Interstice operads of words

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1. The associative symmetric operad

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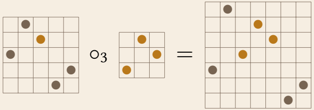
Outline

1. The associative symmetric operad

Composition of permutations

Given two permutations σ and ν, and $i \in [\![|\sigma|]\!]$, let $\sigma \square_i \nu$ be the permutation having as permutation matrix the one obtained by inserting the matrix of ν onto the i-th point of the matrix of σ.

– Example –



$$35412 \circ_3 132 = 3746512$$

This composition is a map

$$\square_i : \mathfrak{S}(n) \times \mathfrak{S}(m) \to \mathfrak{S}(n+m-1), \qquad n, m \geqslant 1, \quad i \in [\![n]\!],$$

where $\mathfrak{S}(n)$ is the set of all permutations of size $n \geqslant 1$.

Algebraic point of view

The pair **Per** $:= (\mathfrak{S}, \square_i)$ is an operad: any permutation σ of size $n \geqslant 1$ is seen as an operator of arity n and the composition map behaves like the usual composition of operators.

One can ask about a minimal generating set of this operad, that is a smallest set $\mathfrak{G}$ such that each permutation σ can be obtained by composing elements of $\mathfrak{G}$ with each other.

– Proposition –

The set $\mathfrak{G}$ of all simple permutations of size $n \geqslant 2$ is a minimal generating set of **Per**.

A permutation σ is simple if no factor of σ of length between 2 and $|\sigma| - 1$ is an interval.

– Examples –

The permutation

- $71435628 = 513426 \square_3 213$ is not simple;
- 4135726 is simple.

The elements of $\mathfrak{G}$ are enumerated, arity by arity, by Sequence **A111111** beginning by

$$0, 2, 0, 2, 6, 46, 338, 2926, 28146, 298526.$$

Linearization of Per

In order to pursue the algebraic study of this operad, we consider its linearized version: from now, **Per** is the linear span $\mathrm{Span}(\mathfrak{S})$ of $\mathfrak{S}$ over a field $\mathbb{K}$ of characteristic zero.

The set $\{\mathsf{E}_\sigma : \sigma \in \mathfrak{S}\}$ is hence a basis of **Per**, called elementary basis. We set also

$$\mathsf{E}_\sigma \circ_i \mathsf{E}_\nu := \mathsf{E}_{\sigma \,\square_i\, \nu}.$$

– Examples –

In **Per**,

$$\mathsf{E}_{35412} \circ_3 \mathsf{E}_{132} = \mathsf{E}_{3746512},$$

$$(\mathsf{E}_{312} - \mathsf{E}_{2341}) \circ_2 (2\mathsf{E}_{12} - \mathsf{E}_{321}) = 2\mathsf{E}_{4123} - \mathsf{E}_{53214} - 2\mathsf{E}_{23451} + \mathsf{E}_{254361}.$$

The main interest to consider such a linearized version of **Per** is that we can consider change of bases, quotients, and linear maps and operad morphisms to other operads.

Partial order on permutations

Let $\preccurlyeq$ be the partial order relation on $\mathfrak{S}$ satisfying $\sigma \preccurlyeq \nu$ if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\nu)$, where $\mathrm{Inv}(\pi) := \{(i, j) : i < j \text{ and } \pi(i) > \pi(j)\}$.

This is the left weak order on permutations.

– Example –

Let $\sigma := 23154$ and $\nu := 25143$.

Since $\mathrm{Inv}(\sigma) = \{(1,3), (2,3), (4,5)\}$ and $\mathrm{Inv}(\nu) = \{(1,3), (2,3), (2,4), (2,5), (4,5)\}$, we have $\sigma \preccurlyeq \nu$.

Let the elements F_σ, $\sigma \in \mathfrak{S}$, of **Per** defined by

$$\mathsf{F}_\sigma := \sum_{\sigma \preccurlyeq \sigma'} \mu_{\preccurlyeq}(\sigma, \sigma') \mathsf{E}_{\sigma'},$$

where $\mu_{\preccurlyeq}$ is the Möbius function of the poset $(\mathfrak{S}, \preccurlyeq)$.

– Example –

$$\mathsf{F}_{4123} = \mathsf{E}_{4123} - \mathsf{E}_{4132} - \mathsf{E}_{4213} + \mathsf{E}_{4321}$$

Alternative basis of Per

By Möbius inversion, for any $\sigma \in \mathfrak{S}$,

$$\mathsf{E}_\sigma = \sum_{\sigma \preccurlyeq \sigma'} \mathsf{F}_{\sigma'}.$$

Therefore, by triangularity, the family $\{\mathsf{F}_\sigma : \sigma \in \mathfrak{S}\}$ is a basis of **Per**.

A natural question now is to describe the composition $\circ_i$ of **Per** on this new basis.

– Theorem [Aguiar, Livernet, 2007] –

For any $\sigma, \nu \in \mathfrak{S}$,

$$\mathsf{F}_\sigma \circ_i \mathsf{F}_\nu = \sum_{\sigma \,\square_i\, \nu \preccurlyeq \pi \preccurlyeq \sigma \,\blacksquare_i\, \nu} \mathsf{F}_\pi,$$

for a certain composition operation $\blacksquare_i$ on $\mathfrak{S}$.

In other terms, the support of any composition expressed in the F-basis, is an interval of the left weak order.

Structures on permutations and motivations

There are other algebraic and combinatorial structures on permutations:

- symmetric groups;
- lattices for the left and right weak order;
- Hopf bialgebra of Malvenuto-Reutenauer (also known as **FQSym**) [**Malvenuto, Reutenauer**, 1994] [**Duchamp, Hivert, Thibon**, 2002];
- dendriform algebra [**Loday**, 2001].

The main motivation of this work is to construct an alternative operad structure on $\mathfrak{S}$.

We obtain

- a new operad on permutations;
- operads on some generalizations of permutations;
- operads on Fuss-Catalan objects.

We introduce several bases for these operads, analogs of the compositions operations $\square_i$ and $\blacksquare_i$, and analogs of the left weak order on permutations.

2. Cliffs and related objects

Cliffs

A range map is a map $\delta : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ and a δ-cliff is a word u on $\mathbb{N}$ such that for all $i \in [|u|]$, $0 \leqslant u(i) \leqslant \delta(i)$.

The length of u is denoted by $\ell(u)$. The size of u is denoted by $|u|$ and is $\ell(u) + 1$.

Let Cl_δ be the set of all δ-cliffs.

– Example –

If $\delta = 102222\ldots$, then

$$\mathsf{Cl}_\delta(5) = \{0000, 0001, 0002, 0010, 0011, \ldots, 1022\}.$$

We will consider mainly two sorts of range maps:

- for any $m \in \mathbb{N}$, $\mathbf{m}$ as the range map satisfying $\mathbf{m}(i) = (i - 1)m$;
- for any $c \in \mathbb{N}$, $\underline{c}$ as the range map satisfying $\underline{c}(i) = c$.

Cliffs and other objects

A δ-hill is a weakly increasing δ-cliff. Let Hi_δ be the set of all δ-hills.

Some sets of cliffs or hills are in one-to-one correspondence with combinatorial families:

- $\mathsf{Cl}_{\underline{1}}(n)$ with integers compositions of n.

– Example –

$1100010 \in \mathsf{Cl}_{\underline{1}}(8) \leftrightarrow (1, 1, 4, 2)$

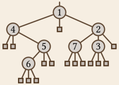
- $\mathsf{Cl}_{\mathbf{1}}(n)$ with permutations of size $n - 1$.

– Example –

$002323 \in \mathsf{Cl}_{\mathbf{1}}(7) \leftrightarrow 436512$

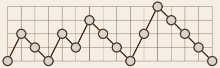
- $\mathsf{Cl}_{\mathbf{m}}(n)$ with incr. $m+1$-ary trees of $n-1$ nodes.

– Example –

$0230228 \in \mathsf{Cl}_{\mathbf{2}}(8) \leftrightarrow$ 

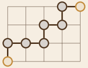
- $\mathsf{Hi}_{\mathbf{m}}(n)$ with m-Dyck paths with $n - 1$ rising steps.

– Example –

$02366 \in \mathsf{Hi}_{\mathbf{2}}(6) \leftrightarrow$ 

- $\mathsf{Hi}_{\underline{c}}(n)$ with N/E paths from $(0, 0)$ to $(n - 1, c)$.

– Example –

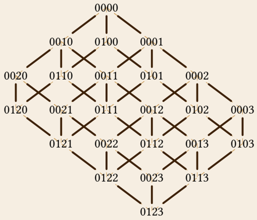
$1123 \in \mathsf{Hi}_{\underline{3}}(5) \leftrightarrow$ 

Cliffs and posets

Let $\preccurlyeq$ be the order relation on Cl_δ such that $u \preccurlyeq v$ if $u(i) \leqslant v(i)$ for all i.

– Example –

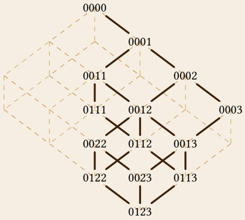
Hasse diagram of $(\mathrm{Cl}_{\mathbf{1}}(5), \preccurlyeq)$:



– Example –

Hasse diagram of $(\mathrm{Hi}_{\mathbf{1}}(5), \preccurlyeq)$:

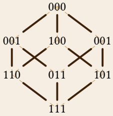
Also known as the Stanley lattice [**Stanley**, 1975].



– Example –

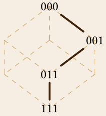
Hasse diagram of $(\mathrm{Cl}_{\underline{1}}(5), \preccurlyeq)$:

This is the Boolean lattice.



– Example –

Hasse diagram of $(\mathrm{Hi}_{\underline{1}}(5), \preccurlyeq)$:



3. Operad structures on cliffs

Interstice operads

For any alphabet A, let

$$\mathbf{I}(A) := \bigoplus_{n \geqslant 1} \operatorname{Span}\left(A^{n-1}\right).$$

The set $\{\mathsf{E}_u : u \in A^*\}$ is a basis of $\mathbf{I}(A)$. The arity of any E_u in $\mathbf{I}(A)$ is $\ell(u) + 1$ where $\ell(u)$ is the length of the word u.

Let the composition $\circ_i$ defined by $\mathsf{E}_u \circ_i \mathsf{E}_v := \mathsf{E}_{u \,\square_i\, v}$ where $u \,\square_i\, v := u(1, i-1)\ v\ u(i, \ell(u))$.

– Example –

For $A := \{a, b, c\}$, we have

$$\mathsf{E}_{aabacb} \circ_4 \mathsf{E}_{cbaa} = \mathsf{E}_{aab\,cbaa\,acb}.$$

in $\mathbf{I}(A)$.

The pair $(\mathbf{I}(A), \circ_i)$ is an operad, called A-interstice operad.

Suboperads of interstice operads

Let the subspace $\mathbf{Cl}_\delta := \mathrm{Span}(\{\mathsf{E}_u : u \in \mathsf{Cl}_\delta\})$ of $\mathbf{I}(\mathbb{N})$.

Two observations:

- when δ is a weakly increasing, if $u, v \in \mathsf{Cl}_\delta$, then $u \,\square_i\, v \in \mathsf{Cl}_\delta$. For this reason, $\mathbf{Cl}_\delta$ is a suboperad of $\mathbf{I}(\mathbb{N})$;
- when δ is not weakly increasing, $\mathbf{Cl}_\delta$ is not a suboperad of $\mathbf{I}(\mathbb{N})$. For instance, if $\delta = 0110\ldots$, even if $01 \in \mathsf{Cl}_\delta$, we have $01 \,\square_2\, 01 = 0101 \notin \mathsf{Cl}_\delta$.

– Objective –

Build substructures of $\mathbf{I}(\mathbb{N})$ on δ-cliffs for the largest possible class of range maps δ.

A first try consists in considering quotients $\mathbf{I}(\mathbb{N})/\mathcal{V}_\delta$, where $\mathcal{V}_\delta := \mathrm{Span}(\{\mathsf{E}_u : u \in \mathbb{N}^* \setminus \mathsf{Cl}_\delta\})$.

This does not work in general. One has a counter-example for $\delta := 0110\ldots$ since $\mathsf{E}_{11} \in \mathcal{V}_\delta$ and $\mathsf{E}_0 \in \mathbf{I}(\mathbb{N})$ but $\mathsf{E}_{11} \circ_1 \mathsf{E}_0 = \mathsf{E}_{011} \notin \mathcal{V}_\delta$.

Quotients of cliff operads

Given a range map δ, let $\bar{\delta}$ be the range map defined by

$$\bar{\delta}(i) := \max\{\delta(1), \dots, \delta(i)\}.$$

– Example –

If $\delta = 10032242\dots$, then $\bar{\delta} = 11133344\dots$.

By construction, $\bar{\delta}$ is weakly increasing, and thus, $\mathbf{Cl}_{\bar{\delta}}$ is an operad.

Let the subspace $\mathcal{V}_\delta := \mathsf{Span}(\{\mathsf{E}_u : u \in \mathsf{Cl}_{\bar{\delta}} \setminus \mathsf{Cl}_\delta\})$ of $\mathbf{Cl}_{\bar{\delta}}$.

– Theorem [Combe, G., 2021] –

For any range map δ, the space $\mathcal{V}_\delta$ is an operad ideal of the operad $\mathbf{Cl}_{\bar{\delta}}$ iff δ is unimodal. Therefore, when δ is unimodal, the space $\mathbf{Cl}_{\bar{\delta}}/\mathcal{V}_\delta$ is an operad.

A range map δ is unimodal if there is no $i_1 < i_2 < i_3$ such that $\delta(i_1) > \delta(i_2) < \delta(i_3)$.

For any unimodal range map δ, we set $\mathbf{Cl}_\delta := \mathbf{Cl}_{\bar{\delta}}/\mathcal{V}_\delta$. The composition of this operad satisfies

$$\mathsf{E}_u \circ_i \mathsf{E}_v = \chi_\delta(u \,\square_i\, v)\mathsf{E}_{u \,\square_i\, v},$$

where $\chi_\delta : \mathbb{N}^* \to \mathbb{K}$ satisfies, for any $u \in \mathbb{N}^*$, $\chi_\delta(u) = 1$ if $u \in \mathsf{Cl}_\delta$ and $\chi_\delta(u) = 0$ otherwise.

Examples of quotients of cliff operads

– Example –

When δ is weakly increasing, we have $\bar{\delta} = \delta$, so that $\mathcal{V}_\delta$ is the null space and $\mathbf{Cl}_\delta$ is a suboperad of $\mathbf{I}(\mathbb{N})$.

– Example –

When $\delta := 1232\dots$, we have $\bar{\delta} = 1233\dots$. In $\mathbf{Cl}_\delta$,

$$\mathsf{E}_{002} \circ_3 \mathsf{E}_{10} = \mathsf{E}_{00102} \quad \text{and} \quad \mathsf{E}_{002} \circ_3 \mathsf{E}_{1311} = 0.$$

– Example –

When $\delta := 00233421\dots$, we have $\bar{\delta} = 00233444\dots$. In $\mathbf{Cl}_\delta$,

$$\mathsf{E}_{0011} \circ_4 \mathsf{E}_{002} = \mathsf{E}_{0010021} \quad \text{and} \quad \mathsf{E}_{0011} \circ_5 \mathsf{E}_{002} = 0.$$

Alternative bases

Due to the partial composition on the E-basis, when δ is unimodal, the E-basis is a set-operad of $\mathbf{Cl}_\delta$ iff δ is weakly increasing.

– Objective –

Show that when δ is unimodal, $\mathbf{Cl}_\delta$ admits a set-operad basis so that this operad is a set-operad.

Let the elements F_u, $u \in \mathrm{Cl}_\delta$, of $\mathbf{Cl}_\delta$ defined by

$$\mathsf{F}_u := \sum_{u \preccurlyeq u'} \mu_{\preccurlyeq}(u, u') \mathsf{E}_{u'}.$$

Let also the elements H_u, $u \in \mathrm{Cl}_\delta$, of $\mathbf{Cl}_\delta$ defined by

$$\mathsf{H}_u := \sum_{u' \preccurlyeq u} \mathsf{F}_{u'}.$$

By Möbius inversion and by triangularity, the families $\{\mathsf{F}_u : u \in \mathrm{Cl}_\delta\}$ and $\{\mathsf{H}_u : u \in \mathrm{Cl}_\delta\}$ are bases of $\mathbf{Cl}_\delta$.

F-basis and composition

– Example –

In $\mathbf{Cl}_{224\ldots}$ we have

$$\mathsf{F}_{1221} = \mathsf{E}_{1221} - \mathsf{E}_{1222} - \mathsf{E}_{1231} - \mathsf{E}_{2221} + \mathsf{E}_{1232} + \mathsf{E}_{2222} + \mathsf{E}_{2231} - \mathsf{E}_{2232}.$$

For any $u, v \in \mathrm{Cl}_\delta$, let $u \blacksquare_i v$ be the word obtained from $u \square_i v$ by setting to the maximal value the letters of u and v which are locally maximal.

– Examples –

Let $\delta := 11321\ldots$. We have

- $1022 \blacksquare_3 101 = 10\,301\,21$;
- $1022 \blacksquare_4 003 = 102\,001\,1$.

– Proposition [Combe, G., 2021] –

For any unimodal range map δ,

$$\mathsf{F}_u \circ_i \mathsf{F}_v = \chi_\delta(u \square_i v) \sum_{u \square_i v \preccurlyeq w \preccurlyeq u \blacksquare_i v} \mathsf{F}_w.$$

– Example –

In $\mathbf{Cl}_{123454\ldots}$,

$$\mathsf{F}_{013} \circ_2 \mathsf{F}_{103} = \mathsf{F}_{010313} + \mathsf{F}_{010314} + \mathsf{F}_{010413} + \mathsf{F}_{010414} + \mathsf{F}_{020313} + \mathsf{F}_{020314} + \mathsf{F}_{020413} + \mathsf{F}_{020414}.$$

H-basis and composition

– Examples –

In $\mathbf{Cl}_{3221\dots}$ we have

$$\mathsf{H}_{2101} = \mathsf{F}_{0000} + \mathsf{F}_{0001} + \mathsf{F}_{0101} + \mathsf{F}_{1001} + \mathsf{F}_{1100} + \mathsf{F}_{1101} + \mathsf{F}_{2000} + \mathsf{F}_{2001} + \mathsf{F}_{2100} + \mathsf{F}_{2101}.$$

The δ-reduction of $u \in \mathbb{N}^*$ is the δ-cliff $\mathrm{r}_\delta(u)$ defined by $(\mathrm{r}_\delta(u))(i) := \min\{u(i), \delta(i)\}$ for all i.

– Example –

- $\mathrm{r}_1(\mathbf{212066}) = 012045$
- $\mathrm{r}_2(\mathbf{212066}) = 012066$

– Proposition [Combe, G., 2021] –

For any unimodal range map δ,

$$\mathsf{H}_u \circ_i \mathsf{H}_v = \mathsf{H}_{\mathrm{r}_\delta(u \blacksquare_i v)}.$$

– Examples –

In $\mathbf{Cl}_{22342\dots}$ we have

- $\mathsf{H}_{01} \circ_3 \mathsf{H}_{\mathbf{221}} = \mathsf{H}_{01341}$;
- $\mathsf{H}_{2033} \circ_3 \mathsf{H}_{\mathbf{12}} = \mathsf{H}_{201422}$.

Summary and continuation

From now, we have constructed three operads, fitting in the diagram

$$\mathbf{I}(\mathbb{N}) \longleftarrow\!\!\prec \mathbf{Cl}_{\bar{\delta}} \twoheadrightarrow \mathbf{Cl}_{\delta}$$

where δ is any unimodal range map.

In this case, $\mathbf{Cl}_\delta$ is an operad and admits three bases: the E-basis, F-basis, and the H-basis. This last one is a set-operad basis, showing that $\mathbf{Cl}_\delta$ is a set-operad.

– Question –

Describe a minimal generating set $\mathfrak{G}_\delta$ of $\mathbf{Cl}_\delta$ and a minimal generating set of the space $\mathcal{R}_\delta$ of nontrivial relations of $\mathbf{Cl}_\delta$.

Minimal generating set

A nonempty δ-cliff w is δ-prime if $w = u \,\square_i\, v$ with $u, v \in \mathsf{Cl}_\delta$ implies $(u, v) \in \{(w, \epsilon), (\epsilon, w)\}$.

We denote by $\mathcal{P}_\delta$ the set of all δ-prime δ-cliffs.

– Examples –

Let $\delta := 12233211\dots$. We have

- $10033 \in \mathcal{P}_\delta$;
- $121332 \in \mathcal{P}_\delta$;
- $11 \notin \mathcal{P}_\delta$ since $11 = 1 \,\square_1\, 1$;
- $11222 \notin \mathcal{P}_\delta$ since $11222 = 122 \,\square_2\, 12$.

From the behavior if the composition of $\mathbf{Cl}_\delta$ over the E-basis, $\{\mathsf{E}_u : u \in \mathcal{P}_\delta\}$ is a minimal generating set of $\mathbf{Cl}_\delta$.

– Examples –

- $\mathfrak{G}_{\underline{2}} = \{\mathsf{E}_0, \mathsf{E}_1, \mathsf{E}_2\}$;
- $\mathfrak{G}_{1221\dots} = \{\mathsf{E}_0, \mathsf{E}_1, \mathsf{E}_{02}, \mathsf{E}_{12}, \mathsf{E}_{022}, \mathsf{E}_{122}\}$;
- $\mathfrak{G}_{\mathbf{1}} = \{\mathsf{E}_0, \mathsf{E}_{01}, \mathsf{E}_{002}, \mathsf{E}_{011}, \mathsf{E}_{012}, \mathsf{E}_{0003}, \mathsf{E}_{0013}, \mathsf{E}_{0021}, \mathsf{E}_{0022}, \mathsf{E}_{0023}, \mathsf{E}_{0103}, \mathsf{E}_{0111}, \mathsf{E}_{0112}, \mathsf{E}_{0113}, \mathsf{E}_{0121}, \mathsf{E}_{0122}, \mathsf{E}_{0123}, \dots\}$.

Finite and infinite presentations

A range map δ is 1-dominated if there is a $k \geqslant 1$ such that for all $k' \geqslant k$, $\delta(1) \geqslant \delta(k')$.

– Examples –

The range map

- $\underline{c}$, $c \in \mathbb{N}$, is 1-dominated;
- $23567732\ldots$ is 1-dominated;
- $21\ldots$ is 1-dominated;
- $\mathbf{m}$, $m \geqslant 1$, is not 1-dominated;
- $01\ldots$ is not 1-dominated;
- $2356773\ldots$ is not 1-dominated.

– Theorem [Combe, G., 2021] –

For any unimodal range map δ, $\mathfrak{G}_\delta$ is finite iff δ is 1-dominated.

– Proposition [Combe, G., 2021] –

For any unimodal range map δ, if δ is not 1-dominated, then $\mathcal{R}_\delta$ is not finitely generated.

Operads on m-increasing trees

Immediately from the definition of **m**-cliffs,

$$\dim \mathbf{Cl_m}(n) = \prod_{i \in [n-1]} 1 + (i-1)m.$$

Since $\dim \mathbf{Cl_1} = (n-1)!$, $\mathbf{Cl_1}$ is an operad on permutations with shifted arities.

– Proposition [Combe, G., 2021] **–**

For any $m \geqslant 0$, $\#\mathfrak{G}_\mathbf{m}(1) = 0$, $\#\mathfrak{G}_\mathbf{m}(2) = 1$, and, for any $n \geqslant 3$,

$$\#\mathfrak{G}_\mathbf{m}(n) = (m/(m+1)) \dim \mathbf{Cl_m}(n).$$

The space $\mathcal{R}_\mathbf{m}$ is not finitely generated.

The sequence of the numbers of generators of $\mathcal{R}_\mathbf{1}$ begins with 0, 0, 1, 2, 7, 33, 185, 1211.

The space $\mathcal{R}_\mathbf{1}$ contains nonhomogeneous and nonquadratic nontrivial relations, as *e.g.*,

$$\mathsf{E}_{002} \circ_3 \mathsf{E}_{01} - (\mathsf{E}_0 \circ_2 \mathsf{E}_0) \circ_3 \mathsf{E}_{012}.$$

Quotient operads

– Objective –

Given a subset $\mathcal{S}$ of Cl_δ, describe a general construction for a substructure of the operad $\mathbf{Cl}_\delta$ on $\mathcal{S}$.

Let the subspace $\mathcal{V}_\mathcal{S} := \mathrm{Span}(\{\mathsf{F}_u : u \in \mathsf{Cl}_\delta \setminus \mathcal{S}\})$ of $\mathbf{Cl}_\delta$.

– Proposition [Combe, G., 2021] –

Let δ be a unimodal range map δ and $\mathcal{S}$ be a nonempty graded subset of Cl_δ. If $\mathcal{S}$ is closed by subword reduction, then $\mathcal{V}_\mathcal{S}$ is an operad ideal of $\mathbf{Cl}_\delta$ and $\mathbf{Cl}_\mathcal{S}$ is a quotient operad of $\mathbf{Cl}_\delta$.

The set $\mathcal{S}$ is closed by subword reduction if for any $w \in \mathcal{S}$, all subwords w' of w satisfy $\mathrm{r}_\delta(w') \in \mathcal{S}$.

Alternative bases and composition

– Theorem [Combe, G., 2021] –

Let δ be a unimodal range map and $\mathcal{S}$ be a nonempty graded subset of Cl_δ such that $\mathcal{S}$ is closed by subword reduction. For any $u, v \in \mathcal{S}$ and $i \in [|u|]$,

$$\mathsf{F}_u \circ_i \mathsf{F}_v = \chi_\delta(u \,\square_i\, v) \sum_{\substack{w \in \mathcal{S} \\ u \,\square_i\, v \preccurlyeq w \preccurlyeq u \,\blacksquare_i\, v}} \mathsf{F}_w. \tag{1}$$

Moreover, when for any $n \geqslant 1$, $\mathcal{S}(n)$ is a sublattice of $\mathsf{Cl}_\delta(n)$, if (1) is different from 0, the support of this element is an interval of the poset $\mathcal{S}$.

Let $\theta_\mathcal{S} : \mathbf{Cl}_\delta \to \mathbf{Cl}_\mathcal{S}$ be the canonical projection map satisfying

$$\theta_\mathcal{S}(\mathsf{F}_w) := \begin{cases} \mathsf{F}_w & \text{if } w \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Let the bases $\{\mathsf{E}_u : u \in \mathcal{S}\}$ and $\{\mathsf{H}_u : u \in \mathcal{S}\}$ of $\mathbf{Cl}_\mathcal{S}$ defined by

$$\mathsf{E}_w := \theta_\mathcal{S}(\mathsf{E}_w) = \sum_{w' \in \mathcal{S}, w \preccurlyeq w'} \mathsf{F}_{w'} \qquad \text{and} \qquad \mathsf{H}_w := \theta_\mathcal{S}(\mathsf{H}_w) = \sum_{w' \in \mathcal{S}, w' \preccurlyeq w} \mathsf{F}_{w'}.$$

Operads on m-Dyck paths

For any unimodal range map δ, let $\mathbf{Hi}_\delta := \mathbf{Cl}_{\mathsf{Hi}_\delta}$. Since Hi_δ is closed by subword reduction, $\mathbf{Hi}_\delta$ is a well-defined operad.

Since $\mathsf{Hi}_{\mathbf{m}}(n)$ is in one-to-one correspondence with $m+1$-ary trees with $n-1$ internal nodes,

$$\dim \mathbf{Hi}_{\mathbf{m}}(n) = \mathrm{cat}_m(n-1) \qquad \text{where} \qquad \mathrm{cat}_m(n) := \frac{1}{mn+1}\binom{mn+n}{n}$$

is the n-th m-Fuss-Catalan number.

– Proposition [Combe, G., 2021] –

If $n \geqslant 2$, then $\#\mathfrak{G}_{\mathsf{Hi}_1}(n) = \mathrm{cat}_1(n-2)$.

The sequence of the numbers of generators of $\mathfrak{G}_{\mathsf{Hi}_2}$ begins with $0, 1, 2, 7, 29, 133, 654, 3383, 18179$.

The sequence of the cardinalities of $\mathcal{R}_{\mathsf{Hi}_1}$ begins by $0, 0, 1, 2, 6, 18, 60, 197$.

The space $\mathcal{R}_{\mathsf{Hi}_1}$ contains nonhomogeneous and nonquadratic nontrivial relations, as *e.g.*,

$$(\mathsf{E}_0 \circ_1 \mathsf{E}_0) \circ_1 \mathsf{E}_{01} - \mathsf{E}_{01} \circ_3 \mathsf{E}_{01}.$$

Operads on paths in c-rectangles

Since $\mathsf{Hi}_{\underline{c}}(n)$ is in one to one correspondence with paths from $(0,0)$ to $(n-1, c)$ made of north and east steps,

$$\dim \mathbf{Hi}_{\underline{c}}(n) = \binom{n+c-1}{c}.$$

– Theorem [Combe, G., 2021] –

The operad $\mathbf{Hi}_{\underline{c}}$ admits the following presentation. It is minimally generated by $\mathfrak{G}_{\mathsf{Hi}_{\underline{c}}} = \{\mathsf{E}_0, \dots, \mathsf{E}_c\}$ and the space of nontrivial relations $\mathcal{R}_{\mathsf{Hi}_{\underline{c}}}$ is generated by

$$\mathsf{E}_a \circ_1 \mathsf{E}_b - \mathsf{E}_b \circ_2 \mathsf{E}_{a'}, \qquad 0 \leqslant b \leqslant c, \;\; 0 \leqslant a, a' \leqslant b,$$

$$\mathsf{E}_b \circ_1 \mathsf{E}_a - \mathsf{E}_a \circ_2 \mathsf{E}_b, \qquad 0 \leqslant b \leqslant c, \;\; 0 \leqslant a \leqslant b$$

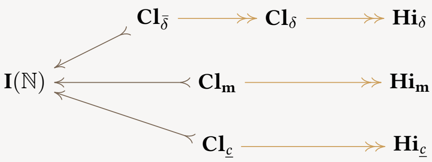
of the free operad generated by $\mathfrak{G}_{\mathsf{Hi}_{\underline{c}}}$.

As a side remarks,

- $\mathbf{Hi}_{\underline{1}}$ is the Koszul dual of the duplicial operad [**Brouder, Frabetti**, 2003];
- $\mathbf{Hi}_{\underline{2}}$ is the Koszul dual of the triplicial operad [**Leroux**, 2011].

Conclusion and open questions

We have introduced a hierarchy of operads on words



where δ is any unimodal range map.

Some of these, like $\mathbf{Cl}_{\mathbf{m}}$ and $\mathbf{Hi}_{\mathbf{m}}$ are very singular objects since they have complicated presentations.

Some questions:

1. provide a sufficient condition for the fact that $\mathcal{R}_{\delta}$ is not finitely generated;
2. describe the presentations of $\mathbf{Cl}_{\mathbf{m}}$ and $\mathbf{Hi}_{\mathbf{m}}$;
3. find morphisms involving the operads $\mathbf{Cl}_{\delta}$ (or its quotients) and other operads arising in algebraic combinatorics (like **Per**, the dendriform operad, the pre-Lie operad, *etc.*).