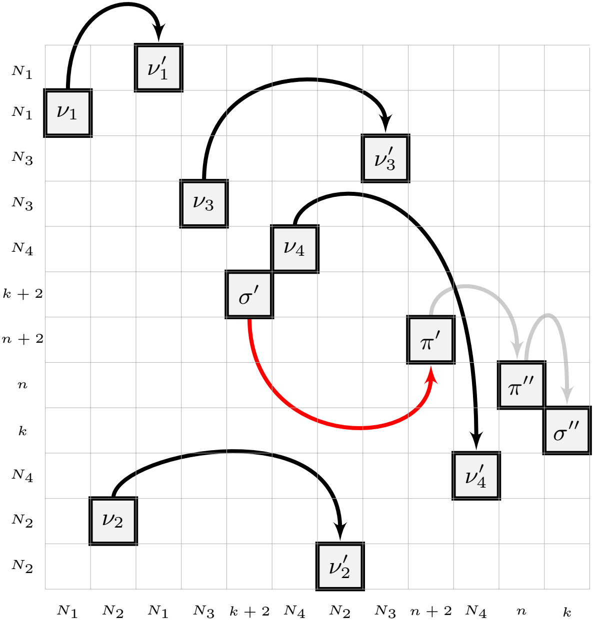

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Enumeration of square permutations?

Algorithms

Given two permutations π and σ, how hard is the problem of deciding whether σ is a square root of π?

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How algebraic properties of the associative algebra $(\mathbb{Q}[S], \bullet)$ or of the coalgebra $(\mathbb{Q}[S], \Delta)$ are connected to combinatorial and algorithmic properties of square permutations?

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