Some combinatorial aspects of combinatory logic

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Outline

1. Terms and rewrite systems
2. Combinatory logic
3. Mockingbird lattices
4. Conclusion and future work

Outline

1. Terms and rewrite systems

Terms

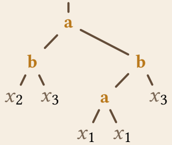
A signature is a graded set $\mathfrak{G} := \bigsqcup_{n \geqslant 0} \mathfrak{G}(n)$ wherein each $\mathbf{a} \in \mathfrak{G}(n)$ is an constant of arity n.

A $\mathfrak{G}$-term is

- either a variable x from the set $\mathbb{X} := \{x_1, x_2, \dots\}$;
- either a pair $(\mathbf{a}, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ where $\mathbf{a} \in \mathfrak{G}(n)$ and each $\mathfrak{t}_i$ is a $\mathfrak{G}$-term.

The set of all $\mathfrak{G}$-terms is denoted by $\mathfrak{T}(\mathfrak{G})$.

– Example –



This is the tree representation of the $\mathfrak{G}$-term

$$(\mathbf{a}, ((\mathbf{b}, (x_1, x_2)), (\mathbf{b}, ((\mathbf{a}, (x_1, x_1)), x_3))))$$

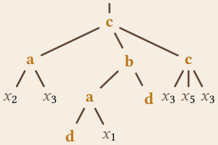
where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$.

More on terms

The frontier of a $\mathfrak{G}$-term $\mathfrak{t}$ is the sequence of the variables appearing in $\mathfrak{t}$.

The ground arity of $\mathfrak{t}$ is the greatest integer n such that x_n is a variable appearing in $\mathfrak{t}$, its arity is the length of its frontier, and its degree is its number of its internal nodes.

– Example –



The frontier of this term is $(x_2, x_3, x_1, x_3, x_5, x_3)$.

Its ground arity is 5, its arity is 6, and its degree is 7.

The term $\mathfrak{t}$ is

- planar if its frontier is $(x_1, \ldots, x_n)$;
- standard if its frontier is a permutation of $(x_1, \ldots, x_n)$;
- linear if there are no multiple occurrences of the same variable in the frontier of $\mathfrak{t}$;
- closed if its arity is 0.

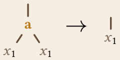
Rewrite systems on terms

A rewrite relation on $\mathfrak{T}(\mathfrak{G})$ is a binary relation $\rightarrow$ on $\mathfrak{T}(\mathfrak{G})$ such that if $\mathfrak{s} \rightarrow \mathfrak{s}'$, then all variables of $\mathfrak{s}'$ appear in $\mathfrak{s}$.

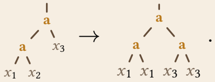
The context closure of $\rightarrow$ is the binary relation $\Rightarrow$ satisfying $\mathfrak{t} \Rightarrow \mathfrak{t}'$ whenever $\mathfrak{t}'$ is obtained by replacing in $\mathfrak{t}$ a factor $\mathfrak{s}$ by $\mathfrak{s}'$ provided that $\mathfrak{s} \rightarrow \mathfrak{s}'$.

– Example –

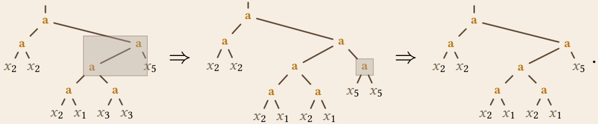
For $\mathfrak{G} := \mathfrak{G}(2) := \{\mathtt{a}\}$, let the rewrite relation $\rightarrow$ defined by



and



We have



A term rewrite system (TRS) is such a pair $(\mathfrak{G}, \rightarrow)$.

More on rewrite systems

Let $\mathcal{S} := (\mathfrak{G}, \to)$ be a TRS.

We define

- $\preccurlyeq$ as the reflexive and transitive closure of $\Rightarrow$;
- $\equiv$ as the reflexive, symmetric, and transitive closure of $\Rightarrow$;
- $\mathrm{G}(T)$ as the digraph on the set $\{\mathfrak{t}' \in \mathfrak{T}(\mathfrak{G}) : \mathfrak{t} \preccurlyeq \mathfrak{t}' \text{ for a } \mathfrak{t} \in T\}$ of vertices and the set $\Rightarrow$ of edges, where T is any subset of $\mathfrak{T}(\mathfrak{G})$.

A term $\mathfrak{t} \in \mathfrak{T}(\mathfrak{G})$ is

- a normal form for $\mathcal{S}$ if there is no edge of source $\mathfrak{t}$ in $\mathrm{G}(\{\mathfrak{t}\})$;
- weakly normalizing in $\mathcal{S}$ if there is at least one normal form in $\mathrm{G}(\{\mathfrak{t}\})$;
- is strongly normalizing in $\mathcal{S}$ if $\mathrm{G}(\{\mathfrak{t}\})$ is finite and acyclic.

When all $\mathfrak{G}$-terms are strongly normalizing, $\mathcal{S}$ is terminating.

If for any $\mathfrak{G}$-term $\mathfrak{t}$, $\mathfrak{t} \preccurlyeq \mathfrak{s}_1$ and $\mathfrak{t} \preccurlyeq \mathfrak{s}_2$ implies the existence of $\mathfrak{t}'$ such that $\mathfrak{s}_1 \preccurlyeq \mathfrak{t}'$ and $\mathfrak{s}_2 \preccurlyeq \mathfrak{t}'$, then $\mathcal{S}$ is confluent.

Properties of rewrite systems

Let $\mathcal{S} := (\mathfrak{G}, \to)$ be a TRS.

One has the following properties (see for instance [**Baader, Nipkow**, 1998]).

– Proposition –

If $\mathcal{S}$ is terminating and confluent, then the set of normal forms of $\mathcal{S}$ is a set of representatives of $\mathfrak{T}(\mathfrak{G})/_{\equiv}$.

– Proposition –

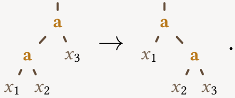
If $\mathcal{S}$ is terminating and confluent, then to decide if $\mathfrak{t} \equiv \mathfrak{t}'$ in $\mathcal{S}$, compute $\mathfrak{s}$ and $\mathfrak{s}'$ as, respectively, the unique formal forms in the $\equiv$-equivalence classes of $\mathfrak{t}$ and $\mathfrak{t}'$ and check if $\mathfrak{s} = \mathfrak{s}'$.

– Proposition –

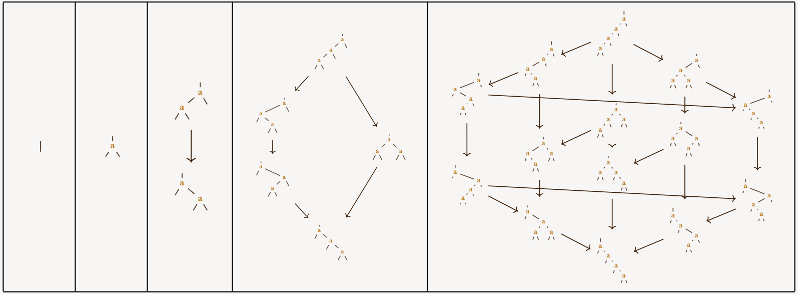
The set of normal forms of $\mathcal{S}$ is the set of all $\mathfrak{G}$-terms avoiding each $\mathfrak{G}$-term $\mathfrak{t}$ where $\mathfrak{t}$ is on the left of a rule of $\mathcal{S}$.

The Tamari rewrite system

Let $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\to$ be the rewrite relation on $\mathfrak{T}(\mathfrak{G})$ defined by



Here are the first graphs $\mathrm{T}(k)$ of planar $\mathfrak{G}$-terms of degrees $k \geqslant 0$:



$\mathrm{T}(0)$	$\mathrm{T}(1)$	$\mathrm{T}(2)$	$\mathrm{T}(3)$	$\mathrm{T}(4)$

The Tamari rewrite system

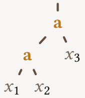
- The binary relation $\to$ is the right rotation operation, an important operation on binary search trees appearing in algorithms handling balanced binary trees [**Adelson-Velsky, Landis**, 1962].

- This TRS is terminating and confluent.

  As consequence, $\preccurlyeq$ is a partial order relation.

  This order endows each set $\mathrm{T}(k)$ with the structure of a lattice, known as Tamari lattice [**Huang, Tamari**, 1962].

- The set of normal forms of this TRS contains all right comb trees, that are the trees avoiding



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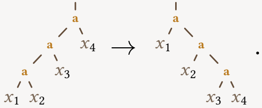
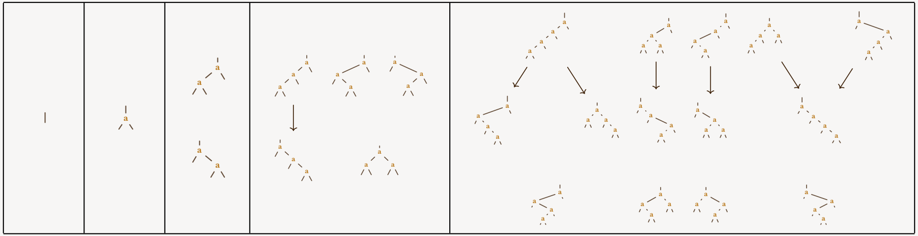
  There is exactly one planar normal form of degree k for any $k \geqslant 0$.

A variant of the Tamari rewrite system

Let $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\to$ be the rewrite relation on $\mathfrak{T}(\mathfrak{G})$ defined by



Here are the first graphs $\mathrm{T}_3(k)$ of planar $\mathfrak{G}$-terms of degrees $k \geqslant 0$:



$\mathrm{T}_3(0)$	$\mathrm{T}_3(1)$	$\mathrm{T}_3(2)$	$\mathrm{T}_3(3)$	$\mathrm{T}_3(4)$

A variant of the Tamari rewrite system

This TRS is terminating but **not confluent**.

The Buchberger completion algorithm [**Knuth, Bendix**, 1970], [**Dotsenko, Khoroshkin**, 2010] is a semi-algorithm taking as input a terminating but not confluent TRS $(\mathfrak{G}, \to)$ and outputting a new rewrite relation $\to'$ such that

- $(\mathfrak{G}, \to')$ is terminating **and confluent**;
- the preorders $\preccurlyeq$ and $\preccurlyeq'$ are equal.

– Theorem [Chenavier, Cordero, G., 2018] –

The normal forms of a completion of the previous TRS can be described as the $\mathfrak{G}$-terms avoiding 11 planar $\mathfrak{G}$-terms of degrees from 3 to 7.

The generating series of the planar normal forms of this TRS, enumerated w.r.t. their arities, is

$$F(t) = \frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right).$$

The first coefficients of this generating series are

$$1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 14, 15, 16, 17, 18, 19, 20.$$

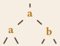
Enumeration of planar terms avoiding factors

– Theorem [G., 2020] –

Let $\mathfrak{G}$ be a signature, and $\mathcal{P}$ be a set of planar $\mathfrak{G}$-terms. The generating series enumerating the planar $\mathfrak{G}$-terms avoiding $\mathcal{P}$ w.r.t. the arity (parameter t) and the degree (parameter q) is $F(\mathcal{P}, \emptyset)$ where, for any set $\mathcal{Q}$ of planar $\mathfrak{G}$-terms,

$$F(\mathcal{P}, \mathcal{Q}) = t + q \sum_{\substack{k \geqslant 1 \\ \mathsf{a} \in \mathfrak{G}(k)}} \sum_{\substack{\ell \geqslant 1 \\ \{\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(\ell)}\} \subseteq M(\mathcal{P} \cup \mathcal{Q}, \mathsf{a}) \\ (\mathcal{S}_1, \dots, \mathcal{S}_k) = \mathcal{R}^{(1)} \dotplus \dots \dotplus \mathcal{R}^{(\ell)}}} (-1)^{1+\ell} \prod_{i \in [k]} F(\mathcal{P}, \mathcal{S}_i).$$

– Example –

For $\mathcal{P} :=$ { 

}, we obtain the algebraic system of equations

$$F(\mathcal{P}, \emptyset) = t + q\, F(\mathcal{P}, \{\mathsf{a}\})\, F(\mathcal{P}, \emptyset) + q\, F(\mathcal{P}, \emptyset)\, F(\mathcal{P}, \{\mathsf{b}\}) - q\, F(\mathcal{P}, \{\mathsf{a}\})\, F(\mathcal{P}, \{\mathsf{b}\}) + q\, F(\mathcal{P}, \emptyset)\, F(\mathcal{P}, \emptyset),$$

$$F(\mathcal{P}, \{\mathsf{a}\}) = t + q\, F(\mathcal{P}, \emptyset)\, F(\mathcal{P}, \emptyset),$$

$$F(\mathcal{P}, \{\mathsf{b}\}) = t + q\, F(\mathcal{P}, \{\mathsf{a}\})\, F(\mathcal{P}, \emptyset) + q\, F(\mathcal{P}, \emptyset)\, F(\mathcal{P}, \{\mathsf{b}\}) - q\, F(\mathcal{P}, \{\mathsf{a}\})\, F(\mathcal{P}, \{\mathsf{b}\}).$$

2. Combinatory logic

Applicative terms

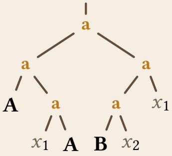
An applicative signature is a signature $\mathfrak{G}$ satisfying $\mathfrak{G} = \mathfrak{G}(0) \sqcup \mathfrak{G}(2)$ where $\mathfrak{G}(2) = \{\mathbf{a}\}$.

An applicative term is a term on an applicative signature.

Each applicative term can be expressed as a bracketed infix expression on $\mathfrak{G}(0) \sqcup \mathbb{X}$ wherein the symbol $\mathbf{a}$ is made implicit and the bracketing is implicit from left to right.

– Example –

On the applicative signature $\mathfrak{G}$ where $\mathfrak{G}(0) = \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$,



$$\leftrightarrow \quad (\mathbf{A}\ \mathbf{a}\ (x_1\ \mathbf{a}\ \mathbf{A}))\ \mathbf{a}\ (((\mathbf{B}\ \mathbf{a}\ x_2)\ \mathbf{a}\ x_1)) \quad \leftrightarrow \quad (\mathbf{A}\ (x_1\ \mathbf{A}))(((\mathbf{B}\ x_2)\ x_1)) \quad \leftrightarrow \quad \mathbf{A}(x_1\ \mathbf{A})(\mathbf{B}\ x_2 x_1).$$

Combinatory logic systems

A combinatory logic system (CLS) is a pair $(\mathfrak{C}, \rightarrow)$ where

- $\mathfrak{C}$ is a finite set;
- $\rightarrow$ is a rewrite relation on the applicative signature $\mathfrak{G}_{\mathfrak{C}}$ where $\mathfrak{G}_{\mathfrak{C}}(0) = \mathfrak{C}$, and for any $\mathbf{X} \in \mathfrak{C}$, there is exactly one rule of the form

$$\mathbf{X}\, x_1 \ldots x_n \rightarrow \mathfrak{t}$$

such that $\mathfrak{t}$ is an $\{\mathbf{a}\}$-term.

– Example –

Let the CLS $(\mathfrak{C}, \rightarrow)$ [**Schönfinkel**, 1924] [**Curry**, 1930] where $\mathfrak{C} := \{\mathbf{I}, \mathbf{K}, \mathbf{S}\}$ and $\rightarrow$ satisfies

- $\mathbf{I}\, x_1 \rightarrow x_1$,
- $\mathbf{K}\, x_1 x_2 \rightarrow x_1$,
- $\mathbf{S}\, x_1 x_2 x_3 \rightarrow x_1 x_3 (x_2 x_3)$.

This CLS is Turing complete: there are algorithms to faithfully translate any λ-term into a term of this CLS. These are abstraction algorithms [**Rosser**, 1955], [**Curry, Feys**, 1958].

Some important combinators

In *To Mock a Mockingbird: and Other Logic Puzzles* [**Smullyan**, 1985], a large number of rules are listed.

Here is a sublist:

- Identity bird: $\mathbf{I}\, x_1 \to x_1$
- Mockingbird: $\mathbf{M}\, x_1 \to x_1 x_1$
- Kestrel: $\mathbf{K}\, x_1 x_2 \to x_1$
- Thrush: $\mathbf{T}\, x_1 x_2 \to x_2 x_1$
- Mockingbird 1: $\mathbf{M_1}\, x_1 x_2 \to x_1 x_1 x_2$
- Warbler: $\mathbf{W}\, x_1 x_2 \to x_1 x_2 x_2$
- Lark: $\mathbf{L}\, x_1 x_2 \to x_1 (x_2 x_2)$
- Owl: $\mathbf{O}\, x_1 x_2 \to x_2 (x_1 x_2)$
- Turing bird: $\mathbf{U}\, x_1 x_2 \to x_2 (x_1 x_1 x_2)$
- Cardinal: $\mathbf{C}\, x_1 x_2 x_3 \to x_1 x_3 x_2$
- Vireo: $\mathbf{V}\, x_1 x_2 x_3 \to x_3 x_1 x_2$
- Bluebird: $\mathbf{B}\, x_1 x_2 x_3 \to x_1 (x_2 x_3)$
- Starling: $\mathbf{S}\, x_1 x_2 x_3 \to x_1 x_3 (x_2 x_3)$
- Jay: $\mathbf{J}\, x_1 x_2 x_3 x_4 \to x_1 x_2 (x_1 x_4 x_3)$

Algorithmic questions

Let $\mathcal{C} := (\mathfrak{C}, \to)$ be a CLS.

– Word problem –

Is there an algorithm to decide, given two terms $\mathfrak{t}$ and $\mathfrak{t}'$ of $\mathcal{C}$, if $\mathfrak{t} \equiv \mathfrak{t}'$? (See [**Baader, Nipkow**, 1998], [**Statman**, 2000].)

- Yes for the CLS on **L** [**Statman**, 1989], [**Sprenger, Wymann-Böni**, 1993].
- Yes for the CLS on **W** [**Sprenger, Wymann-Böni**, 1993].
- Yes for the CLS on $\mathbf{M_1}$ [**Sprenger, Wymann-Böni**, 1993].
- Open for the CLS on **S** [**RTA Problem #97**, 1975].

– Strong normalization problem –

Is there an algorithm to decide, given a term $\mathfrak{t}$ of $\mathcal{C}$, if $\mathfrak{t}$ is strongly normalizing?

- Yes for the CLS on **S** [**Waldmann**, 2000].
- Yes for the CLS on **J** [**Probst, Studer**, 2000].

Combinatorial questions

Let $\mathcal{C} := (\mathfrak{C}, \rightarrow)$ be a CLS.

– Structure of the rewriting graphs –

1. Are all the connected components of the graph $\mathrm{G}(\mathfrak{T}(\mathfrak{G}_{\mathfrak{C}}))$ of $\mathcal{C}$ finite?
2. Understand when $\mathrm{G}(\{\mathfrak{t}\})$ and $\mathrm{G}(\{\mathfrak{t}'\})$ are isomorphic graphs.
3. Is the preorder $\preccurlyeq$ an order relation on $\mathfrak{T}(\mathfrak{G}_{\mathfrak{C}})$?
4. If so, are all intervals $[\mathfrak{t}, \mathfrak{t}']$ lattices?

– Enumerative issues –

1. In the graph $\mathrm{G}(\mathfrak{T}(\mathfrak{G}_{\mathfrak{C}}))$ of $\mathcal{C}$, count w.r.t. their degrees the closed terms that are
   1.1 minimal;
   1.2 maximal (which are thus normal forms);
   1.3 both minimal and maximal (which are thus isolated vertices).
2. Count the $\equiv$-equivalence classes of closed terms w.r.t. their degrees.
3. Count the connected components of $\mathrm{G}(\mathfrak{T}(\mathfrak{G}_{\mathfrak{C}}))$ w.r.t. the minimal degrees of their closed terms.

First properties: termination

A CLS $\mathcal{C}$ can be non-terminating.

There are several possible causes:

- The set $G(\{t\})$ is infinite for a term t of $\mathcal{C}$.

– Example –

In the CLS on **S**, the terms

- $\mathbf{S}(\mathbf{S}\,\mathbf{S})(\mathbf{S}\,\mathbf{S})(\mathbf{S}(\mathbf{S}\,\mathbf{S})(\mathbf{S}\,\mathbf{S}))$ [**Zachos**, 1978];
- $\mathbf{S}\,\mathbf{S}\,\mathbf{S}(\mathbf{S}\,\mathbf{S}\,\mathbf{S})(\mathbf{S}\,\mathbf{S}\,\mathbf{S})$ [**Barendregt**, 1984]

have this property.

- There is a cycle, that is two terms t and t' of $\mathcal{C}$ such that $t \Rightarrow t' \preccurlyeq t$.

– Example –

In the CLS on $\mathbf{F}\,x_1 x_2 \to x_2 x_2$, one has $t \Rightarrow t' \Rightarrow t$ with $t := \mathbf{F}\,\mathbf{F}(\mathbf{F}\,\mathbf{F}\,\mathbf{F})$ and $t' := \mathbf{F}\,\mathbf{F}\,\mathbf{F}(\mathbf{F}\,\mathbf{F}\,\mathbf{F})$.

First properties: confluence and lattices

– Proposition –

Any CLS is confluent.

This is a consequence of the orthogonality [**Rosen**, 1973] of any CLS.

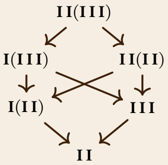
A rewrite relation $\to$ is orthogonal if

- $\mathfrak{t} \to \mathfrak{t}'$ implies that $\mathfrak{t}$ is linear;
- $\mathfrak{t} \to \mathfrak{t}'$, $\mathfrak{s} \to \mathfrak{s}'$, and $\mathfrak{t}$ and $\mathfrak{s}$ overlap, implies $\mathfrak{t} = \mathfrak{s}$.

Some terminating (and confluent) CLS have intervals that are not lattices.

– Example –

In the CLS on $\mathbf{I}\, x_1 \to x_1$, the interval $[\mathbf{II}(\mathbf{III}), \mathbf{II}]$ admits the Hasse diagram



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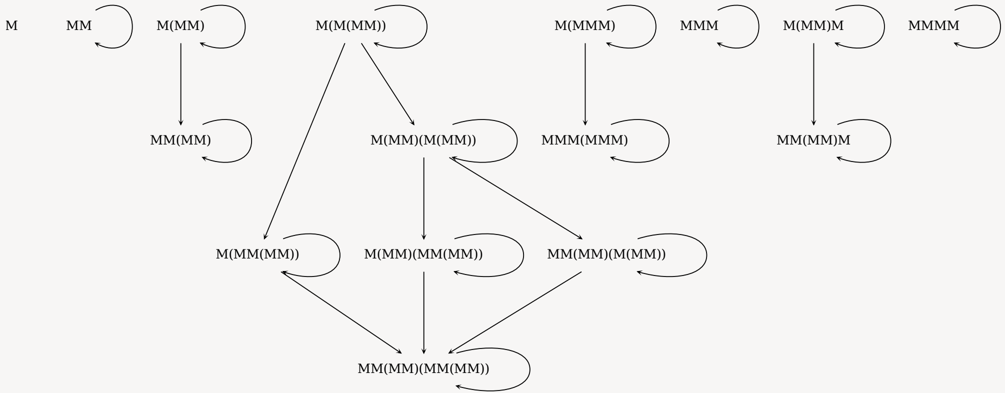
3. Mockingbird lattices

The Mockingbird rewrite graph

Let us focus on the CLS on $\mathbf{M}\, x_1 \to x_1 x_1$.

Here is the graph $\mathsf{G}(\mathfrak{T}(\mathfrak{G}_{\mathfrak{C}}))$ restrained to closed terms of degrees 3 or less:



Termination

Let $\to'$ the rewrite relation defined by $\mathbf{M}(x_1 x_2) \to' (x_1 x_2)(x_1 x_2)$.

– Lemma –

The preorders $\preccurlyeq$ and $\preccurlyeq'$ are equal.

– Lemma –

If $\mathfrak{t} \Rightarrow' \mathfrak{t}'$, then $\mathrm{ht}(\mathfrak{t}) = \mathrm{ht}(\mathfrak{t}')$ and $\deg(\mathfrak{t}) < \deg(\mathfrak{t}')$.

As a consequence, $\preccurlyeq'$ is an order relation.

– Proposition [G., 2021–] –

The relation $\preccurlyeq$ is an order relation on terms on $\mathbf{M}$. Moreover, each connected component of the Hasse diagram of this poset is finite.

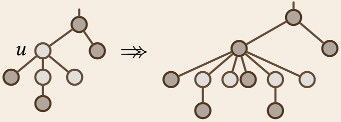
Order relation on black and white trees

A black and white tree (BWT) is a planar rooted tree such that each internal node is either black or white. Let T_{bw} be the set of such trees.

Let $\Rrightarrow$ be the binary relation on T_{bw} such that $\mathfrak{p} \Rrightarrow \mathfrak{q}$ if $\mathfrak{q}$ can be obtained from $\mathfrak{p}$ by selecting a white node u of $\mathfrak{p}$, by turning it into black, and by duplicating its sequence of descendants.

If $\mathfrak{p} \Rrightarrow \mathfrak{q}$, then there are more black nodes in $\mathfrak{q}$ than in $\mathfrak{p}$. Hence, the reflexive and transitive closure $\ll$ of $\Rrightarrow$ is a partial order.

– Example –



– Lemma –

One has $\mathfrak{p} \ll \mathfrak{p}'$ iff

- either $\mathfrak{p} = \circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell)$, $\mathfrak{p}' = \circ(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell)$, and $\mathfrak{p}_i \ll \mathfrak{p}'_i$ for all $i \in [\ell]$;
- or $\mathfrak{p} = \bullet(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell)$, $\mathfrak{p}' = \bullet(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell)$, and $\mathfrak{p}_i \ll \mathfrak{p}'_i$ for all $i \in [\ell]$;
- or $\mathfrak{p} = \circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell)$, $\mathfrak{p}' = \bullet(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell, \mathfrak{p}''_1, \dots, \mathfrak{p}''_\ell)$, and $\mathfrak{p}_i \ll \mathfrak{p}'_i$ and $\mathfrak{p}_i \ll \mathfrak{p}''_i$ for all $i \in [\ell]$.

Lattices on BWT

Let $\wedge$ and $\vee$ be the two partial binary, commutative, and associative operations on T_{bw} defined by

$$\circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell) \wedge \circ(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell) := \circ(\mathfrak{p}_1 \wedge \mathfrak{p}'_1, \dots, \mathfrak{p}_\ell \wedge \mathfrak{p}'_\ell),$$
$$\bullet(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell) \wedge \bullet(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell) := \bullet(\mathfrak{p}_1 \wedge \mathfrak{p}'_1, \dots, \mathfrak{p}_\ell \wedge \mathfrak{p}'_\ell),$$
$$\circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell) \wedge \bullet(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell, \mathfrak{p}''_1, \dots, \mathfrak{p}''_\ell) := \circ(\mathfrak{p}_1 \wedge \mathfrak{p}'_1 \wedge \mathfrak{p}''_1, \dots, \mathfrak{p}_\ell \wedge \mathfrak{p}'_\ell \wedge \mathfrak{p}''_\ell),$$

and

$$\circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell) \vee \circ(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell) := \circ(\mathfrak{p}_1 \vee \mathfrak{p}'_1, \dots, \mathfrak{p}_\ell \vee \mathfrak{p}'_\ell),$$
$$\bullet(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell) \vee \bullet(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell) := \bullet(\mathfrak{p}_1 \vee \mathfrak{p}'_1, \dots, \mathfrak{p}_\ell \vee \mathfrak{p}'_\ell),$$
$$\circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell) \vee \bullet(\mathfrak{p}'_1, \dots, \mathfrak{p}'_\ell, \mathfrak{p}''_1, \dots, \mathfrak{p}''_\ell) := \bullet(\mathfrak{p}_1 \vee \mathfrak{p}'_1, \dots, \mathfrak{p}_\ell \vee \mathfrak{p}'_\ell, \dots, \mathfrak{p}_1 \vee \mathfrak{p}''_1, \dots, \mathfrak{p}_\ell \vee \mathfrak{p}''_\ell).$$

– Proposition [G., 2021–] –

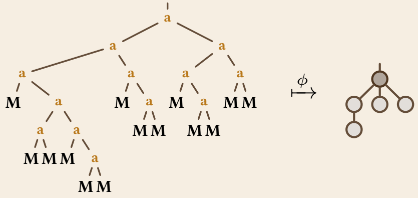
Given a BTW $\mathfrak{p}$, the set $\{\mathfrak{p}' \in T_{\mathrm{bw}} : \mathfrak{p} \ll \mathfrak{p}'\}$ is a lattice for the operations $\wedge$ and $\vee$.

Lattices of closed terms on M

Given a closed term $\mathfrak{t}$ on **M**, let $\phi(\mathfrak{t})$ be the BWT obtained by

1. by adding a root ◉ to $\mathfrak{t}$;
2. by replacing each internal node of $\mathfrak{t}$ having a left child **M** and a right child different from **M** by a ○;
3. by connecting these new nodes following the paths in $\mathfrak{t}$.

– Example –



– Lemma –

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be two closed terms on **M**. One has $\mathfrak{t} \Rightarrow' \mathfrak{t}'$ iff $\phi(\mathfrak{t}) \twoheadrightarrow \phi(\mathfrak{t}')$.

– Theorem [G., 2021–] –

For any closed term $\mathfrak{t}$ on **M**, the poset $(\mathrm{G}(\{\mathfrak{t}\}), \preccurlyeq)$ is a finite lattice.

Maximal and minimal terms

A closed term on $\mathbf{M}$ is maximal iff it avoids the pattern $\mathbf{M}(x_1 x_2)$.

The formal sum F of these trees enumerated w.r.t. their degrees satisfies the equation

$$\mathsf{F} = \mathbf{M} + \mathsf{F}\,\mathbf{M} + (\mathsf{F} - \mathbf{M})(\mathsf{F} - \mathbf{M}).$$

Therefore, the generating series F of these terms satisfies $F = 1 + tF + (F-1)^2$. Its first coefficients are 1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835 and are Motzkin numbers (Seq. **A001006**).

A closed term on $\mathbf{M}$ is minimal iff it avoids the pattern $(x_1 x_2)(x_1 x_2)$.

This leads to the recurrence for the number $a(d)$ of such trees of degree d:

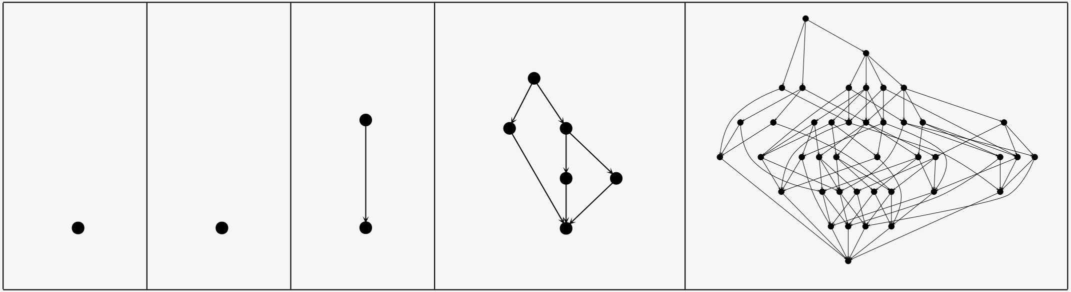
$$a(0) = a(1) = 1, \qquad a(d+1) = b(d) \text{ if } d \text{ is odd}, \qquad a(d+1) = b(d) - a(d/2) \text{ otherwise},$$

where $b(d) := \sum_{0 \leqslant k \leqslant d} a(k)a(d-k)$. The fist numbers are 1, 1, 2, 4, 12, 34, 108, 344, 1136, 3796, 12920.

Mockingbird lattices

The Mockingbird lattice of order $k \geqslant 0$ is the lattice $\mathrm{M}(k)$ on $\mathrm{G}(\{\mathfrak{t}\})$ where $\mathfrak{t}$ is the term $\mathbf{M}(\mathbf{M}(\dots(\mathbf{M}\,\mathbf{M})\dots))$ of degree k.

Here are the Hasse diagrams of these first lattices:



$\mathrm{M}(0)$	$\mathrm{M}(1)$	$\mathrm{M}(2)$	$\mathrm{M}(3)$	$\mathrm{M}(4)$

– Theorem [G., 2021–] –

For any BWT $\mathfrak{p}$, $\{\mathfrak{p}' \in \mathrm{T}_{\mathrm{bw}} : \mathfrak{p} \ll \mathfrak{p}'\}$ is isomorphic to a maximal interval of a Mockingbird lattice.

Minimal and maximal paths

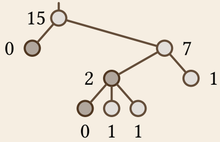
Let $\mathfrak{p}$ be a BWT.

Let $\mathrm{w}(\mathfrak{p})$ be the integer defined recursively as

$$\mathrm{w}(\circ(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell)) = 1 + 2 \sum_{i \in [\ell]} \mathrm{w}(\mathfrak{p}_i),$$

$$\mathrm{w}(\bullet(\mathfrak{p}_1, \dots, \mathfrak{p}_\ell)) = \sum_{i \in [\ell]} \mathrm{w}(\mathfrak{p}_i).$$

– Example –



– Proposition [G., 2021–] –

Let $\mathfrak{t}$ be a closed term on $\mathbf{M}$ and P be the lattice $(\mathrm{G}(\{\mathfrak{t}\}), \preccurlyeq)$.

- A shorted path from $\mathfrak{t}$ to the maximal element of P has as length the number of $\circ$ in $\phi(\mathfrak{t})$.
- A longest path from $\mathfrak{t}$ to the maximal element of P has length $\mathrm{w}(\phi(\mathfrak{t}))$.

Therefore, a longest path in $\mathrm{M}(k)$ for $k \geqslant 1$ as length $2^{k-1} - 1$.

Number of initial intervals

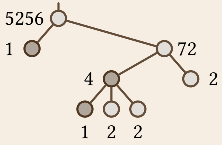
Let $\mathfrak{p}$ be a BWT.

Let $\mathrm{s}(\mathfrak{p})$ be the integer defined recursively as

$$\mathrm{s}(\circ(\mathfrak{p}_1, \dots, \mathfrak{p}_k)) = \left(\prod_{i \in [k]} \mathrm{s}(\mathfrak{p}_i)\right)\left(1 + \prod_{i \in [k]} \mathrm{s}(\mathfrak{p}_i)\right),$$

$$\mathrm{s}(\bullet(\mathfrak{p}_1, \dots, \mathfrak{p}_k)) = \prod_{i \in [k]} \mathrm{s}(\mathfrak{p}_i).$$

– Example –



– Proposition [G., 2021–] –

Let $\mathfrak{t}$ be a closed term on **M** and P be the lattice. $(\mathrm{G}(\{\mathfrak{t}\}), \preccurlyeq)$. Then, $\#P = \mathrm{s}(\phi(\mathfrak{t}))$.

The numbers $a(k)$ of initial intervals in $\mathrm{M}(k)$ satisfies $a(0) = a(1) = 1$, and for $k \geqslant 2$, $a(k) = a(k-1)(1 + a(k-1))$. The first numbers are

$$1, 1, 2, 6, 42, 1806, 3263442, 10650056950806, 113423713055421844361000442.$$

4. Conclusion and future work

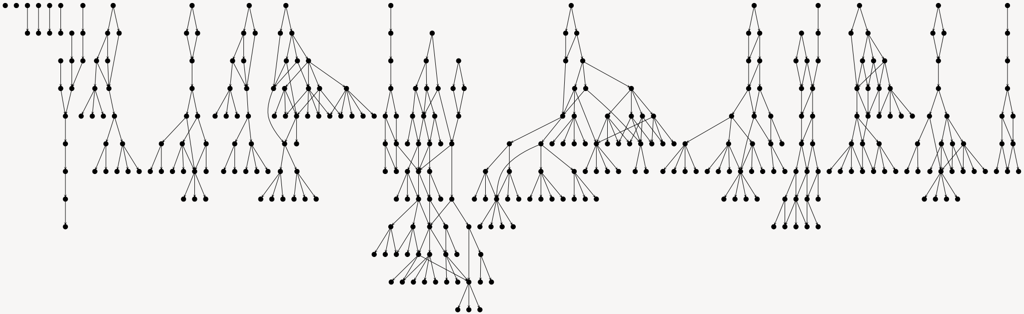
Other CLS: L

When $\#\mathfrak{C} = 1$, one gets already some interesting CLS. Here are experimental data some of these.

For $\mathbf{L}\, x_1 x_2 \to x_1(x_2 x_2)$:

- Minimal terms: 1, 1, 1, 2, 5, 13, 34, 94, 265, 765, 2237, 6632;
- Maximal terms: 1, 1, …;
- Isolated terms: 1, 1, 0, 0, …;
- Equivalence classes : 1, 1, 1, 2, 5, 12, 31, 83;
- Connected components: 1, 1, 1, 2, 4, 9, 22, 60;
- $\to$ is not terminating;
- $\preccurlyeq$ is an order relation;
- Intervals seem lattices.

The rewriting graph of terms on **L** from closed terms of degrees up to 5 and up to 4 rewritings:



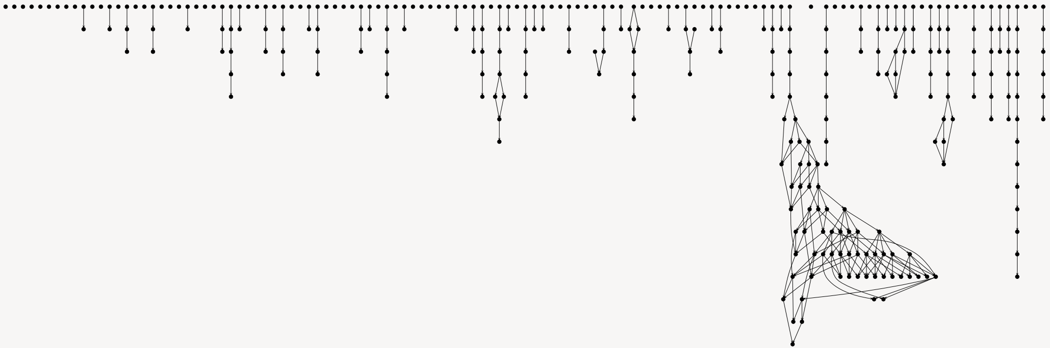
Other CLS: S

For $\mathbf{S}\, x_1 x_2 x_3 \to x_1 x_3 (x_2 x_3)$:

- Minimal terms: 1, 1, 2, 4, 10, 26, 76, 224, 690, 2158, 6882, 22208;
- Maximal terms: 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798 (Motzkin numbers);
- Isolated terms: 1, 1, 2, 3, 7, 15, 37, 87, 218, 546, 1393, 3583;
- Equivalence classes: 1, 1, 2, 4, 10, 27, 78, 234, 722, 2271;
- Connected components: 1, 1, 2, 4, 10, 26, 74, 217, 660, 2053;
- $\to$ is not terminating;
- $\preccurlyeq$ is an order relation [**Bergstra, Klop**, 1979];
- Intervals seem lattices.

Other CLS: S

The rewriting graph of terms on **S** from closed terms of degrees up to 6 and up to 11 rewritings:



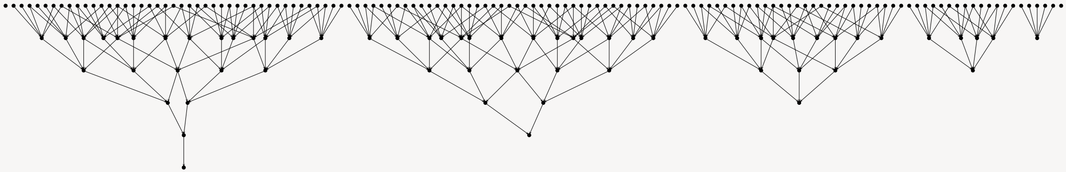
Other CLS: T

For $\mathbf{T}\, x_1 x_2 \to x_2 x_1$:

- Minimal terms: 1, 0, 0, …;
- Maximal terms: 1, 1, …;
- Isolated terms: 1, 0, 0, …;
- Equivalence classes: 1, 1, 2, 3, 4, 5, 6;
- Connected components: 1, 1, …;
- $\to$ is terminating;
- $\preccurlyeq$ is an order relation;
- Interval seem lattices.

Other CLS: T

The rewriting graph of terms on **T** from closed terms of degrees up to 6 and up to 1 rewriting:



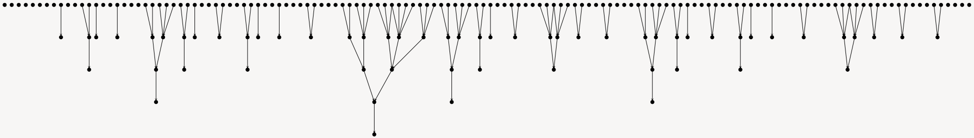
Other CLS: C

For **C** $x_1x_2x_3 \to x_1x_3x_2$:

- Minimal terms: 1, 1, …;
- Maximal terms: 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798 (Motzkin numbers);
- Isolated terms: 1, 1, …;
- Equivalence classes : 1, 1, 2, 5, 13, 33, 83, 209, 531, 1365;
- Connected components : 1, 1, 2, 4, 9, 21, 51, 127, 323, 835 (Motzkin numbers);
- $\to$ is terminating;
- $\preccurlyeq$ is an order relation;
- Interval seem lattices.

Other CLS: C

The rewriting graph of terms on **C** from closed terms of degrees up to 6 and up to 1 rewriting:



Some questions

– Connections with order theory –

Given a CLS $\mathcal{C}$, obtain (necessary/sufficient) conditions for the fact that

1. $\mathcal{C}$ is terminating;
2. when $\preccurlyeq$ is an order relation, all the $\preccurlyeq$-intervals of $\mathcal{C}$ are lattices;

– Normalization –

Provide a generic way to describe all the strongly normalizing terms of a CLS.

– Connection with clone theory –

Realize a CLS as a clone.

Indeed, each CLS $\mathcal{C} := (\mathfrak{C}, \to)$ gives rise to a clone defined as the quotient of the free clone generated by $\mathfrak{G}_{\mathfrak{C}}$ by the clone congruence generated by $\to$. A combinatorial realization of this algebraic structure would provide an encoding of the $\equiv$-equivalence classes of terms of $\mathcal{C}$ compatible with term composition.