Polydendriform algebras

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Outline

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In our context, the operations $\prec$ and $\succ$ have to satisfy some precise relations.

Example: the shuffle algebra

Consider the vector space $\mathbb{Q}\langle \mathtt{a}, \mathtt{b}\rangle$ of noncommutative polynomials.

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Let us endow this space with the shuffle product $\sqcup\!\sqcup$.

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$\sqcup\!\sqcup$ splits into two parts $\prec$ and $\succ$ according to the origin of the last letter of the words.

Example

$$\mathtt{ab} \prec \mathtt{ba} = \mathtt{abab} + \mathtt{baab} + \mathtt{baab}$$

$$\mathtt{ab} \succ \mathtt{ba} = \mathtt{abba} + \mathtt{abba} + \mathtt{baba}$$

Dendriform algebras

A dendriform algebra [**Loday**, 2001] is a $\mathbb{K}$-vector space $\mathcal{V}$ endowed with two operations

$$\prec : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \qquad \text{and} \qquad \succ : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

satisfying, for all $x, y, z \in \mathcal{V}$, the relations

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(x \prec y) \prec z &= x \prec (y \prec z) + x \prec (y \succ z), \\
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Proposition [Loday, 2001]

Let $(\mathcal{V}, \prec, \succ)$ be a dendriform algebra. Then, the operation $\prec + \succ$ is associative.

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In the shuffle algebra, $\prec$ and $\succ$ satisfy

$$u\mathtt{a} \prec v = (u \sqcup\!\sqcup\, v)\mathtt{a} \qquad \text{and} \qquad u \succ v\mathtt{b} = (u \sqcup\!\sqcup\, v)\mathtt{b}.$$

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This is the Ree recursive definition of the shuffle product [**Ree**, 1957], [**Schützenberger**, 1958].

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The dendriform operad [**Loday**, 2001] describes all the dendriform algebras.

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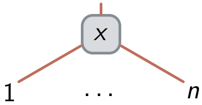
An operator is an object with $n \geqslant 1$ inputs and one output.



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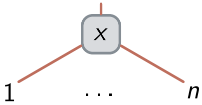

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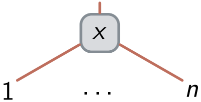

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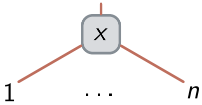

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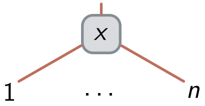

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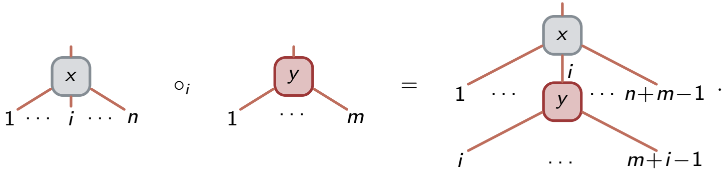
We then obtain a new operator $x \circ_i y$ of arity $n + m - 1$:



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This data has to satisfy some coherence axioms.

Operad axioms

Associativity:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}$$
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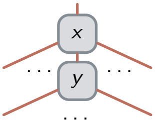

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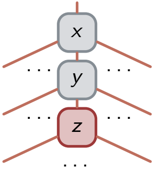

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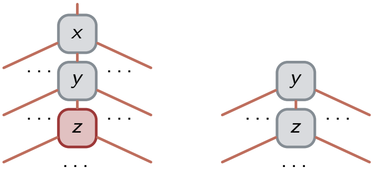

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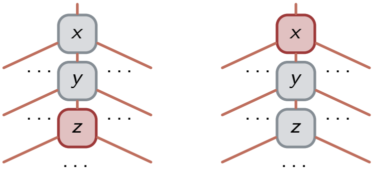

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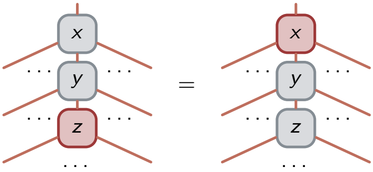

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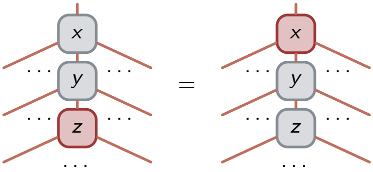

Commutativity:

$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y$
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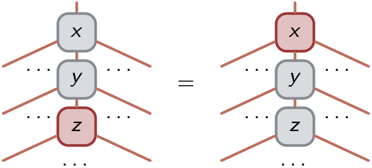

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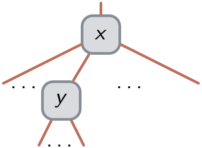

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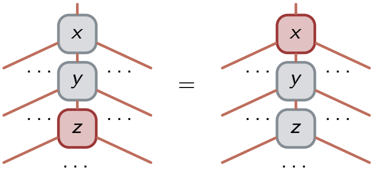

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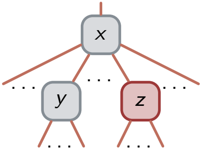

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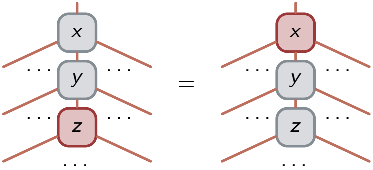

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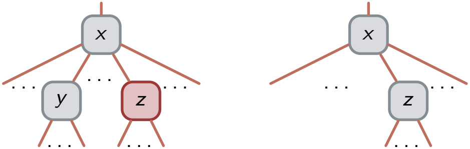

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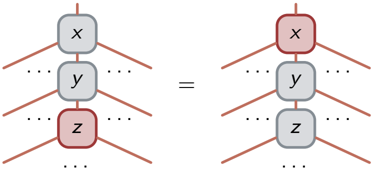

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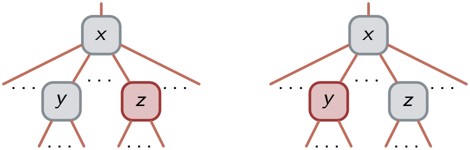

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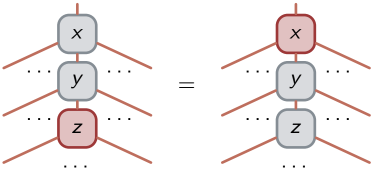

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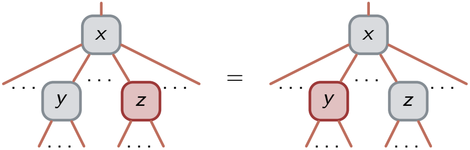

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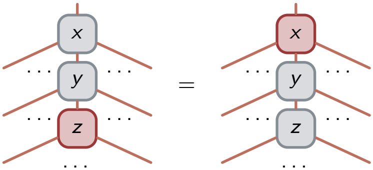

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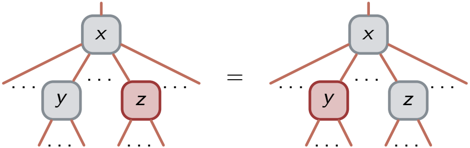

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Operad axioms

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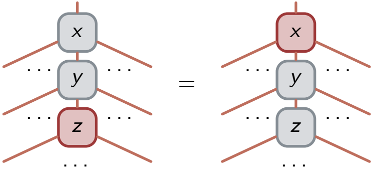

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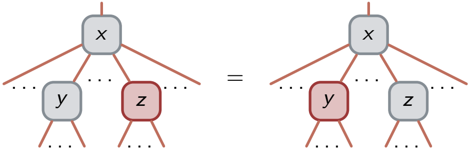

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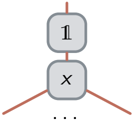

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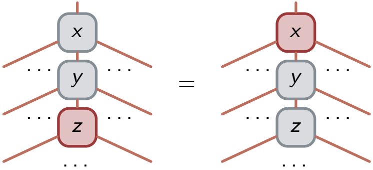

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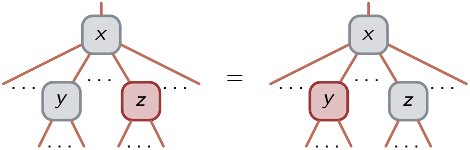

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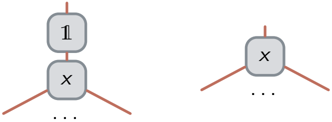

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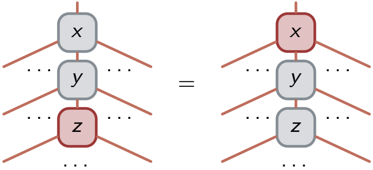

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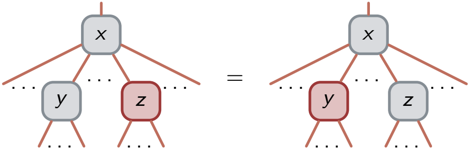

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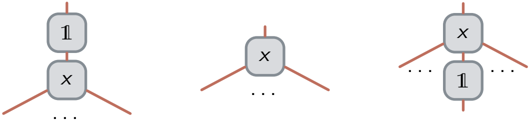

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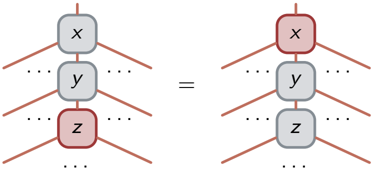

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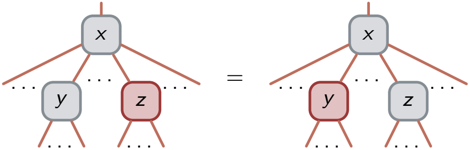

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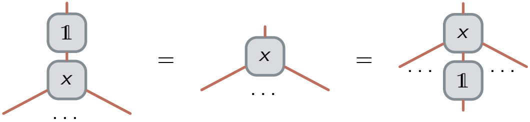

Example: the operad of Motzkin paths

The operad Motz is defined in the following way:

- Motz(n) is the linear span of the Motzkin paths consisting in $n - 1$ steps;
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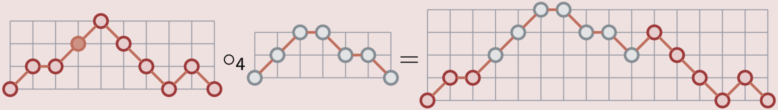

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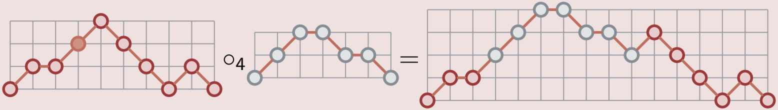

Exercice

Prove that Motz is an operad.

Studying an operad

Given an operad $\mathcal{O}$, one can ask about:

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3. the nontrivial relations between its generators, that are equalities involving partial compositions of elements of G that cannot be expressed by operad axioms.

Example: properties of Motz

1. Hilbert series :

$$\mathcal{H}_{\mathrm{Motz}}(t) = t + t^2 + 2t^3 + 4t^4 + 9t^5 + 21t^6 + 51t^7 + 127t^8 + \cdots$$

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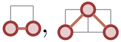
$\{$  $\}$.

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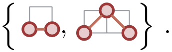

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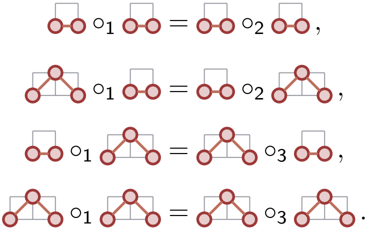

Outline

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Let $G := \sqcup_{n \geqslant 1} G(n)$ be a graded set.

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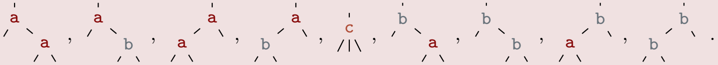
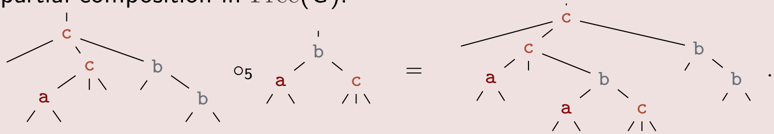


A partial composition in $\mathrm{Free}(G)$:



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- binary when all elements of $\mathfrak{G}_{\mathcal{O}}$ are of arity two;
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Presentations of operads: examples

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Motz admits the presentation $(\mathfrak{G}_{\mathsf{Motz}}, \mathfrak{R}_{\mathsf{Motz}})$ where $\mathfrak{G}_{\mathsf{Motz}} := \mathfrak{G}_{\mathsf{Motz}}(2) \sqcup \mathfrak{G}_{\mathsf{Motz}}(3) := \{\mathtt{a}\} \sqcup \{\mathtt{b}\}$ and $\mathfrak{R}_{\mathsf{Motz}}$ is the space generated by

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Example

The operad DA of directed animals is the operad admitting the presentation $(\mathfrak{G}_{\mathsf{DA}}, \mathfrak{R}_{\mathsf{DA}})$ where $\mathfrak{G}_{\mathsf{DA}} := \mathfrak{G}_{\mathsf{DA}}(2) := \{\mathtt{a}, \mathtt{b}\}$ and $\mathfrak{R}_{\mathsf{DA}}$ is the space generated by

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Implied if there is an orientation $\to$ of $\mathfrak{R}_{\mathcal{O}}$ so that $\to$ is a convergent rewrite rule on the syntax trees of $\mathrm{Free}(\mathfrak{G}_{\mathcal{O}})$ [**Hoffbeck**, 2010].

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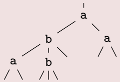

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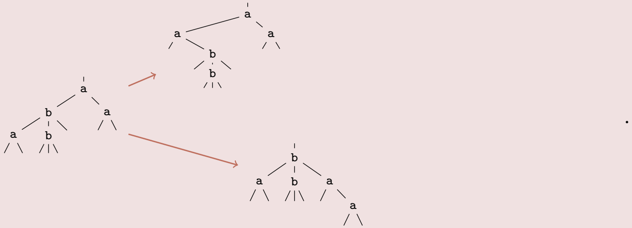

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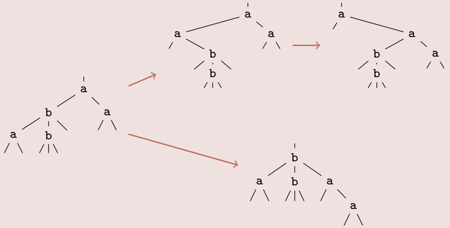

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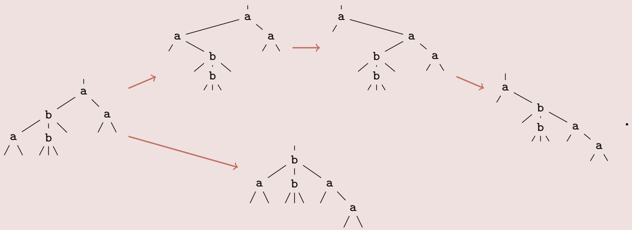

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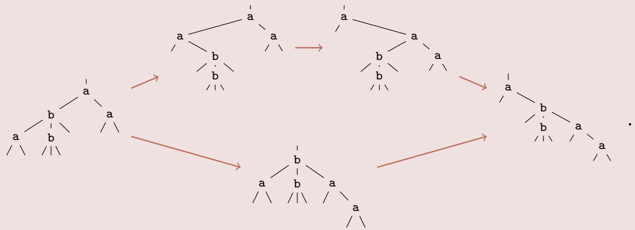

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Let $\mathcal{O}$ be a binary and quadratic operad with presentation $(\mathfrak{G}_{\mathcal{O}}, \mathfrak{R}_{\mathcal{O}})$.

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$$\langle -, - \rangle : \mathrm{Free}(\mathfrak{G}_{\mathcal{O}})(3) \otimes \mathrm{Free}(\mathfrak{G}_{\mathcal{O}})(3) \to \mathbb{K}$$

linearly defined, for all $x, x', y, y' \in \mathfrak{G}_{\mathcal{O}}(2)$, by

$$\langle x \circ_i y, x' \circ_{i'} y' \rangle := \begin{cases} 1 & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1 & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0 & \text{otherwise.} \end{cases}$$

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Then, with knowledge of a presentation of $\mathcal{O}$, one can compute a presentation of $\mathcal{O}^!$.

Properties of Koszul duality

Theorem [**Ginzburg, Kapranov**, 1994]

For any operad $\mathcal{O}$ admitting a binary and quadratic presentation,

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Then, given a Koszul operad $\mathcal{O}$ admitting a binary and quadratic presentation,

$$\text{presentation of } \mathcal{O} \rightsquigarrow \text{presentation of } \mathcal{O}^!,$$

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Outline

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Dendriform operad

The dendriform operad Dendr is the operad admitting the presentation $(\mathfrak{G}_{\mathsf{Dendr}}, \mathfrak{R}_{\mathsf{Dendr}})$ where

$$\mathfrak{G}_{\mathsf{Dendr}} := \mathfrak{G}_{\mathsf{Dendr}}(2) := \{\prec, \succ\},$$

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Theorem [Loday, 2001]

Dendr is a Koszul operad.

Free dendriform algebra

The free dendriform algebra over one generator is the vector space $\mathcal{F}_{\mathsf{Dendr}}$ of binary trees with at least one internal node endowed with the linear operations

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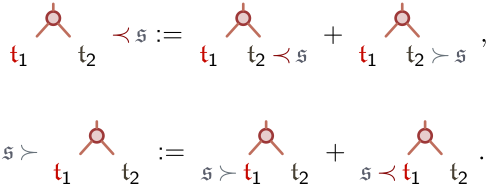
recursively defined, for any binary tree $\mathfrak{s}$ with at least one internal node, and binary trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec | := \mathfrak{s} =: | \succ \mathfrak{s},$$

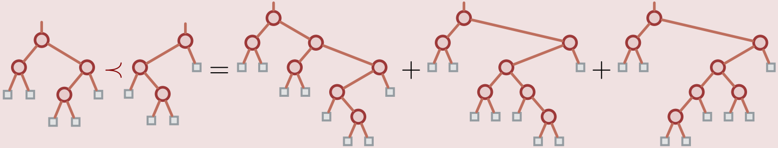
$$| \prec \mathfrak{s} := 0 =: \mathfrak{s} \succ |,$$



Neither $| \prec |$ nor $| \succ |$ are defined.

Free dendriform algebra

Example



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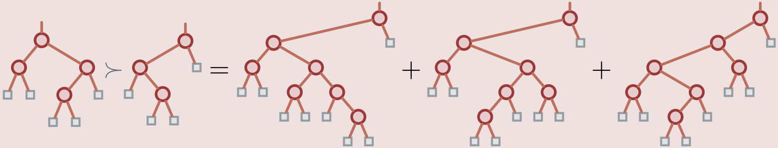

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Diassociative operad

The diassociative operad Dias [**Loday**, 2001] is the operad admitting the presentation $(\mathfrak{G}_{\mathsf{Dias}}, \mathfrak{R}_{\mathsf{Dias}})$ where

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Theorem [**Loday**, 2001]

Dias is a Koszul operad.

Realization of Dias

Dias admits the following realization [G, 2012]:

- Dias(n) is the linear span of the words of length n on $\{0, 1\}$ with exactly one occurrence of 0;
- the partial composition of Dias satisfies

$$u \circ_i v := u_1 \ \ldots \ u_{i-1} \ (u_i \uparrow v_1) \ \ldots \ (u_i \uparrow v_m) \ u_{i+1} \ \ldots \ u_n,$$

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Proposition

Dias is generated by the set $\{01, 10\}$.

Outline

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Theorem [**Loday**, 2001]

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y is of the form

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Koszul dual of Dias

Theorem [Loday, 2001]

Dendr is the Koszul dual of Dias.

Indeed, by definition of Koszul duality,

$$\mathfrak{R}_{\mathsf{Dias}}^{\perp} = \mathrm{Vect}\left(y \in \mathrm{Free}(\mathfrak{G}_{\mathsf{Dias}})(3) : \langle x, y \rangle = 0 \text{ for all } x \in \mathfrak{R}_{\mathsf{Dias}}\right).$$

Let

$$y := \sum_{\mathfrak{t} \in \mathrm{Free}(\mathfrak{G}_{\mathsf{Dias}})(3)} \lambda_{\mathfrak{t}} \mathfrak{t} \;\in\; \mathfrak{R}_{\mathsf{Dias}}^{\perp}.$$

Then, since

$$\begin{aligned}
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Therefore, $\mathfrak{R}_{\mathsf{Dias}}^{\perp}$ is generated by

$$\dashv \circ_1 \vdash - \vdash \circ_2 \dashv, \quad \dashv \circ_1 \dashv - \dashv \circ_2 \dashv - \dashv \circ_2 \vdash, \quad \vdash \circ_1 \dashv + \vdash \circ_1 \vdash - \vdash \circ_2 \vdash,$$

and we recognize dendriform relations.

Dimensions

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$$\mathcal{H}_{\text{Dias}}(-\mathcal{H}_{\text{Dendr}}(-t)) = t.$$

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Hence, the Hilbert series of Dendr is

$$\mathcal{H}_{\text{Dendr}}(t) = \frac{1 - \sqrt{1-4t} - 2t}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots \ .$$

Motivations and goals

- Define a generalization Dendr_γ, $\gamma \in \mathbb{N}$, of Dendr.

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Strategy

Propose a generalization Dias_γ of Dias and then, by Koszul duality, deduce a generalization Dendr_γ of Dendr.

Outline

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- the partial composition of Dias_γ satisfies

$$u \circ_i v := u_1 \; \dots \; u_{i-1} \; (u_i \uparrow v_1) \; \dots \; (u_i \uparrow v_m) \; u_{i+1} \; \dots \; u_n,$$

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Example

$$211\mathbf{2}01 \circ_4 31103 = 2113222301$$

$$11\mathbf{1}101 \circ_3 20 = 1121101$$

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Proposition

Dias_γ is an operad.

First properties of Dias_γ

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Dias_γ is generated by the set $\{0a, a0 : a \in [\gamma]\}$.

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By definition of Dias_γ,

$$\mathcal{H}_{\mathsf{Dias}_\gamma}(t) = \frac{t}{(1-\gamma t)^2} \qquad \text{and} \qquad \dim \mathsf{Dias}_\gamma(n) = n\gamma^{n-1}.$$

γ	Dimensions of Dias_γ
0	1, 0, 0, ...
1	1, 2, 3, 4, 5, 6, 7, 8, ...
2	1, 4, 12, 32, 80, 192, 448, 1024, ...
3	1, 6, 27, 108, 405, 1458, 5103, 17496, ...
4	1, 8, 48, 256, 1280, 6144, 28672, 131072, ...

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Since $\mathsf{Dias}_1 = \mathsf{Dias}$ and Dias_γ is a suboperad of $\mathsf{Dias}_{\gamma+1}$, Dias_γ is a generalization of Dias.

Presentation of Dias_γ

Theorem

Dias_γ admits the presentation $(\mathfrak{G}_{\mathsf{Dias}_\gamma}, \mathfrak{R}_{\mathsf{Dias}_\gamma})$ where

$$\mathfrak{G}_{\mathsf{Dias}_\gamma} := \mathfrak{G}_{\mathsf{Dias}_\gamma}(2) := \{\dashv_a, \vdash_a : a \in [\gamma]\}$$

and $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is generated by

$$\begin{aligned}
&\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, && a, a' \in [\gamma],\\
&\dashv_a \circ_1 \dashv_b - \dashv_a \circ_2 \vdash_b, && a < b \in [\gamma],\\
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\end{aligned}$$

In a more concise way, $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space generated by

$$\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma],$$

$$\begin{aligned}
&\dashv_a \circ_1 \dashv_{a \uparrow a'} - \dashv_a \circ_2 \vdash_{a'}, \qquad && \vdash_a \circ_1 \dashv_{a'} - \vdash_a \circ_2 \vdash_{a \uparrow a'}, \qquad && a, a' \in [\gamma],\\
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\end{aligned}$$

Presentation of Dias_γ

The proof is based upon the existence of a map

$$\mathrm{word} : \mathrm{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right) \to \mathsf{Dias}_\gamma$$

inducing an isomorphism of operads

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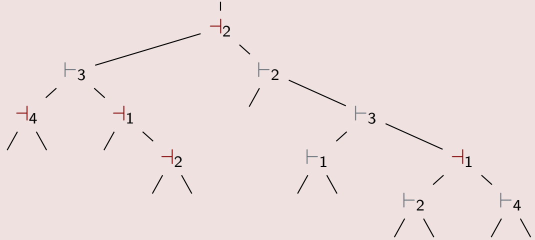
Exemple

In $\mathrm{Free}\left(\mathfrak{G}_{\mathsf{Dias}_5}\right)$,

$\mathfrak{t} :=$



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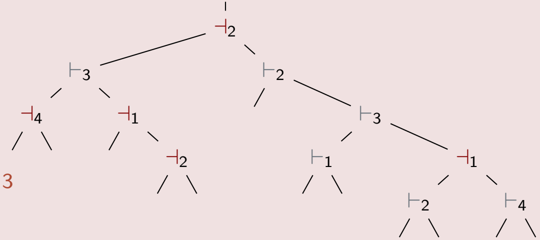

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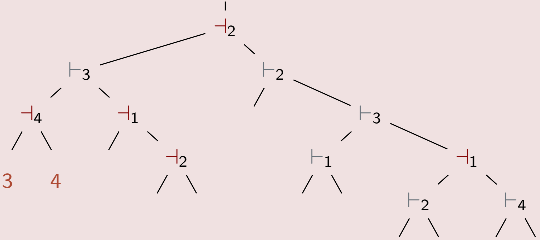

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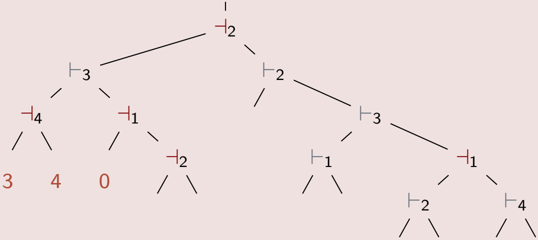

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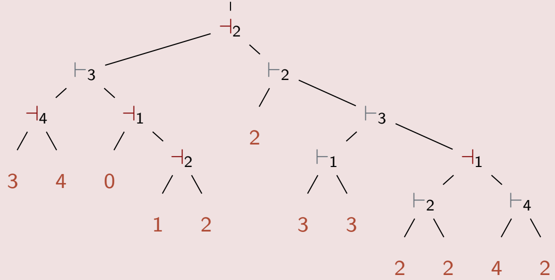

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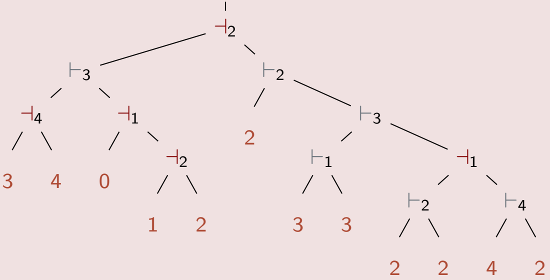

$$\mathrm{word}(\mathfrak{t}) = 340122332242.$$

Koszulity of Dias_γ

Proposition

Dias_γ is a Koszul operad.

Koszulity of Dias_γ

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The proof relies on the orientation $\to$ of $\mathfrak{R}_{\mathrm{Dias}_\gamma}$ satisfying

$$
\begin{aligned}
\vdash_{a'} \circ_2 \dashv_a \quad &\to \quad \dashv_a \circ_1 \vdash_{a'}, && a, a' \in [\gamma],\\
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\end{aligned}
$$

defining a convergent rewrite rule on $\mathrm{Free}(\mathfrak{G}_{\mathrm{Dias}_\gamma})$.

Alternative basis of Dias_γ

Let $\preccurlyeq_\gamma$ be the order relation on the set of words of Dias_γ where $x \preccurlyeq_\gamma y$ if $x_i \leqslant y_i$ for all $i \in [|x|]$.

Example

$$210231 \preccurlyeq_4 220432$$

Let

$$\mathsf{K}_x^{(\gamma)} := \sum_{x \preccurlyeq_\gamma y} \mu_\gamma(x, y) y$$

where μ_γ is the Möbius function of the poset defined by $\preccurlyeq_\gamma$.

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Exemple

$$\mathsf{K}_{102}^{(2)} = 102 - 202$$

$$\mathsf{K}_{102}^{(3)} = 102 - 103 - 202 + 203$$

$$\mathsf{K}_{23102}^{(3)} = 23102 - 23103 - 23202 + 23203 - 33102 + 33103 + 33202 - 33203$$

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$$\mathsf{K}_{102}^{(3)} = 102 - 103 - 202 + 203$$

$$\mathsf{K}_{23102}^{(3)} = 23102 - 23103 - 23202 + 23203 - 33102 + 33103 + 33202 - 33203$$

By triangularity, the $\mathsf{K}_x^{(\gamma)}$ form a basis of Dias_γ.

Alternative basis of Dias_γ

Proposition

On the K-basis, the partial composition map of Dias_γ satisfies

$$\mathsf{K}^{(\gamma)}_{x} \circ_i \mathsf{K}^{(\gamma)}_{y} = \begin{cases} \mathsf{K}^{(\gamma)}_{x \circ_i y} & \text{if } \min(y) > x_i, \\ \sum_{a \in [x_i, \gamma]} \mathsf{K}^{(\gamma)}_{x \circ_{a,i} y} & \text{if } \min(y) = x_i, \\ 0 & \text{otherwise } (\min(y) < x_i). \end{cases}$$

where $x \circ_{a,i} y$ is the word $x \circ_i y$ in which the 0 coming from y is replaced by a.

Alternative basis of Dias_γ

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Exemple

$$\mathsf{K}^{(5)}_{20413} \circ_1 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{3240413}$$

$$\mathsf{K}^{(5)}_{20413} \circ_2 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{2304413}$$

$$\mathsf{K}^{(5)}_{20413} \circ_3 \mathsf{K}^{(5)}_{304} = 0$$

$$\mathsf{K}^{(5)}_{20413} \circ_5 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{2041334} + \mathsf{K}^{(5)}_{2041344} + \mathsf{K}^{(5)}_{2041354}$$

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and $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is generated by

$$\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma],$$

$$\vdash_b \circ_1 \vdash_a, \qquad \dashv_b \circ_2 \dashv_a, \qquad \vdash_b \circ_1 \dashv_a, \qquad \dashv_b \circ_2 \vdash_a, \qquad a < b \in [\gamma],$$

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Its proof uses the identification of $\dashv_a$ with $\mathsf{K}^{(\gamma)}_{0a}$ and of $\vdash_a$ with $\mathsf{K}^{(\gamma)}_{a0}$ together with the previous partial composition rules.

Outline

A generalization of Dendr

Dias_γ admits a binary and quadratic presentation, and thus, has a Koszul dual.

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Let $\mathsf{Dendr}_\gamma := \mathsf{Dias}_\gamma^!$ be the γ-polydendriform operad.

Theorem

Dendr_γ admits the presentation $(\mathfrak{G}_{\mathsf{Dendr}_\gamma}, \mathfrak{R}_{\mathsf{Dendr}_\gamma})$ where

$$\mathfrak{G}_{\mathsf{Dendr}_\gamma} := \mathfrak{G}_{\mathsf{Dendr}_\gamma}(2) := \{\leftharpoonup_a, \rightharpoonup_a : a \in [\gamma]\}$$

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$$\leftharpoonup_a \circ_1 \rightharpoonup_{a'} - \rightharpoonup_{a'} \circ_2 \leftharpoonup_a, \qquad a, a' \in [\gamma],$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \rightharpoonup_b, \qquad \rightharpoonup_a \circ_1 \leftharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma],$$

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$$\leftharpoonup_d \circ_1 \leftharpoonup_d - \sum_{c \in [d]} (\leftharpoonup_d \circ_2 \leftharpoonup_c + \leftharpoonup_d \circ_2 \rightharpoonup_c), \qquad d \in [\gamma],$$

$$\sum_{c \in [d]} (\rightharpoonup_d \circ_1 \rightharpoonup_c + \rightharpoonup_d \circ_1 \leftharpoonup_c) - \rightharpoonup_d \circ_2 \rightharpoonup_d, \qquad d \in [\gamma].$$

Dimensions of Dendr_γ

Since Dias_γ is Koszul, we can compute the dimensions of Dendr_γ from the ones of Dias_γ because

$$\mathcal{H}_{\mathsf{Dias}_\gamma}(-\mathcal{H}_{\mathsf{Dendr}_\gamma}(-t)) = t.$$

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$$\mathcal{H}_{\mathsf{Dendr}_\gamma}(t) = t + 2\gamma t\, \mathcal{H}_{\mathsf{Dendr}_\gamma}(t) + \gamma^2 t\, \mathcal{H}_{\mathsf{Dendr}_\gamma}(t)^2$$

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Proof.

This is a consequence of

$$t = \frac{-\mathcal{H}_{\mathrm{Dendr}_\gamma}(-t)}{(1 + \gamma\, \mathcal{H}_{\mathrm{Dendr}_\gamma}(-t))^2}$$

and the fact that

$$\mathcal{H}_{\mathrm{Dias}_\gamma}(t) = \frac{t}{(1-\gamma t)^2}.$$

□

Dimensions and elements of Dendr_γ

We deduce, from the expression of $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$, that

$$\dim \mathsf{Dendr}_\gamma(n) = \gamma^{n-1} \frac{1}{n+1} \binom{2n}{n}.$$

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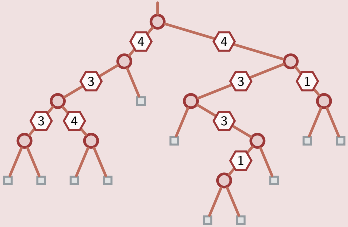
$$\dim \mathsf{Dendr}_\gamma(n) = \gamma^{n-1} \frac{1}{n+1} \binom{2n}{n}.$$

Hence, $\mathsf{Dendr}_\gamma(n)$ is the linear span of γ-edge valued binary trees of size n, that are binary trees with n internal nodes wherein its $n-1$ edges connecting two internal nodes are labeled on $[\gamma]$.

Example



is a 4-edge valued binary tree and a basis element of $\mathsf{Dendr}_6(10)$.

Polydendriform algebras

A Dendr_γ-algebra, called γ-polydendriform algebra is a vector space $\mathcal{V}$ endowed with 2γ binary operations

$$\leftharpoonup_a : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \qquad \text{and} \qquad \rightharpoonup_a : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}, \qquad a \in [\gamma],$$

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satisfying, for all $x, y, z \in \mathcal{V}$, the relations

$$(x \rightharpoonup_{a'} y) \leftharpoonup_a z = x \rightharpoonup_{a'} (y \leftharpoonup_a z), \qquad a, a' \in [\gamma],$$

$$(x \leftharpoonup_b y) \leftharpoonup_a z = x \leftharpoonup_a (y \rightharpoonup_b z), \qquad a < b \in [\gamma],$$

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$$(x \leftharpoonup_d y) \leftharpoonup_d z = \sum_{c \in [d]} x \leftharpoonup_d (y \leftharpoonup_c z) + x \leftharpoonup_d (y \rightharpoonup_c z), \qquad d \in [\gamma],$$

$$\sum_{c \in [d]} (x \rightharpoonup_c y) \rightharpoonup_d z + (x \leftharpoonup_c y) \rightharpoonup_d z = x \rightharpoonup_d (y \rightharpoonup_d z), \qquad d \in [\gamma].$$

γ-split of an associative operation

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Then, γ-polydendriform algebras are adapted to split an associative product $\cdot$ into 2γ parts by

$$\cdot = \leftharpoonup_1 + \rightharpoonup_1 + \leftharpoonup_2 + \rightharpoonup_2 + \cdots + \leftharpoonup_\gamma + \rightharpoonup_\gamma,$$

with the partial sums condition, that is

$$\leftharpoonup_1 + \rightharpoonup_1,$$
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$$\dots,$$

are associative.

Alternative presentation of Dendr_γ

The computation of the Koszul dual of Dias_γ expressed on its presentation $(\mathfrak{G}'_{\mathsf{Dias}_\gamma}, \mathfrak{R}'_{\mathsf{Dias}_\gamma})$ leads to an alternative presentation for Dendr_γ.

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and $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is generated by

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma],$$

$$\prec_a \circ_1 \prec_{a'} - \prec_{a \downarrow a'} \circ_2 \prec_a - \prec_{a \downarrow a'} \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma],$$

$$\succ_{a \downarrow a'} \circ_1 \prec_{a'} + \succ_{a \downarrow a'} \circ_1 \succ_a - \succ_a \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma],$$

where $\downarrow$ denotes the operation min on integers.

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where $\downarrow$ denotes the operation min on integers.

Fact: this presentation of Dendr_γ can also be obtained through the change of basis

$$\prec_b = \sum_{a \in [b]} \leftharpoonup_a, \quad \text{and} \quad \succ_b = \sum_{a \in [b]} \rightharpoonup_a, \qquad b \in [\gamma].$$

Free γ-dendriform algebras

We endow the space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees with linear operations

$$\prec_a, \succ_a : \mathcal{F}_{\mathsf{Dendr}_\gamma} \otimes \mathcal{F}_{\mathsf{Dendr}_\gamma} \to \mathcal{F}_{\mathsf{Dendr}_\gamma}, \qquad a \in [\gamma],$$

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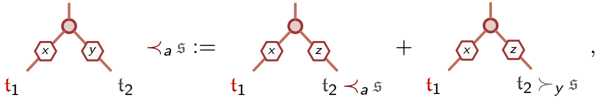
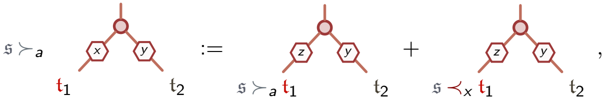
recursively defined, for any γ-edge valued binary tree $\mathfrak{s}$ and any γ-edge valued binary trees or leaves $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec_a | := \mathfrak{s} =: | \succ_a \mathfrak{s},$$

$$| \prec_a \mathfrak{s} := 0 =: \mathfrak{s} \succ_a |,$$



$z := a \downarrow y,$



$z := a \downarrow x.$

Note that neither $| \prec_a |$ nor $| \succ_a |$ are defined.

Free γ-dendriform algebras

Theorem

$\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is the free γ-polydendriform algebra over one generator.

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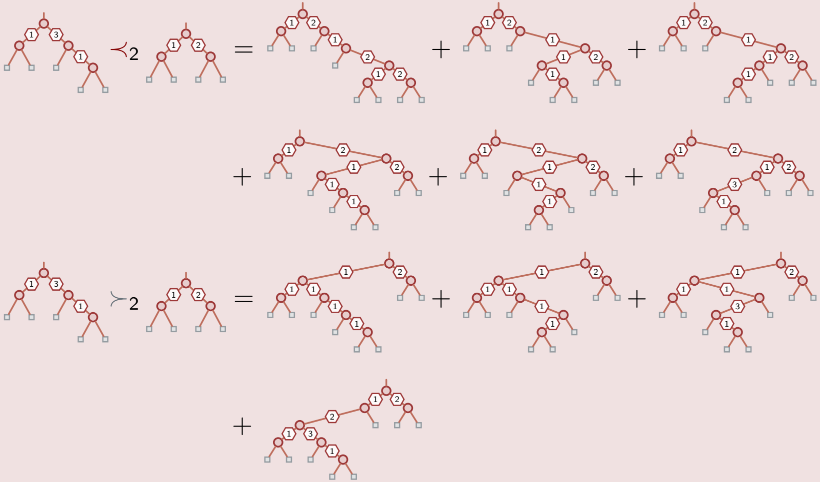

Outline

Example: the polynomial product

Let $A := \{\mathtt{a}_0, \mathtt{a}_1, \dots\}$ be a totally ordered alphabet by $\mathtt{a}_i \leqslant \mathtt{a}_j$ if $i \leqslant j$.

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$$\mathtt{a}_0\mathtt{a}_2\mathtt{a}_0 \cdot \mathtt{a}_1\mathtt{a}_2 = \mathtt{a}_0\mathtt{a}_2\mathtt{a}_0\mathtt{a}_1\mathtt{a}_2$$

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Example

$$a_0a_2a_0 \cdot a_1a_2 = a_0a_2a_0a_1a_2$$

$\cdot$ splits into two parts $\prec$ and $\succ$ according to the origin of the greatest letter:

$$u \prec v := \begin{cases} uv & \text{if } \max(u) > \max(v), \\ 0 & \text{otherwise}, \end{cases} \qquad u \succ v := \begin{cases} uv & \text{if } \max(u) \leqslant \max(v), \\ 0 & \text{otherwise}. \end{cases}$$

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Example: the associative operad

The associative operad As is defined in the following way:

- $\mathrm{As}(n)$ is the one-dimensional space spanned by the abstract operator $\mathfrak{a}_n$ of arity n;
- the partial composition is linearly defined by $\mathfrak{a}_n \circ_i \mathfrak{a}_m := \mathfrak{a}_{n+m-1}$;
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Example

$$\mathfrak{a}_4 \circ_2 \mathfrak{a}_3 = \mathfrak{a}_6$$

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3. Nontrivial relations:

$$\mathfrak{a}_2 \circ_1 \mathfrak{a}_2 = \mathfrak{a}_2 \circ_2 \mathfrak{a}_2.$$

Algebras over an operad

Let $\mathcal{O}$ be an operad.

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must be satisfied, for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$.

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Let $\mathcal{O}$ be an operad.

An $\mathcal{O}$-algebra is a vector space $\mathcal{V}$ where any $x \in \mathcal{O}$ of arity n endows $\mathcal{V}$ with a linear map

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Moreover, if $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of operads, ϕ gives rises to a functor from the category of $\mathcal{O}_2$-algebras to the category of $\mathcal{O}_1$-algebras.

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Recall that As admits the presentation $(\mathfrak{G}_{\mathsf{As}}, \mathfrak{R}_{\mathsf{As}})$ where $\mathfrak{G}_{\mathsf{As}} = \mathfrak{G}_{\mathsf{As}}(2) = \{\mathtt{a}\}$ and $\mathfrak{R}_{\mathsf{As}}$ is generated by $\mathtt{a} \circ_1 \mathtt{a} - \mathtt{a} \circ_2 \mathtt{a}$.

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Using infix notation for the binary operator $\mathtt{a}$, we obtain the relation

$$(x \,\mathtt{a}\, y) \,\mathtt{a}\, z = x \,\mathtt{a}\, (y \,\mathtt{a}\, z),$$

so that As-algebras are associative algebras.

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- the unit of Dendr is the binary tree .

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Theorem [G, 2012]

For any monoid $\mathcal{M}$, $\mathrm{T}\mathcal{M}$ is an operad.