Series on colored operads and combinatorial generation

Samuele Giraudo
LIGM, Université Paris-Est Marne-la-Vallée

Workshop
Category, Homotopy and Rewriting

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Outline

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Production rules on words

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Example

Let $V := \{\mathbf{a}, \mathbf{b}\}$, $T := \{a, b, c\}$, and $P := \{(\mathbf{a}, b), (\mathbf{a}, \mathbf{a}a\mathbf{b}), (\mathbf{b}, ac)\}$. Since

$$b\mathbf{aa} \to b\mathbf{a}a\mathbf{ba} \to bba\mathbf{ba} \to bbaac\mathbf{a},$$

the word $bbaac\mathbf{a}$ is derivable from $b\mathbf{aa}$.

Context-free grammars

A context-free grammar G is a tuple $(V, T, P, \mathbf{s})$ where

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and u has no occurrence of any variable.

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$$\mathbf{a} \to \mathbf{a}a\mathbf{b} \to ba\mathbf{b} \to baac,$$

the word $baac$ is generated by G.

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A-trees

Let A be a finite graded alphabet partitioned into two sets

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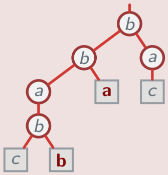
Example

Let $V := \{\mathbf{a}, \mathbf{b}\}$ and $T := \{a, b, c\}$ where $|a| := 1$, $|b| := 2$, and $|c| := 0$. The planar rooted tree



is an A-tree.

Production rules on trees

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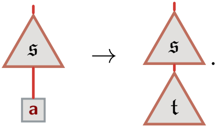

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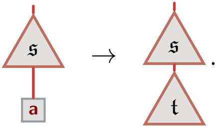

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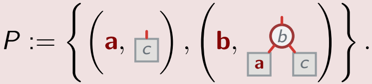
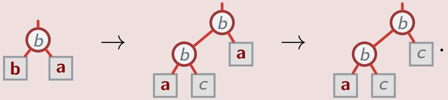


One has the derivation



Regular tree grammars

A regular tree grammar G is a tuple $(V, T, P, \mathbf{s})$ where

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An A-tree $\mathfrak{t}$ is generated by G if

$$\boxed{\mathbf{s}} \to \cdots \to \mathfrak{t}$$

and $\mathfrak{t}$ has no occurrence of any variable (*i.e.,* all leaves of $\mathfrak{t}$ are labeled on T).

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Let G be the regular tree grammar with $V := \{\mathbf{a}, \mathbf{b}\}$, $T := \{a, b, c\}$ where $|a| := 2$, $|b| := 1$, and $|c| := 0$,

$$P := \left\{ \left(\mathbf{a}, c\right), \left(\mathbf{a}, a(\mathbf{b}, \mathbf{b})\right), \left(\mathbf{b}, c\right), \left(\mathbf{b}, b(\mathbf{a})\right) \right\},$$

and $\mathbf{s} := \mathbf{a}$. This grammar generates all alternating unary-binary trees with a nonunary root

Outline

Objectives

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Develop grammars generating any kind of combinatorial objects such as

- words;
- various species of trees;
- integer compositions;
- various species of paths;
- *etc.*

For this, we shall

- rely on colored operad theory;
- develop the notion of formal power series on colored operads.

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Colored operads

In this work any operad $\mathcal{C}$ is

- nonsymmetric;
- set-theoretical:
$$\mathcal{C} = \bigsqcup_{n \geqslant 0} \mathcal{C}(n);$$
- such that $\mathcal{C}(0)$ is empty;
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- such that $\mathcal{C}(0)$ is empty;
- such that all $\mathcal{C}(n)$, $n \geqslant 1$, are finite;
- endowed with a composition map
$$\circ : \mathcal{C}(n) \times \mathcal{C}(m_1) \times \cdots \times \mathcal{C}(m_n) \to \mathcal{C}(m_1 + \cdots + m_n)$$
and equivalent partial composition maps
$$\circ_i : \mathcal{C}(n) \times \mathcal{C}(m) \to \mathcal{C}(n + m - 1), \qquad i \in [n].$$

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In this work any operad $\mathcal{C}$ is

- is colored on a set $\mathscr{C}$ of colors. Color maps are

$$\mathrm{out} : \mathcal{C}(n) \to \mathscr{C}, \qquad n \geqslant 1,$$

and

$$\mathrm{in} : \mathcal{C}(n) \to \mathscr{C}^n, \qquad n \geqslant 1.$$

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Any noncolored operad can be seen as a colored operad on a singleton as set of colors.

Bud operads

Let $\mathcal{O}$ be a noncolored operad and $\mathscr{C}$ be a finite set.

Let $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$, the bud operad of $\mathcal{O}$, be the colored operad defined by

$$\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})(n) := \mathscr{C} \times \mathcal{O}(n) \times \mathscr{C}^n, \qquad n \geqslant 1,$$

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wherein

$$\mathrm{out}((a, x, u)) := a,$$

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and

$$(a, x, u) \circ_i (b, y, v) := (a, x \circ_i y, u \leftarrow_i v),$$

where $u \leftarrow_i v$ is the word obtained by replacing the i-th letter of u by v.

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Proposition

The construction $\mathcal{O} \mapsto \mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$ is a functor from the category of noncolored operads to the category of $\mathscr{C}$-colored operads.

The bud operad of the associative operad

The associative operad As is defined by

- $\mathrm{As}(n) := \{\star_n\}, n \geqslant 1$;
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For any set of colors $\mathscr{C}$,

$$\mathrm{Bud}_{\mathscr{C}}(\mathrm{As}) = \bigsqcup_{n \geqslant 1} \{(a, \star_n, u_1 \dots u_n) : a, u_1, \dots, u_n \in \mathscr{C}\}.$$

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Example

In $\mathrm{Bud}_{\{1,2,3\}}(\mathrm{As})$,

$$(2, \star_4, 3112) \circ_2 (1, \star_3, 233) = (2, \star_6, 323312).$$

The operad of Motzkin paths and its bud operad

The operad of Motzkin paths Motz is defined by

- Motz(n) is the set of the Motzkin paths consisting in $n - 1$ steps;
- the partial composition $x \circ_i y$ of two Motzkin paths consists in replacing the i-th point of x by y.

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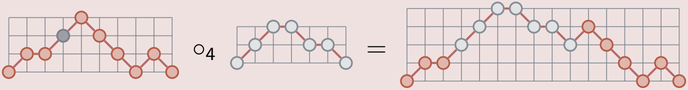

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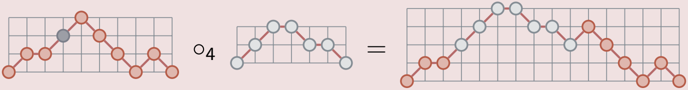

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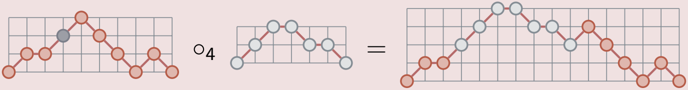

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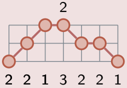

is an element of $\mathrm{Bud}_{\{1,2,3\}}(\text{Motz})$ with 2 as output color and 2213221 as input colors.

Outline

Bud generating systems

A bud generating system $\mathcal{B}$ is a tuple $(\mathcal{O}, \mathscr{C}, \mathfrak{R}, I, T)$ where

- $\mathcal{O}$ is a noncolored operad;
- $\mathscr{C}$ is a finite set of colors;
- $\mathfrak{R}$ is a finite subset of $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$ of rules;
- I is a subset of initial colors of $\mathscr{C}$;
- T is a subset of terminal colors of $\mathscr{C}$.

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provided that there exist $i \in \mathbb{N}$ and $r \in \mathfrak{R}$ such that

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Generation

A element $x \in \mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$ is generated by $\mathcal{B}$ if

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The language of $\mathcal{B}$ is the set $\mathrm{L}(\mathcal{B})$ of all elements generated by $\mathcal{B}$.

Bud generating systems and Motzkin paths

Let the bud generating system $\mathcal{B} := (\mathsf{Motz}, \{1, 2\}, \mathfrak{R}, \{1\}, \{1, 2\})$ where

$\mathfrak{R} := \{(1,$  $, 22), (1,$  $, 111)\}.$

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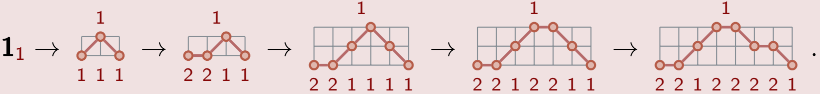

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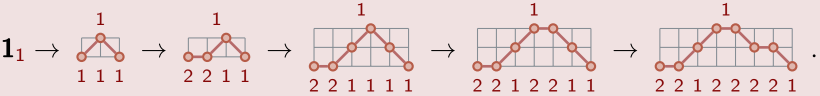

Proposition

$\mathrm{L}(\mathcal{B})$ is in bijection with the set of Motzkin paths with no consecutive horizontal steps.

These Motzkin paths are enumerated by Sequence **A104545**:

$$1, 1, 1, 3, 5, 11, 25, 55, 129, 303, 721, 1743.$$

Synchronous production rules

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$$x_1 \rightsquigarrow x_2$$

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The reflexive and transitive closure of $\rightsquigarrow$ is the synchronous derivation relation.

Synchronous generation

A element $x \in \mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$ is synchronously generated by $\mathcal{B}$ if

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$$x = (\ldots((\mathbf{1}_c \circ [r_{1,1}]) \circ [r_{2,1}, \ldots, r_{2,j_2}]) \ldots) \circ [r_{k,1}, \ldots, r_{k,j_k}]$$

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$$x = (\dots((\mathbf{1}_c \circ [r_{1,1}]) \circ [r_{2,1}, \dots, r_{2,j_2}]) \dots) \circ [r_{k,1}, \dots, r_{k,j_k}]$$

and $\mathrm{in}(x) \in T^*$.

The synchronous language of $\mathcal{B}$ is the set $\mathrm{SL}(\mathcal{B})$ of all elements synchronously generated by $\mathcal{B}$.

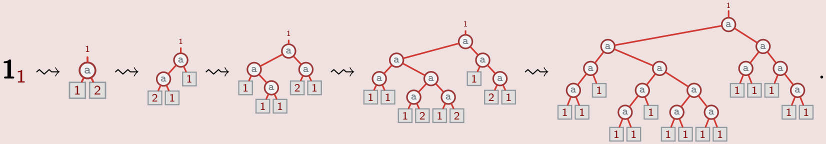
Bud generating systems and balanced binary trees

Let the bud generating system $\mathcal{B} := (\mathrm{Mag}, \{1, 2\}, \mathfrak{R}, \{1\}, \{1\})$ where $\mathrm{Mag} := \mathrm{Free}(\{\mathtt{a}\})$, $|\mathtt{a}| := 2$, and

$\mathfrak{R} := \Big\{ \Big(1,$  $, 11\Big), \Big(1,$  $, 12\Big), \Big(1,$  $, 21\Big), (2, \mathbf{1}, 1) \Big\}.$

Example

There is in $\mathcal{B}$ the sequence of derivations



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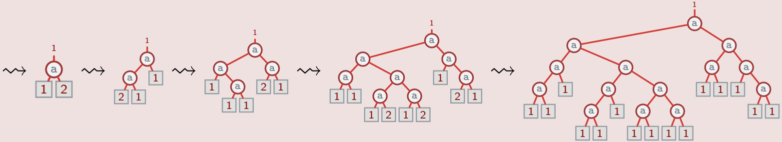
$\mathbf{1}_1 \rightsquigarrow$  .

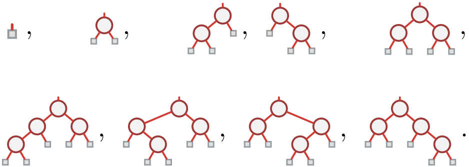
Proposition

$\mathrm{SL}(\mathcal{B})$ is in bijection with the set of balanced binary trees.

Balanced binary trees

A balanced binary tree is a binary tree $\mathfrak{t}$ such that, for each internal node x of $\mathfrak{t}$, the height of the left and of the right subtrees of x differ by at most 1.

First balanced binary trees are



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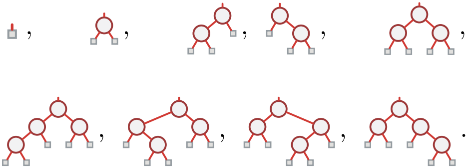

These trees are enumerated by Sequence **A006265**:

$$1, 1, 2, 1, 4, 6, 4, 17, 32, 44, 60, 70.$$

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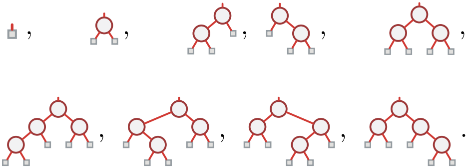

These trees are enumerated by Sequence **A006265**:

$$1, 1, 2, 1, 4, 6, 4, 17, 32, 44, 60, 70.$$

Their generating series is $F(x, 0)$ where

$$F(x, y) = x + F(x^2 + 2xy, x).$$

Outline

Languages of bud generating systems

Proposition

If $\mathcal{B} := (\mathcal{O}, \mathscr{C}, \mathfrak{R}, I, T)$ is a bud generating system,

$$\mathrm{L}(\mathcal{B}) = \{x \in \mathrm{Bud}_{\mathscr{C}}(\mathcal{O}) : x \in \mathcal{C}, \mathrm{out}(x) \in I, \mathrm{in}(x) \in T^*\},$$

where $\mathcal{C}$ is suboperad of $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$ generated by $\mathfrak{R}$.

Languages of bud generating systems

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where $\mathcal{C}$ is suboperad of $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$ generated by $\mathfrak{R}$.

Proposition

If $\mathcal{B}$ is a bud generating system, $\mathrm{SL}(\mathcal{B}) \subseteq \mathrm{L}(\mathcal{B})$.

Simulation of context-free grammars

Let $G := (V, T, P, \mathbf{s})$ be a proper context-free grammar, *i.e.*, for all $(\mathbf{a}, u) \in P$, $|u| \geqslant 1$.

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Let the bud generating system

$$\mathcal{B} := (\mathsf{As}, V \sqcup T, \mathfrak{R}, \{\mathbf{s}\}, T)$$

where

$$\mathfrak{R} := \left\{ \left(\mathbf{a}, \star_{|u|}, u\right) : (\mathbf{a}, u) \in P \right\}.$$

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Proposition

The map $\mathrm{in} : \mathrm{L}(\mathcal{B}) \to W$, where W is the set of words generated by G, is a bijection.

Hence, any proper context-free grammar can be simulated by a bud generating system based on the associative operad.

Simulation of context-free grammars

Example

Let $G := (\{\mathbf{a}, \mathbf{b}\}, \{a, b\}, P, \mathbf{a})$ be the proper context-free grammar where

$$P := \{(\mathbf{a}, a), (\mathbf{a}, b\mathbf{b}), (\mathbf{b}, b), (\mathbf{b}, \mathbf{ab})\}.$$

The bud generating system $\mathcal{B} := (\mathrm{As}, \{\mathbf{a}, \mathbf{b}, a, b\}, \mathfrak{R}, \{\mathbf{a}\}, \{a, b\})$ where

$$\mathfrak{R} := \{(\mathbf{a}, \star_1, a), (\mathbf{a}, \star_2, b\mathbf{b}), (\mathbf{b}, \star_1, b), (\mathbf{b}, \star_2, \mathbf{ab})\}$$

simulates G.

Simulation of context-free grammars

Example

The sequence of derivations

$$\mathbf{a} \to b\mathbf{b} \to b\mathbf{ab} \to bb\mathbf{bb} \to bbb\mathbf{b} \to bbb\mathbf{ab} \to bbba\mathbf{b} \to bbbab$$

in G is interpreted into the sequence of derivations

$$\begin{aligned}\mathbf{1}_{\mathbf{a}} \to (\mathbf{a}, \star_2, b\mathbf{b}) \to (\mathbf{a}, \star_3, b\mathbf{ab}) \to (\mathbf{a}, \star_4, bb\mathbf{bb}) \to (\mathbf{a}, \star_4, bbb\mathbf{b})\\ \to (\mathbf{a}, \star_5, bbb\mathbf{ab}) \to (\mathbf{a}, \star_5, bbba\mathbf{b}) \to (\mathbf{a}, \star_5, bbbab)\end{aligned}$$

in $\mathcal{B}$.

Simulation of regular tree grammars

Let $G := (V, T, P, \mathbf{s})$ be a regular tree grammar.

Let the bud generating system

$$\mathcal{B} := (\mathrm{Free}(T \setminus T(0)), V \sqcup T(0), \mathfrak{R}, \{\mathbf{s}\}, T(0))$$

where

$$\mathfrak{R} := \{(\mathbf{a}, \mathrm{tr}(\mathfrak{t}), \mathrm{fr}(\mathfrak{t})) : (\mathbf{a}, \mathfrak{t}) \in P\},$$

$\mathrm{tr}(\mathfrak{t})$ denoting the tree obtained by forgetting the labels of the leaves of $\mathfrak{t}$ and $\mathrm{fr}(\mathfrak{t})$ denoting the word obtained by reading the labels of the leaves of $\mathfrak{t}$.

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Proposition

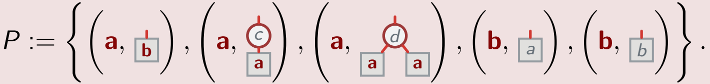
The map $\phi : \mathrm{L}(\mathcal{B}) \to W$ defined by $\phi((\mathbf{a}, \mathfrak{t}, u)) := \mathfrak{t}_u$, where $\mathfrak{t}_u$ is the tree obtained by labeling the leaves of $\mathfrak{t}$ by the letters of u and where W is the set of trees generated by G, is a bijection.

Hence, any regular tree grammar can be simulated by a bud generating system based on free operads.

Simulation of regular tree grammars

Example

Let $G := (\{\mathbf{a}, \mathbf{b}\}, \{a, b, c, d\}, P, \mathbf{a})$ be the regular tree grammar where $|a| := 0$, $|b| := 0$, $|c| := 1$, $|d| := 2$, and



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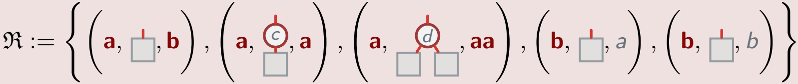
$\mathcal{B} := (\mathrm{Free}(\{c, d\}, \{\mathbf{a}, \mathbf{b}, a, b\}, \mathfrak{R}, \{\mathbf{a}\}, \{a, b\}))$ where



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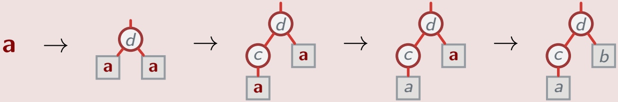
Simulation of regular tree grammars

Example

The sequence of derivations



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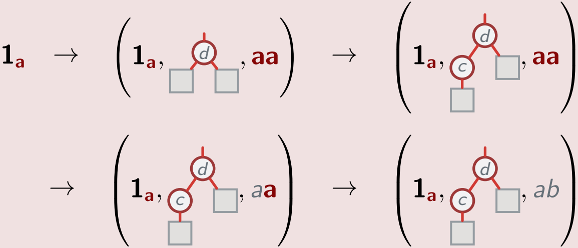

in $\mathcal{B}$.

Outline

Outline

Series on colored operads

Let $\mathcal{C}$ be a colored operad and $\mathbb{K}$ be a field ($\mathbb{K} := \mathbb{Q}(q_0, q_1, \dots)$).

A $\mathcal{C}$-series is a map

$$\mathbf{f} : \mathcal{C} \to \mathbb{K}.$$

The set of all such series is denoted by $\mathbb{K} \langle\langle \mathcal{C} \rangle\rangle$.

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The support of $\mathbf{f}$ is the set

$$\mathrm{Supp}(\mathbf{f}) := \{x \in \mathcal{C} : \langle x, \mathbf{f} \rangle \neq 0\}.$$

Series on colored operads

The set $\mathbb{K}\langle\langle\mathcal{C}\rangle\rangle$ is endowed with the pointwise addition and the multiplication by a scalar, forming a vector space.

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$$\mathbf{f} = \sum_{x \in \mathcal{C}} \langle x, \mathbf{f} \rangle \, x.$$

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The series of colored units is

$$\mathbf{u} := \sum_{c \in \mathscr{C}} \mathbf{1}_c,$$

where $\mathscr{C}$ is the set of colors of $\mathcal{C}$.

Generating series and operads

$\mathcal{C}$-series form generalizations of usual generating series.

There are other ones:

- series on monoids [**Salomaa, Soittola**, 1978];
- series on trees [**Berstel, Reutenauer**, 1982];
- series on operads [**Chapoton**, 2002].

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Several definitions of series have been considered on various sorts of operads:

- algebraic (and symmetric) operads [**Chapoton**, 2002, 2009];
- nonsymmetric algebraic operads [**van der Laan**, 2004];
- nonsymmetric set-operads $\mathcal{O}$ with $\mathcal{O}(1) := \{\mathbf{1}\}$ [**Frabetti**, 2008];
- algebraic (and symmetric) operads [**Loday, Nikolov**, 2013].

Pre-Lie product

The pre-Lie product $\mathbf{f} \curvearrowleft \mathbf{g}$ of two $\mathcal{C}$-series $\mathbf{f}$ and $\mathbf{g}$ is defined by

$$\langle x, \mathbf{f} \curvearrowleft \mathbf{g} \rangle := \sum_{\substack{y,z \in \mathcal{C} \\ i \in [|y|] \\ x = y \circ_i z}} \langle y, \mathbf{f} \rangle \langle z, \mathbf{g} \rangle .$$

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The product $\curvearrowleft$ is

- bilinear;
- totally defined (because all $\mathcal{C}(n)$ are finite);
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Proposition

The pre-Lie product satisfies

$$(\mathbf{f} \curvearrowleft \mathbf{g}) \curvearrowleft \mathbf{h} - \mathbf{f} \curvearrowleft (\mathbf{g} \curvearrowleft \mathbf{h}) = (\mathbf{f} \curvearrowleft \mathbf{h}) \curvearrowleft \mathbf{g} - \mathbf{f} \curvearrowleft (\mathbf{h} \curvearrowleft \mathbf{g}).$$

Hence, $(\mathbb{K} \langle\langle \mathcal{C} \rangle\rangle, \curvearrowleft)$ is a pre-Lie algebra.

Pre-Lie star product

For any $\ell \geqslant 0$, let

$$\mathbf{f}^{\curvearrowleft \ell} := \begin{cases} \mathbf{u} & \text{if } \ell = 0, \\ \mathbf{f}^{\curvearrowleft \ell - 1} \curvearrowleft \mathbf{f} & \text{otherwise.} \end{cases}$$

The $\curvearrowleft$-star of $\mathbf{f}$ is the $\mathcal{C}$-series

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Lemma

If $\mathrm{Supp}(\mathbf{f})(1)$ is $\mathcal{C}$-finitely factorizing, $\mathbf{f}^{\curvearrowleft *}$ is a well-defined series.

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A subset S of $\mathcal{C}(1)$ is $\mathcal{C}$-finitely factorizing if all $x \in \mathcal{C}(1)$ admits finitely many factorizations on S.

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Pre-Lie star product

Proposition

If $\mathbf{f}$ is a $\mathcal{C}$-series such that $\mathrm{Supp}(\mathbf{f})(1)$ is $\mathcal{C}$-finitely factorizing,

$$\langle x, \mathbf{f}^{\curvearrowleft *} \rangle = \delta_{x, \mathbf{1}_{\mathrm{out}(x)}} + \sum_{\substack{y,z \in \mathcal{C} \\ i \in [|y|] \\ x = y \circ_i z}} \langle y, \mathbf{f}^{\curvearrowleft *} \rangle \langle z, \mathbf{f} \rangle .$$

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Proposition

If $\mathbf{f}$ is a $\mathcal{C}$-series such that $\mathrm{Supp}(\mathbf{f})(1)$ is $\mathcal{C}$-finitely factorizing, the equation

$$\mathbf{x} - \mathbf{x} \curvearrowleft \mathbf{f} = \mathbf{u}$$

admits the unique solution $\mathbf{x} = \mathbf{f}^{\curvearrowleft *}$.

Composition product

The composition product $\mathbf{f} \odot \mathbf{g}$ of two $\mathcal{C}$-series $\mathbf{f}$ and $\mathbf{g}$ is defined by

$$\langle x, \mathbf{f} \odot \mathbf{g} \rangle := \sum_{\substack{y, z_1, \dots, z_{|y|} \in \mathcal{C} \\ x = y \circ \left[z_1, \dots, z_{|y|} \right]}} \langle y, \mathbf{f} \rangle \prod_{i \in [|y|]} \langle z_i, \mathbf{g} \rangle .$$

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Proposition

The composition product is associative and hence, $(\mathbb{K} \langle\langle \mathcal{C} \rangle\rangle, \odot)$ is a monoid.

Composition star product

For any $\ell \geqslant 0$, let

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Composition star product

Proposition

If $\mathbf{f}$ is a $\mathcal{C}$-series such that $\mathrm{Supp}(\mathbf{f})(1)$ is $\mathcal{C}$-finitely factorizing,

$$\left\langle x, \mathbf{f}^{\odot_*} \right\rangle = \delta_{x,\mathbf{1}_{\mathrm{out}(x)}} + \sum_{\substack{y, z_1, \ldots, z_{|y|} \in \mathcal{C} \\ x = y \circ \left[z_1, \ldots, z_{|y|}\right]}} \left\langle y, \mathbf{f}^{\odot_*} \right\rangle \prod_{i \in [|y|]} \left\langle z_i, \mathbf{f} \right\rangle .$$

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admits the unique solution $\mathbf{x} = \mathbf{f}^{\odot_*}$.

Outline

Hook generating series

The hook generating series of $\mathcal{B}$ is

$$\mathrm{hook}(\mathcal{B}) := \mathbf{i} \odot \mathbf{r}^{\curvearrowleft *} \odot \mathbf{t},$$

where $\mathbf{r}$ (resp. $\mathbf{i}$, $\mathbf{t}$) is the characteristic series of $\mathfrak{R}$ (resp. $\{\mathbf{1}_c : c \in I\}$, $\{\mathbf{1}_c : c \in T\}$).

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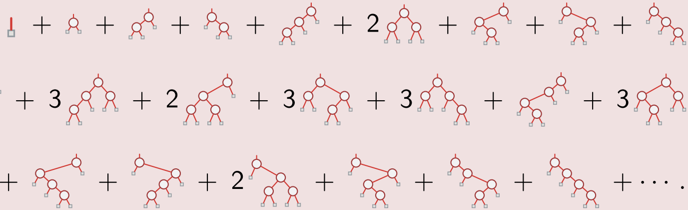
Example

For $\mathcal{B} := (\mathsf{Mag}, \{1\}, \{$  $\}, \{1\}, \{1\})$,

$\operatorname{hook}(\mathcal{B}) =$ 

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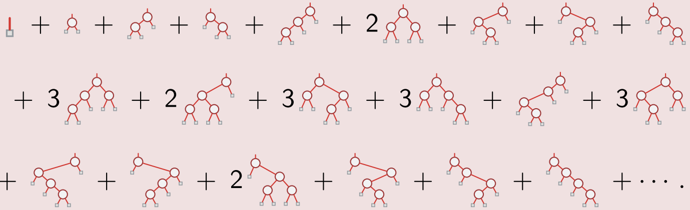
where $\mathbf{r}$ (resp. $\mathbf{i}$, $\mathbf{t}$) is the characteristic series of $\mathfrak{R}$ (resp. $\{\mathbf{1}_c : c \in I\}$, $\{\mathbf{1}_c : c \in T\}$).

Example

For $\mathcal{B} := (\mathrm{Mag}, \{1\}, \{$  $\}, \{1\}, \{1\})$,

$\mathrm{hook}(\mathcal{B}) =$ 

This example explains the name of $\mathrm{hook}(\mathcal{B})$: the coefficients of the above series can be obtained by a hook formula on binary trees [**Knuth**, 1973].

Analogs of the hook statistic

Let $\mathcal{O}$ be an operad, G be a generating set of $\mathcal{O}$, and consider the bud generating system $\mathcal{B}_{\mathcal{O},G} := (\mathcal{O}, \{1\}, G, \{1\}, \{1\})$.

The coefficients $\langle x, \mathrm{hook}(\mathcal{B})\rangle$ define a statistic on the objects of $\mathcal{O}$, analogs to the hook statistic on trees.

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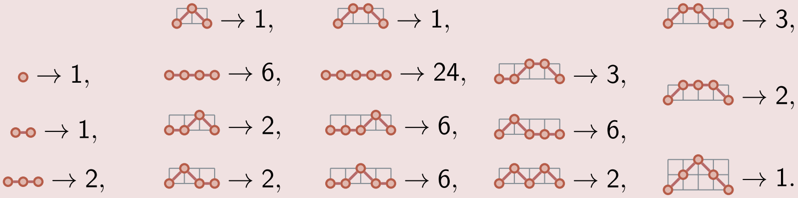
Example

From $\mathcal{B}_{\mathrm{Motz},G}$ with $G := \{$  $\}$, we have an analog of the hook statistic for Motzkin paths:



Derivation graphs

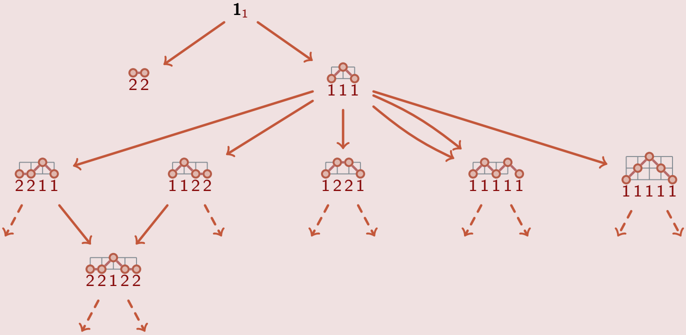
There is a combinatorial interpretation of $\mathrm{hook}(\mathcal{B})$.

The derivation graph of $\mathcal{B}$ is the oriented multigraph $\mathrm{DG}(\mathcal{B})$ with

- the set of elements derivable from $\mathbf{1}_c$, $c \in I$, as set of vertices;
- there is a edge from x_1 to x_2 if $x_1 \rightarrow x_2$.

Example

The derivation graph of $\mathcal{B} := (\mathrm{Motz}, \{1, 2\}, \mathfrak{R}, \{1\}, \{1, 2\})$ where $\mathfrak{R} := \{(1,$  $, 22), (1,$  $, 111)\}$ is



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Derivation graphs and hook generating series

Proposition

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing, for all $x \in \mathrm{L}(\mathcal{B})$, the coefficient $\langle x, \mathrm{hook}(\mathcal{B}) \rangle$ is the number of paths in $\mathrm{DG}(\mathcal{B})$ from a $\mathbf{1}_c$, $c \in I$, to x.

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Theorem

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing,

$$\mathrm{hook}(\mathcal{B}) = \sum_{\substack{\mathfrak{t} \in \mathrm{Free}(\mathfrak{R}) \\ \mathrm{out}(\mathfrak{t}) \in I \\ \mathrm{in}(\mathfrak{t}) \in T^*}} \frac{\deg(\mathfrak{t})!}{\prod\limits_{v \in \mathrm{N}(\mathfrak{t})} \deg(\mathfrak{t}_v)} \mathrm{eval}_{\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})}(\mathfrak{t}).$$

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Proposition

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing,

$$\mathrm{Supp}(\mathrm{hook}(\mathcal{B})) = \mathrm{L}(\mathcal{B}).$$

Synchronous generating series

The synchronous generating series of $\mathcal{B}$ is

$$\mathrm{sync}(\mathcal{B}) := \mathbf{i} \odot \mathbf{r}^{\odot *} \odot \mathbf{t}.$$

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If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing,

$$\mathrm{sync}(\mathcal{B}) = \sum_{\substack{\mathfrak{t} \in \mathrm{Free}_{\mathrm{perf}}(\mathfrak{R}) \\ \mathrm{out}(\mathfrak{t}) \in I \\ \mathrm{in}(\mathfrak{t}) \in T^*}} \mathrm{eval}_{\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})}(\mathfrak{t}).$$

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Proposition

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing,

$$\mathrm{Supp}(\mathrm{sync}(\mathcal{B})) = \mathrm{SL}(\mathcal{B}).$$

Outline

Series of colors

Let $\mathbf{f}$ be a $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-series.

Let

$$\mathfrak{col} : \mathbb{K}\langle\langle \mathrm{Bud}_{\mathscr{C}}(\mathcal{O}) \rangle\rangle \to \mathbb{K}\langle\langle \mathrm{Bud}_{\mathscr{C}}(\mathsf{As}) \rangle\rangle$$

be the map defined by

$$\langle (a, \star_n, u), \mathfrak{col}(\mathbf{f}) \rangle := \sum_{(a,x,u) \in \mathrm{Bud}_{\mathscr{C}}(\mathcal{O})} \langle (a, x, u), \mathbf{f} \rangle .$$

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This series is the series of colors of $\mathbf{f}$.

The series $\mathfrak{col}(\mathbf{f})$ is a version of $\mathbf{f}$ wherein only the colors of the elements of its support are taken into account.

Series of color types

Let

$$\mathfrak{colt} : \mathbb{K} \langle\langle \mathrm{Bud}_{\mathscr{C}}(\mathcal{O}) \rangle\rangle \to \mathbb{K}[[\mathscr{C}]] \otimes \mathbb{K}[[\mathscr{C}]]$$

be the map defined by

$$\langle a \otimes u, \mathfrak{colt}(\mathbf{f}) \rangle := \sum_{\substack{(a,x,v) \in \mathrm{Bud}_{\mathscr{C}}(\mathsf{As}) \\ \mathrm{type}(v) = u}} \langle (a, x, v), \mathfrak{col}(\mathbf{f}) \rangle ,$$

where $\mathrm{type}(v)$ is the commutative image of v.

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The series $\mathfrak{colt}(\mathbf{f})$ is a version of $\mathbf{f}$ wherein only the output colors and the commutative images of the input colors of the elements of its support are taken into account.

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The series $\mathfrak{colt}(\mathbf{f})$ is a version of $\mathbf{f}$ wherein only the output colors and the commutative images of the input colors of the elements of its support are taken into account.

Goal

Obtain a expression for $\mathfrak{colt}(\mathrm{sync}(\mathcal{B}))$, counting the elements synchronously generated by $\mathcal{B}$ with respect to their number input colors.

Synchronous generating series

We consider that $\mathscr{C} := \{c_1, \ldots, c_k\}$ and $\mathfrak{R} := \{r_1, \ldots, r_\ell\}$.

Let $\mathcal{M}^{\mathrm{in}}$ be the $\ell \times k$-matrix defined by

$$\mathcal{M}^{\mathrm{in}}_{i,j} := |\mathrm{in}(r_i)|_{c_j}.$$

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$$\mathcal{M}^{\mathrm{out}}_{i,j} := \begin{cases} 1 & \text{if } \mathrm{out}(r_i) = c_j, \\ 0 & \text{otherwise.} \end{cases}$$

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Example

For $\mathcal{B} := (\mathrm{Mag}, \{1, 2\}, \mathfrak{R}, \{1\}, \{1\})$ where

$\mathfrak{R} := \Big\{ \Big(1,$ 

$, 11\Big), \Big(1,$ 

$, 12\Big), \Big(1,$ 

$, 21\Big), (2, \mathbf{1}, 1) \Big\},$

we have

$$\mathcal{M}^{\mathrm{in}} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{M}^{\mathrm{out}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Synchronous generating series

For any $1 \times k$ matrix α, $\mathscr{C}^{\alpha}$ denotes the monomial ${c_1}^{\alpha_1} \dots {c_k}^{\alpha_k}$.

For any set $S := \{s_1, \dots, s_n\}$ of nonnegative integers, $S!$ denotes the multinomial coefficient $\binom{s_1+\cdots+s_n}{s_1,\dots,s_n}$.

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Theorem

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing, for all $a \in \mathcal{C}$ and $\alpha \in \mathrm{Mat}(1, k)$,

$$\left\langle a \otimes \mathscr{C}^{\alpha}, \mathfrak{colt}\left(\mathbf{r}^{\odot_*}\right)\right\rangle = \delta_{\alpha,\mathrm{type}(a)} + \sum_{\substack{\zeta \in \mathrm{Mat}(1,\ell) \\ \zeta \mathcal{M}^{\mathrm{in}} = \alpha}} \left(\prod_{j \in [k]} \{\zeta_i : \mathcal{M}^{\mathrm{out}}_{i,j} = 1\}!\right) \left\langle a \otimes \mathscr{C}^{\zeta \mathcal{M}^{\mathrm{out}}}, \mathfrak{colt}\left(\mathbf{r}^{\odot_*}\right)\right\rangle.$$

Enumeration of balanced binary trees

The synchronous language of the bud generating system $\mathcal{B} := (\mathsf{Mag}, \{1, 2\}, \mathfrak{R}, \{1\}, \{1\})$ where

$\mathfrak{R} := \Big\{ \Big(1,$ 

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Since

$$\mathcal{M}^{\mathrm{in}} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{M}^{\mathrm{out}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the series $\mathfrak{colt}(\mathbf{r}^{\odot_*})$ satisfies

$$\left\langle 1 \otimes 1^{\alpha_1} 2^{\alpha_2}, \mathfrak{colt}\left(\mathbf{r}^{\odot_*}\right) \right\rangle = \delta_{(\alpha_1, \alpha_2),(1,0)}$$

$$+ \sum_{\substack{2\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = \alpha_1 \\ \zeta_2 + \zeta_3 = \alpha_2}} \binom{\zeta_1 + \zeta_2 + \zeta_3}{\zeta_1, \zeta_2, \zeta_3} \left\langle 1 \otimes 1^{\zeta_1 + \zeta_2 + \zeta_3} 2^{\zeta_4}, \mathfrak{colt}\left(\mathbf{r}^{\odot_*}\right) \right\rangle.$$

Enumeration of balanced binary trees

This leads to the definition of the map

$$f_{\alpha_1,\alpha_2} : \mathbb{N}^2 \setminus \{(0,0)\} \to \mathbb{N}$$

satisfying $f_{1,0} = 1$ and the recurrence formula

$$f_{\alpha_1,\alpha_2} = \sum_{2\zeta_1+\alpha_2+\zeta_4=\alpha_1} \binom{\zeta_1+\alpha_2}{\zeta_1} 2^{\alpha_2} f_{\zeta_1+\alpha_2,\zeta_4}.$$

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The coefficient f_{α_1,α_2} is the coefficient of $1 \otimes 1^{\alpha_1} 2^{\alpha_2}$ in $\mathfrak{colt}(\mathrm{sync}(\mathcal{B}))$.

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Moreover, since $I = \{1\}$ and $T = \{1\}$, the number of balanced binary trees with α_1 leaves is

$$\left\langle 1 \otimes 1^{\alpha_1} 2^0, \mathfrak{colt}(\mathrm{sync}(\mathcal{B}))\right\rangle = \left\langle 1 \otimes 1^{\alpha_1} 2^0, \mathfrak{colt}\left(\mathbf{r}^{\odot_*}\right)\right\rangle = f_{\alpha_1,0}.$$

Outline

The bud operad of free operads

The free operad $\mathrm{Free}(G)$ over G, where G is a graded set, is defined by

- $\mathrm{Free}(G)(n)$ is the set of the planar rooted trees with n leaves and where internal nodes are labeled on G, respecting the arities of the labels;
- $\mathfrak{s} \circ_i \mathfrak{t}$ is tree obtained by grafting the root of $\mathfrak{t}$ to the i-th leaf of $\mathfrak{s}$.

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For any set of colors $\mathscr{C}$,

$$\mathrm{Bud}_{\mathscr{C}}(\mathrm{Free}(G)) = \bigsqcup_{n \geqslant 1} \{(a, \mathfrak{t}, u_1 \dots u_n) : a, u_1, \dots, u_n \in \mathscr{C}, \mathfrak{t} \in \mathrm{Free}(G)(n)\}.$$

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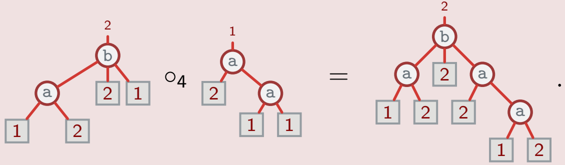
Example

In $\mathrm{Bud}_{\{1,2\}}(\mathrm{Free}(G))$ where $G := \{\mathtt{a}, \mathtt{b}\}$, $|\mathtt{a}| :=: 2$, and $|\mathtt{b}| := 3$,



The bud operad of free operads

In general, $\mathrm{Bud}_{\mathscr{C}}(\mathrm{Free}(G))$ is not a free colored operad.

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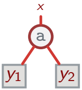
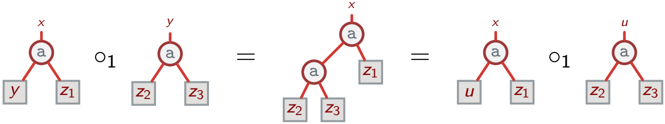
For instance, the colored operad $\mathrm{Bud}_{\{1,2\}}(\mathrm{Free}(\{\mathtt{a}\}))$ with $|\mathtt{a}| := 2$ is generated by the eight corollas

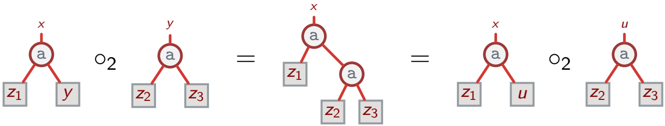


, $x, y_1, y_2 \in \{1, 2\}$,

and are subject to the nontrivial quadratic relations



, $y \neq u$,



, $y \neq u$.

Invertible elements for $\odot$

Proposition

If $\mathrm{Supp}(\mathbf{f})(1) = \{\mathbf{1}_c : c \in \mathscr{C}\} \sqcup S$ for some $\mathcal{C}$-finitely factorizing set S, the equations

$$\mathbf{f} \odot \mathbf{x} = \mathbf{u} \qquad \text{and} \qquad \mathbf{x} \odot \mathbf{f} = \mathbf{u}$$

admit both the unique same solution denoted by $\mathbf{x} = \mathbf{f}^{\odot -1}$.

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Proposition

If $\mathrm{Supp}(\mathbf{f})(1) = \{\mathbf{1}_c : c \in \mathscr{C}\} \sqcup S$ for some $\mathcal{C}$-finitely factorizing set S, $\mathbf{f}^{\odot -1}$ is a well-defined series satisfying

$$\left\langle x, \mathbf{f}^{\odot -1} \right\rangle = \sum_{\substack{\mathfrak{t} \in \mathrm{Free}(\bar{\mathcal{C}}) \\ \mathrm{eval}_{\mathcal{C}}(\mathfrak{t}) = x}} (-1)^{\deg(\mathfrak{t})} \frac{1}{\left\langle \mathbf{1}_{\mathrm{out}(x)}, \mathbf{f} \right\rangle} \prod_{v \in \mathrm{N}(\mathfrak{t})} \frac{\langle \mathrm{lb}(v), \mathbf{f} \rangle}{\prod_{j \in [|v|]} \left\langle \mathbf{1}_{\mathrm{in}_j(v)}, \mathbf{f} \right\rangle}.$$

Therefore, the monoid $(\mathbb{K}\langle\langle \mathcal{C} \rangle\rangle, \odot)$ contains a (large) group formed by the series with a support satisfying the above description.

Syntactic generating series

Let $\mathcal{B} := (\mathcal{O}, \mathscr{C}, \mathfrak{R}, I, T)$ be a bud generating system.

The syntactic generating series of $\mathcal{B}$ is

$$\mathrm{synt}(\mathcal{B}) := \mathbf{i} \odot (\mathbf{u} - \mathbf{r})^{\odot_{-1}} \odot \mathbf{t}.$$

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Lemma

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing, $\mathrm{synt}(\mathcal{B})$ is a well-defined series.

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Lemma

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing, $\mathrm{synt}(\mathcal{B})$ is a well-defined series.

We say that $\mathcal{B}$ is unambiguous if all coefficients of $\mathrm{synt}(\mathcal{B})$ are 0 or 1.

Syntactic generating series

Theorem

If $\mathfrak{R}(1)$ is $\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})$-finitely factorizing,

$$\mathrm{synt}(\mathcal{B}) = \sum_{\substack{\mathfrak{t} \in \mathrm{Free}(\mathfrak{R}) \\ \mathrm{out}(\mathfrak{t}) \in I \\ \mathrm{in}(\mathfrak{t}) \in T^*}} \mathrm{eval}_{\mathrm{Bud}_{\mathscr{C}}(\mathcal{O})}(\mathfrak{t}).$$

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Proposition

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If $\mathcal{B}$ is unambiguous, each element generated by $\mathcal{B}$ admits exactly one treelike expression.