Combinatorial realizations of algebraic structures

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Outline

1. Universal algebra

2. Operads and varieties from monoids

3. Clones and realizations of semigroups

1. Universal algebra

Types of algebraic structures

Combinatorics deals with sets (or spaces) of structured objects:

- monoids;
- groups;
- lattices;
- associative alg.;
- Hopf bialg.;
- Lie alg.;
- pre-Lie alg.;
- dendriform alg.;
- duplicial alg.

Such types of algebras are specified by

1. a collection of operations;
2. a collection of relations between operations.

– Example –

The type of monoids can be specified by

1. the operations $\star$ (binary) and $\mathbb{1}$ (nullary);
2. the relations $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$ and $x \star \mathbb{1} = x = \mathbb{1} \star x$.

Universal algebra

Universal algebra is a formalism to work with such structures.

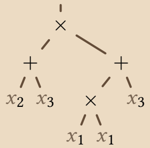
A signature is a graded set $\mathfrak{G} := \bigsqcup_{k \geqslant 0} \mathfrak{G}(k)$ wherein each $\mathbf{a} \in \mathfrak{G}(k)$ is an operation of arity k.

A $\mathfrak{G}$-term is

- either a variable x from the set $\mathbb{X} := \{x_1, x_2, \dots\}$;
- either a pair $(\mathbf{a}, (\mathfrak{t}_1, \dots, \mathfrak{t}_k))$ where $\mathbf{a} \in \mathfrak{G}(k)$ and each $\mathfrak{t}_i$ is a $\mathfrak{G}$-term.

The set of all $\mathfrak{G}$-terms is denoted by $\mathfrak{T}(\mathfrak{G})$.

– Example –



This is the tree representation of the $\mathfrak{G}$-term

$$(\times, ((+, (x_1, x_2)), (+, ((\times, (x_1, x_1)), x_3))))$$

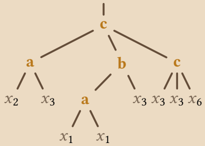
where $\mathfrak{G} := \mathfrak{G}(2) := \{+, \times\}$.

More on terms

Let $\mathfrak{t}$ be a $\mathfrak{G}$-term.

The frontier of $\mathfrak{t}$ is the sequence of all variables appearing in $\mathfrak{t}$.

– Example –



The frontier of this term is $(x_2, x_3, x_1, x_1, x_3, x_3, x_3, x_6)$ and its ground arity is 6.

The ground arity of $\mathfrak{t}$ is the greatest integer n such that x_n is a variable appearing in $\mathfrak{t}$.

The term $\mathfrak{t}$ is

- planar if its frontier is $(x_1, \dots, x_n)$;
- standard if its frontier is a permutation of $(x_1, \dots, x_n)$;
- linear if there are no multiple occurrences of the same variable in the frontier of $\mathfrak{t}$.

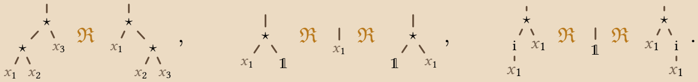
Varieties

A $\mathfrak{G}$-equation is a pair $(\mathfrak{t}, \mathfrak{t}')$ where $\mathfrak{t}$ and $\mathfrak{t}'$ are both $\mathfrak{G}$-terms.

A variety is a pair $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G}$ is a signature and $\mathfrak{R}$ is a set of $\mathfrak{G}$-equations. We denote by $\mathfrak{t} \; \mathfrak{R} \; \mathfrak{t}'$ the fact that $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$.

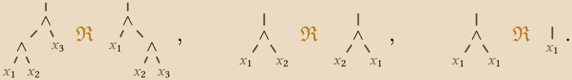
– Example –

The variety of groups is the pair $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(0) \sqcup \mathfrak{G}(1) \sqcup \mathfrak{G}(2)$ with $\mathfrak{G}(0) := \{\mathbb{1}\}$, $\mathfrak{G}(1) := \{\mathtt{i}\}$, and $\mathfrak{G}(2) := \{\star\}$, and $\mathfrak{R}$ is the set of $\mathfrak{G}$-equations satisfying



– Example –

The variety of semilattices is the pair $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\wedge\}$, and $\mathfrak{R}$ is the set of $\mathfrak{G}$-equations satisfying



Algebras of a variety

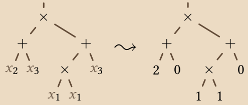
Let $\mathcal{A}$ be a nonempty set. An $\mathcal{A}$-substitution is a map $\sigma : \mathbb{X} \to \mathcal{A}$.
An $\mathcal{A}$-interpretation of a signature $\mathfrak{G}$ is a set

$$\mathfrak{G}_{\mathcal{A}} := \left\{ \mathbf{a}_{\mathcal{A}} : \mathcal{A}^k \to \mathcal{A} : \mathbf{a} \in \mathfrak{G}(k) \text{ for a } k \geqslant 0 \right\}.$$

The evaluation of a $\mathfrak{G}$-term $\mathfrak{t}$ under an $\mathcal{A}$-substitution σ and an $\mathcal{A}$-interpretation $\mathfrak{G}_{\mathcal{A}}$ is defined by induction as

$$\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}) := \begin{cases} \sigma(x) & \text{if } \mathfrak{t} = x \text{ is a variable,} \\ \mathbf{a}_{\mathcal{A}}(\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}_1), \dots, \mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}_k)) & \text{otherwise, where } \mathfrak{t} = (\mathbf{a}, (\mathfrak{t}_1, \dots, \mathfrak{t}_k)). \end{cases}$$

– Example –

$\mathfrak{t} :=$ 

With $\mathcal{A} := \mathbb{N}$, $\mathfrak{G}_{\mathcal{A}}$ defined naturally, and σ satisfying $\sigma(x_1) := 1$, $\sigma(x_2) := 2$, and $\sigma(x_3) := 0$, one obtains $\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}) = 2$.

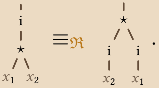
An algebra of a variety $(\mathfrak{G}, \mathfrak{R})$ is a pair $(\mathcal{A}, \mathfrak{G}_{\mathcal{A}})$ where for any $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$ and $\mathcal{A}$-substitution σ, $\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}) = \mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}')$.

Equivalent terms

Two $\mathfrak{G}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathfrak{R}$-equivalent if for all algebras $(\mathcal{A}, \mathfrak{G}_{\mathcal{A}})$ of $(\mathfrak{G}, \mathfrak{R})$ and for all $\mathcal{A}$-substitutions σ, one has $\mathrm{ev}_{\mathcal{A}}^{\sigma}(\mathfrak{t}) = \mathrm{ev}_{\mathcal{A}}^{\sigma}(\mathfrak{t}')$. This property is denoted by $\mathfrak{t} \equiv_{\mathfrak{R}} \mathfrak{t}'$.

– Example –

In the variety of groups,



– Questions –

1. Design an algorithm to decide if two $\mathfrak{G}$-terms are $\equiv_{\mathfrak{R}}$-equivalent. This is known as the word problem.
2. Construct a system of representatives $\mathcal{C}$ of the $\equiv_{\mathfrak{R}}$-equivalence classes. The set $\mathcal{C}$ is a combinatorial realization of the variety.
3. Enumerate the $\equiv_{\mathfrak{R}}$-equivalence classes of (planar/standard/linear) $\mathfrak{G}$-terms w.r.t. their ground arity.

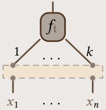
Abstract operations and compositions

To tackle these issues, we need a formalization and an abstraction of the notion of composition of terms in order to consider operations over operations.

An $\mathfrak{G}$-term $\mathfrak{t}$ on the variables $\{x_1, \dots, x_n\}$ is an abstract operation

$$(x_1, \dots, x_n) \mapsto f_{\mathfrak{t}}(x_1, \dots, x_n)$$

depicted as

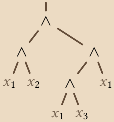


where k is the length of the frontier of $\mathfrak{t}$.

– Example –

For the signature $\mathfrak{G}$ of the variety of semilattices, here is a $\mathfrak{G}$-term seen on the set $\{x_1, \dots, x_4\}$ of variables and the abstract operation it denotes:

$\mathfrak{t} :=$



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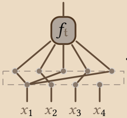

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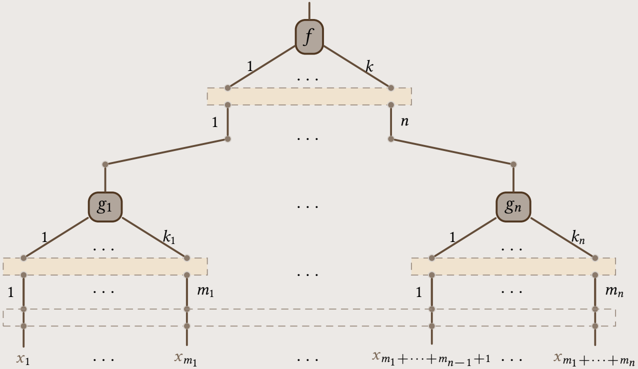
A first paradigm for composition

If f is an abstract operation of arity n and $g_1, \ldots, g_n$ are abstract operations of respective arities $m_1, \ldots, m_n$, then $f \circ [g_1, \ldots, g_n]$ is the abstract operation satisfying

$$(x_1, \ldots, x_{m_1+\cdots+m_n}) \mapsto f\big(g_1(x_1, \ldots, x_{m_1}), \ldots, g_n\big(x_{m_1+\cdots+m_{n-1}+1}, \ldots, x_{m_1+\cdots+m_n}\big)\big).$$

This is the abstract operation depicted as



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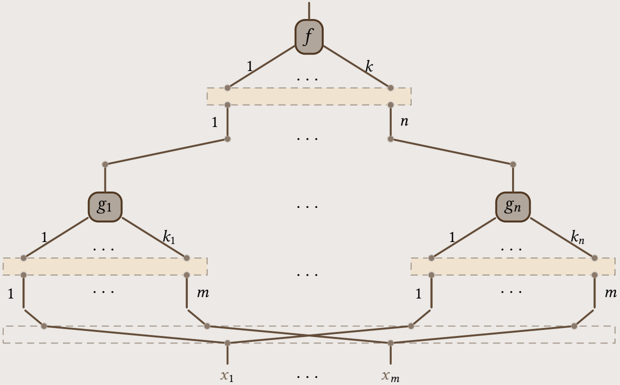
A second paradigm for composition

If f is an abstract operation of arity n and $g_1, \ldots, g_n$ are abstract operations all of arity m, then $f \circledcirc [g_1, \ldots, g_n]$ is the operation satisfying

$$(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

This is the abstract operation depicted as



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Duplicial algebras

A duplicial algebra [**Brouder, Frabetti**, 2003] is a set $\mathcal{A}$ endowed with two binary operations

$$\ll, \gg : \mathcal{A}^2 \to \mathcal{A}$$

satisfying the three relations

$$(x_1 \ll x_2) \ll x_3 = x_1 \ll (x_2 \ll x_3),$$
$$(x_1 \gg x_2) \ll x_3 = x_1 \gg (x_2 \ll x_3),$$
$$(x_1 \gg x_2) \gg x_3 = x_1 \gg (x_2 \gg x_3).$$

– Example –

On $\mathbb{N}^*$, let $\ll$ and $\gg$ be the operations defined by

$$u \ll v := u(v \uparrow_{\max(u)}), \qquad u \gg v := u(v \uparrow_{|u|}).$$

Then, for instance,

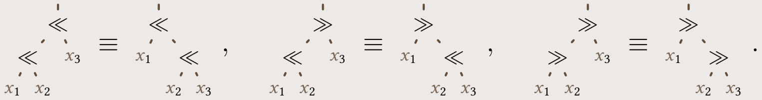
$$0211 \ll 14 = 021136, \qquad 0211 \gg 14 = 021158.$$

This structure is a duplicial algebra [**Novelli, Thibon**, 2013].

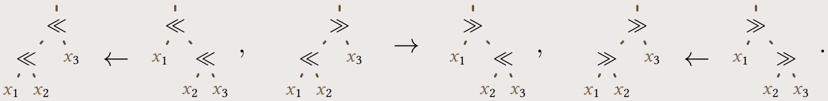
Duplicial operations and equivalence

Let us describe as way to test if two planar duplicial operations are equivalent.

By the duplicial relations, we have



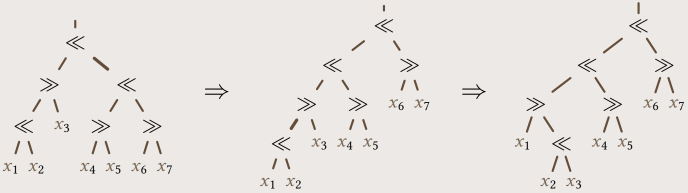
We orient these equations as



in order to obtain a rewrite relation $\Rightarrow$ on the set of all the duplicial operations by performing local moves.

Testing equivalence of duplicial operations

We have for instance the sequence



of rewritings.

– Proposition –

Two planar duplicial operations $\mathfrak{t}$ and $\mathfrak{t}'$ are equivalent iff there is a duplicial operation $\mathfrak{s}$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}$ and $\mathfrak{t}' \overset{*}{\Rightarrow} \mathfrak{s}$.

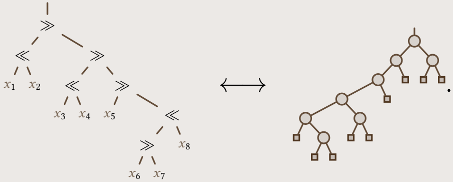
To prove this, we have to establish the fact that $\Rightarrow$ is a terminating and confluent rewrite relation.

Enumerating duplicial operations

– Proposition –

The set of normal forms for $\Rightarrow$ of planar duplicial operations with $n \geqslant 0$ inputs is in one-to-one correspondence with the set of all binary trees with n internal nodes.

A possible bijection puts the following two trees in correspondence:



Therefore, there are

$$\frac{1}{n+1}\binom{2n}{n}$$

pairwise nonequivalent planar duplicial operations with n inputs.

Distributive lattices

A distributive lattice is a set $\mathcal{A}$ endowed with two binary operations

$$\wedge, \vee : \mathcal{A}^2 \to \mathcal{A}$$

satisfying the relations

$$\begin{aligned}
(x_1 \wedge x_2) \wedge x_3 &= x_1 \wedge (x_2 \wedge x_3), & (x_1 \vee x_2) \vee x_3 &= x_1 \vee (x_2 \vee x_3), \\
x_1 \wedge x_2 &= x_2 \wedge x_1, & x_1 \vee x_2 &= x_2 \vee x_1, \\
x_1 \wedge (x_1 \vee x_2) &= x_1, & x_1 \vee (x_1 \wedge x_2) &= x_1, \\
x_1 \vee (x_2 \wedge x_3) &= (x_1 \vee x_2) \wedge (x_1 \vee x_3), & x_1 \wedge (x_2 \vee x_3) &= (x_1 \wedge x_2) \vee (x_1 \wedge x_3).
\end{aligned}$$

– Example –

- On $[n]$, $\vee$ defined as the union and $\wedge$ as the intersection is a finite distributive lattice.
- The set of all Young diagrams is an infinite lattice for the intersection and the union of diagrams.

Combinatorial realization

A normal term is a term $\mathfrak{t}$ expressing as

$$\mathfrak{t} = \mathfrak{s}_1 \vee \ldots \vee \mathfrak{s}_m, \quad m \geqslant 0, \quad \text{where} \quad \mathfrak{s}_i = x_{f_{i,1}} \wedge \ldots \wedge x_{f_{i,k_i}}, \quad k_i \geqslant 1,$$

for any $i, i' \in [k]$, $x_{f_{i,r}} = x_{f_{i,r'}}$ implies $r = r'$, and $\{f_{i,1}, \ldots, f_{i,k_i}\} \subseteq \{f_{i',1}, \ldots, f_{i',k_{i'}}\}$ implies $i = i'$.

– Examples –

$(x_2 \wedge x_3 \wedge x_5) \vee (x_3 \wedge x_7) \vee (x_3 \wedge x_4) \vee x_6$ is a normal term.
$(x_2 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_5)$ is not.

– Proposition –

The set of all sets of sets of positive integers $\{\{f_{1,1}, \ldots, f_{1,k_1}\}, \ldots, \{f_{m,1}, \ldots, f_{m,k_m}\}\}$ satisfying the above properties is a combinatorial realization of the variety of distributive lattices.

Pairwise nonequivalent distributive lattice operations with n inputs are enumerated by the Dedekind numbers whose sequence begins with (only these few terms are known today)

$$1, 2, 5, 19, 167, 7580, 7828353, 2414682040997, 56130437228687557907787.$$

Outline

2. Operads and varieties from monoids

Operads

Nonsymmetric operads provide a formalization of planar operations under the first paradigm for composition.

A nonsymmetric operad is a triple $(\mathcal{O}, \circ, \mathbb{1})$ where

- $\mathcal{O}$ is a graded set
$$\mathcal{O} = \bigsqcup_{n \geqslant 0} \mathcal{O}(n);$$
- $\circ$ is a map
$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n)$$
called full composition map;
- $\mathbb{1}$ is an element of $\mathcal{O}(1)$ called unit.

This data has to satisfy some axioms.

Operad axioms and partial composition maps

The following relations have to be satisfied:

- For all $x \in \mathcal{O}$,

  $$\mathbb{1} \circ [x] = x = x \circ [\mathbb{1}, \ldots, \mathbb{1}].$$

  This says that $\mathbb{1}$ is the identity operation.

- For all $x \in \mathcal{O}(n)$, $y_i \in \mathcal{O}(m_i)$, and $z_{i,j} \in \mathcal{O}$,

  $$(x \circ [y_1, \ldots, y_n]) \circ [z_{1,1}, \ldots, z_{1,m_1}, \ldots, z_{n,1}, \ldots, z_{n,m_n}] = x \circ [y_1 \circ [z_{1,1}, \ldots, z_{1,m_1}], \ldots y_n \circ [z_{n,1}, \ldots, z_{n,m_n}]].$$

  This says that the two ways to compose elements to form an operation having three layers (by starting from top or by starting from bottom) give the same operation.

The partial composition map of $\mathcal{O}$ is the map $\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ where $i \in [n]$ and defined by

$$x \circ_i y := x \circ [\overbrace{\mathbb{1}, \ldots, \mathbb{1}}^{i-1}, y, \overbrace{\mathbb{1}, \ldots, \mathbb{1}}^{n-i}].$$

Free operads

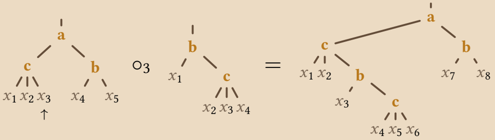
Let $\mathfrak{G}$ be a signature.

The free operad on $\mathfrak{G}$ is the operad $(\mathfrak{P}(\mathfrak{G}), \circ, \mathbb{1})$ where

- $\mathfrak{P}(\mathfrak{G})$ is the set of all planar $\mathfrak{G}$-terms graded by the ground arity;
- $\circ_i$ is defined as follows. The $\mathfrak{G}$-term $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by replacing the variable x_i of $\mathfrak{t}$ by the root of $\mathfrak{s}$, and by setting $(x_1, x_2, \dots)$ for the frontier of the obtained term;
- $\mathbb{1}$ is the $\mathfrak{G}$-term $\underset{x_1}{|}$.

– Example –

By setting $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ where $\mathfrak{G}(2) := \{\mathtt{a}, \mathtt{b}\}$ and $\mathfrak{G}(3) := \{\mathtt{c}\}$, one has

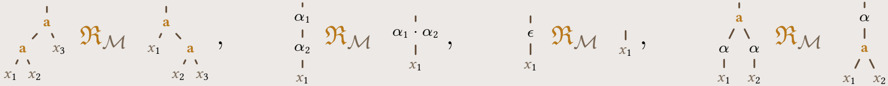


in the free operad $\mathfrak{P}(\mathfrak{G})$.

A variety from a monoid

Let $(\mathcal{M}, \cdot, \epsilon)$ be a monoid.

Let the signature $\mathfrak{G}_{\mathcal{M}} := \mathfrak{G}_{\mathcal{M}}(1) \sqcup \mathfrak{G}_{\mathcal{M}}(2)$ where $\mathfrak{G}_{\mathcal{M}}(1) := \mathcal{M}$ and $\mathfrak{G}_{\mathcal{M}}(2) := \{\mathbf{a}\}$, and let $\mathfrak{R}_{\mathcal{M}}$ be the set of $\mathfrak{G}_{\mathcal{M}}$-equations satisfying



for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

Any algebra of this variety is a semigroup $(\mathcal{A}, \mathbf{a})$ endowed with semigroup endomorphisms $\phi_\alpha : \mathcal{A} \to \mathcal{A}$ with $\alpha \in \mathcal{M}$ and satisfying

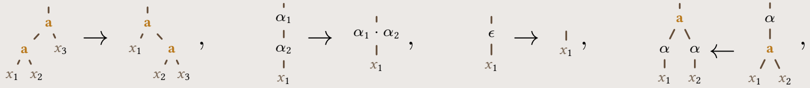
$$\phi_\epsilon(x) = x,$$

$$\phi_{\alpha_1} \circ \phi_{\alpha_2} = \phi_{\alpha_1 \cdot \alpha_2}$$

for any $\alpha_1, \alpha_2 \in \mathcal{M}$ and $x \in \mathcal{M}$.

Orientation of the equations

Let the orientation $\rightarrow$ of $\mathfrak{R}_{\mathcal{M}}$ satisfying

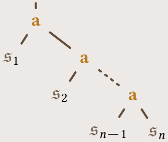


– Proposition –

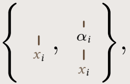
Two planar $\mathfrak{G}_{\mathcal{M}}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are equivalent iff there is a $\mathfrak{G}_{\mathcal{M}}$-term $\mathfrak{s}$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}$ and $\mathfrak{t}' \overset{*}{\Rightarrow} \mathfrak{s}$.

This is a consequence of the fact that $\Rightarrow$ is a convergent rewrite relation.

The set of normal forms for $\Rightarrow$ of planar $\mathfrak{G}_{\mathcal{M}}$-terms is the set of the terms of the form



where $\mathfrak{s}_i \in$



$\alpha_i \in \mathcal{M} \setminus \{\epsilon\}$.

Combinatorial realization

Let $\mathrm{T}\mathcal{M}$ be the set of all words on $\mathcal{M}$, graded by their length, let $\circ_i$ be the partial composition map defined by

$$u \circ_i v := u(1) \dots u(i-1)\ (u(i) \bar{\cdot}\ v)\ u(i+1) \dots u(n),$$

where for any $\alpha \in \mathcal{M}$ and $w \in \mathcal{M}^*$,

$$\alpha \bar{\cdot}\ w := (\alpha \cdot w(1))\ \dots\ (\alpha \cdot w(|w|)),$$

and let $\mathbb{1}$ be ϵ seen as a word of length 1.

– Example –

In $\mathrm{T}(\mathbb{N}, +, 0)$,

$$210\mathbf{0}213 \circ_5 3001 = 2100\,\mathbf{5223}\,13.$$

– Theorem [G., 2015] –

For any monoid $\mathcal{M}$, the triple $(\mathrm{T}\mathcal{M}, \circ, \mathbb{1})$ is an operad.

Moreover, this operad is a combinatorial realization of the variety $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}_{\mathcal{M}})$.

Some suboperads

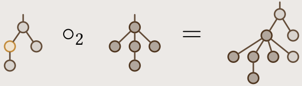
The operads $\mathrm{T}\mathcal{M}$ are large enough to contain a lot of suboperads realizable in combinatorial terms. As main examples:

- For any $m \geqslant 0$, with $\mathcal{M} := (\mathbb{N}, +, 0)$,
  - $\mathbf{PRT}_m$, generated by $\{01, \dots, 0m\}$, on primitive m-Dyck paths;
  - $\mathbf{FCat}_m$, generated by $\{00, 01, \dots, 0m\}$, on m-trees;
  - $\mathbf{Schr}_m$, generated by $\{01, \dots, 0m\} \cup \{00\} \cup \{10, \dots, m0\}$, on some Schröder trees;
  - $\mathbf{Motz}_m$, generated by $\{00, 000, 010, \dots, 0m0\}$, on colored Motzkin paths.

- For any $m \geqslant 0$, with $\mathcal{M} := (\mathbb{Z}/(m+1)\mathbb{Z}, +, 0)$,
  - $\mathbf{Comp}_m$, generated by $\{00, 01, \dots, 0m\}$, on m-words;
  - $\mathbf{DA}_m$, generated by $\{00, 01, \dots, 0(m-1)\}$, on some directed animals.

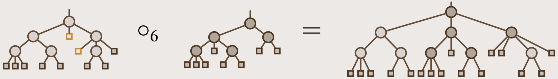
- For any $m \geqslant 0$, $\mathcal{M} := (\mathbb{N}, \max, 0)$,
  - $\mathbf{Dias}_m$, generated by $\{01, \dots, 0m\} \cup \{10, \dots, m0\}$, is the m-pluriassociative operad [**Loday**, 2001] [**G.**, 2016];
  - $\mathbf{Trias}_m$, generated by $\{01, \dots, 0m\} \cup \{00\} \cup \{10, \dots, m0\}$, is the m-pluritriassociative operad [**Loday, Ronco**, 2004] [**G.**, 2016].

Some partial compositions on combinatorial objects

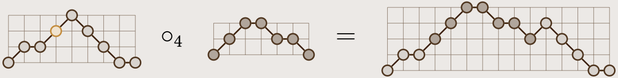


$(\text{in } \mathbf{PRT}_1)$

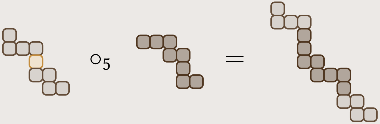
$(\text{in } \mathbf{FCat}_2)$



$(\text{in } \mathbf{Schr}_1)$

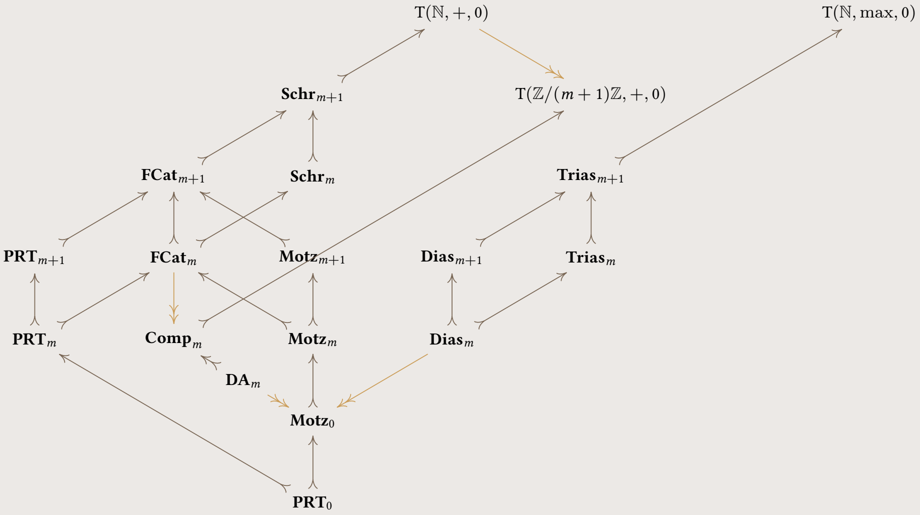


$(\text{in } \mathbf{Motz}_1)$



$(\text{in } \mathbf{Comp}_1)$

Full diagram



3. Clones and realizations of semigroups

Clones

Abstract clones provide a formalization of general operations under the second paradigm for composition.

An abstract clone is a triple $(\mathcal{C}, \circledcirc, \mathbb{1}_{i,n})$ where

- $\mathcal{C}$ is a graded set
$$\mathcal{C} = \bigsqcup_{n \geqslant 0} \mathcal{C}(n);$$
- $\circledcirc$ is a map
$$\circledcirc : \mathcal{C}(n) \times \mathcal{C}(m)^n \to \mathcal{C}(m)$$
called superposition map;
- for each $n \geqslant 0$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is an element of $\mathcal{C}(n)$ called projection.

This data has to satisfy some axioms.

Clone axioms

The following relations have to be satisfied:

- For all $x_i \in \mathcal{C}(m)$,

$$\mathbb{1}_{i,n} \circledcirc [x_1, \dots, x_n] = x_i.$$

  This says that $\mathbb{1}_{i,n}$ is the operation returning its i-th input.

- For all $x \in \mathcal{C}(n)$,

$$x \circledcirc [\mathbb{1}_{1,n}, \dots, \mathbb{1}_{n,n}] = x,$$

  This says that each $\mathbb{1}_{j,n}$, put as j-th input, is an identity operation.

- For all $x \in \mathcal{C}(n)$, $y_i \in \mathcal{C}(m)$, and $z_j \in \mathcal{C}(k)$,

$$(x \circledcirc [y_1, \dots, y_n]) \circledcirc [z_1, \dots, z_m] = x \circledcirc [y_1 \circledcirc [z_1, \dots, z_m], \dots, y_n \circledcirc [z_1, \dots, z_m]].$$

  This says that the two ways to compose elements to form an operation having three layers (by starting from top or by starting from bottom) give the same operation.

Free Clones

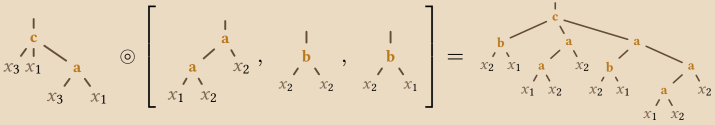
Let $\mathfrak{G}$ be a signature.

The free clone on $\mathfrak{G}$ is the clone $(\mathfrak{T}(\mathfrak{G}), \circledcirc, \mathbb{1}_{i,n})$ where

- $\mathfrak{T}(\mathfrak{G})$ is the set of all $\mathfrak{G}$-terms. Each $\mathfrak{G}$-term $\mathfrak{t}$ is endowed with an integer equal as or greater than its ground arity and called arity;
- $\circledcirc$ is defined as follows. The $\mathfrak{G}$-term $\mathfrak{t} \circledcirc [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ is obtained by replacing each occurrence of a variable x_i of $\mathfrak{t}$ by the root of $\mathfrak{s}_i$ (without any relabeling);
- $\mathbb{1}_{i,n}$ is the term $\overset{|}{x_i}$ of arity n.

– Example –

By setting $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ where $\mathfrak{G}(2) := \{\mathsf{a}, \mathsf{b}\}$ and $\mathfrak{G}(3) := \{\mathsf{c}\}$, one has



in the free clone $\mathfrak{T}(\mathfrak{G})$.

Colored words

Let $(\mathcal{M}, \cdot, \epsilon)$ be a monoid.

Let $\mathrm{W}\mathcal{M}$ be the graded set of all $\mathcal{M}$-colored words defined, for any $n \geqslant 0$, by

$$\mathrm{W}\mathcal{M}(n) := \bigsqcup_{n \geqslant 0} \left\{ \begin{pmatrix} u \\ c \end{pmatrix} : u \in [n]^\ell, c \in \mathcal{M}^\ell, \ell \geqslant 0 \right\}.$$

– Example –

$\begin{pmatrix} 1 & 2 & 1 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is a $\mathbb{Z}/2\mathbb{Z}$-colored word.

Let $\circledcirc$ be the superposition map defined by

$$\begin{pmatrix} u \\ c \end{pmatrix} \circledcirc \left[\begin{pmatrix} v_1 \\ d_1 \end{pmatrix}, \dots, \begin{pmatrix} v_n \\ d_n \end{pmatrix} \right] := \begin{pmatrix} v_{u(1)} & \dots & v_{u(\ell)} \\ (c(1) \cdot d_{u(1)}) & \dots & (c(\ell) \cdot d_{u(\ell)}) \end{pmatrix}.$$

Let finally set $\mathbb{1}_{i,n} := \begin{pmatrix} i \\ \epsilon \end{pmatrix}$.

Clone of colored words

– Example –

In $\mathrm{W}(\mathbb{N}, +, 0)$,

$$\begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \circledcirc \left[\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 1 & 2 & & 1 & 1 & 2 & & 2 & 2 \\ 3 & 0 & 0 & & 4 & 1 & 1 & & 1 & 0 \end{pmatrix}.$$

– Theorem [G., 2020–] –

For any monoid $\mathcal{M}$, $(\mathrm{W}\mathcal{M}, \circledcirc, \mathbb{1}_{i,n})$ is a clone.

The clone $\mathrm{W}\mathcal{M}$ is in fact the clone counterpart of the operad $\mathrm{T}\mathcal{M}$.

This is due to the fact that they have both the same presentation.

Clone of words and congruences

Let us focus on the case where $\mathcal{M}$ is the trivial monoid $\{\epsilon\}$.

Let **Word** := $\mathrm{W}\{\epsilon\}$. We can forget about the colors of the elements of **Word** without any loss of information.

- Let $\equiv_{\mathrm{s}}$ be the equivalence relation on **Word** wherein $u \equiv_{\mathrm{s}} v$ if u and v have both the same sorted version.
- Let $\equiv_{\mathrm{l}}$ (resp. $\equiv_{\mathrm{r}}$) be the equivalence relation on **Word** wherein $u \equiv_{\mathrm{l}} v$ (resp. $u \equiv_{\mathrm{r}} v$) if the versions of u and v obtained by keeping only the leftmost (resp. rightmost) among the multiple occurrences of a same letter are equal.

– Examples –

We have $311322 \equiv_{\mathrm{s}} 131232$, $223111352 \equiv_{\mathrm{l}} 2333315$, $5142144 \equiv_{\mathrm{r}} 552214$.

– Proposition –

The equivalence relations $\equiv_{\mathrm{s}}$, $\equiv_{\mathrm{l}}$, and $\equiv_{\mathrm{r}}$ are clone congruences of **Word**.

Multisets

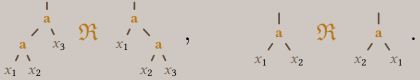
Let **MSet** $:=$ **Word**$/_{\equiv_s}$.

The elements of **MSet** can be seen as multisets of positive integers. By encoding any such multiset $u = \lbag 1^{a(1)}, \dots, n^{a(n)} \rbag$ by the tuple $a = (a(1), \dots, a(n))$, the superposition map of **MSet** expresses as a matrix multiplication

$$a \circledcirc [b_1, \dots, b_n] = \begin{pmatrix} a(1) & \dots & a(n) \end{pmatrix} \begin{pmatrix} b_1(1) & \dots & b_1(m) \\ \vdots & \dots & \vdots \\ b_n(1) & \dots & b_n(m) \end{pmatrix}.$$

– Proposition –

The clone **MSet** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathtt{a}\}$ and $\mathfrak{R}$ satisfies



Therefore, **MSet** is a combinatorial realization of the variety of commutative semigroups.

Arrangements

Let $\mathbf{Arr}_l := \mathbf{Word}/_{\equiv_l}$.

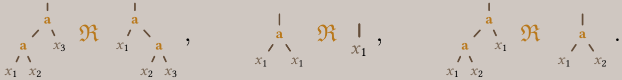
The elements of $\mathbf{Arr}_l(n)$ can be seen as arrangements (words without repetitions) on $[n]$. For any $n \geqslant 0$,

$$\#\mathbf{Arr}_l(n) = \sum_{0 \leqslant k \leqslant n} \frac{n!}{k!}$$

and this sequence starts by 1, 2, 5, 16, 65, 326, 1957, 13700, 109601.

– Proposition –

The clone $\mathbf{Arr}_l$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\mathfrak{R}$ satisfies



The algebra of this variety are left-regular bands, that are idempotent semigroups wherein the operation a satisfies the relation $x_1 \mathsf{a} x_2 \mathsf{a} x_1 = x_1 \mathsf{a} x_2$.

Analogs properties hold for the quotient $\mathbf{Arr}_r := \mathbf{Word}/_{\equiv_r}$, leading to right-regular bands.

Sets

– Lemma –

$\equiv_{\mathrm{s}} \circ \equiv_{\mathrm{l}} = \equiv_{\mathrm{l}} \circ \equiv_{\mathrm{s}}$

Therefore, this composition is a clone congruence of **Word**.

Let us set it as $\equiv_{\mathrm{i}}$ and let **Set** $:=$ **Word**$/_{\equiv_{\mathrm{i}}}$.

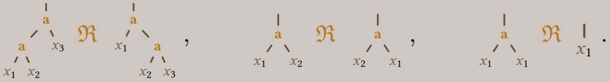
The elements of **Set** can be seen as sets of positive integers. On such objects, the superposition map of **Set** expresses as

$$U \circledcirc [V_1, \dots, V_n] = \bigcup_{j \in U} V_j.$$

Moreover, for any $n \geqslant 0$, $\#\mathbf{Set}(n) = 2^n$.

– Proposition –

The clone **Set** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\mathfrak{R}$ satisfies



Therefore, **Set** is a combinatorial realization of the variety of semilattices.

Arrangements of blocks

Let us consider some intersections involving the congruences $\equiv_{\mathrm{s}}$, $\equiv_{\mathrm{l}}$, and $\equiv_{\mathrm{r}}$.

Let $\equiv_{\mathrm{sl}} := \equiv_{\mathrm{s}} \cap \equiv_{\mathrm{l}}$ and $\mathbf{ArrB}_{\mathrm{l}} := \mathbf{Word}/_{\equiv_{\mathrm{sl}}}$.

The elements of $\mathbf{ArrB}_{\mathrm{l}}(n)$ can be seen as arrangements of possibly empty blocks of repeated letters of $[n]$.

– Examples –

The word 3311115526 is such an element of $\mathbf{ArrB}_{\mathrm{l}}(9)$. The word 222223331112 is not an element of $\mathbf{ArrB}_{\mathrm{l}}$.

– Proposition –

The clone $\mathbf{ArrB}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\mathfrak{R}$ satisfies

$$\mathsf{a}(\mathsf{a}(x_1, x_2), x_3) \;\mathfrak{R}\; \mathsf{a}(x_1, \mathsf{a}(x_2, x_3)), \qquad \mathsf{a}(\mathsf{a}(x_1, x_2), x_1) \;\mathfrak{R}\; \mathsf{a}(\mathsf{a}(x_1, x_1), x_2).$$

Analogs properties hold for the quotient $\mathbf{ArrB}_{\mathrm{r}} := \mathbf{Word}/_{\equiv_{\mathrm{sr}}}$, where $\equiv_{\mathrm{sr}} := \equiv_{\mathrm{s}} \cap \equiv_{\mathrm{r}}$.

Pairs of compatible arrangements

Let $\equiv_{\mathrm{lr}} := \equiv_{\mathrm{l}} \cap \equiv_{\mathrm{r}}$ and $\mathbf{PArr} = \mathbf{Word}/_{\equiv_{\mathrm{lr}}}$.

The elements of $\mathbf{PArr}(n)$ can be seen as pairs (u, v) such that u and v are arrangements on $[n]$, such that j appears in u iff j appears in v.

– Example –

$(3261, 1263)$ is such an element of $\mathbf{PArr}(6)$.

For any $n \geqslant 0$,

$$\#\mathbf{PArr}(n) = \sum_{0 \leqslant k \leqslant n} \frac{n!k!}{(n-k)!}$$

and this sequence starts by 1, 2, 7, 52, 749, 17686, 614227, 29354312, 1844279257.

– Proposition –

The clone **PArr** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\mathfrak{R}$ satisfies

Therefore, **PArr** is a combinatorial realization of the variety of regular bands.

Pairs of compatible arrangements of blocks

Let $\equiv_{\mathrm{slr}} := \equiv_{\mathrm{s}} \cap \equiv_{\mathrm{l}} \cap \equiv_{\mathrm{r}}$ and $\mathbf{PArrB} := \mathbf{Word}/_{\equiv_{\mathrm{slr}}}$.

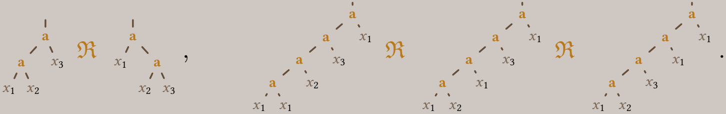
The elements of $\mathbf{PArrB}(n)$ can be seen as pairs (u, v) such that u and v are arrangements of possibly empty blocks of repeated letters on $[n]$, with u and v having the same number of occurrences of any letter.

– Example –

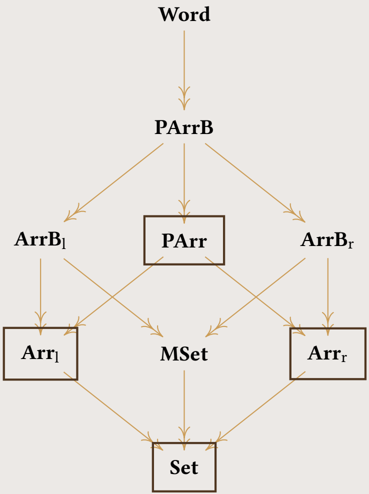
$(3222611, 22211263)$ is such an element of $\mathbf{PArrB}(6)$.

– Proposition –

The clone **PArrB** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\mathsf{a}\}$ and $\mathfrak{R}$ satisfies



Full diagram



Squared clones are combinatorial.