Pattern avoidance in trees, operads, and enumeration

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Outline

Syntax trees and patterns

An alphabet is a graded set $\mathfrak{G} := \bigsqcup_{n \geqslant 1} \mathfrak{G}(n)$.

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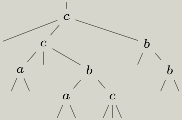
Let $\mathbf{S}(\mathfrak{G})$ be the set of $\mathfrak{G}$-syntax trees defined recursively as

- $| \in \mathbf{S}(\mathfrak{G})$, where $|$ is the leaf;
- if $a \in \mathfrak{G}$ and $\mathfrak{t}_1, \dots, \mathfrak{t}_{|a|} \in \mathbf{S}(\mathfrak{G})$, then $a\,(\,\mathfrak{t}_1, \dots, \mathfrak{t}_{|a|}\,) \in \mathbf{S}(\mathfrak{G})$.

- Example -

Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ such that $\mathfrak{G}(2) = \{a, b\}$ and $\mathfrak{G}(3) = \{c\}$.



denotes the $\mathfrak{G}$-tree

$c\,(\,|, c\,(\,a\,(\,|, |\,), |, b\,(\,a\,(\,|, |\,), c\,(\,|, |, |\,)\,)\,), b\,(\,|, b\,(\,|, |\,)\,)\,)$

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Let $\mathbf{S}(\mathfrak{G})$ be the set of $\mathfrak{G}$-syntax trees defined recursively as

- $\shortmid \in \mathbf{S}(\mathfrak{G})$, where $\shortmid$ is the leaf;
- if $a \in \mathfrak{G}$ and $\mathfrak{t}_1, \dots, \mathfrak{t}_{|a|} \in \mathbf{S}(\mathfrak{G})$, then $a\,(\,\mathfrak{t}_1, \dots, \mathfrak{t}_{|a|}\,) \in \mathbf{S}(\mathfrak{G})$.

Let $\mathfrak{t} = a\,(\,\mathfrak{t}_1, \dots, \mathfrak{t}_{|a|}\,) \in \mathbf{S}(\mathfrak{G})$. Some definitions:

- the degree $\deg(\mathfrak{t})$ of $\mathfrak{t}$ is its number of internal nodes;
- the arity $|\mathfrak{t}|$ of $\mathfrak{t}$ is its number of leaves;
- for any $i \in [|a|]$, $\mathfrak{t}(i)$ is the i-th subtree $\mathfrak{t}_i$ of $\mathfrak{t}$.

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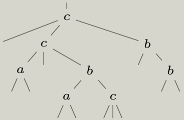

denotes the $\mathfrak{G}$-tree

$c\,(\,\shortmid, c\,(\,a\,(\,\shortmid, \shortmid\,), \shortmid, b\,(\,a\,(\,\shortmid, \shortmid\,), c\,(\,\shortmid, \shortmid, \shortmid\,)\,)\,), b\,(\,\shortmid, b\,(\,\shortmid, \shortmid\,)\,)\,)$

having degree 8 and arity 12.

Compositions of syntax trees

Let $\mathfrak{t}, \mathfrak{s} \in \mathbf{S}(\mathfrak{G})$. For each $i \in [|\mathfrak{t}|]$, the partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ is the tree obtained by grafting the root of $\mathfrak{s}$ onto the i-th leaf of $\mathfrak{t}$.

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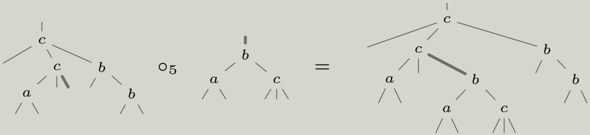

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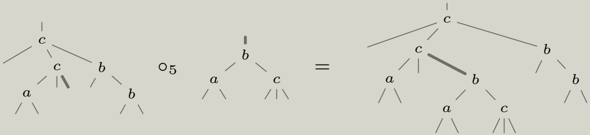

Let $\mathfrak{t}$, $\mathfrak{s}_1$, ..., $\mathfrak{s}_{|\mathfrak{t}|}$ be $\mathfrak{G}$-trees. The full composition $\mathfrak{t} \circ \left[\mathfrak{s}_1, \dots, \mathfrak{s}_{|\mathfrak{t}|}\right]$ is obtained by grafting simultaneously the roots of each $\mathfrak{s}_i$ onto the i-th leaf of $\mathfrak{t}$.

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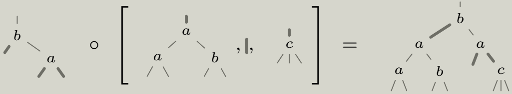

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Factors and prefixes

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If $\mathfrak{t}$ decomposes as

$$\mathfrak{t} = \mathfrak{r} \circ_i \left(\mathfrak{s} \circ \left[\mathfrak{r}_1, \dots, \mathfrak{r}_{[\mathfrak{s}|}\right]\right)$$

for some trees $\mathfrak{r}$, $\mathfrak{r}_1$, ..., $\mathfrak{r}_{|\mathfrak{s}|}$, and $i \in [|\mathfrak{r}|]$, then $\mathfrak{s}$ is a factor of $\mathfrak{t}$.

This property is denoted by $\mathfrak{s} \preccurlyeq_{\mathrm{f}} \mathfrak{t}$.

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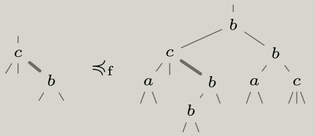

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If in the previous decomposition $\mathfrak{r} = |$, then

$$\mathfrak{t} = \mathfrak{s} \circ \left[\mathfrak{r}_1, \ldots, \mathfrak{r}_{[\mathfrak{s}|}\right],$$

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This property is denoted by $\mathfrak{s} \preccurlyeq_{\mathrm{p}} \mathfrak{t}$.

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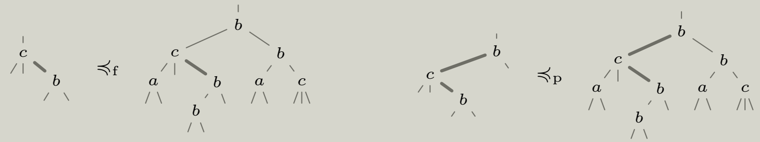

A $\mathfrak{G}$-tree $\mathfrak{t}$ avoids (resp. prefix-avoids) a $\mathfrak{G}$-tree $\mathfrak{s}$ if $\mathfrak{s} \not\preccurlyeq_{\mathrm{f}} \mathfrak{t}$ (resp. $\mathfrak{s} \not\preccurlyeq_{\mathrm{p}} \mathfrak{t}$).

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For any $\mathcal{P} \subseteq \mathbf{S}(\mathfrak{G})$, let

$$\mathrm{A}(\mathcal{P}) := \left\{\mathfrak{t} \in \mathbf{S}(\mathfrak{G}) : \text{ for all } \mathfrak{s} \in \mathcal{P}, \mathfrak{s} \not\preccurlyeq_{\mathrm{f}} \mathfrak{t}\right\}.$$

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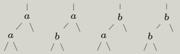
- A () is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$.

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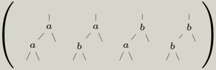
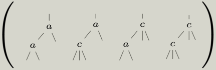
- A  is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \ldots$.
- A  is enumerated by $1, 1, 2, 4, 9, 21, 51, 127, \ldots$ (A001006).

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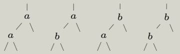
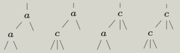
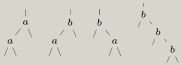
- A () is enumerated by $1, 2, 4, 8, 16, 32, 64, 128, \dots$.
- A () is enumerated by $1, 1, 2, 4, 9, 21, 51, 127, \dots$ (A001006).
- A () is enumerated by $1, 2, 5, 13, 35, 96, 267, 750, \dots$ (A005773).

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Enumerate $\mathrm{A}(\mathcal{P})$ w.r.t. the arities of the trees.

Consistent words

Let $\mathcal{P} \subseteq \mathbf{S}(\mathfrak{G}) \setminus \{\shortmid\}$ and $a \in \mathfrak{G}(k)$.

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A sequence $\mathcal{S} := (\mathcal{S}_1, \ldots, \mathcal{S}_k)$, where each $\mathcal{S}_i$ is a subset of $\mathbf{S}(\mathfrak{G})$, is $\mathcal{P}_a$-consistent if for any $\mathfrak{s} \in \mathcal{P}_a$, there is an $i \in [k]$ such that $\mathfrak{s}(i) \neq \shortmid$ and $\mathfrak{s}(i) \in \mathcal{S}_i$.

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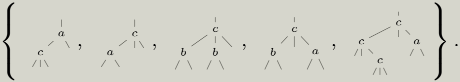

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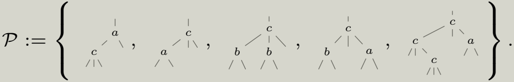
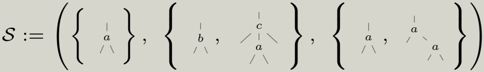


The sequence



is $\mathcal{P}_c$-consistent.

Let $\mathcal{P} \subseteq \mathbf{S}(\mathfrak{G}) \setminus \{\shortmid\}$, $a \in \mathfrak{G}(k)$, and $\mathcal{S} := (\mathcal{S}_1, \ldots, \mathcal{S}_k)$ be a $\mathcal{P}_a$-consistent sequence.

Admissible trees

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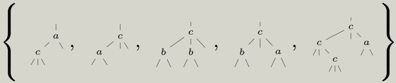
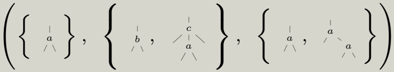
$\mathcal{P} :=$  .

and the $\mathcal{P}_c$-consistent word

$\mathcal{S} :=$  .

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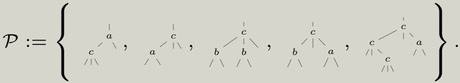

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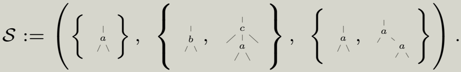

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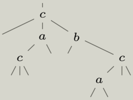

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A $\mathcal{P}_a$-consistent word $\mathcal{S}$ is minimal if for any decomposition $\mathcal{S} = \mathcal{S}' \dot{+} \mathcal{S}''$ where $\mathcal{S}'$ is a $\mathcal{P}_a$-consistent word and $\mathcal{S}''$ is a sequence of subsets of $\mathbf{S}(\mathfrak{G})$, one has $\mathcal{S} = \mathcal{S}'$.

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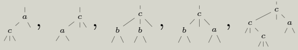
$\mathcal{P} := \Big\{$  $\Big\}$.

Minimal consistent words

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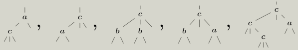
$\mathcal{P} := \Big\{$  $\Big\}.$

We have $\mathfrak{M}(\mathcal{P}_a) = \Big\{ \Big(\Big\{$  $\Big\},\ \emptyset \Big) \Big\}$,

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A $\mathcal{P}_a$-consistent word $\mathcal{S}$ is minimal if for any decomposition $\mathcal{S} = \mathcal{S}' \dot{+} \mathcal{S}''$ where $\mathcal{S}'$ is a $\mathcal{P}_a$-consistent word and $\mathcal{S}''$ is a sequence of subsets of $\mathbf{S}(\mathfrak{G})$, one has $\mathcal{S} = \mathcal{S}'$.

Let $\mathfrak{M}(\mathcal{P}_a)$ be the set of all minimal $\mathcal{P}_a$-consistent words.

- Examples -

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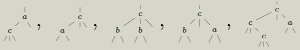
$\mathcal{P} := \{$  $\}$.

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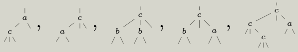
$\mathcal{P} := \{$  $\}$.

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The following assertions are equivalent:

1. $\mathfrak{t}$ prefix-avoids $\mathcal{P}$;
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- Lemma -

Let $\mathcal{P}, \mathcal{Q} \subseteq \mathbf{S}(\mathfrak{G}) \setminus \{|\}$ and $\mathfrak{t}$ be a $\mathfrak{G}$-tree with root labeled by $a \in \mathfrak{G}(k)$.

The following assertions are equivalent:

1. $\mathfrak{t}$ avoids $\mathcal{P}$ and prefix-avoids $\mathcal{Q}$;
2. for all $i \in [k]$, $\mathfrak{t}(i)$ avoid $\mathcal{P}$ and there exists a minimal $(\mathcal{P} \cup \mathcal{Q})_a$-consistent word $\mathcal{S}$ such that $\mathfrak{t}$ is $\mathcal{S}$-admissible.

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \dots)$ and $\mathcal{X}$ be a set.

Formal power series

Let $\mathbb{K}$ be the field $\mathbb{Q}(q_0, q_1, q_2, \ldots)$ and $\mathcal{X}$ be a set.

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The sum notation of $\mathbf{f}$ is

$$\mathbf{f} = \sum_{x \in \mathcal{X}} \langle x, \mathbf{f} \rangle x.$$

Tree series

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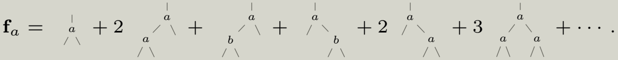
For $x \in \mathfrak{G}$, let $\mathbf{f}_x$ be the $\mathbf{S}(\mathfrak{G})$-series wherein $\langle \mathfrak{t}, \mathbf{f}_x \rangle$ is the number of occurrences of x in $\mathfrak{t}$. For instance,



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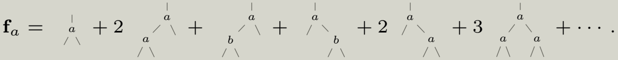

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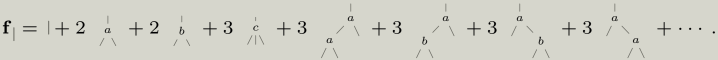

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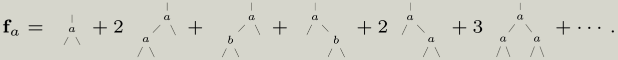

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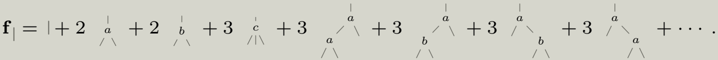

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- Lemma -

Let $\mathcal{X}$ be a set and $\mathcal{S}_1$, ..., $\mathcal{S}_n$, $n \geqslant 0$, be subsets of $\mathcal{X}$.

Then, the characteristic series of $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n$ expresses as

$$\mathbf{f}_{\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n} = \sum_{\substack{\ell \geqslant 1 \\ \{i_1, \ldots, i_\ell\} \subseteq [n]}} (-1)^{\ell+1} \, \mathbf{f}_{\mathcal{S}_{i_1} \cap \cdots \cap \mathcal{S}_{i_\ell}}.$$

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Let $\mathcal{X}$ be a set endowed with a size map $|-|: \mathcal{X} \to \mathbb{N}$.

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A k-ary product $\star : \mathbb{K}\langle\langle \mathcal{X} \rangle\rangle^{\otimes k} \to \mathbb{K}\langle\langle \mathcal{X} \rangle\rangle$ is enumeration-compatible if

$$\mathrm{en}(\star(\mathbf{f}_1, \dots, \mathbf{f}_k)) = \prod_{i \in [k]} \mathrm{en}(\mathbf{f}_i)$$

for all $\mathcal{X}$-series $\mathbf{f}_1$, ..., $\mathbf{f}_k$.

The composition of the $\mathbf{S}(\mathfrak{G})$-series $\mathbf{f}$ and $\mathbf{g}_1$, ..., $\mathbf{g}_n$ is the series

$$\mathbf{f} \bar{\circ} [\mathbf{g}_1, \ldots, \mathbf{g}_n] := \sum_{\substack{\mathfrak{t} \in \mathbf{S}(\mathfrak{G})(n) \\ \mathfrak{s}_1, \ldots, \mathfrak{s}_n \in \mathbf{S}(\mathfrak{G})}} \left(\langle \mathfrak{t}, \mathbf{f} \rangle \prod_{i \in [n]} \langle \mathfrak{s}_i, \mathbf{g}_i \rangle \right) \mathfrak{t} \circ [\mathfrak{s}_1, \ldots, \mathfrak{s}_n] .$$

Observe that this product is linear in all its arguments.

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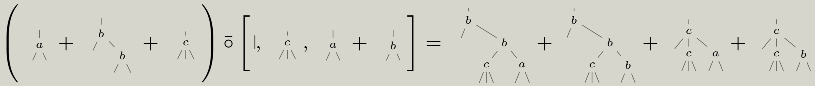

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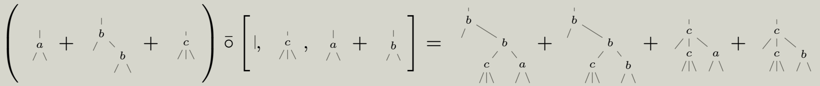

For any $\mathfrak{t} \in \mathbf{S}(\mathfrak{G})(n)$, let $\bar{\circ}_{\mathfrak{t}} : \mathbb{K} \langle\langle \mathbf{S}(\mathfrak{G}) \rangle\rangle^{\otimes n} \to \mathbb{K} \langle\langle \mathbf{S}(\mathfrak{G}) \rangle\rangle$ be the product defined by

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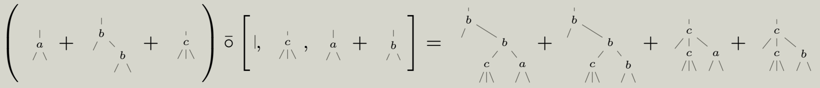

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We describe a system of equations for $\mathbf{F}(\mathcal{P}, \emptyset)$ using sums, multiplications by scalars, and operations $\bar{\circ}_a$, $a \in \mathfrak{G}$.

System of equations

When $\mathfrak{G}$, $\mathcal{P}$, and $\mathcal{Q}$ satisfy some conditions, $\mathbf{F}(\mathcal{P}, \mathcal{Q})$ expresses as an inclusion-exclusion formula involving simpler terms $\mathbf{F}(\mathcal{P}, \mathcal{S}_i)$.

- Theorem [G., 2019] -

The series $\mathbf{F}(\mathcal{P}, \mathcal{Q})$ satisfies

$$\mathbf{F}(\mathcal{P}, \mathcal{Q}) = \mathbb{I} + \sum_{\substack{k \geqslant 1 \\ a \in \mathfrak{G}(k)}} \sum_{\substack{\ell \geqslant 1 \\ \left\{\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(\ell)}\right\} \subseteq \mathfrak{M}((\mathcal{P} \cup \mathcal{Q})_a) \\ (\mathcal{S}_1, \dots, \mathcal{S}_k) = \mathcal{R}^{(1)} \dotplus \dots \dotplus \mathcal{R}^{(\ell)}}} (-1)^{1+\ell} \, \bar{\circ}_a \left(\mathbf{F}(\mathcal{P}, \mathcal{S}_1), \dots, \mathbf{F}(\mathcal{P}, \mathcal{S}_k)\right).$$

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This leads to a system of equations for the generating series of $\mathrm{A}(\mathcal{P})$.

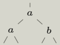
Indeed, the generating series of $\mathrm{A}(\mathcal{P})$ is the series $F(\mathcal{P},\emptyset)$ where

$$F(\mathcal{P},\mathcal{Q}) = z + \sum_{\substack{k\geqslant 1 \\ a\in\mathfrak{G}(k)}} \sum_{\substack{\ell\geqslant 1 \\ \left\{\mathcal{R}^{(1)},\dots,\mathcal{R}^{(\ell)}\right\}\subseteq\mathfrak{M}((\mathcal{P}\cup\mathcal{Q})_a) \\ (\mathcal{S}_1,\dots,\mathcal{S}_k)=\mathcal{R}^{(1)}\dot{+}\cdots\dot{+}\mathcal{R}^{(\ell)}}} (-1)^{1+\ell} \prod_{i\in[k]} F(\mathcal{P},\mathcal{S}_i).$$

System of equations

- Example -

Let

$$\mathcal{P} := \left\{ \right.$$  $$\left. \right\}.$$

We obtain the system of formal power series of trees

$$\begin{aligned}
\mathbf{F}(\mathcal{P}, \emptyset) &= \mid + \,\bar{\circ}_a\left(\mathbf{F}(\mathcal{P}, \{a\}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_a\left(\mathbf{F}(\mathcal{P}, \emptyset), \mathbf{F}(\mathcal{P}, \{b\})\right) \\
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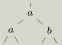
$\mathcal{P} := \Big\{$  $\Big\}.$

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This leads to the system of generating series

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System of equations

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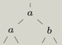
$$\mathcal{P} := \left\{ \right.$$  $$\left. \right\}.$$

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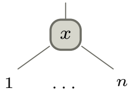
As a consequence, $F(\mathcal{P},\emptyset)$ satisfies

$$z - F(\mathcal{P},\emptyset) + (2+z)F(\mathcal{P},\emptyset)^2 - F(\mathcal{P},\emptyset)^3 + F(\mathcal{P},\emptyset)^4 = 0.$$

Operads and enumeration

An operator is an entity having $n \geqslant 1$ inputs and a single output:



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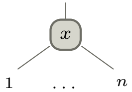

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1. selecting an input of x specified by its position i;
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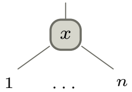

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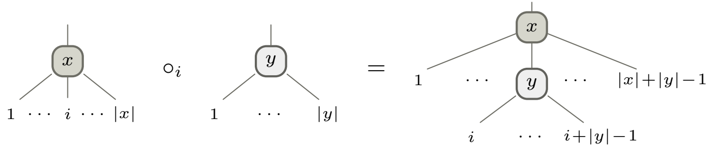
This produces a new operator



of arity $|x| + |y| - 1$.

Operads

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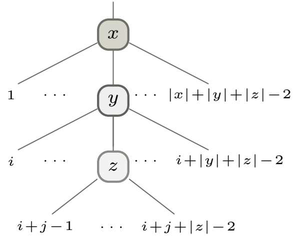
This data has to satisfy some axioms.

The associativity **relation**

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$

says that the pictured operation can be constructed from top to bottom or from bottom to top.



Operad axioms

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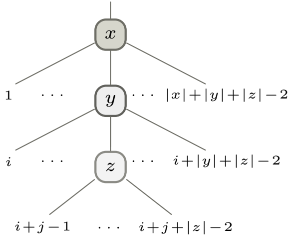

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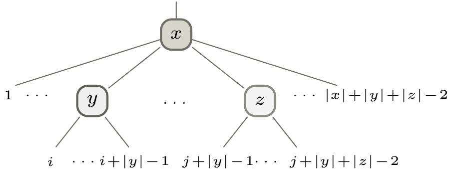

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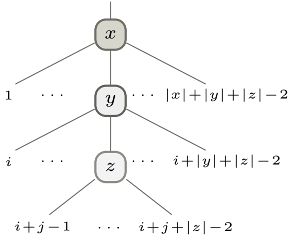

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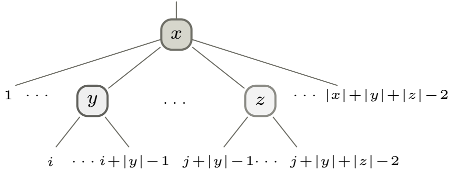
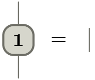


The unitality relation

$$\mathbf{1} \circ_1 x = x = x \circ_i \mathbf{1}$$
$$1 \leqslant i \leqslant |x|$$

says that $\mathbf{1}$ is the identity map.



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Let $\mathfrak{G}$ be an alphabet.

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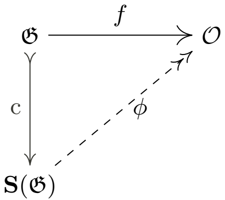
Free operads satisfy the following universality property.

For any alphabet $\mathfrak{G}$, any operad $\mathcal{O}$, and any map $f : \mathfrak{G} \to \mathcal{O}$ preserving the arities, there exists a unique operad morphism $\phi : \mathbf{S}(\mathfrak{G}) \to \mathcal{O}$ such that $f = \phi \circ \mathrm{c}$.



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- For any $u \in \mathrm{T}\mathcal{M}(n)$ and $v \in \mathrm{T}\mathcal{M}(m)$,

$$u \circ_i v := u_1 \dots u_{i-1}\ (u_i \star v_1) \dots (u_i \star v_m)\ u_{i+1} \dots u_n.$$

- Example -

In $\mathrm{T}(\mathbb{N}, +, 0)$,

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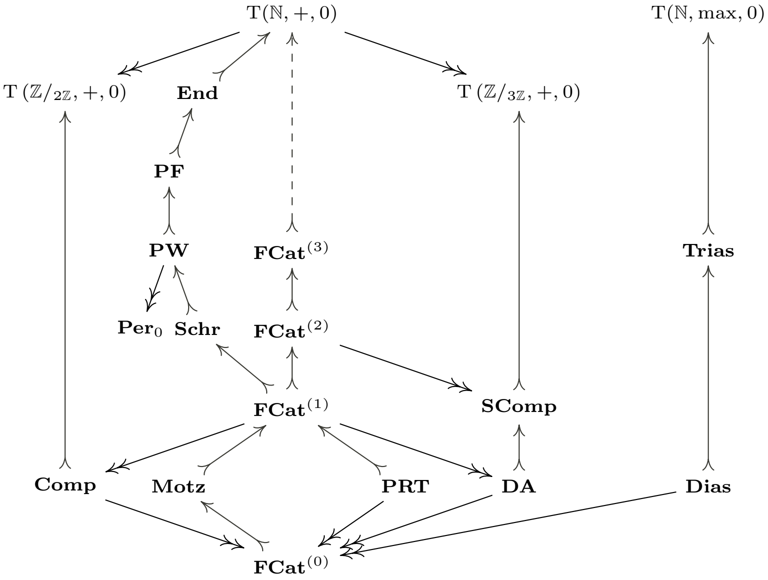
- Theorem [G., 2015] -

For any monoid $\mathcal{M}$, $\mathrm{T}\mathcal{M}$ is an operad.

Some combinatorial suboperads

Monoid	Operad	Generators	First dimensions	Comb. objects
$(\mathbb{N}, +, 0)$	**End**	--	$1, 4, 27, 256, 3125$	Endofunctions
	PF	--	$1, 3, 16, 125, 1296$	Parking functions
	PW	--	$1, 3, 13, 75, 541$	Packed words
	$\mathbf{Per}_0$	--	$1, 2, 6, 24, 120$	Permutations
	PRT	01	$1, 1, 2, 5, 14, 42$	Planar rooted trees
	$\mathbf{FCat}^{(m)}$	00, 01, ..., 0m	Fuss-Catalan numbers	m-trees
	Schr	00, 01, 10	$1, 3, 11, 45, 197$	Schröder trees
	Motz	00, 010	$1, 1, 2, 4, 9, 21, 51$	Motzkin words
$(\mathbb{Z}/_{2\mathbb{Z}}, +, 0)$	**Comp**	00, 01	$1, 2, 4, 8, 16, 32$	Compositions
$(\mathbb{Z}/_{3\mathbb{Z}}, +, 0)$	**DA**	00, 01	$1, 2, 5, 13, 35, 96$	Directed animals
	SComp	00, 01, 02	$1, 3, 27, 81, 243$	Seg. compositions
$(\mathbb{N}, \max, 0)$	**Dias**	01, 10	$1, 2, 3, 4, 5$	Some bin. words
	Trias	00, 01, 10	$1, 3, 7, 15, 31$	Some bin. words



T(ℕ, +, 0)
T(ℕ, max, 0)
T (ℤ/2ℤ, +, 0)
End
T (ℤ/3ℤ, +, 0)
PF
PW
FCat^(3)
Trias
Per_0
Schr
FCat^(2)
FCat^(1)
SComp
Comp
Motz
PRT
DA
Dias
FCat^(0)

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A congruence of $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ preserving the arities and such that $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all $i \in [|x|]$.

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A presentation of $\mathcal{O}$ is a pair $(\mathfrak{G}, \equiv)$ such that $\mathfrak{G}$ is an alphabet and $\equiv$ is a congruence of $\mathcal{O}$ satisfying

$$\mathcal{O} \simeq \mathbf{S}(\mathfrak{G})/_{\equiv}.$$

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A basis of $\mathcal{O}$ is a subset $\mathcal{B}$ of $\mathbf{S}(\mathfrak{G})$ such that for any $[\mathfrak{t}]_{\equiv} \in \mathbf{S}(\mathfrak{G})/_{\equiv}$, there exists a unique $\mathfrak{s} \in \mathcal{B}$ such that $\mathfrak{s} \in [\mathfrak{t}]_{\equiv}$.

Operads, presentations, and patterns

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- Link between bases and pattern avoidance -

In most cases, $\mathcal{B}$ can be described as the set of $\mathfrak{G}$-trees avoiding a subset $\mathcal{P}_{\mathcal{B}}$ of $\mathbf{S}(\mathfrak{G})$.

Let $\mathcal{X} = \bigsqcup_{n \geqslant 1} \mathcal{X}(n)$ be a family of combinatorial objects, and consider that we want to describe the generating series

$$\mathrm{en}\left(\mathbf{f}_{\mathcal{X}}\right) = \sum_{n \geqslant 1} \#\mathcal{X}(n) z^n.$$

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2. exhibiting a presentation $(\mathfrak{G}, \equiv)$ of $\mathcal{O}_{\mathcal{X}}$ and a basis $\mathcal{B}$;
3. computing the series $\mathbf{F}\left(\mathcal{P}_{\mathcal{B}}, \emptyset\right)$ where $\mathcal{P}_{\mathcal{B}}$ is a set of $\mathfrak{G}$-trees satisfying $\mathrm{A}\left(\mathcal{P}_{\mathcal{B}}\right) = \mathcal{B}$.

Operads and enumeration

Let $\mathcal{X} = \bigsqcup_{n \geqslant 1} \mathcal{X}(n)$ be a family of combinatorial objects, and consider that we want to describe the generating series

$$\mathrm{en}\left(\mathbf{f}_{\mathcal{X}}\right) = \sum_{n \geqslant 1} \# \mathcal{X}(n) z^n.$$

- Operads as tools for enumeration -

The approach using operads consists in

1. endowing $\mathcal{X}$ with the structure of an operad $\mathcal{O}_{\mathcal{X}}$;
2. exhibiting a presentation $(\mathfrak{G}, \equiv)$ of $\mathcal{O}_{\mathcal{X}}$ and a basis $\mathcal{B}$;
3. computing the series $\mathbf{F}\left(\mathcal{P}_{\mathcal{B}}, \emptyset\right)$ where $\mathcal{P}_{\mathcal{B}}$ is a set of $\mathfrak{G}$-trees satisfying $\mathrm{A}\left(\mathcal{P}_{\mathcal{B}}\right) = \mathcal{B}$.

By the previous results,

$$\mathbf{F}\left(\mathcal{P}_{\mathcal{B}}, \emptyset\right) = \mathbf{f}_{\mathcal{X}},$$

so that $\mathrm{en}(\mathbf{F}\left(\mathcal{P}_{\mathcal{B}}, \emptyset\right))$ is the generating series of $\mathcal{X}$.

Examples

Let $\textbf{Schr}$ be the suboperad of $\mathrm{T}(\mathbb{N}, +, 0)$ generated by $\mathfrak{G} := \{00, 01, 10\}$.

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- Proposition -

The set **Schr** contains exactly all the words u having at least a 0 and, for any letter $u_i \geqslant 1$, there is a letter $u_j = u_i - 1$ and the letters of the factor of u between u_i and u_j is made of letters $u_k \geqslant u_i$.

Schröder trees

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Moreover, the set $\mathbf{Schr}(n)$ is in one-to-one correspondence with the set of Schröder trees with $n + 1$ leaves.

The first dimensions of $\mathbf{Schr}$ are hence

$$1, 3, 11, 45, 197, 903, 4279, 20793 \qquad (\mathbf{A001003}).$$

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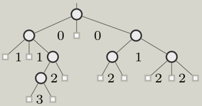
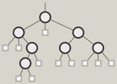
- Example -

$1132002122 \quad \longleftrightarrow$



$\longleftrightarrow$



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The operad **Schr** admits the presentation $(\mathfrak{G}, \equiv)$ where $\equiv$ is the smallest operad congruence satisfying

$$\mathrm{c}(00) \circ_1 \mathrm{c}(00) \equiv \mathrm{c}(00) \circ_2 \mathrm{c}(00),$$
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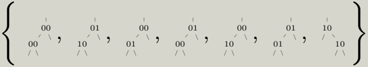
- Proposition -

The set of the $\mathfrak{G}$-trees avoiding the set

$\mathcal{P} :=$ 

is a basis of **Schr**.

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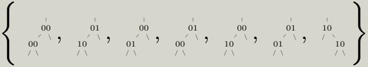

is a basis of **Schr**.

The characteristic series of this basis satisfies

$$\mathbf{F}(\mathcal{P}, \emptyset) = | + \bar{\circ}_{00}\left(\mathbf{F}(\mathcal{P}, \mathfrak{G}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \mathfrak{G}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_{10}\left(\mathbf{F}(\mathcal{P}, \emptyset), \mathbf{F}(\mathcal{P}, \{10\})\right),$$

$$\mathbf{F}(\mathcal{P}, \mathfrak{G}) = |,$$

$$\mathbf{F}(\mathcal{P}, \{10\}) = | + \bar{\circ}_{00}\left(\mathbf{F}(\mathcal{P}, \mathfrak{G}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \mathfrak{G}), \mathbf{F}(\mathcal{P}, \emptyset)\right).$$

The generating series of these trees enumerated w.r.t. their arities (parameter z) and the numbers of internal nodes by types (parameters q_a, $a \in \mathfrak{G}$) satisfies

$$z + (z(q_{00} + q_{01} + q_{10}) - 1)\, F(\mathcal{P}, \emptyset) + (z(q_{00}q_{10} + q_{01}q_{10}))\, F(\mathcal{P}, \emptyset)^2 = 0.$$

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$$\begin{aligned} F(\mathcal{P}, \emptyset) = {} & z + (q_{00} + q_{01} + q_{10})\,z^2 \\ & + \left(q_{00}^2 + 2q_{00}q_{01} + 3q_{00}q_{10} + q_{01}^2 + 3q_{01}q_{10} + q_{10}^2\right) z^3 \\ & + \left(q_{00}^3 + 3q_{00}^2q_{01} + 6q_{00}^2q_{10} + 3q_{00}q_{01}^2 + 12q_{00}q_{01}q_{10} + 6q_{00}q_{10}^2 \right. \\ & \left. + q_{01}^3 + 6q_{01}^2q_{10} + 6q_{01}q_{10}^2 + q_{10}^3\right) z^4 + \cdots . \end{aligned}$$

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The coefficients of the bivariate series obtained by specializing q_{10} and q_{01} (resp. q_{00} and q_{01}) to 1 are the ones of Triangle A126216 (resp. Triangle A114656).

m-trees

For any $m \geqslant 0$, let $\mathbf{FCat}^{(m)}$ be the suboperad of $\mathrm{T}(\mathbb{N}, +, 0)$ generated by $\mathfrak{G} := \{00, 01, \ldots, 0m\}$.

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The set $\mathbf{FCat}^{(m)}$ contains exactly all the words u satisfying $u_1 = 0$ and $u_i \in [0, u_{i-1} + m]$ for all valid positions i and $i - 1$.

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Moreover, the set $\mathbf{FCat}^{(m)}(n)$ is in one-to-one correspondence with the set of planar rooted trees with wherein all their n internal nodes have $m + 1$ children.

The dimensions of $\mathbf{FCat}^{(m)}$ are hence Fuss-Catalan numbers.

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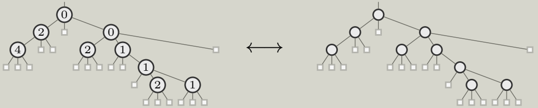
- Example -

For $m := 2$,

$024021121 \longleftrightarrow$



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The operad $\mathbf{FCat}^{(m)}$ admits the presentation $(\mathfrak{G}, \equiv)$ where $\equiv$ is the smallest operad congruence satisfying

$$\mathrm{c}\,(0k_3) \circ_1 \mathrm{c}\,(0k_1) \equiv \mathrm{c}\,(0k_1) \circ_2 \mathrm{c}\,(0k_2)\,, \qquad k_3 = k_1 + k_2.$$

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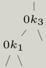
$\mathcal{P} := \Big\{$  $: 0 \leqslant k_1 \leqslant k_3 \leqslant m \Big\}$

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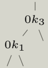

$$\left. : 0 \leqslant k_1 \leqslant k_3 \leqslant m \right\}$$

is a basis of $\mathbf{FCat}^{(m)}$.

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$$\mathbf{F}(\mathcal{P}, \emptyset) = | + \sum_{0 \leqslant k \leqslant m} \bar{\circ}_{0k} \left(\mathbf{F}\left(\mathcal{P}, \mathcal{Q}_k\right), \mathbf{F}(\mathcal{P}, \emptyset) \right),$$

$$\mathbf{F}\left(\mathcal{P}, \mathcal{Q}_k\right) = | + \sum_{k < k' \leqslant m} \bar{\circ}_{0k'} \left(\mathbf{F}\left(\mathcal{P}, \mathcal{Q}_{k'}\right), \mathbf{F}(\mathcal{P}, \emptyset) \right),$$

where

$$\mathcal{Q}_k := \{00, 01, \ldots, 0k\}.$$

The generating series of these trees enumerated w.r.t. their arities (parameter z) and the numbers of internal nodes by types (parameters q_a, $a \in \mathfrak{G}$) satisfies

$$-F(\mathcal{P}, \emptyset) + z \prod_{0 \leqslant k \leqslant m} (q_{0k} F(\mathcal{P}, \emptyset) + 1) = 0.$$

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One has, for $m := 1$,

$$\begin{aligned} F(\mathcal{P}, \emptyset) = z &+ (q_{00} + q_{01})\, z^2 \\ &+ \left(q_{00}^2 + 2q_{00}q_{01} + 2q_{01}^2\right) z^3 + \left(q_{00}^3 + 3q_{00}^2 q_{01} + 5q_{00}q_{01}^2 + 5q_{01}^3\right) z^4 \\ &+ \left(q_{00}^4 + 4q_{00}^3 q_{01} + 9q_{00}^2 q_{01}^2 + 14q_{00}q_{01}^3 + 14q_{01}^4\right) z^5 \\ &+ \left(q_{00}^5 + 5q_{00}^4 q_{01} + 14q_{00}^3 q_{01}^2 + 28q_{00}^2 q_{01}^3 + 42q_{00}q_{01}^4 + 42q_{01}^5\right) z^6 + \cdots, \end{aligned}$$

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The coefficients of the bivariate series obtained by specializing q_{10} (resp. q_{00}) to 1 are the ones of Triangle A033184 (resp. Triangle A009766).

Directed animals

Let $\mathbf{DA}$ be the suboperad of $(\mathbb{Z}/_{3\mathbb{Z}}, +, 0)$ generated by $\mathfrak{G} := \{00, 01\}$.

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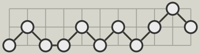
- Example -

$01001010121 \longleftrightarrow 1\bar{1}01\bar{1}1\bar{1}11\bar{1} \longleftrightarrow$ 

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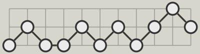

Since prefixes of Motzkin paths are in one-to-one correspondence with directed animals on the square lattice [Gouyou-Beauchamps, Viennot, 1988], $\mathbf{DA}$ is an operad on such objects.

The first dimensions of $\mathbf{DA}$ are

$$1, 2, 5, 13, 35, 96, 267, 750, 2123 \qquad (\mathbf{A005773}).$$

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The operad **DA** admits the presentation $(\mathfrak{G}, \equiv)$ where $\equiv$ is the smallest operad congruence satisfying

$$\mathrm{c}(00) \circ_1 \mathrm{c}(00) \equiv \mathrm{c}(00) \circ_2 \mathrm{c}(00),$$

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$$(\mathrm{c}(00) \circ_1 \mathrm{c}(01)) \circ_2 \mathrm{c}(01) \equiv (\mathrm{c}(01) \circ_2 \mathrm{c}(01)) \circ_3 \mathrm{c}(01).$$

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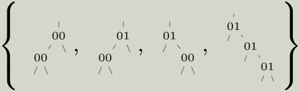

is a basis of $\mathbf{DA}$.

Directed animals

The characteristic series of the previous basis of $\mathbf{DA}$ satisfies

$$
\begin{aligned}
\mathbf{F}(\mathcal{P}, \emptyset) &= \shortmid + \bar{\circ}_{00}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right)\right), \\
\mathbf{F}(\mathcal{P}, \{00\}) &= \shortmid + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right)\right), \\
\mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right) &= \shortmid + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, 01, \mathfrak{t}\}\right)\right), \\
\mathbf{F}\left(\mathcal{P}, \{00, 01, \mathfrak{t}\}\right) &= \shortmid.
\end{aligned}
$$

where $\mathfrak{t} := \mathrm{c}(01) \circ_2 \mathrm{c}(01)$.

The characteristic series of the previous basis of $\mathbf{DA}$ satisfies

$$\begin{aligned}
\mathbf{F}(\mathcal{P},\emptyset) &= | + \bar{\circ}_{00}\left(\mathbf{F}(\mathcal{P},\{00\}),\mathbf{F}(\mathcal{P},\emptyset)\right) + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P},\{00\}),\mathbf{F}\left(\mathcal{P},\{00,\mathfrak{t}\}\right)\right),\\
\mathbf{F}(\mathcal{P},\{00\}) &= | + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P},\{00\}),\mathbf{F}\left(\mathcal{P},\{00,\mathfrak{t}\}\right)\right),\\
\mathbf{F}\left(\mathcal{P},\{00,\mathfrak{t}\}\right) &= | + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P},\{00\}),\mathbf{F}\left(\mathcal{P},\{00,01,\mathfrak{t}\}\right)\right),\\
\mathbf{F}\left(\mathcal{P},\{00,01,\mathfrak{t}\}\right) &= |.
\end{aligned}$$

where $\mathfrak{t} := \mathrm{c}(01) \circ_2 \mathrm{c}(01)$.

The generating series of these trees enumerated w.r.t. their arities (parameter z) and the numbers of internal nodes by types (parameters q_a, $a \in \mathfrak{G}$) satisfies

$$F(\mathcal{P},\emptyset) = \frac{1 - \sqrt{1 - 2zq_{01} - 3z^2q_{01}^2} - z(2q_{00} + q_{01})}{2z\left(q_{00}^2 + q_{00}q_{01} + q_{01}^2\right) - 2q_{00}}$$

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$$\begin{aligned}
\mathbf{F}(\mathcal{P}, \emptyset) &= | + \bar{\circ}_{00} \left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_{01} \left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right)\right), \\
\mathbf{F}(\mathcal{P}, \{00\}) &= | + \bar{\circ}_{01} \left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right)\right), \\
\mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right) &= | + \bar{\circ}_{01} \left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, 01, \mathfrak{t}\}\right)\right), \\
\mathbf{F}\left(\mathcal{P}, \{00, 01, \mathfrak{t}\}\right) &= |.
\end{aligned}$$

where $\mathfrak{t} := \mathrm{c}(01) \circ_2 \mathrm{c}(01)$.

The generating series of these trees enumerated w.r.t. their arities (parameter z) and the numbers of internal nodes by types (parameters q_a, $a \in \mathfrak{G}$) satisfies

$$F(\mathcal{P}, \emptyset) = \frac{1 - \sqrt{1 - 2zq_{01} - 3z^2 q_{01}^2} - z(2q_{00} + q_{01})}{2z\left(q_{00}^2 + q_{00}q_{01} + q_{01}^2\right) - 2q_{00}}$$

One has

$$\begin{aligned}
F(\mathcal{P}, \emptyset) = z &+ (q_{00} + q_{01})\, z^2 + \left(q_{00}^2 + 2q_{00}q_{01} + 2q_{01}^2\right) z^3 \\
&+ \left(q_{00}^3 + 3q_{00}^2 q_{01} + 5q_{00}q_{01}^2 + 4q_{01}^3\right) z^4 + \left(q_{00}^4 + 4q_{00}^3 q_{01} + 9q_{00}^2 q_{01}^2 + 12q_{00}q_{01}^3 + 9q_{01}^4\right) z^5 \\
&+ \left(q_{00}^5 + 5q_{00}^4 q_{01} + 14q_{00}^3 q_{01}^2 + 25q_{00}^2 q_{01}^3 + 30q_{00}q_{01}^4 + 21q_{01}^5\right) z^6 + \cdots .
\end{aligned}$$

The characteristic series of the previous basis of $\mathbf{DA}$ satisfies

$$\begin{aligned}
\mathbf{F}(\mathcal{P}, \emptyset) &= | + \bar{\circ}_{00}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}(\mathcal{P}, \emptyset)\right) + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right)\right), \\
\mathbf{F}(\mathcal{P}, \{00\}) &= | + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right)\right), \\
\mathbf{F}\left(\mathcal{P}, \{00, \mathfrak{t}\}\right) &= | + \bar{\circ}_{01}\left(\mathbf{F}(\mathcal{P}, \{00\}), \mathbf{F}\left(\mathcal{P}, \{00, 01, \mathfrak{t}\}\right)\right), \\
\mathbf{F}\left(\mathcal{P}, \{00, 01, \mathfrak{t}\}\right) &= |.
\end{aligned}$$

where $\mathfrak{t} := \mathrm{c}(01) \circ_2 \mathrm{c}(01)$.

The generating series of these trees enumerated w.r.t. their arities (parameter z) and the numbers of internal nodes by types (parameters q_a, $a \in \mathfrak{G}$) satisfies

$$F(\mathcal{P}, \emptyset) = \frac{1 - \sqrt{1 - 2zq_{01} - 3z^2 q_{01}^2} - z(2q_{00} + q_{01})}{2z\left(q_{00}^2 + q_{00}q_{01} + q_{01}^2\right) - 2q_{00}}$$

One has

$$\begin{aligned}
F(\mathcal{P}, \emptyset) = z &+ \left(q_{00} + q_{01}\right) z^2 + \left(q_{00}^2 + 2q_{00}q_{01} + 2q_{01}^2\right) z^3 \\
&+ \left(q_{00}^3 + 3q_{00}^2 q_{01} + 5q_{00}q_{01}^2 + 4q_{01}^3\right) z^4 + \left(q_{00}^4 + 4q_{00}^3 q_{01} + 9q_{00}^2 q_{01}^2 + 12q_{00}q_{01}^3 + 9q_{01}^4\right) z^5 \\
&+ \left(q_{00}^5 + 5q_{00}^4 q_{01} + 14q_{00}^3 q_{01}^2 + 25q_{00}^2 q_{01}^3 + 30q_{00}q_{01}^4 + 21q_{01}^5\right) z^6 + \cdots .
\end{aligned}$$

The coefficients of the bivariate series obtained by specializing q_{10} (resp. q_{00}) to 1 are the ones of Triangle A064189 (resp. Triangle A026300).