From PROs to combinatorial Hopf algebras

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Contents

Outline

Operads

(Nonsymmetric set-)operad: triple $(\mathcal{O}, \circ_i, \mathbf{1})$ where:

$\mathcal{O}$ is a graded set

$$\mathcal{O} := \biguplus_{n \geqslant 1} \mathcal{O}(n);$$

$\circ_i$ is a composition map

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1), \qquad n, m \geqslant 1, i \in [n];$$

$\mathbf{1}$ is an element of $\mathcal{O}(1)$, called unit.

This data has to satisfy axioms.

Operadic axioms

For all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $z \in \mathcal{O}$,

Associativity:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \qquad i \in [n], j \in [m];$$

Commutativity:

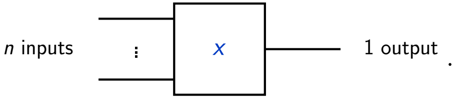
$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \qquad 1 \leqslant i < j \leqslant n;$$

Unitarity:

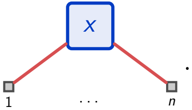
$$\mathbf{1} \circ_1 x = x = x \circ_i \mathbf{1}, \qquad i \in [n].$$

Trees and elements of operads

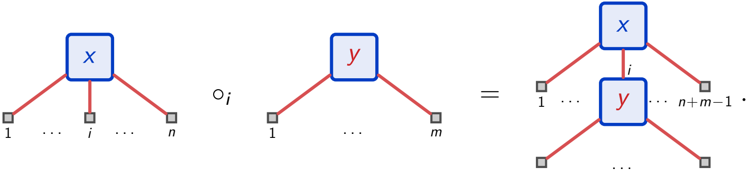
Element x of $\mathcal{O}(n)$ $\rightsquigarrow$ operator of arity n:



Operator x of arity n $\rightsquigarrow$ planar rooted tree with n leaves:



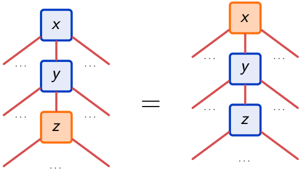
Composition map $\rightsquigarrow$ tree grafting:



Trees and operadic axioms

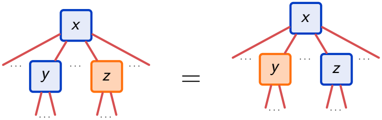
Associativity:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$$



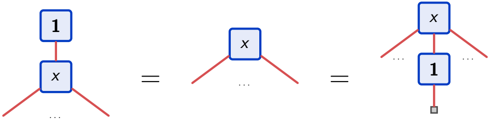
Commutativity:

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y$$



Unitarity:

$$\mathbf{1} \circ_1 x = x = x \circ_i \mathbf{1}$$



The associative operad

Assoc: associative operad.

Elements of **Assoc**(n): one formal symbol α_n.

Composition:

$$\alpha_n \circ_i \alpha_m := \alpha_{n+m-1}.$$

Example

$$\alpha_4 \circ_2 \alpha_3 = \alpha_6, \qquad \alpha_1 \circ_1 \alpha_1 = \alpha_1, \qquad \alpha_4 \circ_4 \alpha_1 = \alpha_4$$

Free operads

$G := \uplus_{n \geqslant 1} G(n)$: a graded set.

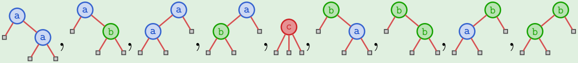
$\mathcal{O}_G$: free operad on G.

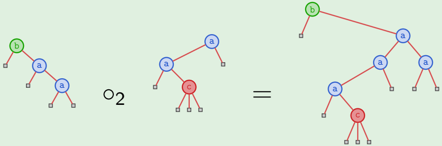
Elements of $\mathcal{O}_G(n)$: G-labeled planar rooted trees with n leaves.

Composition $S \circ_i T$: graft the root of T on the ith leaf of S.

Example

Let $G := G(2) \uplus G(3)$ with $G(2) := \{a, b\}$ and $G(3) := \{c\}$.

$\mathcal{O}_G(3) = \Big\{$  $\Big\}$



Outline

Combinatorial Hopf algebras

Combinatorial Hopf algebra: triple $(\mathcal{H}, \cdot, \Delta)$ where:

$\mathcal{H}$ is a graded $\mathbb{K}$-vector space

$$\mathcal{H} := \bigoplus_{n \geqslant 0} \mathcal{H}_n$$

s.t. $\dim \mathcal{H}_0 = 1$ and the $\mathcal{H}_n$ are finite-dimensional;

$\cdot : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is a graded associative product;

$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a graded coassociative coproduct.

This data has to satisfy some axioms including

$$\Delta(x \cdot y) = \Delta(x)\Delta(y), \qquad x, y \in \mathcal{H}.$$

The shuffle/deconcatenation Hopf algebra

Let $\mathcal{H} := \bigoplus_{n \geqslant 0} \mathrm{Vect}\left(\{\mathtt{a}, \mathtt{b}\}^n\right)$.

Shuffle product $\sqcup\!\sqcup$ on $\mathcal{H}$.

Example

$$\begin{aligned}\mathtt{ab} \sqcup\!\sqcup\, \mathtt{aa} &= \mathtt{abaa} + \mathtt{aaba} + \mathtt{aaab} + \mathtt{aaba} + \mathtt{aaab} + \mathtt{aaab} \\ &= 3\,\mathtt{aaab} + 2\,\mathtt{aaba} + \mathtt{abaa}\end{aligned}$$

Deconcatenation coproduct Δ on $\mathcal{H}$.

Example

$$\Delta(\mathtt{baa}) = \epsilon \otimes \mathtt{baa} + \mathtt{b} \otimes \mathtt{aa} + \mathtt{ba} \otimes \mathtt{a} + \mathtt{baa} \otimes \epsilon$$

Theorem

$(\mathcal{H}, \sqcup\!\sqcup, \Delta)$ is a combinatorial Hopf algebra. [**Malvenuto, Reutenauer**, 1993]

Outline

The natural Hopf algebra of an operad

Operads

Combinatorial Hopf algebras

From operads to Hopf algebras

The natural Hopf algebra of an operad

$\mathcal{O}$: an operad s.t. $\mathcal{O}(1) = \{\mathbf{1}\}$.
Set $\mathcal{O}^+$ as $\mathcal{O} \setminus \mathcal{O}(1)$.

$H(\mathcal{O})$: the natural Hopf algebra of $\mathcal{O}$.

Vector space:

$$H(\mathcal{O}) := \mathrm{Vect}\left(S_M : M \text{ finite multiset of elements of } \mathcal{O}^+\right).$$

Product:

$$S_{M_1} \cdot S_{M_2} := S_{M_1 \cup M_2}.$$

Coproduct: the unique algebra morphism satisfying

$$\Delta\left(S_{\{\{x\}\}}\right) := \sum_{\substack{y, z_1, \dots, z_\ell \in \mathcal{O} \\ y \circ [z_1, \dots, z_\ell] = x}} S_{\{\{y\}\}} \otimes S_{\{\{z_1, \dots, z_\ell\}\}}.$$

Gradation: the degree of $S_{\{\{x\}\}}$ is $n - 1$ if $x \in \mathcal{O}(n)$.

The natural Hopf algebra of an operad

Some properties:

$H(\mathcal{O})$ is commutative but not cocommutative in general;

$H(\mathcal{O})$ is free as a commutative algebra;

Algebraic generators of $H(\mathcal{O})$: $S_{\{\{x\}\}}$, where $x \in \mathcal{O}^+$.

Construction considered in several works as [**van der Laan**, 2004], [**Chapoton, Livernet**, 2007], [**Frabetti**, 2008], [**Méndez, Liendo**, 2013].

The natural Hopf algebra of **Assoc**

Bases of H(**Assoc**): indexed by finite multisets of elements of **Assoc**$^+$.

Indexes encoded by nonincreasing words on $\mathbb{N} \setminus \{0, 1\}$.

Example

$$S_{\{\{\alpha_2,\alpha_2,\alpha_4,\alpha_5\}\}} \longrightarrow S_{5422}$$

Degree	Basis elements of H(**Assoc**)
0	S_ϵ
1	S_2
2	S_3, S_{22}
3	S_4, S_{32}, S_{222}
4	S_5, S_{42}, S_{33}, S_{322}, S_{2222}

Example

$$S_{22} \cdot S_{32} = S_{3222}$$

Example

$$\Delta(S_4) = S_\epsilon \otimes S_4 + S_2 \otimes S_{22} + 2S_2 \otimes S_3 + 3S_3 \otimes S_2 + S_4 \otimes S_\epsilon$$

H(**Assoc**) is the Faà di Bruno Hopf algebra *FdB*.

Outline

PROs

PRO: quadruple $(\mathcal{P}, *, \circ, \mathbf{1}_p)$ where:

$\mathcal{P}$ is a bigraded set

$$\mathcal{P} := \biguplus_{p \geqslant 0} \biguplus_{q \geqslant 0} \mathcal{P}(p, q);$$

$*$ is a horizontal composition map

$$* : \mathcal{P}(p, q) \times \mathcal{P}(p', q') \to \mathcal{P}(p + p', q + q'), \qquad p, p', q, q' \geqslant 0;$$

$\circ$ is a vertical composition map

$$\circ : \mathcal{P}(q, r) \times \mathcal{P}(p, q) \to \mathcal{P}(p, r), \qquad p, q, r \geqslant 0;$$

$\mathbf{1}_p$ is for any $p \geqslant 0$ an element of $\mathcal{P}(p, p)$, called unit of arity p.

This data has to satisfy axioms.

PROs axioms

For all $x, y, z, t \in \mathcal{P}$, when they make sense, the following six relations must be satisfied:

Horizontal associativity:

$$(x * y) * z = x * (y * z);$$

Vertical associativity:

$$(x \circ y) \circ z = x \circ (y \circ z);$$

Interchange relation:

$$(x \circ y) * (z \circ t) = (x * z) \circ (y * t);$$

Unitarity relations:

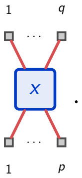
$$\mathbf{1}_p * \mathbf{1}_q = \mathbf{1}_{p+q};$$

$$x * \mathbf{1}_0 = x = \mathbf{1}_0 * x;$$

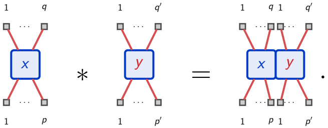
$$x \circ \mathbf{1}_p = x = \mathbf{1}_q \circ x.$$

Operators and elements of PROs

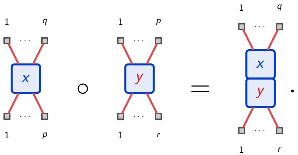
Element x of $\mathcal{P}(p, q)$ $\rightsquigarrow$ operator with p inputs and q outputs:



Horizontal composition:



Vertical composition:



The PRO of maps

Map: PRO of maps.

Elements of $\mathbf{Map}(p, q)$: maps from $[p]$ to $[q]$ (encoded by words).

Horizontal composition: shifted concatenation.

Vertical composition: map composition.

Example

Let $x := 3115 \in \mathbf{Map}(4, 5)$ and $y := 133 \in \mathbf{Map}(3, 9)$. Then,

$$x * y = 3115688.$$

Example

Let $x := 1224244 \in \mathbf{Map}(7, 6)$ and $y := 3312 \in \mathbf{Map}(4, 7)$. Then,

$$x \circ y = 2212.$$

Free PROs

$G := \uplus_{p \geqslant 1} \uplus_{q \geqslant 1} G(p, q)$: a bigraded set.

$\mathcal{P}_G$: free PRO on G.

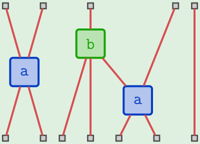
Elements of $\mathcal{P}_G(p, q)$: operators built from formal operators labeled on G with p inputs and q outputs.

Horizontal composition: concatenation of operators.

Vertical composition: composition of operators.

Example

Let $G := G(2, 2) \uplus G(3, 1)$ with $G(2, 2) := \{a\}$ and $G(3, 1) := \{b\}$. Then,



is an element of $\mathcal{P}_G(7, 5)$.

Outline

Maximal decompositions

$\mathcal{P}$: a free PRO.

Maximal decomposition $\mathrm{dec}(x)$ of $x \in \mathcal{P}$: word $(y_1, \ldots, y_\ell)$ s.t.

$$x = y_1 * \cdots * y_\ell$$

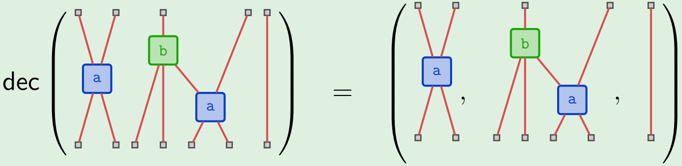
where ℓ is maximal and $y_i \neq \mathbf{1}_0$ for all $i \in [\ell]$.

Since $(\mathcal{P}, *, \mathbf{1}_0)$ is free as a monoid, $\mathrm{dec}(x)$ is unique.

Example

$$\mathrm{dec}(\mathbf{1}_0) = \epsilon, \quad \mathrm{dec}(\mathbf{1}_1) = (\mathbf{1}_1), \quad \mathrm{dec}(\mathbf{1}_2) = (\mathbf{1}_1, \mathbf{1}_1)$$

Example



Reduced elements

$\mathcal{P}$: a free PRO.

$x \in \mathcal{P}$ is reduced if all letters of $\mathrm{dec}(x)$ are different from $\mathbf{1}_1$.

Reduction $\mathrm{red}(x)$ of x:

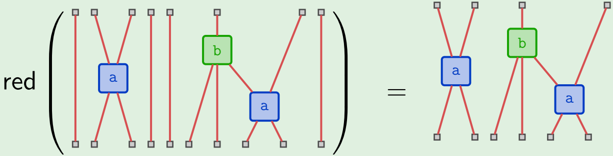
$$\mathrm{red}(x) := z_1 * \cdots * z_k,$$

where $(z_1, \ldots, z_k)$ is the longest subword of $\mathrm{dec}(x)$ s.t. $z_i \neq \mathbf{1}_1$, $i \in [k]$.

Example

$\mathbf{1}_0$ is reduced; $\mathbf{1}_1$ and $\mathbf{1}_2$ are not reduced; $\mathrm{red}(\mathbf{1}_9) = \mathbf{1}_0$.

Example



The bialgebra of a free PRO

$\mathcal{P}$: a free PRO.

Set $\mathrm{red}(\mathcal{P})$ as the set of reduced elements of $\mathcal{P}$.

$\mathrm{H}(\mathcal{P})$: the bialgebra of $\mathcal{P}$.

Vector space:

$$\mathrm{H}(\mathcal{P}) := \mathrm{Vect}\left(\mathbf{S}_x : x \in \mathrm{red}(\mathcal{P})\right).$$

Product:

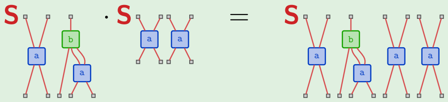
$$\mathbf{S}_x \cdot \mathbf{S}_y := \mathbf{S}_{x*y}.$$

Coproduct:

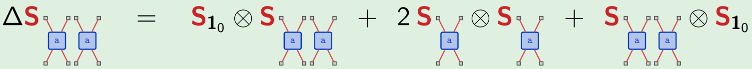
$$\Delta\left(\mathbf{S}_x\right) := \sum_{\substack{y,z\in\mathcal{P}\\ y\circ z=x}} \mathbf{S}_{\mathrm{red}(y)} \otimes \mathbf{S}_{\mathrm{red}(z)}.$$

The bialgebra of a free PRO

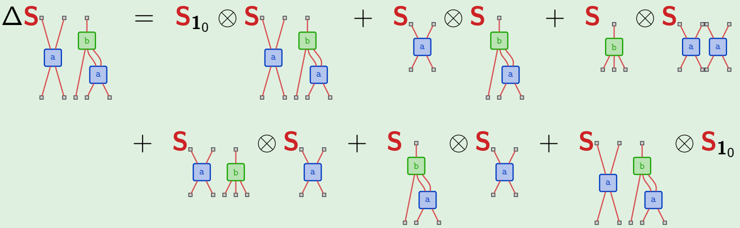
Example



Example



Example



First properties

Theorem

Let $\mathcal{P}$ be a free PRO. Then, $\mathsf{H}(\mathcal{P})$ is a bialgebra.

An element $x \in \mathcal{P}$ is indecomposable if $|\operatorname{dec}(x)| = 1$.

Proposition

Let $\mathcal{P}$ be a free PRO. Then, $\mathsf{H}(\mathcal{P})$ is freely generated as an algebra by the $\mathbf{S}_x$, where the x are indecomposable and reduced elements of $\mathcal{P}$.

In general, $\mathsf{H}(\mathcal{P})$ is neither commutative nor cocommutative.

The coproduct of $\mathsf{H}(\mathcal{P})$ is not multiplicity-free on the $\mathbf{S}$ basis.

Gradation

$\mathcal{P}_G$: free PRO on G.

$w : G \to \mathbb{N}$: map.

w-weight $\omega_w(x)$ of $x \in \mathcal{P}_G$:

$$\omega_w(x) := \sum_{g \in x} w(g).$$

Proposition

Let $\mathcal{P}_G$ be a free PRO on G and $w : G \to \mathbb{N}$ be a map. If the following two conditions are satisfied:

1. for any $g \in G$, $w(g) \geqslant 1$;
2. for any $n \geqslant 1$, the fiber $w^{-1}(n)$ is finite;

then, $\mathsf{H}(\mathcal{P})$ endowed with the gradation

$$\mathsf{H}(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathrm{Vect}\left(\mathbf{S}_x : x \in \mathrm{red}(\mathcal{P}) \text{ and } \omega_w(x) = n\right)$$

is a combinatorial Hopf algebra.

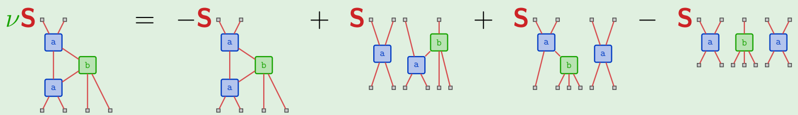
Antipode

Proposition

Let $\mathcal{P}$ be a free PRO. Then, for any reduced x of $\mathcal{P}$ of different from $\mathbf{1}_0$, the antipode ν of $\mathsf{H}(\mathcal{P})$ satisfies

$$\nu(\mathsf{S}_x) = \sum_{\substack{x_1, \dots, x_\ell \in \mathcal{P}, \ell \geqslant 1 \\ x_1 \circ \dots \circ x_\ell = x \\ \mathrm{red}(x_i) \neq \mathbf{1}_0, i \in [\ell]}} (-1)^\ell \, \mathsf{S}_{\mathrm{red}(x_1 * \dots * x_\ell)}.$$

Example



Outline

Congruences of PROs

$\mathcal{P}$: a PRO.

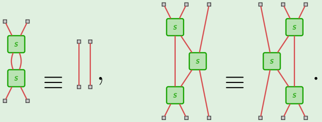
Congruence of PRO: equivalence relation $\equiv$ on $\mathcal{P}$ s.t.

1. $x \equiv x'$ and $x \in \mathcal{P}(p, q)$ imply $x' \in \mathcal{P}(p, q)$;
2. $x \equiv x'$ and $y \equiv y'$ imply $x * y \equiv x' * y'$;
3. $x \equiv x'$ and $y \equiv y'$ and $x \circ y$ well-defined imply $x \circ y \equiv x' \circ y'$.

$\mathcal{P}/_{\equiv}$: quotient of $\mathcal{P}$ by $\equiv$ defined in the usual way.

Example

Let **Per** $:= \mathcal{P}/_{\equiv}$ be the quotient of the free PRO $\mathcal{P}$ on $G := G(2,2) := \{s\}$ by the finest congruence $\equiv$ satisfying



Per is the PRO of permutations and $\equiv$ is a congruence of PROs.

Good congruences of PROs

$\mathcal{P}$: a free PRO.

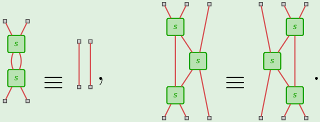
Good congruence of $\mathcal{P}$: congruence $\equiv$ on $\mathcal{P}$ s.t.

1. for any $x \in \text{red}(\mathcal{P})$, all the elements of $[x]_\equiv$ are reduced;
2. for any $x, y \in \text{red}(\mathcal{P})$ s.t. $x \equiv y$, $\text{dec}(x)$ and $\text{dec}(y)$ have the same length ℓ and for any $i \in [\ell]$, $x_i \equiv y_i$.

$\mathcal{Q}$ is a good PRO if it is the quotient of a free PRO by a good congruence.

Example

Per is the quotient of the free PRO on $G := G(2, 2) := \{s\}$ by the finest congruence $\equiv$ satisfying



Per is **not** a good PRO since $\equiv$ does not satisfy 1.

The bialgebra of a good PRO

$\mathcal{P}$: a free PRO.

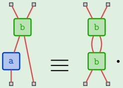
$\equiv$: a good congruence of $\mathcal{P}$.

For any $x \in \mathrm{red}(\mathcal{P})$, set

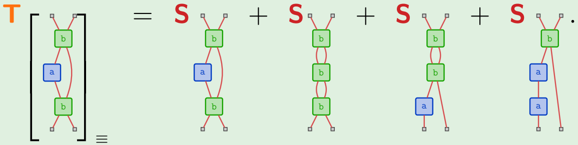
$$\mathbf{T}_{[x]_\equiv} := \sum_{y \in [x]_\equiv} \mathbf{S}_y .$$

Example

Let $\mathcal{P}$ be the quotient of the free PRO on $G := G(1,1) \uplus G(2,2)$ with $G(1,1) := \{\mathtt{a}\}$ and $G(2,2) := \{\mathtt{b}\}$ by the finest congruence $\equiv$ satisfying



One has



The bialgebra of a good PRO

Theorem

Let $\mathcal{P}$ be a free PRO and $\equiv$ be a good congruence of $\mathcal{P}$. Then, the family

$$\left\{\mathbf{T}_{[x]_\equiv} : x \in \mathrm{red}(\mathcal{P})\right\}$$

spans a sub-bialgebra of $\mathrm{H}(\mathcal{P})$, denoted by $\mathrm{H}\left(\mathcal{P}/_\equiv\right)$.

Some easy properties:

$\mathrm{H}\left(\mathcal{P}/_\equiv\right)$ is freely generated as an algebra by the $\mathbf{T}_{[x]_\equiv}$, where the $[x]_\equiv$ are $\equiv$-equivalence classes of indecomposable and reduced elements of $\mathcal{P}$.

If $\mathrm{H}(\mathcal{P})$ is graded, and, for any $\equiv$-equivalence class $[x]_\equiv$ of reduced elements of $\mathcal{P}$, all elements of $[x]_\equiv$ have the same degree, then $\mathrm{H}\left(\mathcal{P}/_\equiv\right)$ is graded.

Outline

The PRO of an operad

$\mathcal{O}$: an operad.

$\mathsf{R}(\mathcal{O})$: the PRO of $\mathcal{O}$ [**Markl**, 2006].

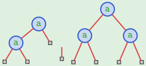
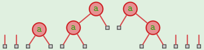
Elements: finite sequences of elements of $\mathcal{O}$.

Horizontal composition: concatenation of sequences.

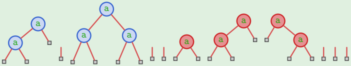
Vertical composition: extension of the composition map of $\mathcal{O}$.

Example

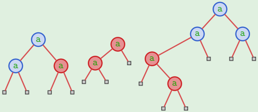
Let $\mathcal{O}$ be the free operad on $G := G(2) := \{\mathtt{a}\}$. Set

$x :=$  and $y :=$ .

Then, one has $x \in \mathsf{R}(\mathcal{O})(8, 3)$, $y \in \mathsf{R}(\mathcal{O})(13, 8)$,

$x * y =$ ,

and

$x \circ y =$ .

The natural Hopf algebra of an operad

Lemma

Let $\mathcal{O}$ be an operad s.t. $\mathcal{O}(1) = \{\mathbf{1}\}$. Then, $\mathsf{R}(\mathcal{O})$ is a good PRO.

Proposition

Let $\mathcal{O}$ be an operad s.t. $\mathcal{O}(1) = \{\mathbf{1}\}$. Then, the bialgebras $H(\mathcal{O})$ and $\mathsf{H}(\mathsf{R}(\mathcal{O}))/_{Com}$ are isomorphic, where *Com* is the vector space generated by

$$\mathbf{T}_x \cdot \mathbf{T}_y - \mathbf{T}_y \cdot \mathbf{T}_x,$$

where x and y are elements of $\mathsf{R}(\mathcal{O})$ with 1 as output arity.

Example

$$FdB = H(\mathbf{Assoc}) = \mathsf{H}(\mathsf{R}(\mathbf{Assoc}))/_{Com}$$

Outline

The PRO of a monoid

$\mathcal{M}$: a monoid.

$\mathrm{B}(\mathcal{M})$: PRO of $\mathcal{M}$.

Elements: finite sequences of elements of $\mathcal{M}$.

Horizontal composition: concatenation of sequences.

Vertical composition:

$$x_1 \dots x_p \circ y_1 \dots y_p := (x_1 \bullet y_1) \dots (x_p \bullet y_p),$$

where $\bullet$ is the product of $\mathcal{M}$.

Example

Let $\mathbb{N}$ be the additive monoid of nonnegative integers. Set

$$x := 002501 \qquad \text{and} \qquad y := 200111.$$

Then, one has $x \in \mathcal{M}(\mathbb{N})(6,6)$, $y \in \mathcal{M}(\mathbb{N})(6,6)$,

$$x * y = 002501200111,$$

and

$$x \circ y = 202612.$$

The Hopf algebra of a monoid

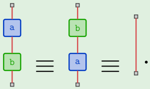
Lemma

Let $\mathcal{M}$ be a monoid that does not contain any nontrivial subgroup. Then, $\mathsf{B}(\mathcal{M})$ is a good PRO.

Example

Let $\mathbb{Z}$ be the additive monoid of integers. $\mathbb{Z}$ admits the presentation $\mathbb{Z} = \langle \mathtt{a}, \mathtt{b} : \mathtt{ab} = \mathtt{ba} = 1 \rangle$.

$\mathsf{B}(\mathbb{Z})$ is the quotient of the free PRO $\mathcal{P}$ on $G := G(1,1) := \{\mathtt{a}, \mathtt{b}\}$ by the finest congruence $\equiv$ satisfying



$\mathbb{Z}$ is a group and $\equiv$ is **not** a good congruence.

Example

$\mathsf{B}(\mathbb{N})$ is a good PRO.

Outline

Examples

- Noncommutative symmetric functions
- Noncommutative Faà di Bruno algebra and its deformations
- Hopf algebras of planar rooted forests
- Hopf algebras of heaps of pieces

The PRO of ladders

Lad: free PRO on $G := G(1,1) := \{\text{a}\}$.

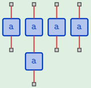
Lemma

The morphism of PROS $\phi : \textbf{Lad} \to \text{B}(\mathbb{N})$ satisfying

$$\phi\left(\quad\right) = 1$$

is an isomorphism.

Example



$$\xrightarrow{\phi} \quad 1211$$

The reduced elements of **Lad** are encoded by words on $\mathbb{N} \setminus \{0\}$.

Gradation: $w(\text{a}) := 1$. Degree of a word: the sum of its letters.

Noncommutative symmetric functions

H(**Lad**) is the Hopf algebra **Sym** of noncommutative symmetric functions over the **S** basis.

Degree	Basis elements of H(**Lad**)
0	$\mathbf{S}_\epsilon$
1	$\mathbf{S}_1$
2	$\mathbf{S}_2$, $\mathbf{S}_{11}$
3	$\mathbf{S}_3$, $\mathbf{S}_{21}$, $\mathbf{S}_{12}$, $\mathbf{S}_{111}$

Example

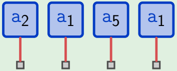
$$\begin{aligned}\Delta\mathbf{S}_{21} &= \mathbf{S}_\epsilon \otimes \mathbf{S}_{21} + \mathbf{S}_1 \otimes \mathbf{S}_2 + \mathbf{S}_1 \otimes \mathbf{S}_{11} \\ &+ \mathbf{S}_{11} \otimes \mathbf{S}_1 + \mathbf{S}_2 \otimes \mathbf{S}_1 + \mathbf{S}_2 \otimes \mathbf{S}_\epsilon\end{aligned}$$

The PRO of positive integers

Pos: free PRO on $G := G(1,0) := \{\mathtt{a}_n : n \geqslant 1\}$.

The reduced elements of **Pos** are encoded by words on $\mathbb{N} \setminus \{0\}$.

Example



$\longrightarrow$ 2151

Gradation: $w(\mathtt{a}_n) := n$. Degree of a word: the sum of its letters.

Noncommutative symmetric functions

H(**Pos**) is **Sym** over the Φ basis.

Degree	Basis elements of H(**Pos**)
0	$\mathbf{S}_\epsilon$
1	$\mathbf{S}_1$
2	$\mathbf{S}_2, \quad \mathbf{S}_{11}$
3	$\mathbf{S}_3, \quad \mathbf{S}_{21}, \quad \mathbf{S}_{12}, \quad \mathbf{S}_{111}$

Example

$$\Delta\mathbf{S}_{21} = \mathbf{S}_\epsilon \otimes \mathbf{S}_{21} + \mathbf{S}_1 \otimes \mathbf{S}_2 + \mathbf{S}_2 \otimes \mathbf{S}_1 + \mathbf{S}_{21} \otimes \mathbf{S}_\epsilon$$

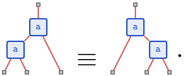
Outline

Examples

- Noncommutative symmetric functions
- Noncommutative Faà di Bruno algebra and its deformations
- Hopf algebras of planar rooted forests
- Hopf algebras of heaps of pieces

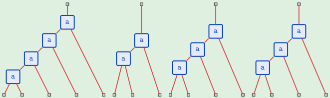
The PRO R(**Assoc**)

The PRO R(**Assoc**) is the quotient of the free PRO on $G := G(2,1) := \{\mathtt{a}\}$ by the finest congruence $\equiv$ satisfying



The reduced elements of R(**Assoc**) are encoded by words on $\mathbb{N} \setminus \{0,1\}$.

Example



$$\longrightarrow \quad \alpha_5\alpha_3\alpha_4\alpha_4 \quad \longrightarrow \quad 5344$$

Gradation: $w(\mathtt{a}) := 1$. Degree of a word: sum of its letters minus its length.

Example

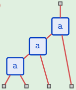
$$\omega_w(5344) = (5+3+4+4) - 4 = 12$$

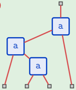
Noncommutative Faà di Bruno Hopf algebra

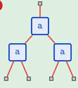
H(R(**Assoc**)) is the noncommutative Faà di Bruno Hopf algebra **FdB**.

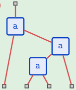
Degree	Basis elements of H(R(**Assoc**))
0	$\mathbf{T}_{\epsilon}$
1	$\mathbf{T}_{2}$
2	$\mathbf{T}_{3}$, $\mathbf{T}_{22}$
3	$\mathbf{T}_{4}$, $\mathbf{T}_{32}$, $\mathbf{T}_{23}$, $\mathbf{T}_{222}$

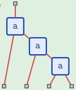
Example

$\mathbf{T}_5 = \mathbf{S}$ 

$+ \mathbf{S}$ 

$+ \mathbf{S}$ 

$+ \mathbf{S}$ 

$+ \mathbf{S}$ 

Example

$$\Delta(\mathbf{T}_5) = \mathbf{T}_{\epsilon} \otimes \mathbf{T}_5 + \mathbf{T}_2 \otimes \mathbf{T}_{23} + \mathbf{T}_2 \otimes \mathbf{T}_{32} + 3\,\mathbf{T}_3 \otimes \mathbf{T}_3 + 3\,\mathbf{T}_3 \otimes \mathbf{T}_{22} + 4\,\mathbf{T}_4 \otimes \mathbf{T}_2 + \mathbf{T}_5 \otimes \mathbf{T}_{\epsilon}$$

The PRO R($\mathbf{Assoc}_\gamma$)

γ: a positive integer.

$\mathbf{Assoc}_\gamma$: suboperad of **Assoc** generated by $\alpha_{\gamma+1}$.

Example

$$\mathbf{Assoc}_1 = \mathbf{Assoc} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \dots\}$$
$$\mathbf{Assoc}_2 = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9, \dots\}$$

The reduced elements of R($\mathbf{Assoc}_\gamma$) are encoded by words on $\{k\gamma + 1 : k \geqslant 1\}$.

Example

In $\mathbf{Assoc}_2$,

$$\alpha_3\alpha_3\alpha_9\alpha_5 \quad \longrightarrow \quad 3395.$$

Gradation: $w(\alpha_\gamma) := 1$. Degree of a letter ℓ: $\frac{\ell-1}{\gamma}$. Degree of a word: sum of the degrees of its letters.

Foissy's deformation of **FdB**

Degree	Basis elements of $\mathsf{H}(\mathsf{R}(\mathbf{Assoc}_2))$
0	$\mathbf{T}_{\epsilon}$
1	$\mathbf{T}_{3}$
2	$\mathbf{T}_{5}$, $\mathbf{T}_{33}$
3	$\mathbf{T}_{7}$, $\mathbf{T}_{53}$, $\mathbf{T}_{35}$, $\mathbf{T}_{333}$

γ-deformation of **FdB** [**Foissy**, 2008]: $\mathbf{FdB}_\gamma$, $\gamma \in \mathbb{R}$.

$\mathbf{FdB}_0$ is the Hopf algebra of noncommutative symmetric functions.

$\mathbf{FdB}_1$ is the noncommutative Faà di Bruno Hopf algebra.

All the $\mathbf{FdB}_\gamma$, $\gamma \in \mathbb{R} \setminus \{0\}$ are isomorphic.

Proposition

For any integer $\gamma \geqslant 1$, the Hopf algebras $\mathsf{H}(\mathsf{R}(\mathbf{Assoc}_\gamma))$ and $\mathbf{FdB}_\gamma$ are isomorphic.

Outline

Examples

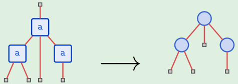
- Noncommutative symmetric functions
- Noncommutative Faà di Bruno algebra and its deformations
- Hopf algebras of planar rooted forests
- Hopf algebras of heaps of pieces

The PRO of planar rooted forests

PRF: free PRO on $G := \uplus_{n \geqslant 1} G(n, 1) := \uplus_{n \geqslant 1} \{\mathtt{a}_n\}$.

The elements of **PRF** are encoded by forests of planar rooted leafy trees.

Example

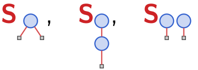


The reduced elements of **PRF** are encoded by planar rooted forests with no empty tree.

Gradation: $w(\mathtt{a}_n) := n$. Degree of a forest: number of edges.

A Hopf algebra on planar rooted forests

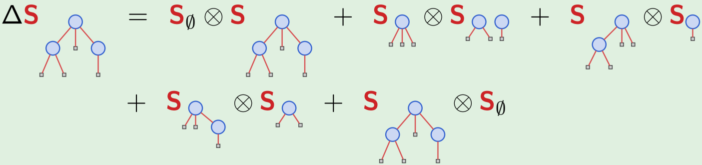
H(**PRF**): Hopf algebra of forests of planar rooted leafy trees.

Degree	Basis elements of H(**PRF**)
0	$\mathbf{S}_{\emptyset}$
1	
2	

First dimensions:

1, 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378.

Example



Outline

Examples

- Noncommutative symmetric functions
- Noncommutative Faà di Bruno algebra and its deformations
- Hopf algebras of planar rooted forests
- Hopf algebras of heaps of pieces

The PRO of heaps of pieces

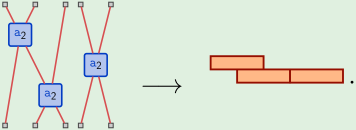
γ: a positive integer.

$\mathbf{Hp}_\gamma$: free PRO on $G := G(\gamma, \gamma) := \{a_\gamma\}$.

The elements of $\mathbf{Hp}_\gamma$ are encoded by heaps of pieces of length γ.

Example

In $\mathbf{Hp}_2$,

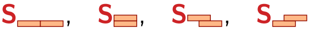


The reduced elements of $\mathbf{Hp}_\gamma$ are encoded by connected heaps of pieces of length γ.

Gradation: $w(a_\gamma) := 1$. Degree of a heap of pieces: number of pieces.

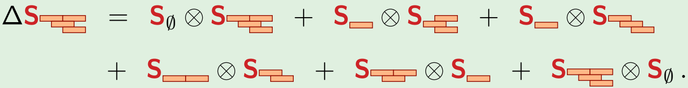
A Hopf algebra of heaps of pieces

$\mathsf{H}(\mathbf{Hp}_{\gamma})$: Hopf algebra of connected heaps of pieces of length γ.

Degree	Basis elements of $\mathsf{H}(\mathbf{Hp}_2)$
0	$\mathbf{S}_{\emptyset}$
1	
2	

Example

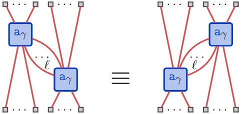
In $\mathsf{H}(\mathbf{Hp}_2)$,



The PRO of friable heaps of pieces

γ: a positive integer.

$\mathbf{FHp}_\gamma$: quotient of $\mathbf{Hp}_\gamma$ by the finest congruence $\equiv$ satisfying



$, \qquad \ell \in [\gamma - 1].$

Gradation: the gradation of $\mathbf{Hp}_\gamma$.

Proposition

The PRO $\mathbf{FHp}_\gamma$ is isomorphic to the sub-PRO of $\mathsf{B}(\mathbb{N})$ generated by 1^γ.

The elements of $\mathbf{FHp}_\gamma$ are encoded by connected heaps of pieces of length 1.

Example

In $\mathbf{FHp}_2$,

$[$  $]_\equiv \quad \leftrightarrow \quad 231 \quad \longrightarrow$  .

A Hopf algebra of friable heaps of pieces

H($\mathbf{FHp}_\gamma$): Hopf algebra of connected heaps of friable pieces of length γ.

Degree	Basis elements of H($\mathbf{FHp}_2$)
0	$\mathbf{T}_\emptyset$
1	**T**
2	**T**, **T**, **T**

Example

In H($\mathbf{FHp}_2$),

T $=$ **S** $+$ **S** $+$ **S** .

Example

In H($\mathbf{FHp}_2$),

Δ**T** $=$ $\mathbf{T}_\emptyset \otimes$ **T** $+$ **T** $\otimes$ **T** $+$ **T** $\otimes$ **T**

$+$ **T** $\otimes$ **T** $+$ **T** $\otimes$ **T** $+$ **T** $\otimes \mathbf{T}_\emptyset$.

Summary

PRO	Hopf algebra
Lad	**Sym** (**S** basis)
Pos	**Sym** (Φ basis)
R(**Assoc**)	**FdB**
R($\mathbf{Assoc}_\gamma$)	$\mathbf{FdB}_\gamma$, $\gamma \in \mathbb{N} \setminus \{0\}$
PRF	Hopf algebra of planar rooted forests
$\mathbf{Hp}_\gamma$	Hopf algebra of heaps of pieces of length γ
$\mathbf{FHp}_\gamma$	Hopf algebra of friable heaps of pieces of length γ