Some combinatorial structures related to operads

Samuele Giraudo

LIGM, Université Paris-Est Marne-la-Vallée

Séminaire Philippe Flajolet

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Outline

Types of algebraic structures and operads

From monoids to operads

Operads as tools for enumeration

Pairs of graded graphs

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Types of algebraic structures

Algebraic combinatorics deals with sets (or spaces) of structured objects:

- monoids;
- groups;
- lattices;
- associative alg.;
- Hopf bialg.;
- Lie alg.;
- pre-Lie alg.;
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— Example —

The type of monoids can be specified by

1. the operations $\star$ (binary) and $\mathbb{1}$ (nullary);
2. the relations $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$ and $x \star \mathbb{1} = x = \mathbb{1} \star x$.

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  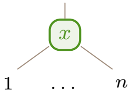

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  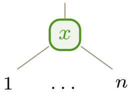

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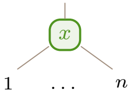

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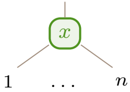

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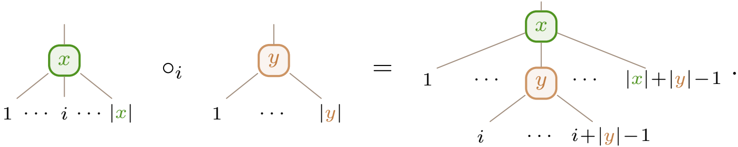

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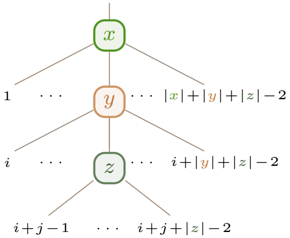
This data has to satisfy some axioms.

Operad axioms

The associativity relation

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$

says that the pictured operation can be constructed from top to bottom or from bottom to top.



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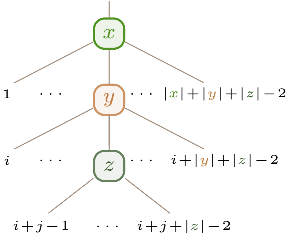

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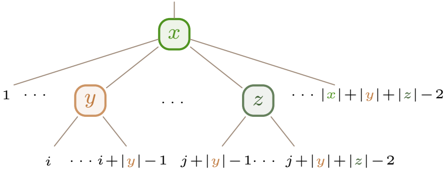

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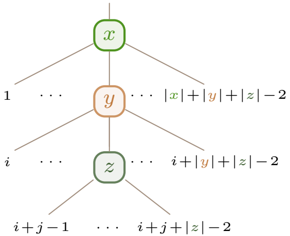

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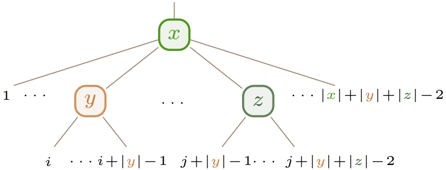
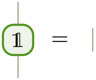


The unitality relation

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}$$
$$1 \leqslant i \leqslant |x|$$

says that $\mathbb{1}$ is the identity map.



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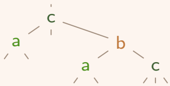

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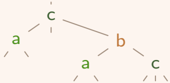

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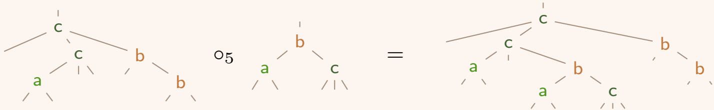

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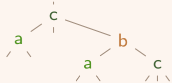

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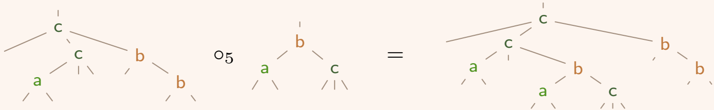

- The unit is the leaf $\shortmid$.

Algebras over operads

Let $\mathcal{O}$ be an operad. An algebra over $\mathcal{O}$ is a space $\mathcal{V}$ equipped, for all $x \in \mathcal{O}(n)$, with linear maps

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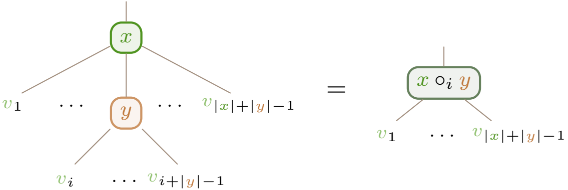

holds for any $x, y \in \mathcal{O}$, $i \in [|x|]$, and $v_1, \ldots, v_{|x|+|y|-1} \in \mathcal{V}$.

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In the same way, there are operads for

- Lie alg.;
- pre-Lie alg. [Chapoton, Livernet, 2001];
- dendriform alg. [Loday, 2001];
- duplicial alg. [Loday, 2008];
- diassociative alg. [Loday, 2001];
- brace alg.

Scope of operads

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Endowing a set of combinatorial objects with an operad structure helps to

- highlight elementary building block for the objects;
- build combinatorial structures on the objects (posets, lattices, *etc.*);
- enumerative prospects and discovery of statistics.

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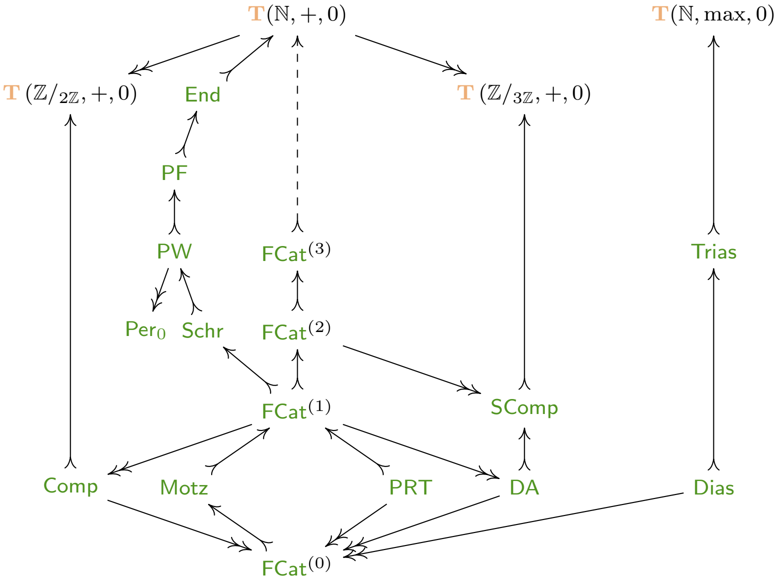
— Theorem [G., 2015] —

For any monoid $\mathcal{M}$, $\mathbf{T}\mathcal{M}$ is an operad.

Some combinatorial suboperads

Monoid	Operad	Generators	First dimensions	Combinatorial objects
$(\mathbb{N}, +, 0)$	End	–	$1, 4, 27, 256, 3125$	Endofunctions
	PF	–	$1, 3, 16, 125, 1296$	Parking functions
	PW	–	$1, 3, 13, 75, 541$	Packed words
	Per_0	–	$1, 2, 6, 24, 120$	Permutations
	PRT	01	$1, 1, 2, 5, 14, 42$	Planar rooted trees
	$\mathsf{FCat}^{(m)}$	$00, 01, \dots, 0m$	Fuß-Catalan numbers	m-trees
	Schr	$00, 01, 10$	$1, 3, 11, 45, 197$	Schröder trees
	Motz	$00, 010$	$1, 1, 2, 4, 9, 21, 51$	Motzkin words
$(\mathbb{Z}/_{2\mathbb{Z}}, +, 0)$	Comp	$00, 01$	$1, 2, 4, 8, 16, 32$	Compositions
$(\mathbb{Z}/_{3\mathbb{Z}}, +, 0)$	DA	$00, 01$	$1, 2, 5, 13, 35, 96$	Directed animals
	SComp	$00, 01, 02$	$1, 3, 27, 81, 243$	Seg. compositions
$(\mathbb{N}, \max, 0)$	Dias	$01, 10$	$1, 2, 3, 4, 5$	Bin. words with exact. one 0
	Trias	$00, 01, 10$	$1, 3, 7, 15, 31$	Bin. words with at least one 0

Diagram of operads



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- $\mathsf{Comp}(2) = \{00, 01\}$;
- $\mathsf{Comp}(3) = \{000 = 00 \circ_1 00 = 00 \circ_2 00, \qquad 001 = 01 \circ_1 00 = 00 \circ_2 01,$
  $010 = 00 \circ_1 01 = 01 \circ_2 01, \qquad 011 = 01 \circ_1 01 = 01 \circ_2 00\}$.

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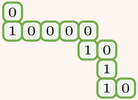
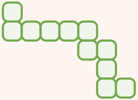
For any $n \geqslant 1$, $\mathsf{Comp}(n)$ is the set of all the words of length n on $\{0, 1\}$ beginning by 0.

There is a one-to-one correspondence between $\mathsf{Comp}(n)$ and the set of all ribbon diagrams with n boxes (0: new box at right, 1: new box below).

— Example —

$01000010110 \quad \longleftrightarrow$ 

$\longleftrightarrow$ 

Operad of integer compositions

Under this realization, the partial composition of Comp is described as follows.

The ribbon $\mathfrak{r} \circ_i \mathfrak{s}$ is obtained by inserting $\mathfrak{s}$ (resp. the transpose of $\mathfrak{s}$) into the ith box of $\mathfrak{r}$ when this box is (resp. is not) the highest of its column.

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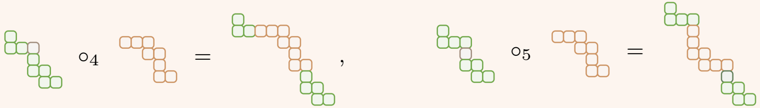

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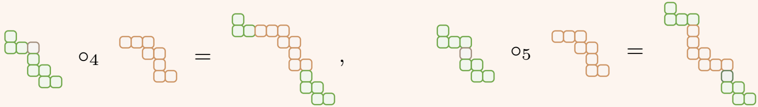
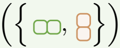


— Proposition [G., 2015] **—**

The operad Comp is the quotient of **F**  by the finest operad congruence $\equiv$ satisfying

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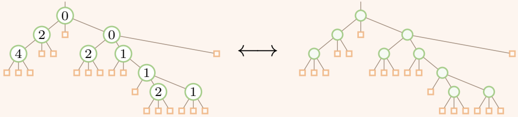
One-to-one correspondence between $\mathsf{FCat}^{(m)}(n)$ and the set of all m-trees (planar rooted trees where internal nodes have $m+1$ children) by inserting iteratively a node on the leaf specified by the letter (from right to left).

— Example —

When $m = 2$,

$024021121 \longleftrightarrow$



Generalization of the Stanley poset

There is a byproduct: for any $u, v \in \mathsf{FCat}^{(m)}(n)$, we set $u \preceq v$ if $u_i \leqslant v_i$ for all $i \in [n]$.

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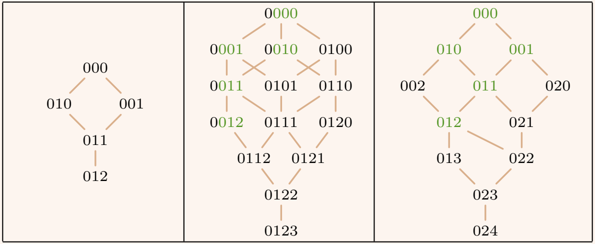

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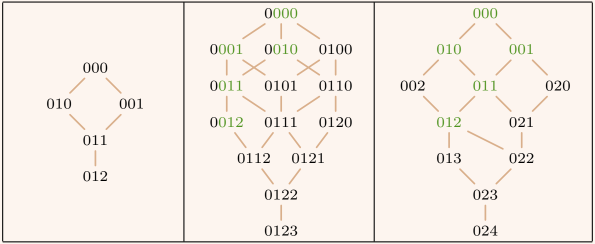

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The $\left(\mathsf{FCat}^{(1)}(n), \preceq\right)$, $n \geqslant 1$, are the Stanley posets (usually described in terms of inclusion of Dyck paths).

Generalization of the Stanley poset

There are injections

$$\iota : \mathsf{FCat}^{(m)}(n) \to \mathsf{FCat}^{(m+1)}(n), \quad \iota(u) := u,$$
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- number of intervals
  - $m = 1$: $1, 3, 14, 84, 594, 4719, \dots$ (**A005700**)
  - $m = 2$: $1, 6, 66, 1001, 18564, 395352, \dots$ (unknown)
  - $m = 3$: $1, 10, 200, 5700, 210894, \dots$ (unknown)

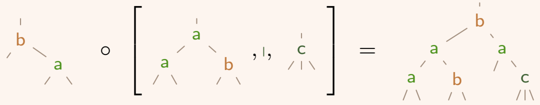
Outline

Operads as tools for enumeration

Factors and prefixes

Let $\mathfrak{t}, \mathfrak{s}_1, \ldots, \mathfrak{s}_{|\mathfrak{t}|}$ be $\mathfrak{G}$-trees. The tree $\mathfrak{t} \circ \left[\mathfrak{s}_1, \ldots, \mathfrak{s}_{|\mathfrak{t}|}\right]$ is obtained by grafting simultaneously the roots of each $\mathfrak{s}_i$ onto the ith leaf of $\mathfrak{t}$.

— Example —



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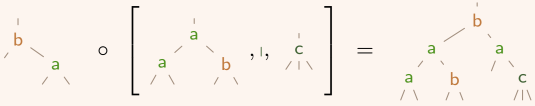

If $\mathfrak{t}$ writes as $\mathfrak{t} = \mathfrak{r} \circ_i \left(\mathfrak{s} \circ \left[\mathfrak{r}_1, \ldots, \mathfrak{r}_{[\mathfrak{s}|}\right]\right)$ for some trees $\mathfrak{s}, \mathfrak{r}, \mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|}$, and $i \in [|\mathfrak{r}|]$, $\mathfrak{s}$ is a factor of $\mathfrak{t}$ (denoted by $\mathfrak{s} \preccurlyeq_{\mathrm{f}} \mathfrak{t}$).

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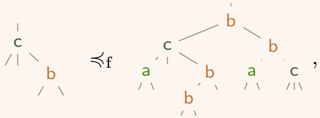

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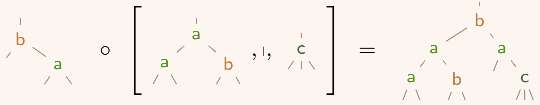

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When moreover $\mathfrak{r} = \mid$, $\mathfrak{s}$ is a prefix of $\mathfrak{t}$ (denoted by $\mathfrak{s} \preccurlyeq_{\mathrm{p}} \mathfrak{t}$).

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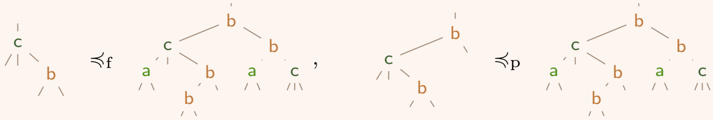

Rewrite systems on trees

A rewrite rule is a binary relation $\rightarrow$ on $\mathbf{F}(\mathfrak{G})$ such that $\mathfrak{s} \rightarrow \mathfrak{s}'$ implies $|\mathfrak{s}| = |\mathfrak{s}'|$.

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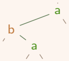
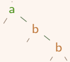
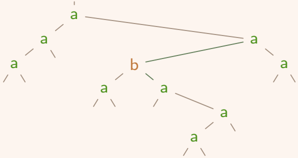
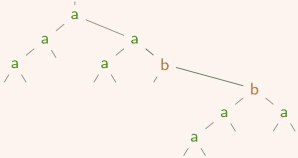
If $\rightarrow$ is the rewrite rule satisfying  $\rightarrow$ , one has

 $\Rightarrow$  .

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Let $(\mathbf{F}(\mathfrak{G}), \to)$ be a rewrite system.

A normal form for $\Rightarrow$ is a tree $\mathfrak{t}$ such that there is no $\mathfrak{t}'$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$.
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— Proposition —

Let $(\mathbf{F}(\mathfrak{G}), \to)$ be a rewrite system. If $\Rightarrow$ is terminating and confluent, then for any $n \geqslant 1$, $\mathcal{N}_{\to}(n)$ is

- in a one-to-one correspondence with the connected components of the graph $(\mathbf{F}(\mathfrak{G})(n), \Rightarrow)$;
- the set of all $\mathfrak{G}$-trees of arity n factor-avoiding $\mathcal{P}_{\to}$, the set of the left members of $\to$.

Tamari lattices

Let $(\mathbf{F}(\{a\}), \to)$ be the rewrite system defined by  $\to$  .

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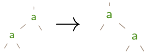
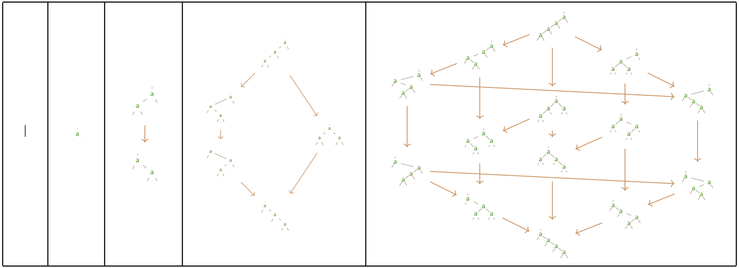
Let $(\mathbf{F}(\{\mathsf{a}\}), \to)$ be the rewrite system defined by  .

First graphs $(\mathbf{F}(\{\mathsf{a}\})(n), \Rightarrow)$:



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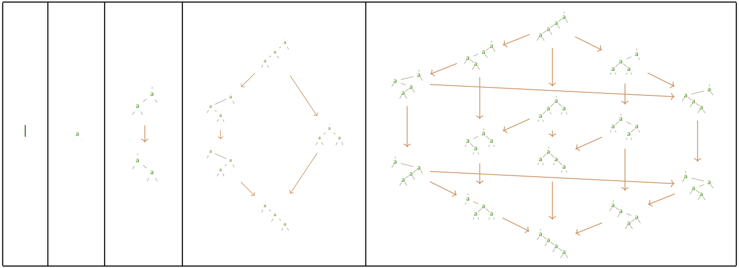

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Properties:

- $\Rightarrow$ is terminating and confluent;
- $\mathcal{N}_{\to}$ is the set of the trees factor-avoiding  (right comb trees);
- Numbers of connected components of the graphs: $1, 1, 1, 1, \ldots$.

A variant of Tamari lattices

Let $(\mathbf{F}(\{\mathsf{a}\}), \to)$ be the rewrite system defined by 

$\to$ 

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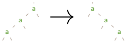
A variant of Tamari lattices

Let $(\mathbf{F}(\{\mathsf{a}\}), \rightarrow)$ be the rewrite system defined by  .

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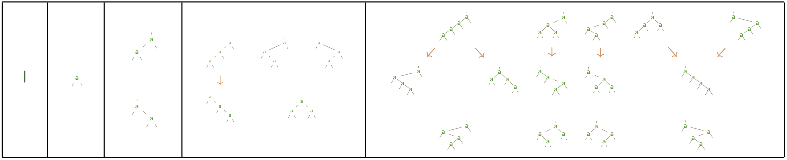

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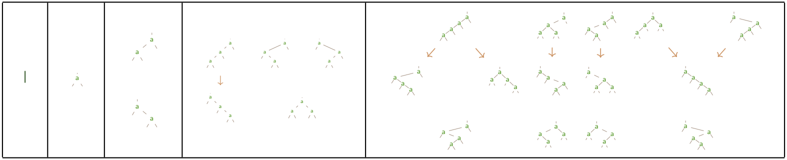

1	2	3	4	5

— Theorem [Chenavier, Cordero, G., 2018] —

- $\Rightarrow$ is terminating but **not** confluent;
- $\mathcal{N}_{\to}$ can be described as the set of the $\{a\}$-trees avoiding eleven patterns;

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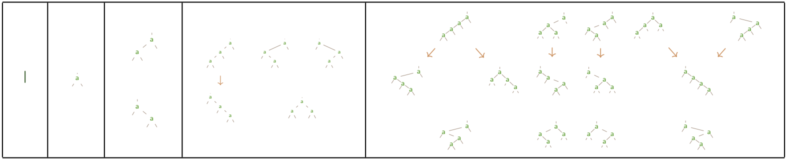

1	2	3	4	5

— Theorem [Chenavier, Cordero, G., 2018] —

- $\Rightarrow$ is terminating but **not** confluent;
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- Numbers of connected components of the graphs:

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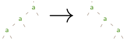
A variant of Tamari lattices

Let $(\mathbf{F}(\{\mathsf{a}\}), \to)$ be the rewrite system defined by  .

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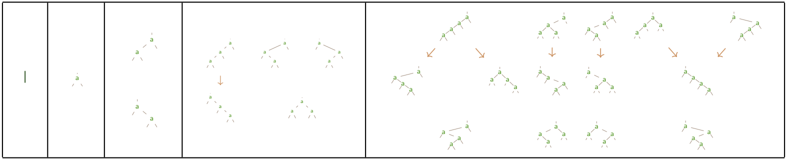

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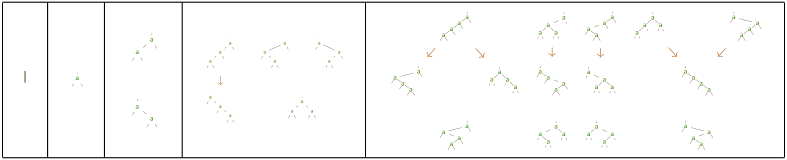

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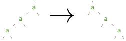
Let $(\mathbb{F}(\{a\}), \to)$ be the rewrite system defined by  .

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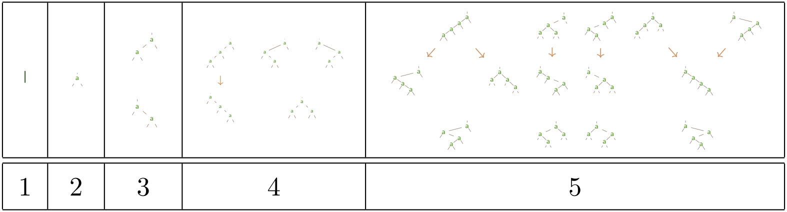

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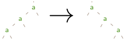
Let $(\mathbb{F}(\{a\}), \rightarrow)$ be the rewrite system defined by  .

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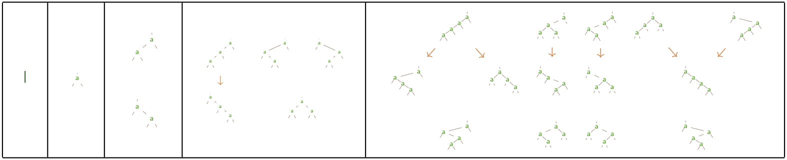

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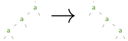
Let $(\mathbf{F}(\{\mathsf{a}\}), \to)$ be the rewrite system defined by  .

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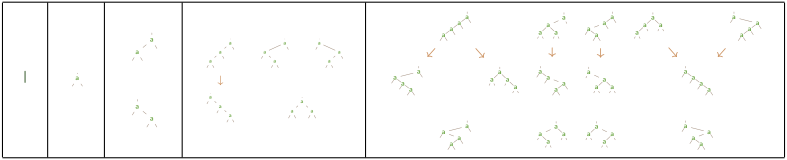

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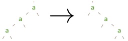
Let $(\mathbf{F}(\{a\}), \to)$ be the rewrite system defined by  .

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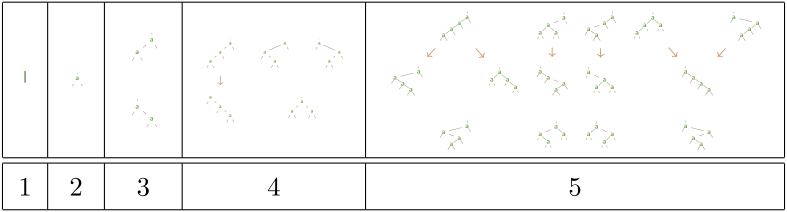

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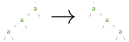
Let $(\mathbf{F}(\{\mathsf{a}\}), \to)$ be the rewrite system defined by  .

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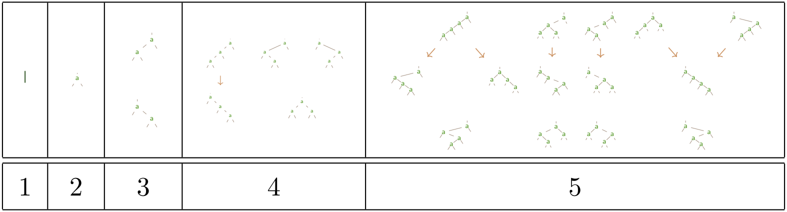

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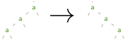
Let $(\mathbb{F}(\{\mathsf{a}\}), \to)$ be the rewrite system defined by  .

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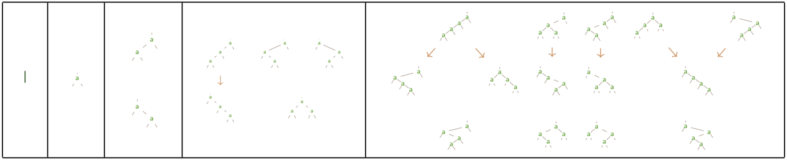

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and its generating function is

$$\frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right).$$

Pattern avoidance and enumeration

Given a set $\mathcal{P} \subseteq \mathbf{F}(\mathfrak{G})$, let $\mathrm{A}(\mathcal{P})$ be the set of all $\mathfrak{G}$-trees factor-avoiding all patterns of $\mathcal{P}$.

Counting the elements of $\mathrm{A}(\mathcal{P})$ w.r.t. the arity is a natural question.

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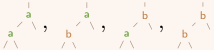
- For $\mathcal{P} :=$ { 

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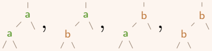
- For $\mathcal{P} :=$ {  }, $\mathrm{A}(\mathcal{P})$ is enumerated by

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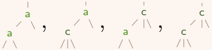
- For $\mathcal{P} :=$ {  }, $\mathrm{A}(\mathcal{P})$ is enumerated by

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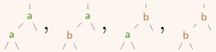
- For $\mathcal{P} := \{$  $\}$, $\mathrm{A}(\mathcal{P})$ is enumerated by

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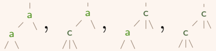
- For $\mathcal{P} := \{$  $\}$, $\mathrm{A}(\mathcal{P})$ is enumerated by

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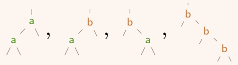
- For $\mathcal{P} := \{$  $\}$, $\mathrm{A}(\mathcal{P})$ is enumerated by

$$1, 2, 5, 13, 35, 96, 267, 750, \ldots (\textbf{A005773}).$$

Formal power series of trees

For any $\mathcal{P}, \mathcal{Q} \subseteq \mathbf{F}(\mathfrak{G})$, let

$$\mathbf{f}(\mathcal{P}, \mathcal{Q}) := \sum_{\substack{\mathfrak{t} \in \mathbf{F}(\mathfrak{G}) \\ \forall \mathfrak{s} \in \mathcal{P}, \mathfrak{s} \not\preccurlyeq_{\mathrm{f}} \mathfrak{t} \\ \forall \mathfrak{s} \in \mathcal{Q}, \mathfrak{s} \not\preccurlyeq_{\mathrm{p}} \mathfrak{t}}} \mathfrak{t}$$

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the series $\mathbf{f}(\mathcal{P}, \mathcal{Q})$ contains all the enumerative data about the trees factor-avoiding $\mathcal{P}$.

System of equations

When $\mathfrak{G}$, $\mathcal{P}$, and $\mathcal{Q}$ satisfy some conditions, $\mathbf{f}(\mathcal{P}, \mathcal{Q})$ expresses as an inclusion-exclusion formula involving simpler terms $\mathbf{f}(\mathcal{P}, \mathcal{S}_i)$.

— Theorem [G., 2017] —

The series $\mathbf{f}(\mathcal{P}, \mathcal{Q})$ satisfies

$$\mathbf{f}(\mathcal{P}, \mathcal{Q}) = \shortmid + \sum_{\substack{k \geqslant 1 \\ \mathsf{a} \in \mathfrak{G}(k)}} \sum_{\substack{\ell \geqslant 1 \\ \left\{\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(\ell)}\right\} \subseteq \mathrm{C}((\mathcal{P} \cup \mathcal{Q})_{\mathsf{a}}) \\ (\mathcal{S}_1, \dots, \mathcal{S}_k) = \mathcal{R}^{(1)} \dotplus \cdots \dotplus \mathcal{R}^{(\ell)}}} (-1)^{1+\ell} \mathsf{a} \bar{\circ} \left[\mathbf{f}(\mathcal{P}, \mathcal{S}_1), \dots, \mathbf{f}(\mathcal{P}, \mathcal{S}_k)\right].$$

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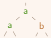
This leads to a system of equations for the generating series of $\mathrm{A}(\mathcal{P})$.

— Example —

For $\mathcal{P} := \Big\{$  $\Big\}$, we obtain the system of formal power series of trees

$$\begin{aligned}
\mathbf{f}(\mathcal{P}, \emptyset) &= | + \mathsf{a} \bar{\circ} [\mathbf{f}(\mathcal{P}, \{\mathsf{a}\}), \mathbf{f}(\mathcal{P}, \emptyset)] + \mathsf{a} \bar{\circ} [\mathbf{f}(\mathcal{P}, \emptyset), \mathbf{f}(\mathcal{P}, \{\mathsf{b}\})] - \mathsf{a} \bar{\circ} [\mathbf{f}(\mathcal{P}, \{\mathsf{a}\}), \mathbf{f}(\mathcal{P}, \{\mathsf{b}\})] \\
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Operads for enumeration

Let $\mathcal{O}$ be an operad admitting a presentation $(\mathfrak{G}, \equiv)$. A rewrite rule $\to$ on $\mathbf{F}(\mathfrak{G})$ is a faithful orientation of $\equiv$ if

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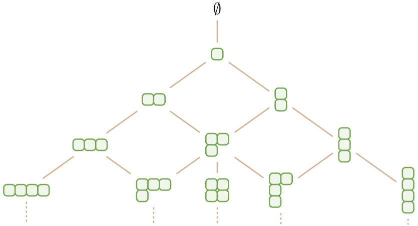
3. computing the series $\mathbf{f}\,(\mathcal{P}_{\to}, \emptyset)$.

Outline

Pairs of graded graphs

Young lattice

The Young lattice is a lattice on the set of all integer partitions. Its Hasse diagram is



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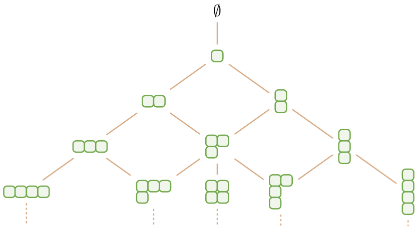

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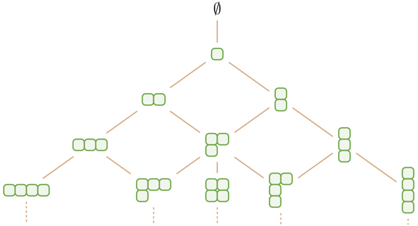

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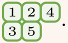
— Example —

The saturated chain

is in correspondence with the standard Young tableau



Graded graphs

A graded graph is a pair (G, U) where $G := \bigsqcup_{d \geqslant 0} G(d)$ is a graded set and U is a linear map

$$\mathrm{U} : \mathbb{K}\langle G(d) \rangle \to \mathbb{K}\langle G(d+1) \rangle, \qquad d \geqslant 0.$$

This map sends any $x \in G$ to its next vertices (with multiplicities).

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$$\mathrm{U} : \mathbb{K}\langle G(d)\rangle \to \mathbb{K}\langle G(d+1)\rangle, \qquad d \geqslant 0.$$

This map sends any $x \in G$ to its next vertices (with multiplicities).

Classical examples include

- the Young lattice [**Stanley**, 1988];
- the bracket tree [**Fomin**, 1994];
- the composition poset [**Björner, Stanley**, 2005];
- the Fibonacci lattice [**Fomin**, 1988], [**Stanley**, 1988].

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Other relations can be considered, like quantum duality [**Lam**, 2010] or filtered duality [**Patrias, Pylyavskyy**, 2018].

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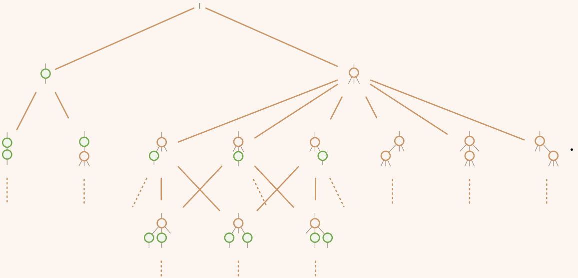
— Example —

For $\mathfrak{G} = \{ \cdot, \cdot \}$, the graph $(\mathbf{F}(\mathfrak{G}), \mathrm{U})$ is



A second graded graph from free operads

Let $(\mathbf{F}(\mathfrak{G}), \mathrm{V})$ be the graded graph defined from the adjoint $\mathrm{V}^\star$ of V by

$$\mathrm{V}^\star(|) := 0, \qquad \mathrm{V}^\star(\mathsf{a}\bar{\circ}[\mathfrak{s}, |, \dots, |]) := \mathfrak{s},$$

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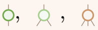
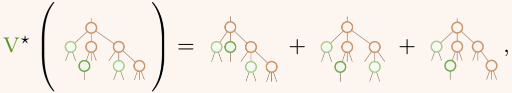
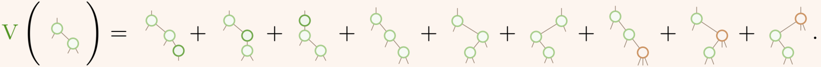
For $\mathfrak{G} = \Big\{$  $\Big\}$,





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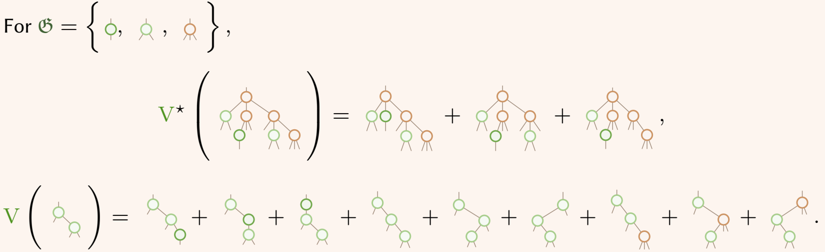
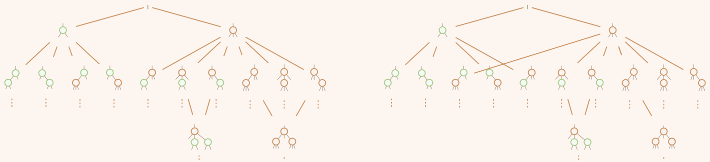


This graph be seen as a generalization of the bracket tree [**Fomin**, 1994] defined on binary trees.

Dual graded graphs from free operads

— Example —

For $\mathfrak{G} = \{$ (binary green generator), (ternary orange generator) $\}$, the pair $(\mathbf{F}(\mathfrak{G}), \mathrm{U}, \mathrm{V})$ is



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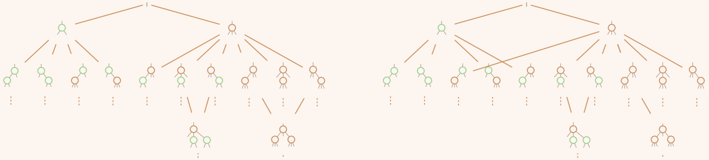
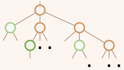


For any $\mathfrak{t} \in \mathbf{F}(\mathfrak{G})$, let $\alpha(\mathfrak{t})$ be the number of leaves of $\mathfrak{t}$ that are not in a first subtree of any internal node of $\mathfrak{t}$.

— Example —

 $\overset{\alpha}{\longmapsto} 5$

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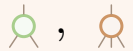
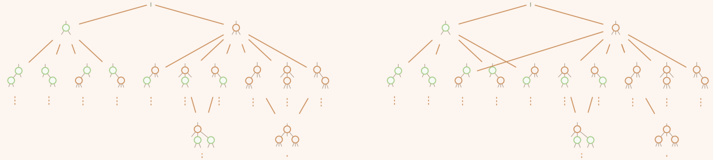
For $\mathfrak{G} = \Big\{$  $\Big\}$, the pair $(\mathbf{F}(\mathfrak{G}), \mathrm{U}, \mathrm{V})$ is



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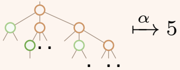

— Theorem [G., 2018] —

When $\mathfrak{G}(1) = \emptyset$, $(\mathbf{F}(\mathfrak{G}), \mathrm{U}, \mathrm{V})$ is ϕ-diagonal dual for the linear map satisfying

$$\phi(\mathfrak{t}) = (\#\mathfrak{G})\, \alpha(\mathfrak{t})\, \mathfrak{t}.$$

Generalization to operads

An operad $\mathcal{O}$ is homogeneous if $\mathcal{O}(1)$ is trivial and its presentation $(\mathfrak{G}, \equiv)$ is so that $\mathfrak{t} \equiv \mathfrak{t}'$ implies that $\mathfrak{t}$ and $\mathfrak{t}'$ have the same degree.

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Let the graphs $(\mathcal{O}, \mathrm{U})$ and $(\mathcal{O}, \mathrm{V})$ defined by

$$\mathrm{U}(x) := \sum_{\substack{\mathsf{a} \in \mathfrak{G} \\ i \in [|x|]}} x \circ_i \mathsf{a}, \qquad \mathrm{V}(x) := \sum_{\substack{y \in \mathcal{O} \\ \exists (\mathfrak{s}, \mathfrak{t}) \in \mathrm{ev}^{-1}(x) \times \mathrm{ev}^{-1}(y) \\ \langle \mathfrak{t}, \mathrm{V}(\mathfrak{s}) \rangle \neq 0}} y.$$

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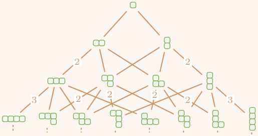
For some operads $\mathcal{O}$, the pair $(\mathcal{O}, \mathrm{U}, \mathrm{V})$ is ϕ-diagonal dual while for others, is not.

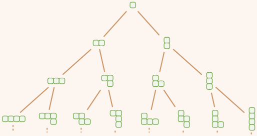
Some pairs of graded graphs from operads

— Example —

The pair (Comp, U, V) is 2-dual.

The graded graph (Comp, U) is the Hasse diagram of the composition poset [Bjöner, Stanley, 2005].





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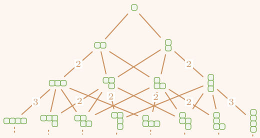
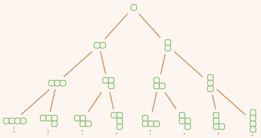
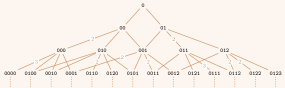
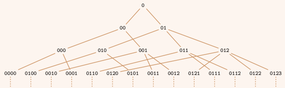




— Example —

The pair $\left(\mathsf{FCat}^{(m)}, \mathrm{U}, \mathrm{V}\right)$ is $m+1$-diagonal dual.





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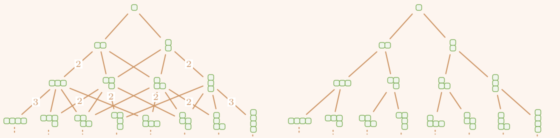

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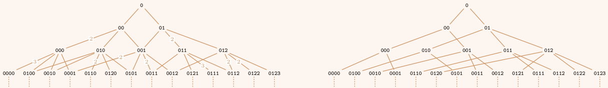

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