Graded graphs and operads

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Outline

Graded graphs

Graded graphs of syntax trees

Graded graphs from operads

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Young lattice

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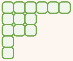
— Example —

$533111 \quad \leftrightarrow$ 

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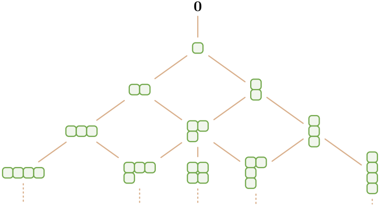

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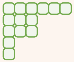
533111 $\leftrightarrow$ 

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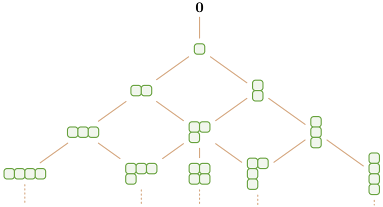

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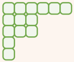
533111 $\leftrightarrow$ 

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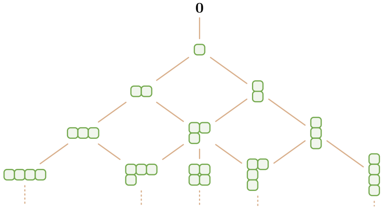

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The path

is in correspondence with the standard Young tableau

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}.$$

Sum of squares formula

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This identity admits a proof [**Stanley**, 1988], [**Sagan**, 2001] based on up and down operators of the Young lattice, defined respectively by

$$\mathrm{U}(\lambda) := \sum_{\substack{\lambda' \\ \lambda \to \lambda'}} \lambda' \qquad \text{and} \qquad \mathrm{U}^\star(\lambda) := \sum_{\substack{\lambda' \\ \lambda' \to \lambda}} \lambda'.$$

These operators are linear maps on the linear span of all integer partitions (over a field $\mathbb{K}$ of characteristic 0) and are adjoint.

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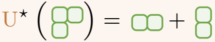

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Proof of the sum of squares formula

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for any $\lambda \vdash n$, we obtain

$$\mathrm{U}^{\star n}\mathrm{U}^n(\mathbf{0}) = \mathrm{U}^{\star n}\left(\sum_{\lambda \vdash n} f_\lambda \lambda\right) = \sum_{\lambda \vdash n} f_\lambda \mathrm{U}^{\star n}(\lambda) = \sum_{\lambda \vdash n} f_\lambda^2\, \mathbf{0}.$$

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Moreover, by induction hypothesis,

$$\begin{aligned}
\mathrm{U}^{\star n}\mathrm{U}^{n}(\mathbf{0}) &= \mathrm{U}^{\star n-1}\mathrm{U}^{\star}\mathrm{U}^{n}(\mathbf{0}) \\
&= \mathrm{U}^{\star n-1}\left(n\mathrm{U}^{n-1} + \mathrm{U}^{n}\mathrm{U}^{\star}\right)(\mathbf{0}) \\
&= \left(n\mathrm{U}^{\star n-1}\mathrm{U}^{n-1}\right)(\mathbf{0}) + \left(\mathrm{U}^{\star n-1}\mathrm{U}^{n}\mathrm{U}^{\star}\right)(\mathbf{0}) \\
&= \left(n\mathrm{U}^{\star n-1}\mathrm{U}^{n-1}\right)(\mathbf{0}) \\
&= n(n-1)!\,\mathbf{0} \\
&= n!\,\mathbf{0}.
\end{aligned}$$

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For this, we start with a graded set $G := \bigsqcup_{d \geqslant 0} G(d)$ and a linear map

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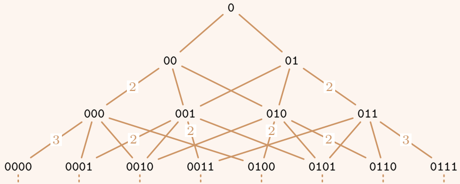
— Example —

Let G be the set of all words on $\{0, 1\}$ starting with 0 and graded by the length.
The map U satisfying

$$\mathrm{U}(u) := \sum_{i \in [|u|]} u_1 \dots u_{i-1} \theta(u_i) u_{i+1} \dots u_{|u|},$$

where θ satisfies $\theta(0) = 00 + 01$ and $\theta(1) = 11 + 10$, defines the graded graph



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— Lemma —

When (G, U) is a multigraph (all weights are in $\mathbb{N}$),

$$(I - \mathrm{U})^{-1}(\mathbf{0}) = \left(\sum_{n \geqslant 0} \mathrm{U}^n\right)(\mathbf{0}) = \sum_{x \in G} \mathrm{h}_{\mathrm{U}}(x)\, x,$$

where $\mathrm{h}_{\mathrm{U}}(x)$ is the number of paths from $\mathbf{0}$ to x.

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Other relations can be considered, like quantum duality [**Lam**, 2010] or filtered duality [**Patrias, Pylyavskyy**, 2018].

Returning paths in graded graphs

— Proposition —

Let $(G, \mathrm{U}, \mathrm{V})$ be a pair of ϕ-diagonal dual graded graphs. For any $n \geqslant 0$,

$$\mathrm{V}^\star \mathrm{U}^n = \mathrm{U}^n \mathrm{V}^\star + \sum_{\substack{k_1, k_2 \geqslant 0 \\ k_1 + k_2 = n-1}} \mathrm{U}^{k_1} \phi \mathrm{U}^{k_2}.$$

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When there is an $r \in \mathbb{K}$ such that, for all $x \in G$, $\phi(x) = rx$, this gives

$$\mathrm{V}^{\star n}\mathrm{U}^{n} = \prod_{j \in [n]} \left(\mathrm{U}\mathrm{V}^{\star} + jrI\right).$$

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A returning path is a pair (p_1, p_2) s.t. p_1 is a path in (G, U) from $\mathbf{0}$ to x and p_2 is a path in $\mathrm{V}^\star$ from x to $\mathbf{0}$. It is to x what a pair of standard Young tableaux of shape λ is to the integer partition λ in the Young lattice.

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— Lemma —

When $(G, \mathrm{U}, \mathrm{V})$ is a connected multigraph, for any $n \geqslant 0$,

$$\mathrm{V}^{\star n} \mathrm{U}^n(\mathbf{0}) = \sum_{x \in G(n)} \mathrm{h}_\mathrm{U}(x) \mathrm{h}_\mathrm{V}(x)\, \mathbf{0}.$$

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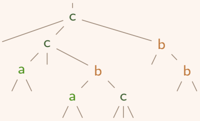
— Example —

Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ such that $\mathfrak{G}(2) = \{\mathsf{a}, \mathsf{b}\}$ and $\mathfrak{G}(3) = \{\mathsf{c}\}$.

Here is a $\mathfrak{G}$-tree having degree 8 and arity 12:



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Compositions of syntax trees

Let $\mathfrak{t}, \mathfrak{s} \in \mathbf{F}(\mathfrak{G})$. For each $i \in [|\mathfrak{t}|]$, the partial composition $\mathfrak{t} \circ_i \mathfrak{s}$ is the tree obtained by grafting the root of $\mathfrak{s}$ onto the ith leaf of $\mathfrak{t}$.

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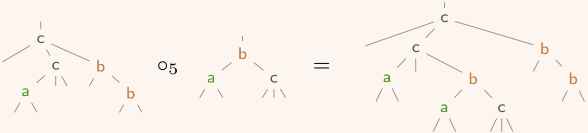

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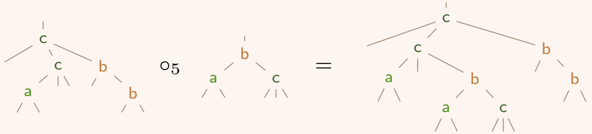

Let $\mathfrak{t}, \mathfrak{s}_1, \dots, \mathfrak{s}_{|\mathfrak{t}|}$ be $\mathfrak{G}$-trees. The full composition $\mathfrak{t} \circ \left[\mathfrak{s}_1, \dots, \mathfrak{s}_{|\mathfrak{t}|}\right]$ is obtained by grafting simultaneously the roots of each $\mathfrak{s}_i$ onto the ith leaf of $\mathfrak{t}$.

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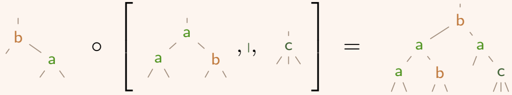

A first graded graph on syntax trees

For any alphabet $\mathfrak{G}$, let $(\mathbf{F}(\mathfrak{G}), \mathrm{U})$ be the graded graph where

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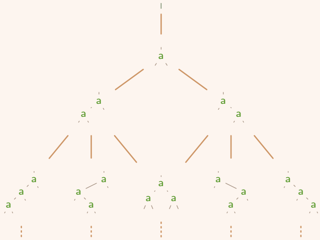
For $\mathfrak{G} = \{\mathsf{a}\}$ with $|\mathsf{a}| = 2$, the graph $(\mathbf{F}(\mathfrak{G}), \mathrm{U})$ is



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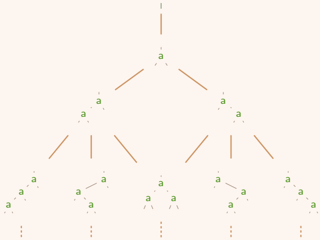
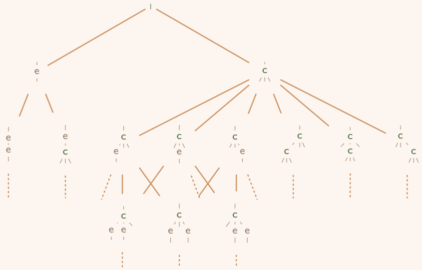


— Example —

For $\mathfrak{G} = \{\mathsf{e}, \mathsf{c}\}$ with $|\mathsf{e}| = 1$ and $|\mathsf{c}| = 3$, the graph $(\mathbf{F}(\mathfrak{G}), \mathrm{U})$ is



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Let $\mathcal{N}_{\max}(\mathfrak{t})$ be the set of all internal nodes of the tree $\mathfrak{t}$ having only leaves as children.

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If $\mathfrak{G} = \{\mathsf{a}\}$ with $|\mathsf{a}| \geqslant 2$, the graded graph $(\mathbf{F}(\mathfrak{G}), \mathrm{U})$ is ϕ-diagonal self-dual for the linear map $\phi : \mathbb{K}\langle \mathbf{F}(\mathfrak{G}) \rangle \to \mathbb{K}\langle \mathbf{F}(\mathfrak{G}) \rangle$ satisfying

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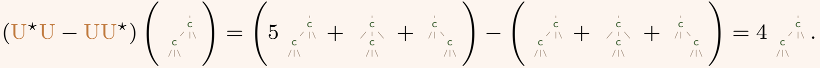

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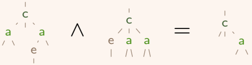

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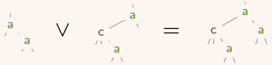

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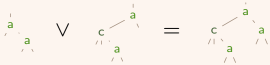

— Proposition —

All $\mathfrak{G}$-prefix posets are meet-semilattices for $\wedge$.

All intervals $[\mathfrak{s}, \mathfrak{t}]$ of these posets are distributive lattices for $\wedge$ and $\vee$.

Prefix poset intervals

Let $\mathfrak{s}$ and $\mathfrak{t}$ be two $\mathfrak{G}$-trees such that $\mathfrak{s} \preceq \mathfrak{t}$. The difference $\mathfrak{t} \setminus \mathfrak{s}$ is the ordered forest $\left(\mathfrak{r}_1, \ldots, \mathfrak{r}_{|\mathfrak{s}|}\right)$ such that the $\mathfrak{r}_i$ are the unique trees satisfying $\mathfrak{t} = \mathfrak{s} \circ \left[\mathfrak{r}_1, \ldots, \mathfrak{r}_{[\mathfrak{s}|}\right]$.

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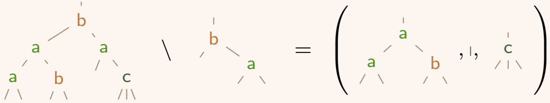

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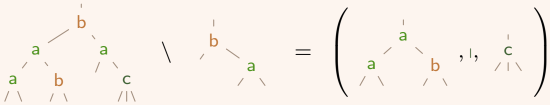

— Proposition —

Any interval $[\mathfrak{s}, \mathfrak{t}]$ is isomorphic as a poset to a Cartesian product of initial intervals. More precisely,

$$[\mathfrak{s}, \mathfrak{t}] \simeq [\shortmid, \mathfrak{r}_1] \times \cdots \times [\shortmid, \mathfrak{r}_{|\mathfrak{s}|}]$$

where $\left(\mathfrak{r}_1, \dots, \mathfrak{r}_{|\mathfrak{s}|}\right)$ is the forest $\mathfrak{t} \setminus \mathfrak{s}$.

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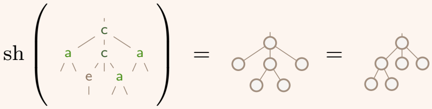

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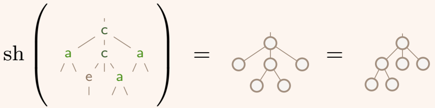

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The intervals $[\mathfrak{s}, \mathfrak{t}]$ and $[\mathfrak{s}', \mathfrak{t}']$ are isomorphic as posets iff

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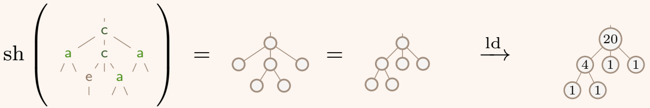

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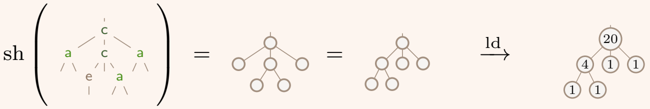

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The generating series $\mathcal{I}_{\mathbf{F}(\mathfrak{G})}(q, t)$ of all the intervals $[\mathfrak{s}, \mathfrak{t}]$ enumerated with respect to $\deg(\mathfrak{s})$ (parameter q) and $\deg(\mathfrak{t})$ (parameter t) satisfies

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Moreover,

$$\mathcal{I}_{\mathbf{F}(\mathfrak{G})}(1,t) = 1 + 2t + 6t^2 + 21t^3 + 80t^4 + 322t^5 + 1348t^6 + \cdots$$

is the generating series of the vertices of the multiplihedra (Sequence **A121988**).

Enumeration of paths

A standard labeling of a tree $\mathfrak{t}$ is a labeled version of $\mathfrak{t}$ where the internal nodes are bijectively labeled on $\{1, \ldots, \deg(\mathfrak{t})\}$ and the label of a node is smaller than the ones of its children.

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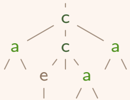
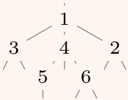
The tree 

admits the standard labeling 

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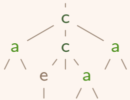
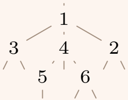
The tree 

admits the standard labeling 

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The hook-length formula for trees [**Knuth**, 2005]

$$\mathrm{h}(\mathfrak{t}) := \frac{\deg(\mathfrak{t})!}{\prod_{u \in \mathcal{N}(\mathfrak{t})} \deg\left(\mathfrak{t}(u)\right)}$$

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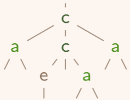
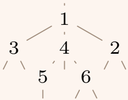
The tree 

admits the standard labeling 

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For any alphabet $\mathfrak{G}$,

$$(I - \mathrm{U})^{-1}(\mid) = \sum_{\mathfrak{t} \in \mathbf{F}(\mathfrak{G})} \mathrm{h}(\mathfrak{t})\, \mathfrak{t}.$$

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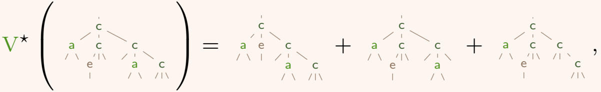

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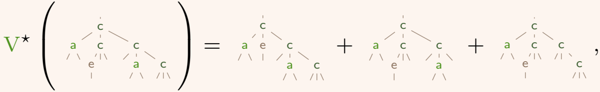

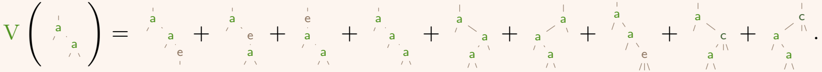

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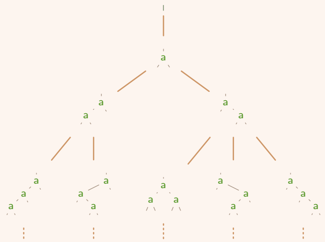

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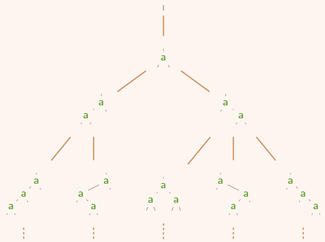

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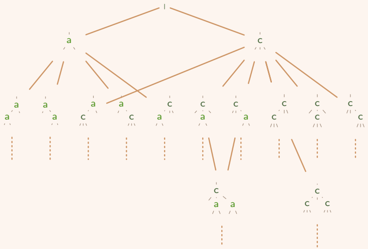
For $\mathfrak{G} = \{a, c\}$ with $|a| = 2$ and $|c| = 3$, the graph $(\mathbf{F}(\mathfrak{G}), \mathrm{V})$ is not a tree.



Dual graded graphs from free operads

For any $\mathfrak{t} \in \mathbf{F}(\mathfrak{G})$, let $\alpha(\mathfrak{t})$ be the number of leaves of $\mathfrak{t}$ that are not in a first subtree of any internal node of $\mathfrak{t}$.

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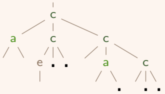

$\overset{\alpha}{\longmapsto} 5$

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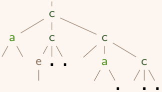
— Example —



$\overset{\alpha}{\longmapsto} 5$

— Theorem [G., 2018] **—**

When $\mathfrak{G}(1) = \emptyset$, $(\mathbf{F}(\mathfrak{G}), \mathrm{U}, \mathrm{V})$ is ϕ-diagonal dual for the linear map satisfying

$$\phi(\mathfrak{t}) := (\#\mathfrak{G})\, \alpha(\mathfrak{t})\; \mathfrak{t}$$

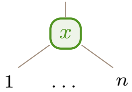
for any $\mathfrak{G}$-tree $\mathfrak{t}$.

Outline

Graded graphs from operads

Abstract operators

An abstract operator is a device



having $n = |x|$ inputs and 1 output.

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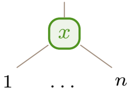
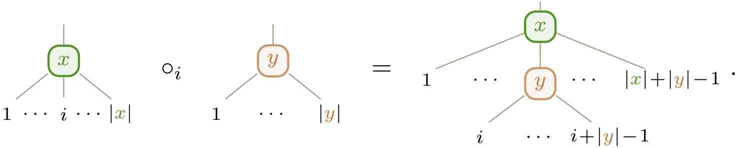


having $n = |x|$ inputs and 1 output.

Such operators can be composed. If x and y are two operators, the composition $x \circ_i y$ is the operator obtained by grafting the output of y onto the ith input of x:



Operads

Operads are algebraic structures formalizing the notion of abstract operators and their composition.

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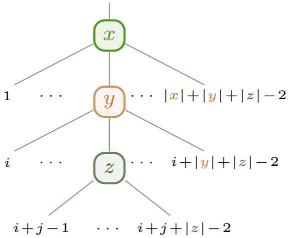
This data has to satisfy some axioms.

Operad axioms

The associativity relation

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$
$$1 \leqslant i \leqslant |x|, 1 \leqslant j \leqslant |y|$$

says that the pictured operator can be constructed from top to bottom or from bottom to top.



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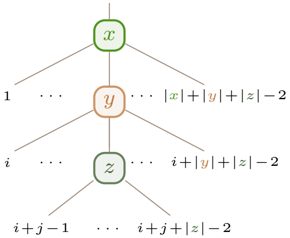

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$$(x \circ_i y) \circ_{j+|y|-1} z = (x \circ_j z) \circ_i y$$
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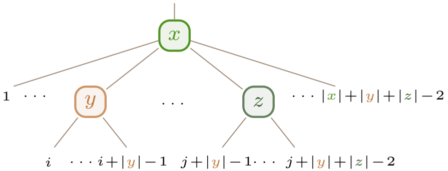

says that the pictured operator can be constructed from left to right or from right to left.

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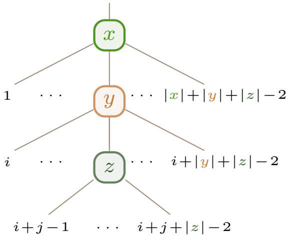

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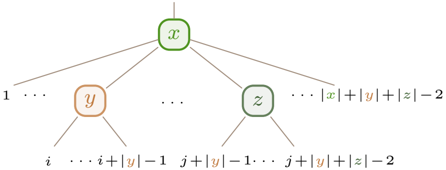

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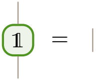
The unitality relation

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}$$
$$1 \leqslant i \leqslant |x|$$



says that $\mathbb{1}$ is the identity map.

Some operads

— Example —

The free operad on an alphabet $\mathfrak{G}$ is the set $\mathbf{F}(\mathfrak{G})$ of all $\mathfrak{G}$-trees endowed with the partial composition $\circ_i$.

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— Example —

The diassociative operad Dias is the operad wherein for any $n \geqslant 1$

$$\mathsf{Dias}(n) := \{1^{\alpha_1} 0 1^{\alpha_2} : \alpha_1 + 1 + \alpha_2 = n\},$$

and

$$1^{\alpha_1}\, 0\, 1^{\alpha_2} \circ_i 1^{\beta_1}\, 0\, 1^{\beta_2} := \begin{cases} 1^{\beta_1+\beta_2}\, 1^{\alpha_1} 0\, 1^{\alpha_2} & \text{if } i \leqslant \alpha_1, \\ 1^{\alpha_1}\, 1^{\beta_1}\, 0\, 1^{\beta_2}\, 1^{\alpha_2} & \text{if } i = \alpha_1 + 1, \\ 1^{\alpha_1}\, 0\, 1^{\alpha_2}\, 1^{\beta_1+\beta_2} & \text{otherwise.} \end{cases}$$

An operad on Motzkin paths

Let Motz be an operad wherein:

- $\mathsf{Motz}(n)$ is the set of all Motzkin paths with n points.

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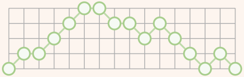

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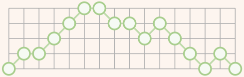 is a Motzkin path of arity 16.

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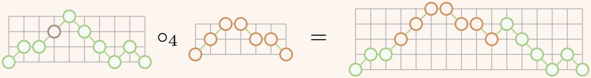

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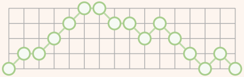

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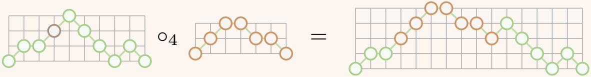

- The unit is ○.

Presentation of Motz

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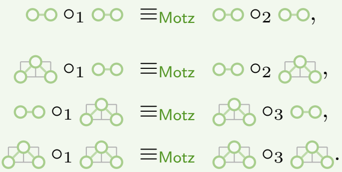
— Proposition [G., 2015] —

The operad Motz admits the presentation $(\mathfrak{G}_{\mathsf{Motz}}, \equiv_{\mathsf{Motz}})$ where

and $\equiv$ is the smallest operad congruence satisfying



A first graded graph from operads

An operad $\mathcal{O}$ is homogeneous if $\mathcal{O}(1) = \{\mathbb{1}\}$, all $\mathcal{O}(n)$, $n \geqslant 1$, are finite, and $\mathcal{O}$ admits the presentation $(\mathfrak{G}, \equiv)$ wherein $\mathfrak{G}$ is finite and $\mathfrak{t} \equiv \mathfrak{t}'$ implies $\deg(\mathfrak{t}) = \deg(\mathfrak{t}')$.

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Let $(\mathcal{O}, \mathrm{U})$ be the graded graph where

$$\mathrm{U}(x) := \sum_{\substack{\mathsf{a} \in \mathfrak{G} \\ i \in [|x|]}} x \circ_i \mathsf{a}.$$

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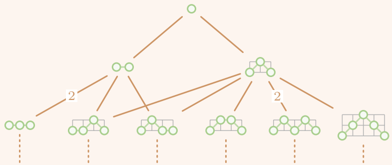

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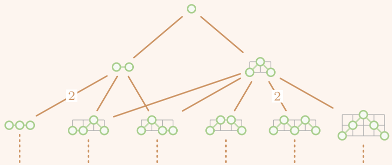

— Example —

The graded graph (As, U) is

A second graded graph from operads

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$$\mathrm{V}(x) := \sum_{\substack{y \in \mathcal{O} \\ \exists (\mathfrak{s}, \mathfrak{t}) \in \mathrm{ev}^{-1}(x) \times \mathrm{ev}^{-1}(y) \\ \langle \mathfrak{t}, \mathrm{V}(\mathfrak{s}) \rangle \neq 0}} y.$$

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— Example —

In (Motz, V),

$\mathrm{V}\left(\right.$  $\left.\right) =$  $+$  $+$ .

Indeed, among others,

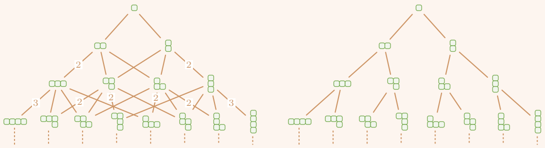
-  admits the factorization  $\circ_1$ ;
-  admits the factorization ( $\circ_1$ ) $\circ_4$ ;
- in $(\mathbf{F}(\mathfrak{G}_{\mathsf{Motz}}), \mathrm{V})$, the tree ( $\circ_1$ ) $\circ_4$  appears in $\mathrm{V}\left(\right.$  $\circ_1$  $\left.\right)$.

Some pairs of graded graphs from operads

— Example —

The pair (Comp, U, V) is 2-dual.

The graded graph (Comp, U) is the composition poset [Bjöner, Stanley, 2005].



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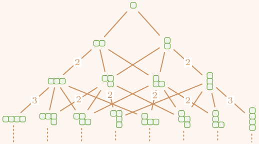

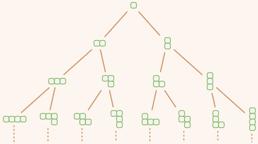

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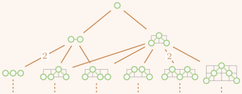
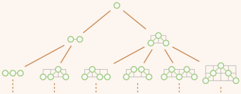
The pair (Motz, U, V) is ϕ-diagonal dual





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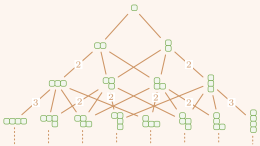
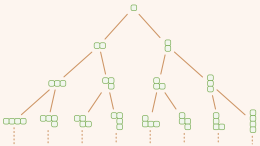

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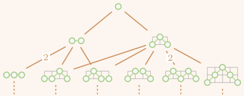
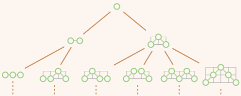
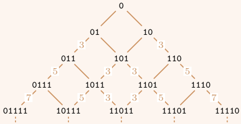
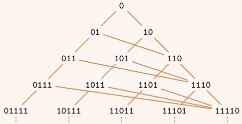



— Example —

The pair (Dias, U, V) is not ϕ-diagonal dual.





Experimental data

— Example: operad As —

Number of elements by degree: $1, 1, 1, 1, 1, 1, \ldots$

The pair $(\mathsf{As}, \mathrm{U}, \mathrm{V})$ is dual.

U-hook series:

$$(I - \mathrm{U})^{-1} (\star_1) = \star_1 + \star_2 + 2 \star_3 + 6 \star_4 + 24 \star_5 + 120 \star_6 + \cdots$$

Number of initial U-paths by length: $1, 1, 2, 6, 24, 120, \ldots$

V-hook series:

$$(I - \mathrm{V})^{-1} (\star_1) = \star_1 + \star_2 + \star_3 + \star_4 + \star_5 + \star_6 + \cdots$$

Number of initial V-paths by length: $1, 1, 1, 1, 1, 1, \ldots$

Number of returning UV-paths by length: $1, 1, 2, 6, 24, 120, \ldots$

Experimental data

— Example: operad Comp —

Number of elements by degree: $1, 2, 4, 8, 16, 32, \ldots$

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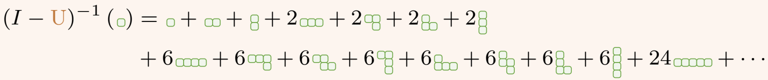
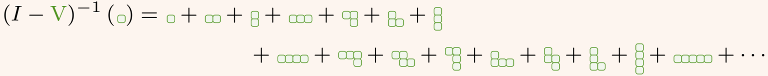


Number of initial U-paths by length: $1, 2, 8, 48, 384, 3840, \ldots$ (Sequence **A000165**)

V-hook series:



Number of initial V-paths by length: $1, 2, 4, 8, 16, 32, \ldots$

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Experimental data

— Example: operad Motz —

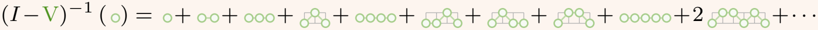
Number of elements by degree: $1, 2, 6, 22, 90, 394, \ldots$ (Sequence **A006318**)

The pair $(\mathsf{Motz}, \mathrm{U}, \mathrm{V})$ is ϕ-diagonal dual.

U-hook series:

Number of initial U-paths by length: $1, 2, 10, 82, 938, 13778, \ldots$ (Sequence **A112487**)

V-hook series:



Number of initial V-paths by length: $1, 2, 6, 26, \ldots$

Number of returning UV-paths by length: $1, 2, 10, 94, 1446, \ldots$

Questions and perspectives

- Necessarily condition on $\mathcal{O}$ for the ϕ-diagonal duality of $(\mathcal{O}, \mathrm{U}, \mathrm{V})$?

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Questions and perspectives

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- Colored versions of the graph $(\mathcal{O}, \mathrm{U})$ to turn it into a tree and use it for (uniform) random generation.