Tree rewriting and enumeration

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September 17, 2018

Outline

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Trees, patterns, and rewrite systems

Syntax trees

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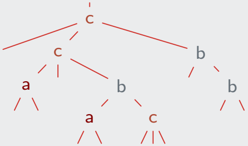
Example

Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ such that $\mathfrak{G}(2) = \{\mathsf{a}, \mathsf{b}\}$ and $\mathfrak{G}(3) = \{\mathsf{c}\}$.

Here is a $\mathfrak{G}$-tree:



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We set $\mathbf{F}(\mathfrak{G})(n)$ as the set of the $\mathfrak{G}$-trees of arity n.

Therefore,

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Remark: since each $\mathfrak{G}(n)$ is finite, if if $\mathfrak{G}(1) = \emptyset$, then all $\mathbf{F}(\mathfrak{G})(n)$ are finite.

Partial composition

Let $\mathfrak{t}, \mathfrak{s} \in \mathbf{F}(\mathfrak{G})$.

For each $i \in [|\mathfrak{t}|]$, $\mathfrak{t} \circ_i \mathfrak{s}$ is the tree obtained by grafting the root of a copy of $\mathfrak{s}$ onto the ith leaf of $\mathfrak{t}$.

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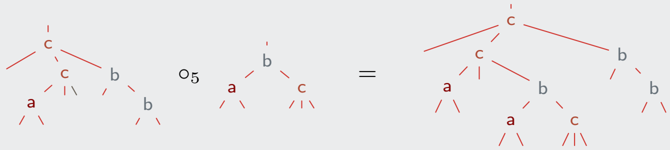

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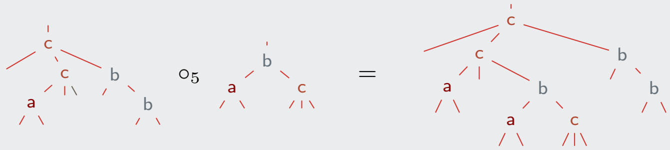

Therefore, $\circ_i$ is a map

$$\circ_i : \mathbf{F}(\mathfrak{G})(n) \times \mathbf{F}(\mathfrak{G})(m) \to \mathbf{F}(\mathfrak{G})(n+m-1)$$

where $i \in [n]$ and $1 \leqslant m$, called partial composition map.

Complete composition

Let $\mathfrak{t}, \mathfrak{s}_1, \ldots, \mathfrak{s}_{|\mathfrak{t}|} \in \mathbf{F}(\mathfrak{G})$.

The $\mathfrak{t} \circ \left[\mathfrak{s}_1, \ldots, \mathfrak{s}_{|\mathfrak{t}|}\right]$ is obtained by grafting simultaneously the roots of copies of the $\mathfrak{s}_i$ onto the ith leaves of $\mathfrak{t}$.

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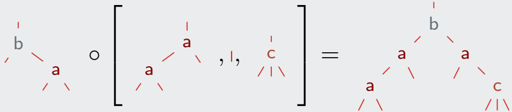

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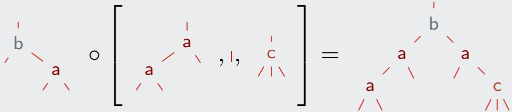

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where $1 \leqslant n$ and $1 \leqslant m_1, \dots, m_n$, called complete composition map.

Patterns and occurrences

Let $\mathfrak{t}, \mathfrak{s} \in \mathbf{F}(\mathfrak{G})$. A $\mathfrak{G}$-tree $\mathfrak{t}$ admits an occurrence of a $\mathfrak{G}$-tree $\mathfrak{s}$ if one can put $\mathfrak{s}$ onto $\mathfrak{t}$ by superimposing the root of $\mathfrak{s}$ and a node of $\mathfrak{t}$ and leaves of $\mathfrak{s}$ with leaves of nodes of $\mathfrak{t}$.

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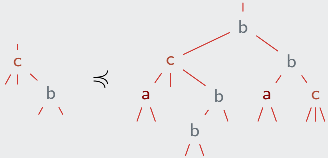

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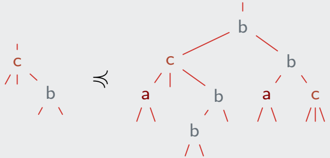

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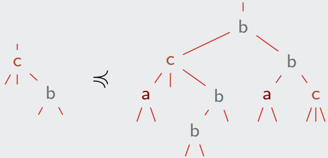

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More formally, $\mathfrak{s} \preccurlyeq \mathfrak{t}$ holds if there exist $\mathfrak{r}, \mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|} \in \mathbf{F}(\mathfrak{G})$ and $i \in [|\mathfrak{r}|]$ such that

$$\mathfrak{t} = \mathfrak{r} \circ_i \left(\mathfrak{s} \circ \left[\mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|}\right]\right).$$

Pattern avoidance

Given a set $\mathcal{P} \subseteq \mathbf{F}(\mathfrak{G})$, let $\mathrm{A}(\mathcal{P})$ be the set of all $\mathfrak{G}$-trees avoiding all patterns of $\mathcal{P}$.

Counting the elements of $\mathrm{A}(\mathcal{P})$ w.r.t. the arity is a usual question.

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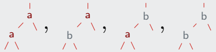
- For $\mathcal{P} :=$ {  }, $\mathrm{A}(\mathcal{P})$ is enumerated by

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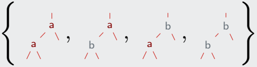
- For $\mathcal{P} :=$ 

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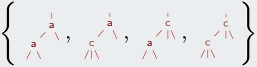
- For $\mathcal{P} :=$ 

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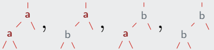
- For $\mathcal{P} := \Big\{$ 

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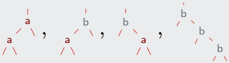
- For $\mathcal{P} := \Big\{$ 

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$$1, 2, 5, 13, 35, 96, 167, 750, \ldots .$$

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A rewrite rule is a binary relation $\rightarrow$ on $\mathbf{F}(\mathfrak{G})$ such that $\mathfrak{s} \rightarrow \mathfrak{s}'$ implies $|\mathfrak{s}| = |\mathfrak{s}'|$.

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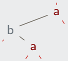
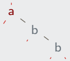
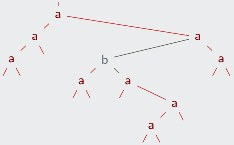
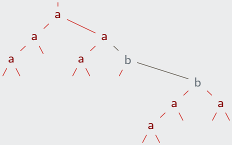
Example

If $\rightarrow$ is the rewrite rule satisfying 

$\rightarrow$ 

, one has



$\Rightarrow$ 

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Rewrite systems

Let $\rightarrow$ a rewrite rule on $\mathfrak{G}$-trees and $\Rightarrow$ be the rewrite relation induced by $\rightarrow$.

Let us define

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Two trees $\mathfrak{t}$ and $\mathfrak{t}'$ are linked if $\mathfrak{t} \overset{*}{\Leftrightarrow} \mathfrak{t}'$. Let $\mathbf{F}(\mathfrak{G})/_{\overset{*}{\Leftrightarrow}}$ be the set of all $\overset{*}{\Leftrightarrow}$-equivalence classes.

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A normal form for $\Rightarrow$ is a tree $\mathfrak{t}$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{t}'$ implies $\mathfrak{t} = \mathfrak{t}'$. Let $\mathcal{N}_{\Rightarrow}$ be the set of all normal forms.

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Theorem (Diamond property)

If $\Rightarrow$ is terminating and locally confluent, then $\Rightarrow$ is confluent.

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Theorem (Diamond property)

If $\Rightarrow$ is terminating and locally confluent, then $\Rightarrow$ is confluent.

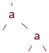
Proposition

Let $\rightarrow$ be a rewrite rule on $\mathbf{F}(\mathfrak{G})$. If $\Rightarrow$ is terminating and confluent, then $\mathcal{N}_{\Rightarrow}$ is

- the set of all $\mathfrak{G}$-trees avoiding the left members of $\rightarrow$;
- in a one-to-one correspondence respecting the arities with $\mathbf{F}(\mathfrak{G})/_{\overset{*}{\Leftrightarrow}}$.

Tamari lattices

Let $\to$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by 

$\to$ 

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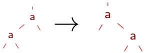
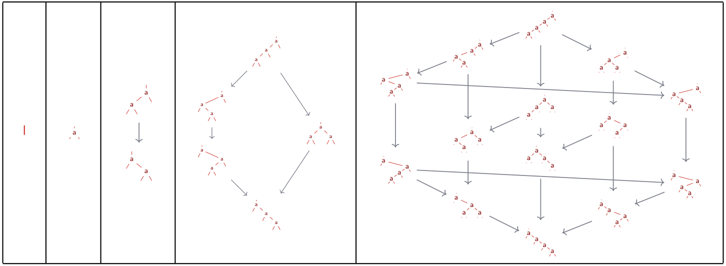
Let $\to$ be the rewrite rule on $\mathbf{F}(\{a\})$ defined by  .

First graphs $(\mathbf{F}(\{a\})(n), \Rightarrow)$:



1	2	3	4	5

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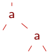
Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  $\rightarrow$  .

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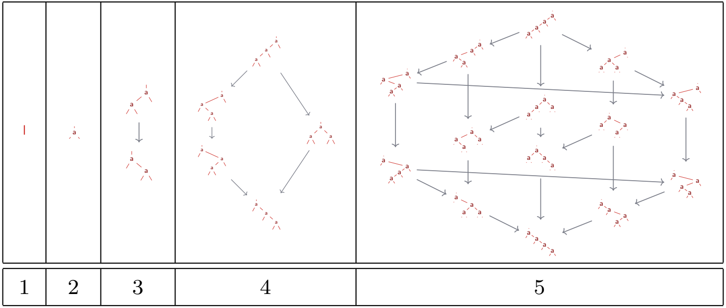

Properties:

- $\Rightarrow$ is terminating and confluent;
- $\mathcal{N}_{\Rightarrow}$ is the set of the trees avoiding , that are right comb trees;
- The sequence $\left(\mathbf{F}(\{\mathsf{a}\})/_{\overset{*}{\Leftrightarrow}}(n)\right)_{n \geqslant 1}$ is $1, 1, 1, 1, \ldots$.

A variant of Tamari lattices

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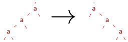
Let $\to$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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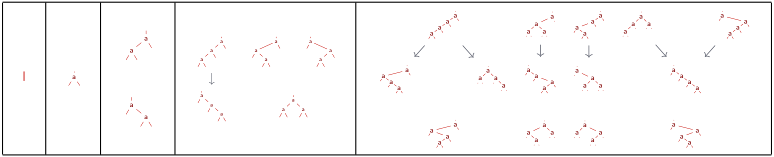

1	2	3	4	5

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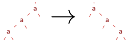
Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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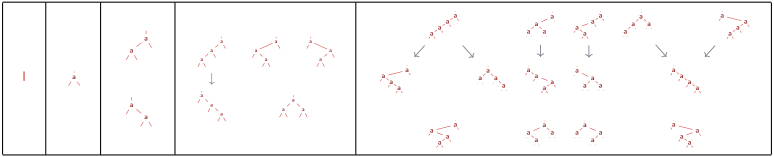

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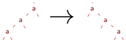
Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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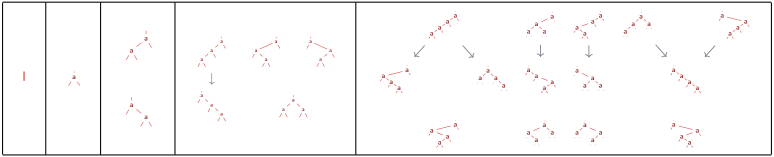

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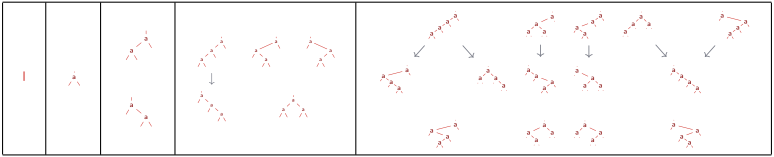

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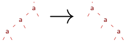
Let $\to$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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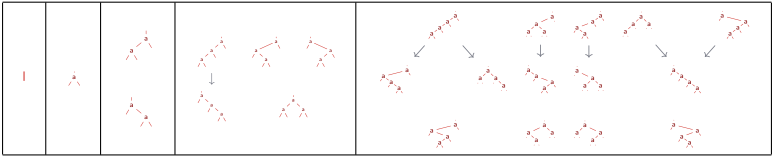

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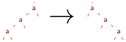
Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{a\})$ defined by  .

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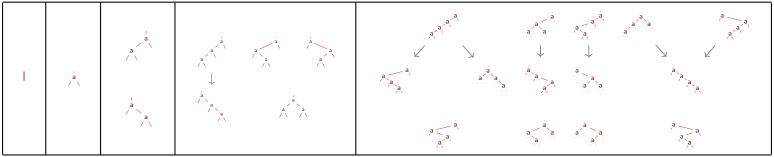

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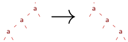
Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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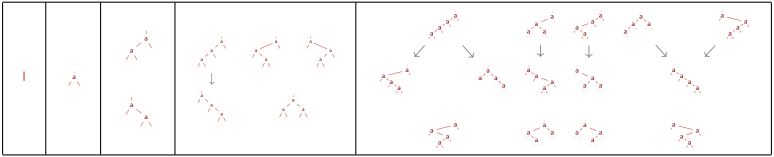

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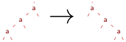
Let $\to$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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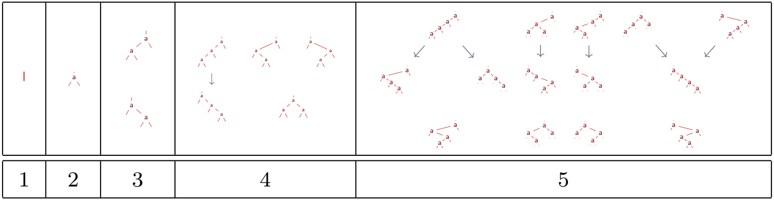

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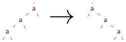
Let $\to$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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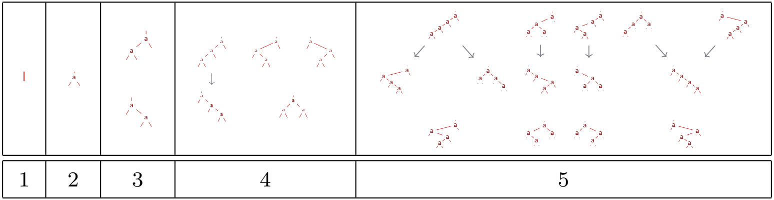

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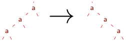
Let $\rightarrow$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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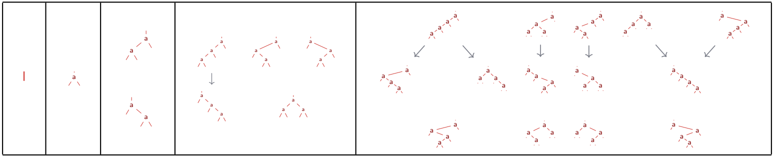

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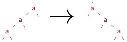
Let $\to$ be the rewrite rule on $\mathbf{F}(\{\mathsf{a}\})$ defined by  .

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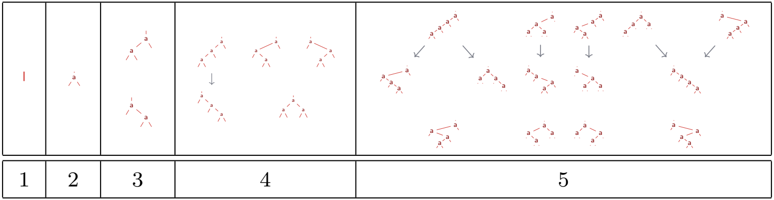

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and its generating function is

$$\frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right).$$

Outline

Tree series and pattern avoidance

Space of tree series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and $\mathfrak{G}$ be a set of letters.

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$$\langle \mathfrak{t}, \lambda \mathbf{f} \rangle := \lambda \langle \mathfrak{t}, \mathbf{f} \rangle ,$$

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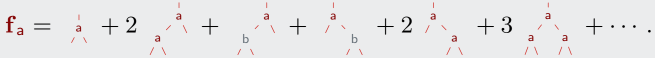
The sum notation of $\mathbf{f}$ is

$$\mathbf{f} = \sum_{\mathfrak{t} \in \mathbf{F}(\mathfrak{G})} \langle \mathfrak{t}, \mathbf{f} \rangle \, \mathfrak{t}.$$

Some tree series

Example

For $x \in \mathfrak{G}$, let $\mathbf{f}_x$ be the $\mathbf{F}(\mathfrak{G})$-series wherein $\langle \mathfrak{t}, \mathbf{f}_x \rangle$ is the number of occurrences of x in $\mathfrak{t}$. For instance,



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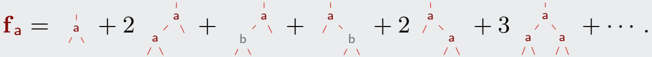

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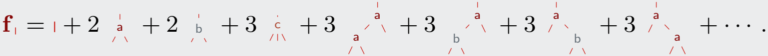

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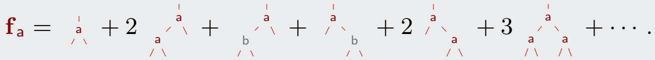

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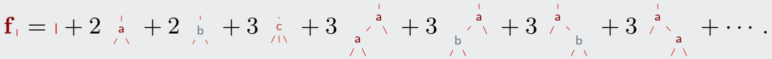

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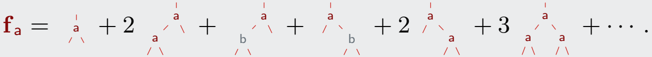

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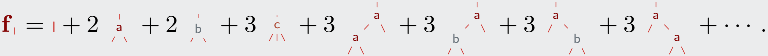

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Evaluation and generating series

Let $\mathcal{S}$ be a set of $\mathfrak{G}$-trees.

The characteristic series of $\mathcal{S}$ is the $\mathbf{F}(\mathfrak{G})$-série

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The evaluation map

$$\mathrm{ev} : \mathbb{K}\langle\langle \mathbf{F}(\mathfrak{G}) \rangle\rangle \to \mathbb{K}\langle\langle t \rangle\rangle$$

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$$\mathrm{ev}(\mathfrak{t}) = t^{|\mathfrak{t}|}.$$

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One has

$$\mathrm{ev}(\mathbf{f}_{\mathcal{S}}) = \sum_{\mathfrak{t} \in \mathcal{S}} t^{|\mathfrak{t}|} = \sum_{n \geqslant 1} \# \{\mathfrak{t} \in \mathcal{S} : |\mathfrak{t}| = n\} \, t^n = \mathcal{G}_{\mathcal{S}}(t)$$

where $\mathcal{G}_{\mathcal{S}}(t)$ is the generating series of $\mathcal{S}$, enumerating its elements w.r.t. the arity.

Composition of tree series

The composition of the $\mathbf{F}(\mathfrak{G})$-series $\mathbf{f}$ and $\mathbf{g}_1, \dots, \mathbf{g}_n$ is the series

$$\mathbf{f} \,\bar{\circ}\, [\mathbf{g}_1, \dots, \mathbf{g}_n] := \sum_{\substack{\mathfrak{t} \in \mathbf{F}(\mathfrak{G})(n) \\ \mathfrak{s}_1, \dots, \mathfrak{s}_n \in \mathbf{F}(\mathfrak{G})}} \left(\langle \mathfrak{t}, \mathbf{f} \rangle \prod_{i \in [n]} \langle \mathfrak{s}_i, \mathbf{g}_i \rangle \right) \mathfrak{t} \circ [\mathfrak{s}_1, \dots, \mathfrak{s}_n] .$$

Observe that this product is linear in all its arguments.

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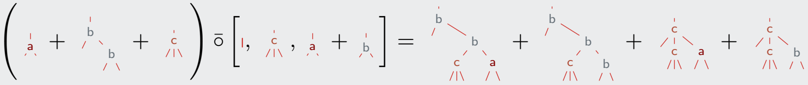

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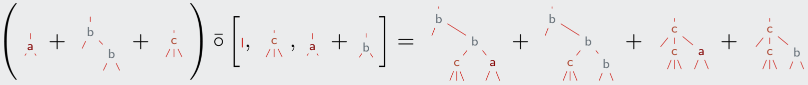

For all $\mathfrak{t} \in \mathbf{F}(\mathfrak{G})(n)$ and all $\mathbf{F}(\mathfrak{G})$-series $\mathfrak{g}_1, \dots, \mathfrak{g}_n$,

$$\operatorname{ev}(\mathfrak{t} \,\bar{\circ}\, [\mathfrak{g}_1, \dots, \mathfrak{g}_n]) = \prod_{i \in [n]} \operatorname{ev}(\mathfrak{g}_i) .$$

Tree series avoiding patterns

Let $\mathcal{P} \subseteq \mathbf{F}(\mathfrak{G})$ and set

$$\mathbf{f}(\mathcal{P}) := \mathbf{f}_{\mathrm{A}(\mathcal{P})} = \sum_{\substack{\mathfrak{t} \in \mathbf{F}(\mathfrak{G}) \\ \forall \mathfrak{s} \in \mathcal{P}, \mathfrak{s} \not\preccurlyeq \mathfrak{t}}} \mathfrak{t}$$

as the series of the $\mathfrak{G}$-trees avoiding all patterns of $\mathcal{P}$.

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When $\mathfrak{G}(1) = \emptyset$, each $\mathbf{F}(\mathfrak{G})(n)$ is finite and thus, there is a finite number of $\mathfrak{G}$-trees of arity n avoiding $\mathcal{P}$. Therefore, the series

$$\mathrm{ev}(\mathbf{f}(\mathcal{P})) = \mathcal{G}_{\mathrm{A}(\mathcal{P})}(t)$$

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Goal

Given $\mathfrak{G}$ and $\mathcal{P} \subseteq \mathbf{F}(\mathfrak{G})$, provide an expression for $\mathbf{f}(\mathcal{P})$.

Occurrences at root

A $\mathfrak{G}$-tree $\mathfrak{t}$ admits an occurrence of a $\mathfrak{G}$-tree $\mathfrak{s}$ at root if there exists $\mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|} \in \mathbf{F}(\mathfrak{G})$ and $i \in [|\mathfrak{r}|]$ such that

$$\mathfrak{t} = \mathfrak{s} \circ \left[\mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|}\right].$$

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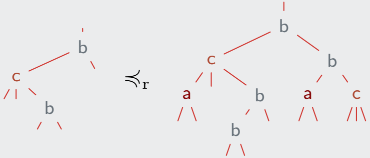

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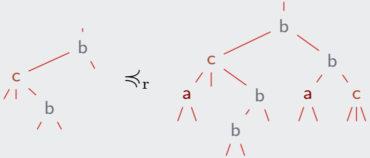

Assume that $\mathfrak{t} = \mathsf{a} \circ [\mathfrak{t}_1, \dots, \mathfrak{t}_k]$ and $\mathfrak{s} = \mathsf{a} \circ [\mathfrak{s}_1, \dots, \mathfrak{s}_k]$ where $\mathsf{a} \in \mathfrak{G}(k)$.

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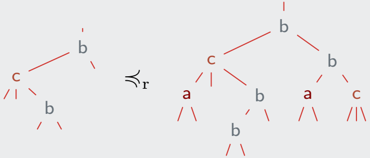

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Then, $\mathfrak{s} \not\preccurlyeq_{\mathrm{r}} \mathfrak{t}$ if and only if there exists an $i \in [k]$ such that $\mathfrak{s}_i \not\preccurlyeq_{\mathrm{r}} \mathfrak{t}_i$.

Admissible words

Let $\mathcal{P} \subseteq \mathbf{F}(\mathfrak{G})$ and $\mathsf{a} \in \mathfrak{G}(k)$. Let

$$\mathcal{P}_{\mathsf{a}} := \left\{ \mathfrak{s} \in \mathcal{P} : \mathsf{a} \preccurlyeq_{\mathrm{r}} \mathfrak{s} \right\}.$$

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A $\mathfrak{G}$-tree $\mathfrak{t} = \mathsf{a} \circ [\mathfrak{t}_1, \ldots, \mathfrak{t}_k]$ avoids at root all patterns of $\mathcal{P}$ if for all patterns $\mathfrak{s} = \mathsf{a} \circ [\mathfrak{s}_1, \ldots, \mathfrak{s}_k] \in \mathcal{P}_{\mathsf{a}}$, there is an $i \in [k]$ such that $\mathfrak{s}_i \not\preccurlyeq_{\mathrm{r}} \mathfrak{t}_i$.

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A word $(S_1, \dots, S_k)$ where letters are sets of $\mathfrak{G}$-trees different from $|$ is $\mathcal{P}_{\mathsf{a}}$-admissible if for any $\mathfrak{s} \in \mathcal{P}_{\mathsf{a}}$, there is an $i \in [k]$ such that $\mathfrak{s}_i \in S_i$.

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Let $\mathcal{P} :=$ 

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- $\left(\left\{\right.$  $\left.\right\}, \emptyset, \left\{\right.$  $\left.\right\}\right)$ is;

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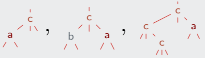
Let $\mathcal{P} :=$ {  }. In terms of $\mathcal{P}_{\mathsf{c}}$-admissibility, the word

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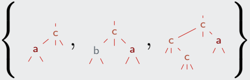
Let $\mathcal{P} :=$  . In terms of $\mathcal{P}_{\mathsf{c}}$-admissibility, the word

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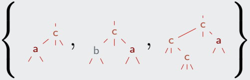
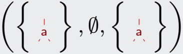
Let $\mathcal{P} :=$  . In terms of $\mathcal{P}_{\mathsf{c}}$-admissibility, the word

-  is;
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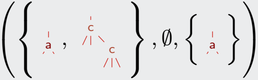
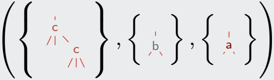
-  is;
-  is not.

Minimal admissible words

The union of two words $(S_1, \ldots, S_k)$ and $(S'_1, \ldots, S'_k)$ of sets of trees is defined by

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A $\mathcal{P}_{\mathsf{a}}$-admissible word u is minimal if any decomposition $u = v \oplus v'$ where v is a $\mathcal{P}_{\mathsf{a}}$-admissible word and v' is a word of sets of trees implies $u = v$.

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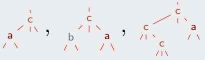
Let $\mathcal{P} :=$ {  }. In terms of minimality, as a $\mathcal{P}_{\mathsf{c}}$-admissible word,

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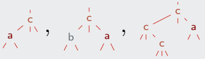
Let $\mathcal{P} :=$ {  }. In terms of minimality, as a $\mathcal{P}_{\mathsf{c}}$-admissible word,

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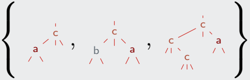
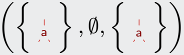
Let $\mathcal{P} :=$ . In terms of minimality, as a $\mathcal{P}_{\mathsf{c}}$-admissible word,

-  is;
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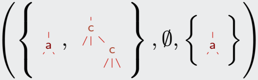
-  is not;

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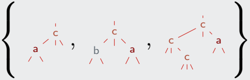
Let $\mathcal{P} :=$ 

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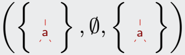
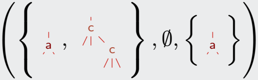
-  is;
-  is not;
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Back to tree series

Let $\mathcal{P}, \mathcal{R} \subseteq \mathbf{F}(\mathfrak{G})$.

Let the tree series

$$\mathbf{f}(\mathcal{P}, \mathcal{R}) := \sum_{\substack{\mathfrak{t} \in \mathbf{F}(\mathfrak{G}) \\ \forall \mathfrak{s} \in \mathcal{P}, \mathfrak{s} \not\preccurlyeq \mathfrak{t} \\ \forall \mathfrak{s} \in \mathcal{R}, \mathfrak{s} \not\preccurlyeq_{\mathrm{r}} \mathfrak{t}}} \mathfrak{t}$$

of the $\mathfrak{G}$-trees avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.

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of the $\mathfrak{G}$-trees avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.

If $(S_1, \ldots, S_k)$ is a $(\mathcal{P} \cup \mathcal{R})_{\mathsf{a}}$-admissible word, $\mathsf{a} \bar{\circ} [\mathbf{f}(\mathcal{P}, S_1), \ldots, \mathbf{f}(\mathcal{P}, S_k)]$ is the characteristic series of all the $\mathfrak{G}$-trees $\mathfrak{t} = \mathsf{a} \circ [\mathfrak{t}_1, \ldots, \mathfrak{t}_k]$ such that all $\mathfrak{t}_i$ avoid $\mathcal{P}$ and avoid S_i at root.

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of the $\mathfrak{G}$-trees avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.

If $(S_1, \dots, S_k)$ is a $(\mathcal{P} \cup \mathcal{R})_{\mathsf{a}}$-admissible word, $\mathsf{a} \bar{\circ} \left[\mathbf{f}\left(\mathcal{P}, S_1\right), \dots, \mathbf{f}\left(\mathcal{P}, S_k\right)\right]$ is the characteristic series of all the $\mathfrak{G}$-trees $\mathfrak{t} = \mathsf{a} \circ [\mathfrak{t}_1, \dots, \mathfrak{t}_k]$ such that all $\mathfrak{t}_i$ avoid $\mathcal{P}$ and avoid S_i at root.

Moreover, the support of the tree series

$$\sum_{(S_1, \dots, S_k) \in \mathrm{M}\left((\mathcal{P} \cup \mathcal{R})_{\mathsf{a}}\right)} \mathsf{a} \bar{\circ} \left[\mathbf{f}\left(\mathcal{P}, S_1\right), \dots, \mathbf{f}\left(\mathcal{P}, S_k\right)\right]$$

is the set of all $\mathfrak{G}$-trees with root labeled by a and avoiding $\mathcal{P}$ and avoiding $\mathcal{R}$ at root.

System of equations

Observe that for any $\mathcal{R}, \mathcal{R}' \subseteq \mathbf{F}(\mathfrak{G})$, the characteristic series of the $\mathfrak{G}$-trees avoiding $\mathcal{P}$, and avoiding $\mathcal{R}$ or $\mathcal{R}'$ at root is

$$\mathbf{f}(\mathcal{P}, \mathcal{R}) + \mathbf{f}\left(\mathcal{P}, \mathcal{R}'\right) - \mathbf{f}\left(\mathcal{P}, \mathcal{R} \cup \mathcal{R}'\right).$$

Therefore, the description of $\mathbf{f}(\mathcal{P}, \mathcal{R})$ uses the inclusion-exclusion principle.

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Therefore, the description of $\mathbf{f}(\mathcal{P}, \mathcal{R})$ uses the inclusion-exclusion principle.

Theorem [G., 2017–]

For any set $\mathfrak{G}$ of letters and $\mathcal{P}, \mathcal{R} \subseteq \mathbf{F}(\mathfrak{G})$,

$$\mathbf{f}(\mathcal{P}, \mathcal{R}) = \shortmid + \sum_{\substack{k \geqslant 1 \\ \mathsf{a} \in \mathfrak{G}(k)}} \; \sum_{\substack{\ell \geqslant 1 \\ \{u_1, \dots, u_\ell\} \subseteq \mathrm{M}\left((\mathcal{P} \cup \mathcal{R})_{\mathsf{a}}\right) \\ (S_1, \dots, S_k) := u_1 \oplus \cdots \oplus u_\ell}} (-1)^{1+\ell} \; \mathsf{a} \,\bar{\circ}\, \left[\mathbf{f}\left(\mathcal{P}, S_1\right), \dots, \mathbf{f}\left(\mathcal{P}, S_k\right)\right].$$

System of equations

Observe that for any $\mathcal{R}, \mathcal{R}' \subseteq \mathbf{F}(\mathfrak{G})$, the characteristic series of the $\mathfrak{G}$-trees avoiding $\mathcal{P}$, and avoiding $\mathcal{R}$ or $\mathcal{R}'$ at root is

$$\mathbf{f}(\mathcal{P}, \mathcal{R}) + \mathbf{f}(\mathcal{P}, \mathcal{R}') - \mathbf{f}(\mathcal{P}, \mathcal{R} \cup \mathcal{R}').$$

Therefore, the description of $\mathbf{f}(\mathcal{P}, \mathcal{R})$ uses the inclusion-exclusion principle.

Theorem [G., 2017–]

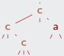
For any set $\mathfrak{G}$ of letters and $\mathcal{P}, \mathcal{R} \subseteq \mathbf{F}(\mathfrak{G})$,

$$\mathbf{f}(\mathcal{P}, \mathcal{R}) = \mathsf{I} + \sum_{\substack{k \geqslant 1 \\ \mathsf{a} \in \mathfrak{G}(k)}} \sum_{\substack{\ell \geqslant 1 \\ \{u_1, \dots, u_\ell\} \subseteq \mathrm{M}\left((\mathcal{P} \cup \mathcal{R})_{\mathsf{a}}\right) \\ (S_1, \dots, S_k) := u_1 \oplus \cdots \oplus u_\ell}} (-1)^{1+\ell} \ \mathsf{a} \bar{\circ} \left[\mathbf{f}(\mathcal{P}, S_1), \dots, \mathbf{f}(\mathcal{P}, S_k)\right].$$

Since in particular $\mathbf{f}(\mathcal{P}) = \mathbf{f}(\mathcal{P}, \emptyset)$ this provides a system of equations describing $\mathbf{f}(\mathcal{P})$.

Main equation for the previous example

Example

Let $\mathcal{P} := \Big\{$  ,  ,  $\Big\}$.

One has $\mathrm{M}(\mathcal{P}_{\mathsf{a}}) = \mathrm{M}(\mathcal{P}_{\mathsf{b}}) = \{(\emptyset, \emptyset)\}$ and

$\mathrm{M}(\mathcal{P}_{\mathsf{c}}) = \Big\{ \Big(\big\{$  $\big\}, \emptyset, \big\{$  $\big\} \Big), \Big(\Big\{$  ,  ,  $\Big\}, \emptyset, \emptyset \Big) \Big\}$.

Therefore,

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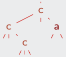
, 

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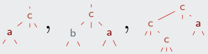
Let $\mathcal{P} := \Bigg\{$  $\Bigg\}$.

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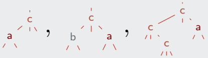
Let $\mathcal{P} :=$ $\Bigg\{$  $\Bigg\}$.

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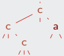
Let $\mathcal{P} := \Big\{$  ,  ,  $\Big\}$.

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Example: directed animals

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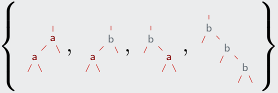
Let $\mathcal{P} :=$  .

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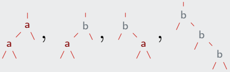
Let $\mathcal{P} := \Big\{$  $\Big\}$.

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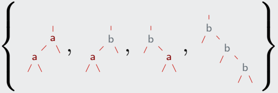
Let $\mathcal{P} :=$ .

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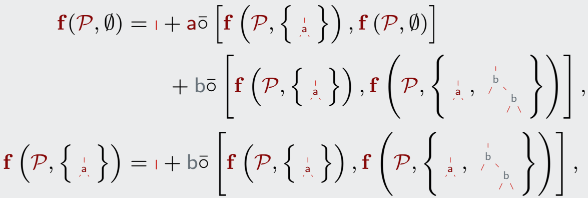

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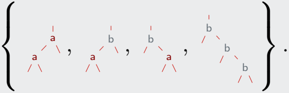
Let $\mathcal{P} :=$ 

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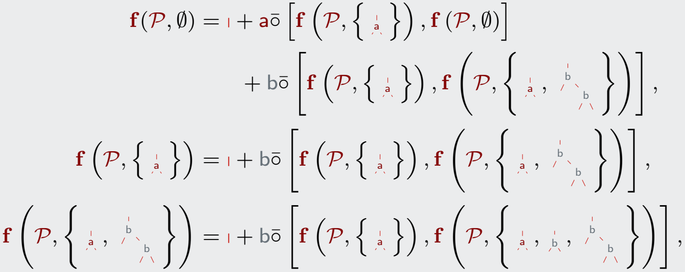

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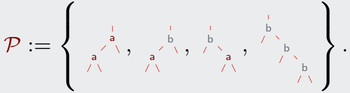

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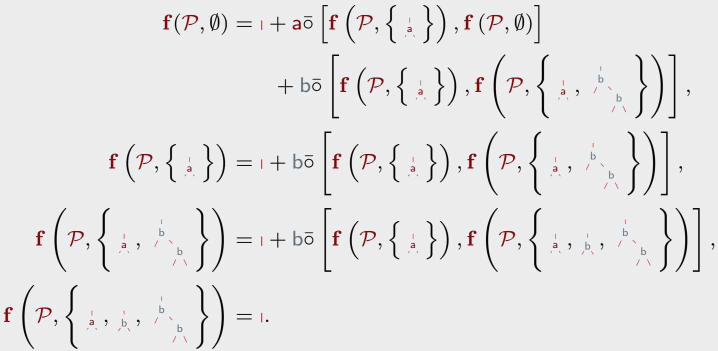

Example: directed animals

Example

By evaluating each member of the previous system, one obtains the system

$$\begin{aligned} \mathcal{G}_{\mathcal{S}}(t) &= t + \mathcal{G}_{\mathcal{S}_1}(t)\mathcal{G}_{\mathcal{S}}(t) + \mathcal{G}_{\mathcal{S}_1}(t)\mathcal{G}_{\mathcal{S}_2}(t), \\ \mathcal{G}_{\mathcal{S}_1}(t) &= t + \mathcal{G}_{\mathcal{S}_1}(t)\mathcal{G}_{\mathcal{S}_2}(t), \\ \mathcal{G}_{\mathcal{S}_2}(t) &= t + \mathcal{G}_{\mathcal{S}_1}(t)\mathcal{G}_{\mathcal{S}_3}(t), \\ \mathcal{G}_{\mathcal{S}_3}(t) &= t \end{aligned}$$

for the generating series $\mathcal{G}_{\mathcal{S}}(t)$ of directed animals.

This leads to

$$t + (3t - 1)\mathcal{G}_{\mathcal{S}}(t) + (3t - 1)\mathcal{G}_{\mathcal{S}}(t)^2 = 0,$$

an algebraic equation satisfied by $\mathcal{G}_{\mathcal{S}}(t)$.

Some remarks

The previous result includes, as special cases:

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Other systems of equations have been described for enumerating trees avoiding patterns in [**Khoroshkin, Piontkovski**, 2012].

Computing admissible words

Given a $\mathfrak{G}$-tree $\mathfrak{t} = \mathsf{a} \circ [\mathfrak{t}_1, \ldots, \mathfrak{t}_k]$, let the element

$$\phi(\mathfrak{t}) := \sum_{\substack{i \in [k] \\ \mathfrak{t}_i \neq \shortmid}} \left(\underbrace{\emptyset, \ldots, \emptyset}_{i-1}, \{\mathfrak{t}_i\}, \underbrace{\emptyset, \ldots, \emptyset}_{k-i} \right)$$

of the free module $\mathbb{B}\left\langle \left(2^{\mathbf{F}(\mathfrak{G})}\right)^k \right\rangle$ on the Boolean semiring $\mathbb{B}$.

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Let the linear combination

$$e_{\mathcal{P}_\mathsf{a}} := \bigoplus_{\mathfrak{t} \in \mathcal{P}_\mathsf{a}} \phi(\mathfrak{t}),$$

containing all $\mathcal{P}_\mathsf{a}$-admissible words (and other $\mathcal{P}_\mathsf{a}$-admissible words).

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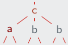
, 

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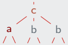
Let $\mathcal{P} :=$ {  ,  }. One has $e_{\mathcal{P}_\mathsf{a}} = e_{\mathcal{P}_\mathsf{b}} = (\emptyset, \emptyset)$ and

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Example

Let $\mathcal{P} := \left\{ \mathsf{c} \circ [\mathsf{a}, \mathsf{a}], \; \mathsf{c} \circ [\mathsf{a}, \mathsf{b}, \mathsf{b}] \right\}$. One has $e_{\mathcal{P}_\mathsf{a}} = e_{\mathcal{P}_\mathsf{b}} = (\emptyset, \emptyset)$ and

$$e_{\mathcal{P}_\mathsf{c}} = \Big(\big(\{\mathsf{a}\}, \emptyset, \emptyset \big) + \big(\emptyset, \mathsf{a}, \emptyset \big) \Big) \oplus \Big(\big(\{\mathsf{a}\}, \emptyset, \emptyset \big) + \big(\emptyset, \mathsf{b}, \emptyset \big) + \big(\emptyset, \emptyset, \mathsf{b} \big) \Big)$$

$$= \big(\{\mathsf{a}\}, \emptyset, \emptyset \big) + \big(\{\mathsf{a}\}, \{\mathsf{b}\}, \emptyset \big) + \big(\{\mathsf{a}\}, \emptyset, \{\mathsf{b}\} \big)$$

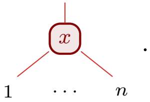
$$+ \big(\{\mathsf{a}\}, \{\mathsf{a}\}, \emptyset \big) + \big(\emptyset, \{\mathsf{a}, \mathsf{b}\}, \emptyset \big) + \big(\emptyset, \{\mathsf{a}\}, \{\mathsf{b}\} \big).$$

Outline

Operads and enumeration

Operators

An operator is an entity having $n \geqslant 1$ inputs and a single output:



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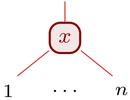

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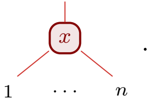

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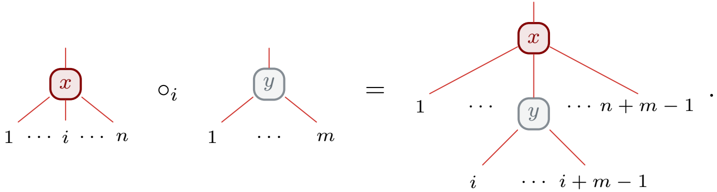
This produces a new operator $x \circ_i y$ of arity $n + m - 1$:



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A (nonsymmetric set-theoretic) operad is a triple $(\mathcal{O}, \circ_i, \mathbb{1})$ where

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This data has to satisfy some axioms.

Operad axioms

Associativity:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$

Operad axioms

Associativity:

$(x \circ_i y)$

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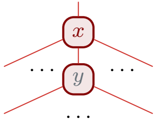

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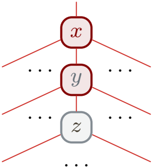

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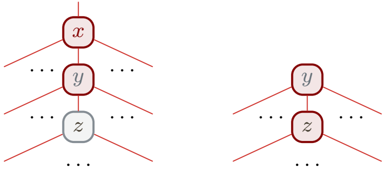

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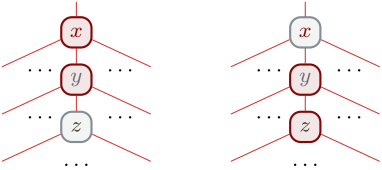

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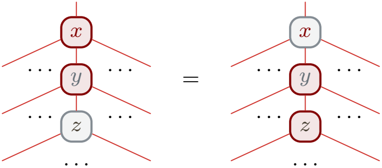

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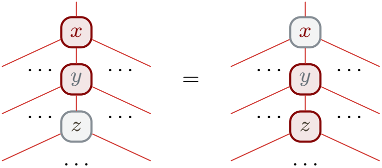

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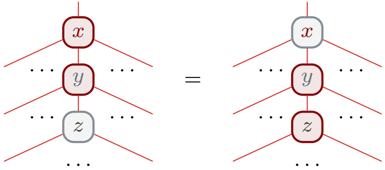

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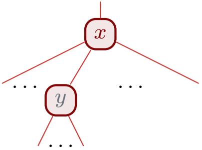

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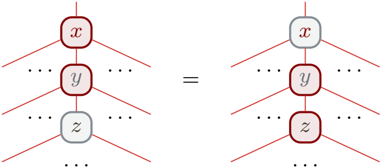

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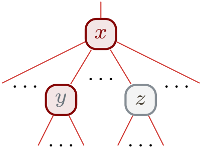

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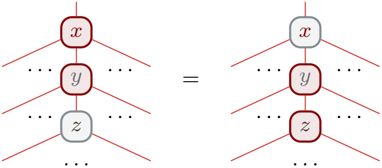

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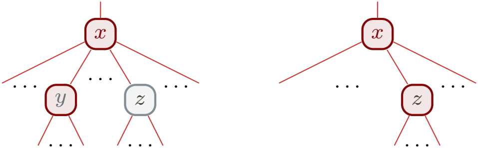

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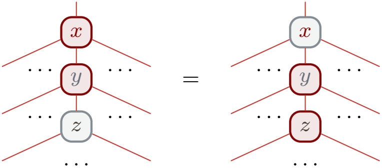

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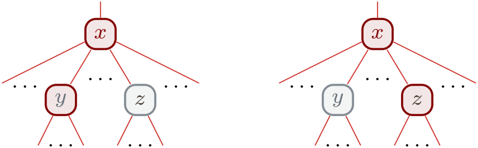

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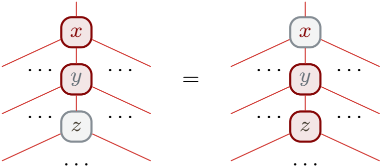

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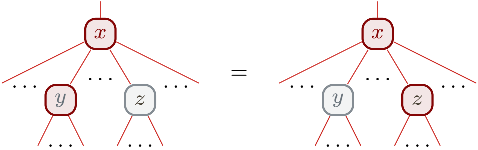

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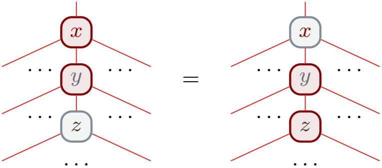

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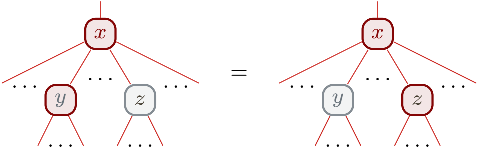

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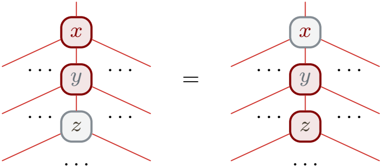

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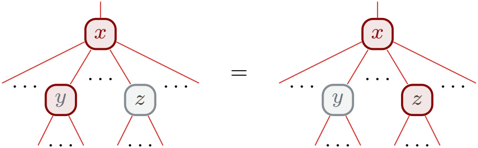

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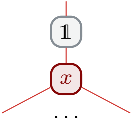

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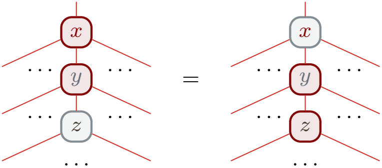

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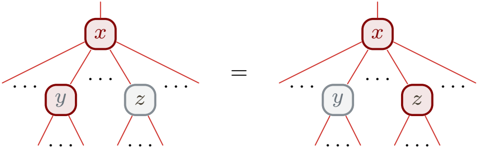

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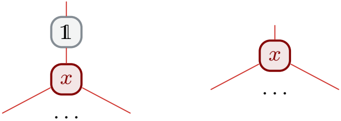

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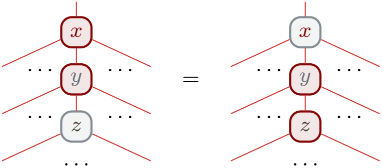

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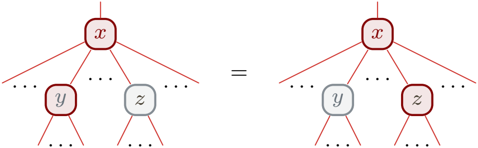

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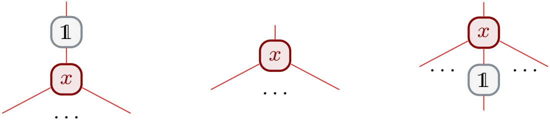

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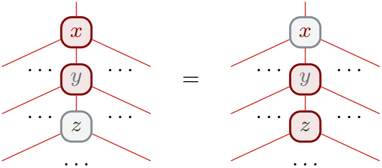

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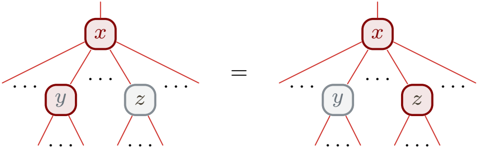

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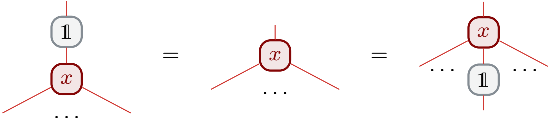

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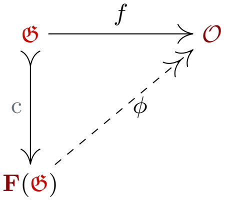
Free operads satisfy the following universality property.

For any set $\mathfrak{G}$ of letters, any operad $\mathcal{O}$, and any map $f : \mathfrak{G} \to \mathcal{O}$ respecting the arities, there exists a unique operad morphism $\phi : \mathbf{F}(\mathfrak{G}) \to \mathcal{O}$ such that $f = \phi \circ \mathrm{c}$.



An operad on paths

Let **Paths** be the operad wherein:

- **Paths**(n) is the set of all paths with n points, that are words $u_1 \dots u_n$ of elements of $\mathbb{N}$.

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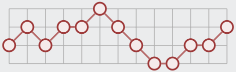

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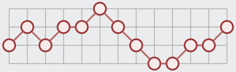

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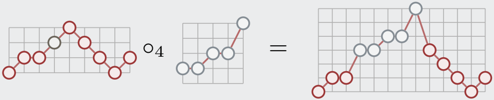

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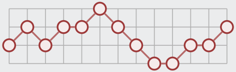

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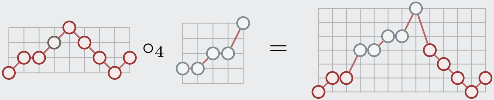
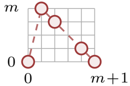


$$011232101 \circ_4 11224 = 0113344632101$$

- The unit is the path 0, depicted as ○, having arity 1.

Suboperad on m-Dyck paths

Let for any $m \geqslant 0$ the suboperad $m\mathbf{Dyck}$ of $\mathbf{Paths}$ generated by $\mathfrak{G}_{m\mathbf{Dyck}} := \mathfrak{G}_{m\mathbf{Dyck}}(m+2) := \{\mathfrak{g}_m\}$ where

$$\mathfrak{g}_m := 0\, m\, (m-1)\, \ldots\, 1\, 0 =$$ 

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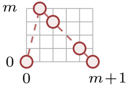
$$\mathfrak{g}_m := 0\, m\, (m-1)\, \ldots\, 1\, 0 =$$ 

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Example

The elements of $2\mathbf{Dyck}$ are, by definition, the paths obtained by composing $\mathfrak{g}_m$ with itself.

- $2\mathbf{Dyck}(1) = \{$  $\}$;
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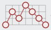
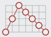
- $2\mathbf{Dyck}(7) = \{$ , ,  $\}$;
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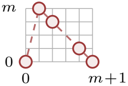
$$\mathfrak{g}_m := 0\, m\, (m-1) \ldots 1\, 0 =$$ 

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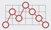
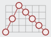
- $2\mathbf{Dyck}(7) = \{$ , ,  $\}$;
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Proposition

For any $m \geqslant 0$ and $n \geqslant 1$, $m\mathbf{Dyck}(n)$ is the set of all m-Dyck paths of length $n - 1$.

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The presentation $(\mathfrak{G}, \equiv)$ is

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- quadratic when $\equiv$ is generated as an operad congruence by an equivalence relation on trees concentrated in degree 2.

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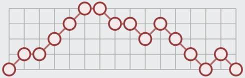
This says that all relations in higher degrees are consequence of this single one and the operad axioms.

An operad on Motzkin paths

Let **Motz** be an operad wherein:

- **Motz**(n) is the set of all Motzkin paths with n points.

Example



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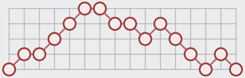 is a Motzkin path of arity 16.

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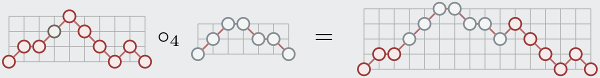

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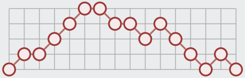 is a Motzkin path of arity 16.

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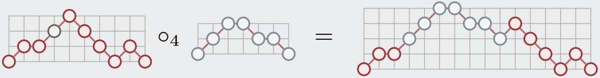

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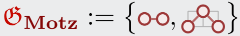
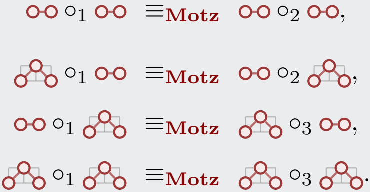


and $\equiv$ is the smallest operad congruence satisfying



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Let $\mathcal{O}$ be an operad admitting a presentation $(\mathfrak{G}, \equiv)$. If

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Such a PBW basis of $\mathcal{O}$ can be described as the set of the trees avoiding the trees appearing as left members for $\to$.

PBW basis of Motz

Let $\to$ be the rewrite rule on $\mathbf{F}\left(\mathfrak{G}_{\mathbf{Motz}}\right)$ defined by

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Example

A normal form for $\Rightarrow$ and the Motzkin path in correspondence with it:

An operad on cyclic paths

For any $\ell \geqslant 1$, let $\ell\mathbf{CPaths}$ be the operad wherein:

- $\ell\mathbf{CPaths}(n)$ is the set of all paths with n points having height smaller than ℓ, that are words $u_1 \dots u_n$ of elements of $\{0, \dots, \ell - 1\}$.

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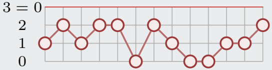

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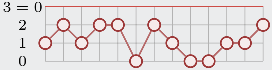

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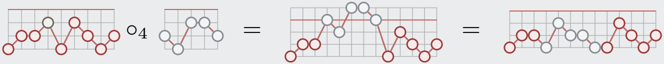
In $3\mathbf{CPaths}$,



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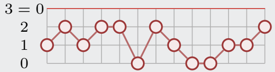

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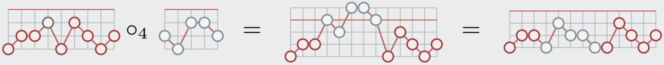

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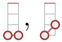
$$\mathfrak{G}_{\mathbf{DA}} := \left\{ \right.$$  $$\left. \right\} .$$

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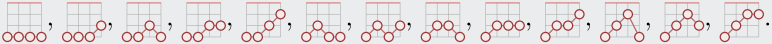

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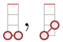

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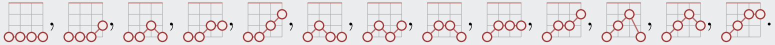


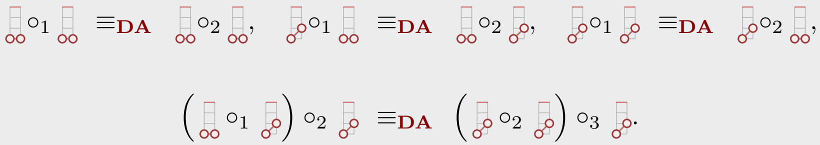
Proposition

For any $n \geqslant 1$, $\mathbf{DA}(n)$ is in one-to-one correspondence with the set of directed animals of size n.

Presentation and PBW basis of $\mathbf{DA}$

Proposition

$\mathbf{DA}$ admits the presentation $(\mathfrak{G}_{\mathbf{DA}}, \equiv)$ where $\equiv_{\mathbf{DA}}$ is the smallest congruence of $\mathbf{F}(\mathfrak{G}_{\mathbf{DA}})$ satisfying



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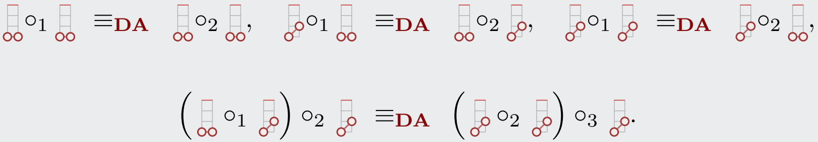

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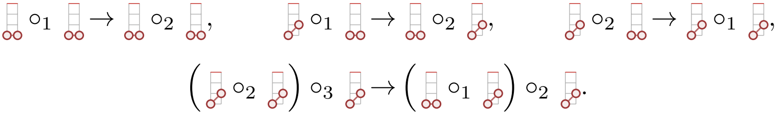

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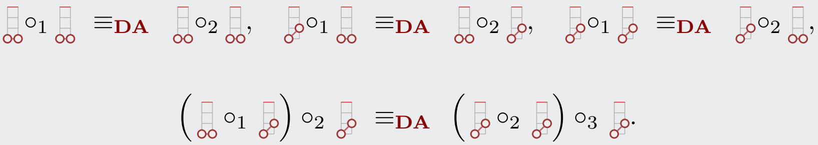

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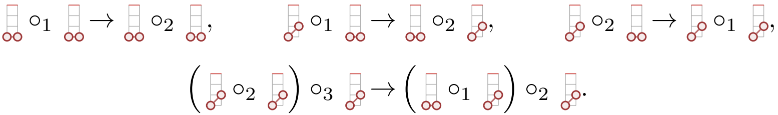
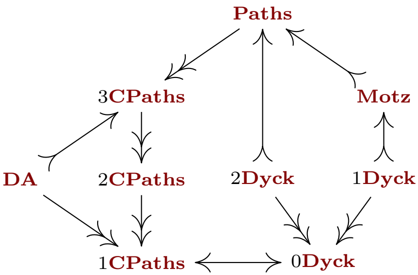


The rewrite relation $\Rightarrow$ induced by $\to$ is terminating and confluent. The $\mathfrak{G}_{\mathbf{DA}}$-trees avoiding the left members of $\to$ form a PBW basis of $\mathbf{DA}$.

Overview of the considered operads



Dimensions:

n	1	2	3	4	5	6	7	8	9	10
Paths	∞									
0**Dyck**	1	1	1	1	1	1	1	1	1	1
1**Dyck**	1	0	1	0	2	0	5	0	14	0
2**Dyck**	1	0	0	1	0	0	3	0	0	12
Motz	1	1	2	4	9	21	51	127	323	835
2**CPaths**	2	4	8	16	32	64	128	256	512	1024
3**CPaths**	3	9	27	81	243	729	2187	6561	19683	59049
DA	1	2	5	13	35	96	267	750	2123	6046

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Finally, $\mathrm{ev}(\mathbf{f}(\mathcal{P}))$ is the Hilbert series of $\mathcal{O}$ and the generating series of $\mathcal{S}$.

Hilbert series of $m\mathbf{Dyck}$

The set $\mathcal{B}$ of the $\mathfrak{G}_{m\mathbf{Dyck}}$-trees avoiding

$$\mathcal{P} := \{\mathfrak{g}_m \circ_1 \mathfrak{g}_m\}$$

is a PBW basis of $m\mathbf{Dyck}$.

The characteristic series of $\mathcal{B}$ is $\mathbf{f}(\mathcal{P}, \emptyset)$ where

$$\mathbf{f}(\mathcal{P}, \emptyset) = | + \mathfrak{g}_m \bar{\circ} \left[\mathbf{f}(\mathcal{P}, \{\mathfrak{g}_m\}), \underbrace{\mathbf{f}(\mathcal{P}, \emptyset), \dots, \mathbf{f}(\mathcal{P}, \emptyset)}_{m+1} \right],$$

$$\mathbf{f}(\mathcal{P}, \{\mathfrak{g}_m\}) = |.$$

By setting $\mathcal{H}(t) := \mathrm{ev}(\mathbf{f}(\mathcal{P}, \emptyset))$, the Hilbert series $\mathcal{H}(t)$ of $m\mathbf{Dyck}$ satisfies

$$t - \mathcal{H}(t) + t\mathcal{H}(t)^{m+1} = 0.$$

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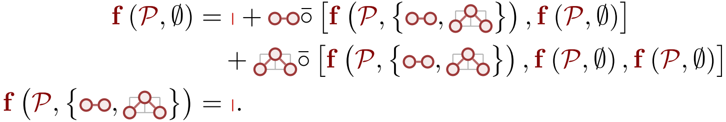

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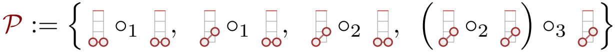

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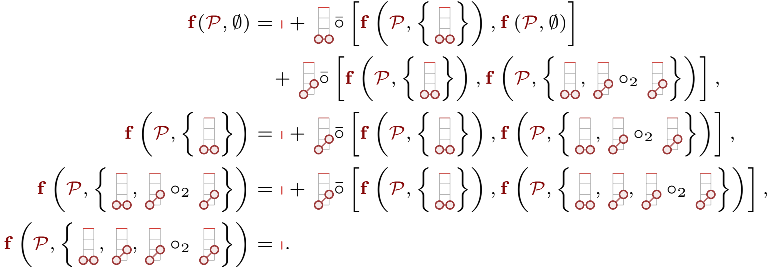

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