Combinatoire algébrique et calcul symbolique

Focus scientifique — *Logique combinatoire et treillis*

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Some combinatorial objects

Words

Let $A := \{\mathbf{a}, \mathbf{b}\}$ be an alphabet and A^* be the combinatorial collection of the **words** on A.

$$A^*(0) = \{\epsilon\}, \qquad A^*(1) = \{\mathbf{a}, \mathbf{b}\}, \qquad A^*(2) = \{\mathbf{aa}, \mathbf{ab}, \mathbf{ba}, \mathbf{bb}\}.$$

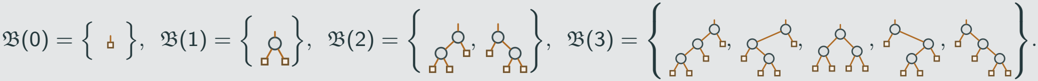
Permutations

Let $\mathfrak{S}$ be the combinatorial collection of the **permutations**.

$$\mathfrak{S}(0) = \{\epsilon\}, \qquad \mathfrak{S}(1) = \{1\}, \qquad \mathfrak{S}(2) = \{12, 21\}, \qquad \mathfrak{S}(3) = \{123, 132, 213, 231, 312, 321\}.$$

Binary trees

Let $\mathfrak{B}$ be the combinatorial collection of the **binary trees**.



Combinatorial collections

A **combinatorial collection** (**CC**) is a set C endowed with a map

$$|-|: C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, the set $C(n) := \{x \in C : |x| = n\}$ is finite.

Some usual questions

1. **Enumerate** the elements of $C(n)$;
2. **Generate exhaustively** the elements of $C(n)$;
3. **Randomly generate** an element of $C(n)$;
4. **Design transformations** between two CCs.

Examples

1. $\#\mathfrak{B}(n) = \frac{1}{n+1}\binom{2n}{n}$;
2. Gray codes on binary trees [**Proskurowski, Ruskey**, 1985];
3. Rémy algorithm [**Rémy**, 1985];
4. Insertion of $\sigma \in \mathfrak{S}$ into a binary search tree.

Combinatory logic

Combinatory logic is a model of computation [Schönfinkel, 1924].

Its objects are **terms**.

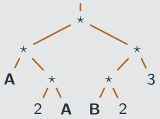
They are binary trees where leaves are labeled by constants $\mathbf{X}$ or by variables $i \in \{1, 2, \ldots\}$.

A computation step consists in transforming a term by applying a local transformation specified by **rewrite rules**.

Example

$\mathbf{A}(2\,\mathbf{A})(\mathbf{B}\,2\,3)$ is a term on the constants $\mathbf{A}$ and $\mathbf{B}$.

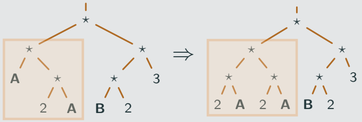
Its tree representation is



Example

Let the rule $\mathbf{A}\,1 \to 1\,1$.

$\big(\underline{\mathbf{A}(2\,\mathbf{A})}\big)(\mathbf{B}\,2\,3) \Rightarrow \big(\underline{(2\,\mathbf{A})(2\,\mathbf{A})}\big)(\mathbf{B}\,2\,3)$ holds.



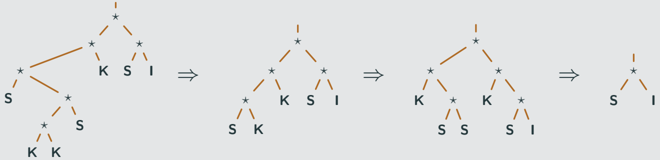
The S, K, I-system

Let the system [**Curry**, 1930] made on the three constants **S**, **K**, and **I**, satisfying

$$\mathbf{S}\,1\,2\,3 \to 1\,3\,(2\,3), \quad \mathbf{K}\,1\,2 \to 1, \quad \mathbf{I}\,1 \to 1.$$

Example

Here is a sequence of computation:



This CLS is Turing-complete: there are algorithms to emulate any λ-term by a term of this CLS. These are **abstraction algorithms** [**Rosser**, 1955], [**Curry, Feys**, 1958].

Some important combinators

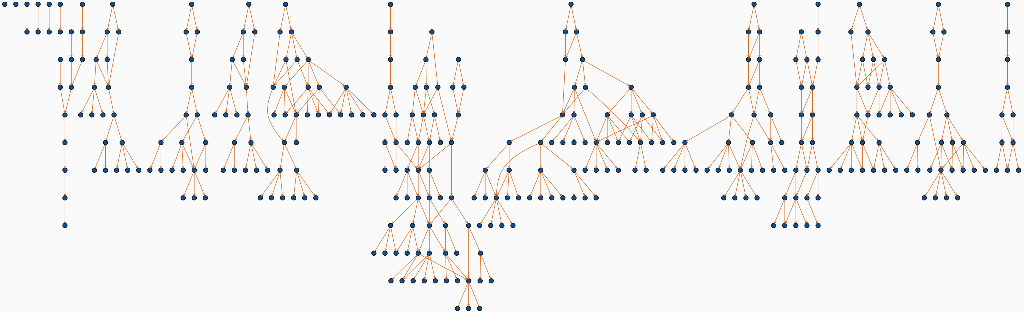
In *To Mock a Mockingbird: and Other Logic Puzzles* [**Smullyan**, 1985], a large number of constants with rules are listed, forming the **Enchanted forest of combinator birds**.

Here is a sublist:

- **Identity bird**: $\mathbf{I}\,1 \to 1$
- **Mockingbird**: $\mathbf{M}\,1 \to 1\,1$
- **Kestrel**: $\mathbf{K}\,1\,2 \to 1$
- **Thrush**: $\mathbf{T}\,1\,2 \to 2\,1$
- **Mockingbird 1**: $\mathbf{M_1}\,1\,2 \to 1\,1\,2$
- **Warbler**: $\mathbf{W}\,1\,2 \to 1\,2\,2$
- **Lark**: $\mathbf{L}\,1\,2 \to 1\,(2\,2)$
- **Owl**: $\mathbf{O}\,1\,2 \to 2\,(1\,2)$
- **Turing bird**: $\mathbf{U}\,1\,2 \to 2\,(1\,1\,2)$
- **Cardinal**: $\mathbf{C}\,1\,2\,3 \to 1\,3\,2$
- **Vireo**: $\mathbf{V}\,1\,2\,3 \to 3\,1\,2$
- **Bluebird**: $\mathbf{B}\,1\,2\,3 \to 1\,(2\,3)$
- **Starling**: $\mathbf{S}\,1\,2\,3 \to 1\,3\,(2\,3)$
- **Jay**: $\mathbf{J}\,1\,2\,3\,4 \to 1\,2\,(1\,4\,3)$

Rewrite graph of L

For $\mathbf{L}\,1\,2 \to 1\,(2\,2)$, here is rewriting the graph of terms on $\mathbf{L}$ from closed terms of degrees up to 5 and up to 4 rewritings:

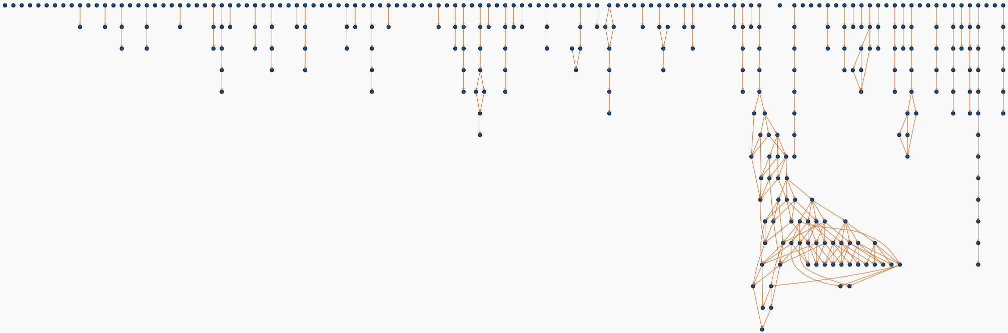


This is a poset.

Interval are conjectured to be lattices.

Rewrite graph of S

For $\mathbf{S}\,1\,2\,3 \to 1\,3\,(2\,3)$, here is the rewriting graph of terms on $\mathbf{S}$ from closed terms of degrees up to 6 and up to 11 rewritings:



This is a poset (consequence of [**Bergstra, Klop**, 1979]).

Interval are conjectured to be lattices.

Algorithmic questions

Let $\mathcal{C}$ be a system.

Word problem

Is there an algorithm to decide, given two terms t and t' of $\mathcal{C}$, if t and t' belong to the same connected component? (See [**Baader, Nipkow**, 1998], [**Statman**, 2000].)

- Yes for the system on **L** [**Statman**, 1989], [**Sprenger, Wymann-Böni**, 1993].
- Yes for the system on **W** [**Sprenger, Wymann-Böni**, 1993].
- Yes for the system on $\mathbf{M_1}$ [**Sprenger, Wymann-Böni**, 1993].
- **Open** for the system on **S** [**RTA Problem #97**, 1975].

Strong normalization problem

Is there an algorithm to decide, given a term t of $\mathcal{C}$, if t if every computation starting from t terminates?

- Yes for the system on **S** [**Waldmann**, 2000].
- Yes for the system on **J** [**Probst, Studer**, 2000].

Combinatorial questions

If $\mathcal{C}$ is a system, let

- $\mathfrak{T}$ be the set of the terms of $\mathcal{C}$;
- $\ll$ be the reflexive and transitive closure of $\Rightarrow$;
- $\mathrm{G}(\mathfrak{t})$ be the set $\{\mathfrak{t}' \in \mathfrak{T} : \mathfrak{t} \ll \mathfrak{t}'\}$.

Structure of the rewrite graphs

1. Is the preorder $\ll$ an order relation on $\mathfrak{T}$?
2. If so, are all subposets $(\mathrm{G}(\mathfrak{t}), \ll)$ lattices for all $\mathfrak{t} \in \mathfrak{T}$?
3. Are all the connected components of the rewrite graph of $\mathcal{C}$ finite?
4. Understand when $(\mathrm{G}(\mathfrak{t}), \Rightarrow)$ and $(\mathrm{G}(\mathfrak{t}'), \Rightarrow)$ are isomorphic graphs.

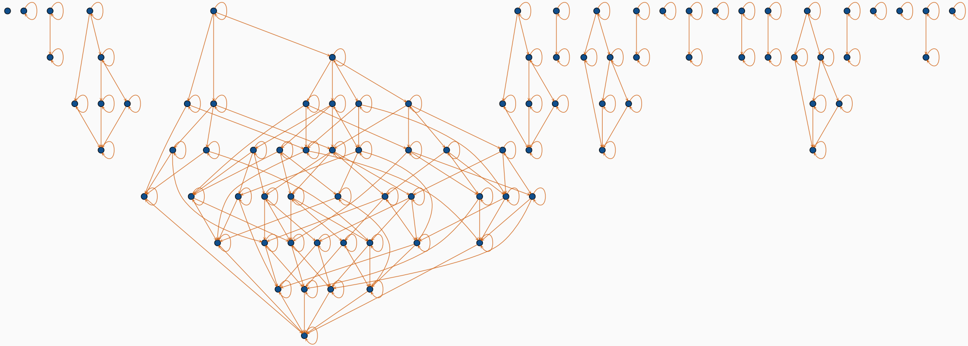
Enumerative issues

1. Enumerate the connected components of $\mathcal{C}$ w.r.t. some size notions;
2. When for $\mathfrak{t}$, the connected component of $\mathfrak{t}$ is finite, compute its number of elements and of edges;
3. When for $\mathfrak{t}$, $(\mathrm{G}(\mathfrak{t}), \ll)$ is a poset, enumerate its intervals.

The Mockingbird system

The **Mockingbird system** is made on the constant **M** satisfying $\mathbf{M}\,1 \to 1\,1$.

Here is a part of its rewrite graph restrained on closed terms of degrees 4 or less:



Lattices of duplicative forests

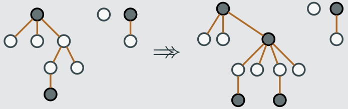
A **duplicative forest** is a forest of planar rooted trees where nodes are either black ● or white ○. Let $\mathcal{D}^*$ be the set of all duplicative forests.

For $\mathfrak{f}, \mathfrak{g} \in \mathcal{D}^*$, $\mathfrak{f} \twoheadrightarrow \mathfrak{g}$ if $\mathfrak{g}$ is obtained by blackening a white node of $\mathfrak{f}$ and by duplicating its sequence of descendants.

The reflexive and transitive closure $\lll$ of $\twoheadrightarrow$ is an order relation.

Let $\mathcal{D}^*(\mathfrak{f}) := \{\mathfrak{f}' \in \mathcal{D}^* : \mathfrak{f} \lll \mathfrak{f}'\}$.

Example

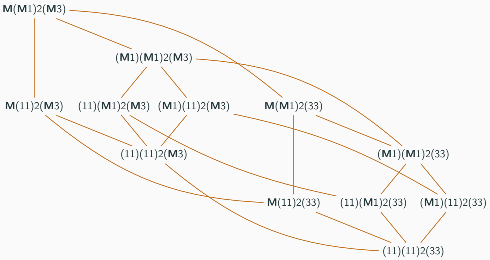


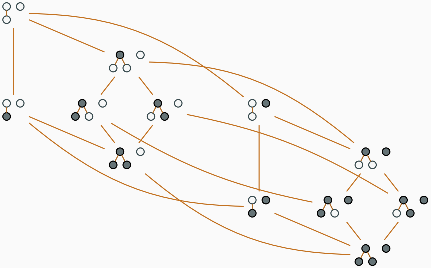
Theorem [G., 2022]

- For any $\mathfrak{f} \in \mathcal{D}^*$, $\mathcal{D}^*(\mathfrak{f})$ is a lattice.
- For any **M**-term $\mathfrak{t}$, the poset $(\mathrm{G}(\mathfrak{t}), \ll)$ is isomorphic to $(\mathcal{D}^*(\mathfrak{f}), \lll)$ for a certain $\mathfrak{f} \in \mathcal{D}^*$.

An example of a duplicative forest lattice

Here are an interval of the Mockingbird system poset and its corresponding interval of the lattice of duplicative forests:





Some enumerative results

For any duplicative forest $\mathfrak{f}$, let the series $\mathbf{gr}(\mathfrak{f}) = \sum_{\mathfrak{f}' \in \mathcal{D}^*(\mathfrak{f})} \mathfrak{f}'$ and $\mathbf{ns}(\mathfrak{f}) = \mathbf{gr}(\mathbf{gr}(\mathfrak{f}))$.

Example

Theorem [G., 2022]

The generating series $F(z)$ of the cardinalities of $\mathbb{M}(d)$, the lattices of terms from $\mathbf{M}(\mathbf{M}\dots(\mathbf{MM})\dots))$ of degrees $d \geq 0$, satisfies

$$F(z) = 1 + zF(z) + z(F(z) \boxtimes F(z)).$$

The generating series $G(z)$ of intervals of $\mathbb{M}(d)$ satisfies $G(z) = G_1(z)$ where for any $k \geq 1$,

$$G_k(z) = 1 + z(G_k(z) \boxtimes G_k(z)) + z \sum_{0 \leq i \leq k} \binom{k}{i} G_{k+i}(z).$$

Coefficients of $F(z)$: $1, 1, 2, 6, 42, 1806, 3263442, \dots$ (Sequence **A007018**).

Coefficients of $G(z)$: $1, 1, 3, 17, 371, 144513, 20932611523, \dots$.