Clone realizations of semigroup varieties

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Outline

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1. Introduction and overview

$\mathcal{M}$-semigroups

– Definition [G., 2015] –

Let $(\mathcal{M}, \cdot, \epsilon)$ be a monoid. An $\mathcal{M}$-**semigroup** is a set S endowed with a binary operation $\star : S \times S \to S$ and unary operations $\theta_\alpha : S \to S$, $\alpha \in \mathcal{M}$, satisfying

$$(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3), \qquad \theta_{\alpha_1}(\theta_{\alpha_2}(x_1)) = \theta_{\alpha_1 \cdot \alpha_2}(x_1),$$

$$\theta_\alpha(x_1 \star x_1) = \theta_\alpha(x_1) \star \theta_\alpha(x_1), \qquad \theta_\epsilon(x_1) = x_1.$$

Such structures (and variations) appear quite often in combinatorics.

– Example –

Let $\mathcal{M} := (\mathbb{N}, \max, 0)$, $S := \mathbb{N}^*$ (the set of sequences of nonnegative integers), $\star$ be the concatenation product, and θ_α be the map sending any word to its subword made of the letters nonsmaller than α. For instance,

$$\theta_2(0015213 \star 41200) = \theta_2(001521341200) = 52342.$$

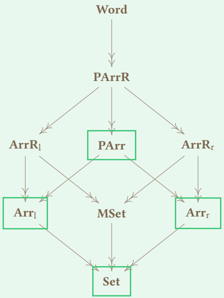
An $\mathcal{M}$-semigroup is hence a semigroup $(S, \star)$ endowed with semigroups endomorphisms θ_α for any $\alpha \in \mathcal{M}$, and such that the map $(x, \alpha) \mapsto \theta_\alpha(x)$ is a monoid action of $\mathcal{M}$ on S.

Overview

– Goals –

- Introduce a clone having, as algebras, $\mathcal{M}$-semigroups and provide a combinatorial realization of it.
- Study this clone and understand what it contains (as quotients or sub-clones).

We will use **terms**, **rewrite systems**, and **clone** theory.



Clone	Combinatorial objects	Realized variety
Word	Monochrome words	Semigroups
MSet	Multisets	Commutative semigroups
Arr$_l$	Arrangements	Left-regular bands
Set	Sets	Semilattices
ArrR$_l$	Arrangements of runs	Ass. and $x_1x_2x_1 = x_1x_1x_2$
PArr	Pairs of compatible arr.	Regular bands
PArrR	Pairs of comp. arr. of runs	Ass. and $x_1x_1x_2x_3x_1 = x_1x_2x_1x_3x_1 = x_1x_2x_3x_1x_1$

2. Varieties, clones, and rewrite systems

Terms

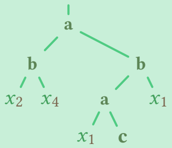
A **signature** is a graded set $\mathfrak{G} := \bigsqcup_{n \geqslant 0} \mathfrak{G}(n)$ wherein each $\mathbf{a} \in \mathfrak{G}(n)$ is an **constant** of arity n.

A $\mathfrak{G}$-**term** is

- either a **variable** x from the set $\mathbb{X}_k := \{x_1, \dots, x_k\}$ for a $k \geqslant 0$;
- either a pair $(\mathbf{a}, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ where $\mathbf{a} \in \mathfrak{G}(n)$ and each $\mathfrak{t}_i$ is a $\mathfrak{G}$-term.

The set of all $\mathfrak{G}$-terms is denoted by $\mathfrak{T}(\mathfrak{G})$.

– Example –



This is the tree representation of the $\mathfrak{G}$-term

$$(\mathbf{a}, ((\mathbf{b}, (x_2, x_4)), (\mathbf{b}, ((\mathbf{a}, (x_1, \mathbf{c})), x_1))))$$

where $\mathfrak{G} := \mathfrak{G}(0) \sqcup \mathfrak{G}(2)$ with $\mathfrak{G}(0) := \{\mathbf{c}\}$ and $\mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$.

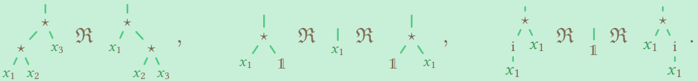
Varieties

A $\mathfrak{G}$-**equation** is a pair $(\mathfrak{t}, \mathfrak{t}')$ where $\mathfrak{t}$ and $\mathfrak{t}'$ are both $\mathfrak{G}$-terms.

A **variety** is a pair $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G}$ is a signature and $\mathfrak{R}$ is a set of $\mathfrak{G}$-equations. We denote by $\mathfrak{t} \, \mathfrak{R} \, \mathfrak{t}'$ the fact that $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$.

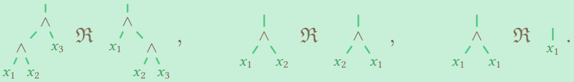
– Example –

The **variety of groups** is the pair $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(0) \sqcup \mathfrak{G}(1) \sqcup \mathfrak{G}(2)$ with $\mathfrak{G}(0) := \{\mathbb{1}\}$, $\mathfrak{G}(1) := \{\mathrm{i}\}$, and $\mathfrak{G}(2) := \{\star\}$, and $\mathfrak{R}$ is the set of $\mathfrak{G}$-equations satisfying



– Example –

The **variety of semilattices** is the pair $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\wedge\}$, and $\mathfrak{R}$ is the set of $\mathfrak{G}$-equations satisfying



Substitutions, interpretations, and evaluations

Let $\mathcal{A}$ be a nonempty set. An $\mathcal{A}$**-substitution** is a map $\sigma : \mathbb{X} \to \mathcal{A}$, where $\mathbb{X} := \{x_1, x_2, \ldots\}$.

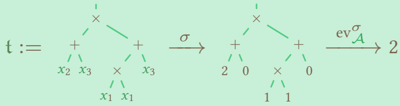
An $\mathcal{A}$**-interpretation** of a signature $\mathfrak{G}$ is a set

$$\mathfrak{G}_{\mathcal{A}} := \left\{ \mathbf{a}_{\mathcal{A}} : \mathcal{A}^k \to \mathcal{A} : \mathbf{a} \in \mathfrak{G}(k), k \geqslant 0 \right\}.$$

The **evaluation** $\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t})$ of a $\mathfrak{G}$-term $\mathfrak{t}$ is the element of $\mathcal{A}$ defined recursively by

$$\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}) := \begin{cases} \sigma(x) & \text{if } \mathfrak{t} = x \text{ is a variable,} \\ \mathbf{a}_{\mathcal{A}}(\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}_1), \ldots, \mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}_k)) & \text{otherwise, where } \mathfrak{t} = (\mathbf{a}, (\mathfrak{t}_1, \ldots, \mathfrak{t}_k)). \end{cases}$$

– Example –



With $\mathcal{A} := \mathbb{N}$, $\mathfrak{G}_{\mathcal{A}}$ defined naturally, and σ satisfying $\sigma(x_1) := 1$, $\sigma(x_2) := 2$, and $\sigma(x_3) := 0$, one obtains $\mathrm{ev}^{\sigma}_{\mathcal{A}}(\mathfrak{t}) = 2$.

Algebras of a variety

An **algebra** of a variety $(\mathfrak{G}, \mathfrak{R})$ is a pair $(\mathcal{A}, \mathfrak{G}_{\mathcal{A}})$ such that, for any $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$ and any $\mathcal{A}$-substitution σ, we have $\mathrm{ev}_{\mathcal{A}}^{\sigma}(\mathfrak{t}) = \mathrm{ev}_{\mathcal{A}}^{\sigma}(\mathfrak{t}')$.

– Example –

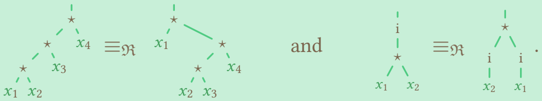
Any algebra of the variety $(\mathfrak{G}, \mathfrak{R})$ of groups is a set $\mathcal{A}$ endowed with three operations $\mathbb{1}$ (nullary), i (unary), and $\star$ (binary), such that, for all $x_1, x_2, x_3 \in \mathcal{A}$,

$$(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3), \qquad x_1 \star \mathbb{1} = x_1 = \mathbb{1} \star x_1, \qquad \mathrm{i}(x_1) \star x_1 = \mathbb{1} = x_1 \star \mathrm{i}(x_1).$$

Two $\mathfrak{G}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are **$\mathfrak{R}$-equivalent** if for all algebras $(\mathcal{A}, \mathfrak{G}_{\mathcal{A}})$ of $(\mathfrak{G}, \mathfrak{R})$ and for all $\mathcal{A}$-substitutions σ, one has $\mathrm{ev}_{\mathcal{A}}^{\sigma}(\mathfrak{t}) = \mathrm{ev}_{\mathcal{A}}^{\sigma}(\mathfrak{t}')$. This property is denoted by $\mathfrak{t} \equiv_{\mathfrak{R}} \mathfrak{t}'$.

– Example –

In the variety of groups,



Clones

Abstract clones [Cohn, 1965] provide a framework to study varieties.

An **abstract clone** is a triple $(\mathcal{C}, \circledcirc, \mathbb{1}_{i,n})$ where

- $\mathcal{C}$ is a graded set $\mathcal{C} = \bigsqcup_{n \geqslant 0} \mathcal{C}(n)$;
- $\circledcirc$ is a map $\circledcirc : \mathcal{C}(n) \times \mathcal{C}(m)^n \to \mathcal{C}(m)$ called **superposition map**;
- for each $n \geqslant 0$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is an element of $\mathcal{C}(n)$ called **projection**.

The following relations have to hold:

- for all $x_i \in \mathcal{C}(m)$,
$$\mathbb{1}_{i,n} \circledcirc [x_1, \ldots, x_n] = x_i;$$
- for all $x \in \mathcal{C}(n)$,
$$x \circledcirc [\mathbb{1}_{1,n}, \ldots, \mathbb{1}_{n,n}] = x;$$
- for all $x \in \mathcal{C}(n)$, $y_i \in \mathcal{C}(m)$, and $z_j \in \mathcal{C}(k)$,
$$(x \circledcirc [y_1, \ldots, y_n]) \circledcirc [z_1, \ldots, z_m] = x \circledcirc [y_1 \circledcirc [z_1, \ldots, z_m], \ldots, y_n \circledcirc [z_1, \ldots, z_m]].$$

Free clones

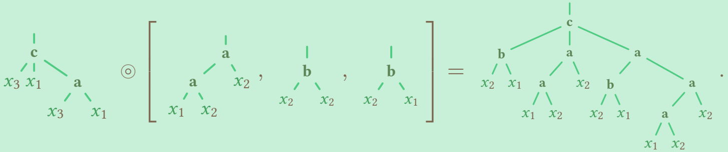
Let $\mathfrak{G}$ be a signature.

The **free clone** on $\mathfrak{G}$ is the clone $(\mathfrak{T}(\mathfrak{G}), \circledcirc, \mathbb{1}_{i,n})$ where

- for any $n \geqslant 0$, $\mathfrak{T}(\mathfrak{G})(n)$ is the set of all $\mathfrak{G}$-terms on $\mathbb{X}_n$;
- $\circledcirc$ is defined as follows. The $\mathfrak{G}$-term $\mathfrak{t} \circledcirc [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ is obtained by replacing each occurrence of a variable x_i of $\mathfrak{t}$ by the root of $\mathfrak{s}_i$;
- $\mathbb{1}_{i,n}$ is the term  of arity n.

– Example –

By setting $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ where $\mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$ and $\mathfrak{G}(3) := \{\mathbf{c}\}$, in the free clone $\mathfrak{T}(\mathfrak{G})$, one has



Clone realizations of varieties

A **clone congruence** of a clone $\mathcal{C}$ is an equivalence relation $\equiv$ on $\mathcal{C}$ compatible with the superposition map, that is, for any $x, x' \in \mathcal{C}(n)$ and $y_1, y'_1, \dots, y_n, y'_n \in \mathcal{C}(m)$, if $x \equiv x'$ and $y_1 \equiv y'_1, \dots, y_n \equiv y'_n$, then

$$x \circledcirc [y_1, \dots, y_n] \equiv x' \circledcirc [y'_1, \dots, y'_n].$$

For any variety $(\mathfrak{G}, \mathfrak{R})$, the $\mathfrak{R}$-equivalence relation $\equiv_{\mathfrak{R}}$ is a clone congruence of $\mathfrak{T}(\mathfrak{G})$.

A **presentation** of a clone $\mathcal{C}$ is a variety $(\mathfrak{G}, \mathfrak{R})$ such that

$$\mathcal{C} \simeq \mathfrak{T}(\mathfrak{G})/_{\equiv_{\mathfrak{R}}}.$$

Conversely, we say in this case that $\mathcal{C}$ is a **clone realization** of $(\mathfrak{G}, \mathfrak{R})$ (see [**Neumann**, 1970]).

An **algebra** on $\mathcal{C}$ is an algebra of the variety $(\mathfrak{G}, \mathfrak{R})$.

Term realizations

Let $(\mathfrak{G}, \mathfrak{R})$ be a variety.

The **term realization** of $(\mathfrak{G}, \mathfrak{R})$ is the clone constructed from an orientation $\to$ of $\mathfrak{R}$ such that

- $\mathcal{C}(n)$ is in one-to-one correspondence with the set of the normal forms on $\mathbb{X}_n$ for $\to$;
- to compute $\mathfrak{t} \circledcirc [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ in $\mathcal{C}$, compute this term in the free clone on $\mathfrak{G}$ and then consider the unique normal form for $\to$ reachable from it;
- the projections of $\mathcal{C}$ are the normal forms reachable from the terms consisting in one leaf.

Term realizations allow us to decide the **word problem**: to decide if two $\mathfrak{G}$-terms are $\equiv_{\mathfrak{R}}$-equivalent, just compare their normal forms [**Baader, Nipkow**, 1998]. This can be undecidable.

In this context, **completion algorithms** are important [**Knuth, Bendix**, 1970].

They can also be used to construct free objects of the category of the algebras of $(\mathfrak{G}, \mathfrak{R})$.

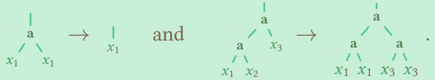
Rewrite systems on terms

A **rewrite relation** on $\mathfrak{T}(\mathfrak{G})$ is a binary relation $\to$ on $\mathfrak{T}(\mathfrak{G})$ such that if $\mathfrak{s} \to \mathfrak{s}'$, then $\mathfrak{s}$ and $\mathfrak{s}'$ are two terms on $\mathbb{X}_n$ for an $n \geqslant 0$.

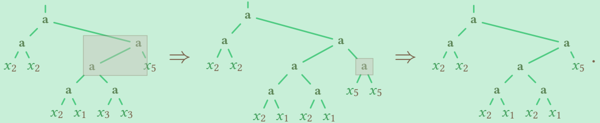
The **context closure** of $\to$ is the binary relation $\Rightarrow$ satisfying $\mathfrak{t} \Rightarrow \mathfrak{t}'$ whenever $\mathfrak{t}'$ is obtained by replacing in $\mathfrak{t}$ a factor $\mathfrak{s}$ by $\mathfrak{s}'$ provided that $\mathfrak{s} \to \mathfrak{s}'$.

– Example –

For $\mathfrak{G} := \mathfrak{G}(2) := \{\mathbf{a}\}$, let the rewrite relation $\to$ defined by



We have



Termination and confluence

Let $\to$ be a rewrite relation on $\mathfrak{T}(\mathfrak{G})$.

Let $(\mathfrak{G}, \mathfrak{R})$ be a variety. A rewrite relation $\to$ on $\mathfrak{T}(\mathfrak{G})$ is an **orientation** of $\mathfrak{R}$ if for any $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$, we have either $\mathfrak{t} \to \mathfrak{t}'$ or $\mathfrak{t}' \to \mathfrak{t}$.

A **normal form** for $\to$ is a $\mathfrak{G}$-term $\mathfrak{t}$ such that there is no $\mathfrak{G}$-term $\mathfrak{t}'$ satisfying $\mathfrak{t} \Rightarrow \mathfrak{t}'$.

When there is no infinite chain $\mathfrak{t}_0 \Rightarrow \mathfrak{t}_1 \Rightarrow \cdots$, the rewrite relation $\to$ is **terminating**.

If $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}_1$ and $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}_2$ implies the existence of $\mathfrak{t}'$ such that $\mathfrak{s}_1 \overset{*}{\Rightarrow} \mathfrak{t}'$ and $\mathfrak{s}_2 \overset{*}{\Rightarrow} \mathfrak{t}'$, then $\to$ is **confluent**.

Some properties:

- For any two $\mathfrak{G}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \equiv_{\mathfrak{R}} \mathfrak{t}'$ iff $\mathfrak{t} \overset{*}{\Leftrightarrow} \mathfrak{t}'$.
- If $\to$ is terminating and confluent, then $\mathfrak{t} \equiv_{\mathfrak{R}} \mathfrak{t}'$ iff there is a normal form $\mathfrak{s}$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}$ and $\mathfrak{t}' \overset{*}{\Rightarrow} \mathfrak{s}$.

Example: duplicial algebras

Let the variety $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\ll, \gg\}$ and $\mathfrak{R}$ is the set $\mathfrak{G}$-equations satisfying

$$(x_1 \ll x_2) \ll x_3 \;\mathfrak{R}\; x_1 \ll (x_2 \ll x_3), \qquad (x_1 \ll x_2) \gg x_3 \;\mathfrak{R}\; x_1 \gg (x_2 \ll x_3), \qquad (x_1 \gg x_2) \gg x_3 \;\mathfrak{R}\; x_1 \gg (x_2 \gg x_3).$$

The algebras of this variety are **duplicial algebras** [**Brouder, Frabetti**, 2003].

– Example –

On $\mathbb{N}^+$ (the set of nonempty sequences of nonnegative integers), let $\ll$ and $\gg$ be the operations defined by

$$u \ll v := u\big(v \uparrow_{\max(u)}\big), \qquad u \gg v := u\big(v \uparrow_{|u|}\big).$$

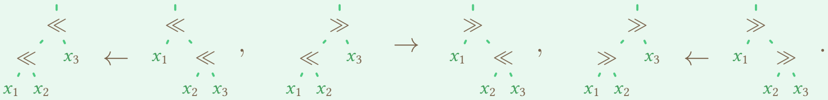
Then, for instance,

$$0211 \ll 14 = 021136, \qquad 0211 \gg 14 = 021158.$$

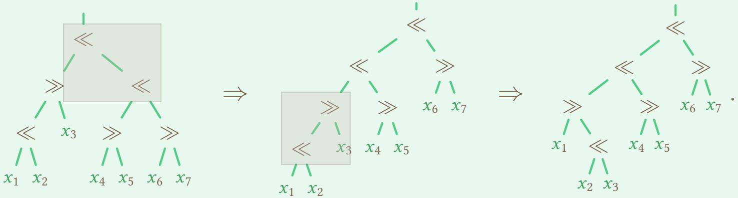
This structure is a duplicial algebra [**Novelli, Thibon**, 2013].

Orientation of duplicial relations

Let the orientation $\to$ of $\mathfrak{R}$ defined by



We have for instance the following sequence of rewritings:



– Proposition [Loday, 2008] **–**

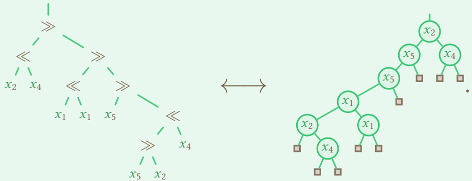
The rewrite relation $\to$ is terminating and confluent.

Encoding duplicial operations

– Proposition [Loday, 2008] **–**

The set of normal forms for $\rightarrow$ with $n \geqslant 0$ inputs is in one-to-one correspondence with the set of all binary trees with n internal nodes where internal nodes are decorated on $\mathbb{X}$.

A possible bijection puts the following two trees in correspondence:



Therefore, there are

$$\frac{1}{n+1}\binom{2n}{n}k^n$$

pairwise nonequivalent duplicial operations with n inputs on variables of $\mathbb{X}_k$.

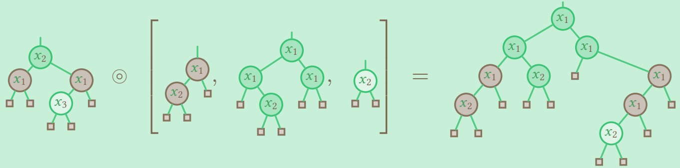
Clone realization

Thanks to previous properties, we obtain a clone realization of the variety of duplicial algebras.

Let **Dup** be the clone such that

- for any $n \geqslant 0$, $\mathbf{Dup}(n)$ is the set of the binary trees where internal nodes are decorated on $\mathbb{X}_n$;
- for any such trees $\mathfrak{t}$ and $\mathfrak{s}_1, \dots, \mathfrak{s}_n$, the superposition $\mathfrak{t} \circledcirc [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ is obtained by replacing in $\mathfrak{t}$ each node u labeled by x_i by $\mathfrak{s}_i$ and by grafting onto the leftmost (resp. rightmost) leaf of $\mathfrak{s}_i$ the left (resp. right) child of u.

– Example –



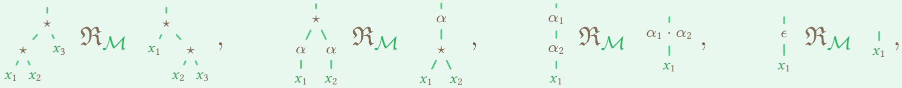
3. Clone of colored words

The variety of $\mathcal{M}$-semigroups

Let $(\mathcal{M}, \cdot, \epsilon)$ be a monoid.

Let the variety $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}_{\mathcal{M}})$ defined by

- $\mathfrak{G}_{\mathcal{M}} := \mathfrak{G}_{\mathcal{M}}(1) \sqcup \mathfrak{G}_{\mathcal{M}}(2)$ where $\mathfrak{G}_{\mathcal{M}}(1) := \mathcal{M}$ and $\mathfrak{G}_{\mathcal{M}}(2) := \{\star\}$;
- $\mathfrak{R}_{\mathcal{M}}$ is the set of $\mathfrak{G}_{\mathcal{M}}$-equations satisfying

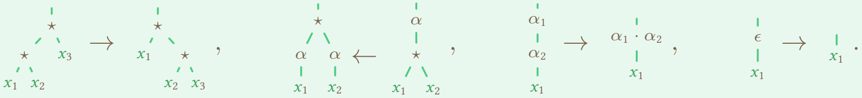


for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

This is the variety of $\mathcal{M}$-semigroups in the sense that any $\mathcal{M}$-semigroup is an algebra of $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}_{\mathcal{M}})$ and any algebra of $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}_{\mathcal{M}})$ is an $\mathcal{M}$-semigroup.

Orientation of the equations

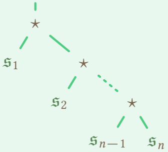
Let the orientation $\to$ of $\mathfrak{R}_{\mathcal{M}}$ satisfying



– Proposition [G., 2020–] –

For any monoid $\mathcal{M}$, the orientation $\to$ of $\mathfrak{R}_{\mathcal{M}}$ is terminating and confluent.

The set of normal forms for $\to$ of planar $\mathfrak{G}_{\mathcal{M}}$-terms is the set of the terms avoiding the left members of $\to$. These are the terms of the form



where $\mathfrak{s}_i = \begin{smallmatrix} | \\ x_{k_i} \end{smallmatrix}$ or $\mathfrak{s}_i = \begin{smallmatrix} | \\ \alpha_{k_i} \\ | \\ x_{k_i} \end{smallmatrix}$, for $\alpha_{k_i} \in \mathcal{M} \setminus \{\epsilon\}$.

Colored words

Let $(\mathcal{M}, \cdot, \epsilon)$ be a monoid.

Let $\mathrm{W}\mathcal{M}$ be the graded set of all $\mathcal{M}$-**colored words** defined for any $n \geqslant 0$ by

$$\mathrm{W}\mathcal{M}(n) := \bigsqcup_{\ell \geqslant 1} \left\{ \binom{u}{c} : (u, c) \in [n]^\ell \times \mathcal{M}^\ell \right\}.$$

– Example –

$\begin{pmatrix} 1 & 2 & 1 & 6 \\ \epsilon & \mathbf{ab} & \mathbf{bab} & \mathbf{b} \end{pmatrix}$ is a $(\{\mathbf{a}, \mathbf{b}\}^*, ., \epsilon)$-colored word of arity 6 (or greater).

Let $\circledcirc$ be the superposition map defined by

$$\binom{u}{c} \circledcirc \left[\binom{v_1}{d_1}, \dots, \binom{v_n}{d_n} \right] := \begin{pmatrix} v_{u(1)} & \dots & v_{u(\ell)} \\ \left(c(1) \bar{\cdot} d_{u(1)}\right) & \dots & \left(c(\ell) \bar{\cdot} d_{u(\ell)}\right) \end{pmatrix}$$

where for any $\alpha \in \mathcal{M}$ and $w \in \mathcal{M}^*$, $\alpha \bar{\cdot} w := (\alpha \cdot w(1)) \ \dots \ (\alpha \cdot w(|w|))$.

Let also $\mathbb{1}_{i,n} := \binom{i}{\epsilon}$.

Clone of colored words

– Example –

In $\mathrm{W}(\{\mathbf{a}, \mathbf{b}\}^*, ., \epsilon)$,

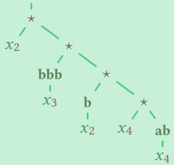
$$\begin{pmatrix} 2 & 2 & 3 \\ \mathbf{ba} & \mathbf{aa} & \epsilon \end{pmatrix} \circledcirc \left[\begin{pmatrix} 2 & 1 \\ \mathbf{b} & \mathbf{aa} \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ \mathbf{bbb} & \epsilon & \mathbf{b} \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ \mathbf{aa} & \mathbf{a} \end{pmatrix} \right] = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\ \mathbf{ba}.\mathbf{bbb} & \mathbf{ba}.\epsilon & \mathbf{ba}.\mathbf{b} & \mathbf{aa}.\mathbf{bbb} & \mathbf{aa}.\epsilon & \mathbf{aa}.\mathbf{b} & \epsilon.\mathbf{aa} & \epsilon.\mathbf{a} \end{pmatrix}.$$

– Theorem [G., 2020–] –

For any monoid $\mathcal{M}$, $(\mathrm{W}\mathcal{M}, \circledcirc, \mathbb{1}_{i,n})$ is a clone and is a clone realization of the variety $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}_{\mathcal{M}})$.

– Example –

Here is a normal for $\to$ of the variety $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{R}_{\mathcal{M}})$ where $\mathcal{M}$ is the monoid $(\{\mathbf{a}, \mathbf{b}\}^*, ., \epsilon)$ and the $\mathcal{M}$-colored word in correspondence:



$$\longleftrightarrow \begin{pmatrix} 2 & 3 & 2 & 4 & 4 \\ \epsilon & \mathbf{bbb} & \mathbf{b} & \epsilon & \mathbf{ab} \end{pmatrix}.$$

Clone of monochrome words

Let us focus on the case where $\mathcal{M}$ is the trivial monoid $\{\epsilon\}$.

Let **Word** $:= \mathrm{W}\{\epsilon\}$. We can forget about the colors of the elements of **Word** without any loss of information.

For any $n \geqslant 0$, **Word**(n) is the set of the nonempty words on the alphabet $[n]$.

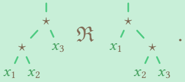
– Example –

In **Word**,

$$311434 \circledcirc [221, 33, 2, 1] \quad = \quad 2\ 221\ 221\ 1\ 2\ 1 \quad = \quad 2221221121$$

– Proposition [G., 2020–] –

The clone **Word** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, **Word** is a clone realization of the variety of semigroups.

Congruences on Word

Let $\equiv_{\mathrm{st}}$ be the equivalence relation on **Word** wherein $u \equiv_{\mathrm{st}} v$ if u and v have both the same **sorted** version.

– Examples –

$47 \equiv_{\mathrm{st}} 74, \quad 311322 \equiv_{\mathrm{st}} 131232, \quad 211 \not\equiv_{\mathrm{st}} 122$

Let $\equiv_{\mathrm{lo}}$ (resp. $\equiv_{\mathrm{ro}}$) be the equivalence relation on **Word** wherein $u \equiv_{\mathrm{lo}} v$ (resp. $u \equiv_{\mathrm{ro}} v$) if the versions of u and v obtained by keeping only the **leftmost** (resp. **rightmost**) among the multiple occurrences of a same letter are equal.

– Examples –

$223111352 \equiv_{\mathrm{lo}} 2333315, \quad 5142144 \equiv_{\mathrm{ro}} 552214,$

$3113 \not\equiv_{\mathrm{lo}} 113$

Let $\equiv_{\mathrm{ll}}$ (resp. $\equiv_{\mathrm{rl}}$) be the equivalence relation on **Word** wherein $u \equiv_{\mathrm{ll}} v$ (resp. $u \equiv_{\mathrm{rl}} v$) if $u_1 = v_1$ (resp. $u_{|u|} = v_{|v|}$).

– Examples –

$1 \equiv_{\mathrm{ll}} 12, \quad 3114 \equiv_{\mathrm{ll}} 32233, \quad 211535 \equiv_{\mathrm{rl}} 5$

– Proposition [G., 2020–] –

The equivalence relations $\equiv_{\mathrm{st}}$, $\equiv_{\mathrm{lo}}$, $\equiv_{\mathrm{ro}}$, $\equiv_{\mathrm{ll}}$, and $\equiv_{\mathrm{rl}}$ are clone congruences of **Word**.

Multisets

Let **MSet** $:= \mathbf{Word}/_{\equiv_{\mathrm{st}}}$.

For any $n \geqslant 0$, the elements of $\mathbf{MSet}(n)$ can be seen as nonempty multisets on $[n]$. By encoding any such multiset $M = \lbag 1^{a(1)}, \dots, n^{a(n)} \rbag$ by the tuple $a = (a(1), \dots, a(n))$, the superposition map of **MSet** expresses as a matrix multiplication

$$a \circledcirc [b_1, \dots, b_n] = \begin{pmatrix} a(1) & \dots & a(n) \end{pmatrix} \begin{pmatrix} b_1(1) & \dots & b_1(m) \\ \vdots & \dots & \vdots \\ b_n(1) & \dots & b_n(m) \end{pmatrix}.$$

– Proposition [G., 2020–] –

The clone **MSet** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies

Therefore, **MSet** is a clone realization of the variety of **commutative semigroups**.

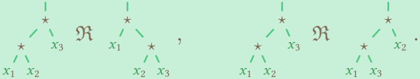
Rooted multisets

Let $\equiv := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{ll}}$ and $\mathbf{RMSet}_{\mathrm{l}} := \mathbf{Word}/_{\equiv}$.

For any $n \geqslant 0$, the elements of $\mathbf{RMSet}_{\mathrm{l}}(n)$ can be seen as pairs (M, i) where M is a nonempty multiset on $[n]$ and $i \in M$.

– Proposition [G., 2021–] –

The clone $\mathbf{RMSet}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, $\mathbf{RMSet}_{\mathrm{l}}$ is a clone realization of the variety of **right-commutative semigroups**, that are semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_3 = x_1 \star x_3 \star x_2$.

Analog properties hold for the quotient $\mathbf{RMSet}_{\mathrm{r}} := \mathbf{Word}/_{\equiv'}$, where $\equiv' := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{rl}}$.

Pairs of integers

Let $\equiv := \equiv_{\text{ll}} \cap \equiv_{\text{rl}}$ and $\textbf{PInt} := \textbf{Word}/_{\equiv}$.

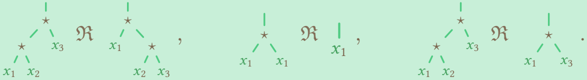
For any $n \geqslant 0$, the set $\textbf{PInt}(n)$ can be identified with $[n]^2$. The superposition of **PInt** expresses as

$$(i, i') \circledcirc [(j_1, j'_1), \dots, (j_n, j'_n)] = (j_i, j'_{i'}).$$

Moreover, $\#\textbf{PInt}(n) = n^2$.

– Proposition [G., 2021–] –

The clone **PInt** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, **PInt** is a clone realization of the variety of **rectangular bands**, that are idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_3 = x_1 \star x_3$.

Arrangements

Let $\mathbf{Arr}_{\mathrm{l}} := \mathbf{Word}/_{\equiv_{\mathrm{lo}}}$.

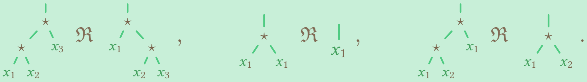
For any $n \geqslant 0$, the elements of $\mathbf{Arr}_{\mathrm{l}}(n)$ can be seen as nonempty arrangements (nonempty words without repetitions) on $[n]$. Moreover,

$$\#\mathbf{Arr}_{\mathrm{l}}(n) = \sum_{0 \leqslant k \leqslant n-1} \frac{n!}{k!}$$

and this sequence starts by 0, 1, 4, 15, 64, 325, 1956, 13699, 109600 (Sequence **A007526**).

– Proposition [G., 2020–] –

The clone $\mathbf{Arr}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, $\mathbf{Arr}_{\mathrm{l}}$ is a clone realization of the variety of **left-regular bands**, that are idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_1 = x_1 \star x_2$.

Analog properties hold for the quotient $\mathbf{Arr}_{\mathrm{r}} := \mathbf{Word}/_{\equiv_{\mathrm{ro}}}$, leading to **right-regular bands**.

Sets

– Lemma [G., 2020–] –

$$\equiv_{\mathrm{st}} \circ \equiv_{\mathrm{lo}} = \equiv_{\mathrm{lo}} \circ \equiv_{\mathrm{st}}$$

Therefore, this composition is a clone congruence of **Word**.

Let us set it as $\equiv_{\mathrm{in}}$ and let **Set** $:=$ **Word**$/_{\equiv_{\mathrm{in}}}$.

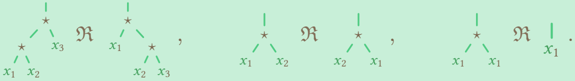
For any $n \geqslant 0$, the elements of $\mathbf{Set}(n)$ can be seen as nonempty subsets of $[n]$. On such objects, the superposition map of **Set** expresses as

$$U \circledcirc [V_1, \dots, V_n] = \bigcup_{j \in U} V_j.$$

Moreover, $\#\mathbf{Set}(n) = 2^n - 1$.

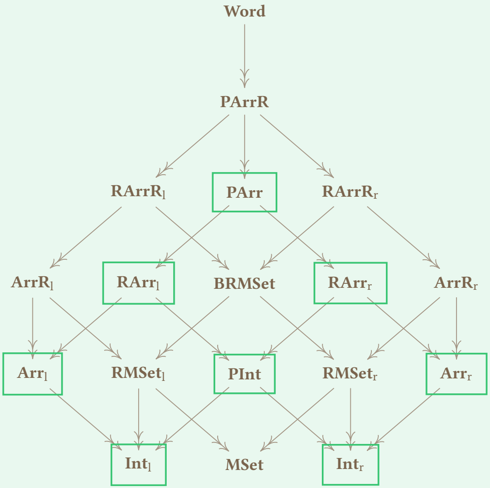
– Proposition [G., 2020–] –

The clone **Set** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, **Set** is a clone realization of the variety of **semilattices**.

Diagram of clones



Maps are surjective clone morphisms.

The framed clones are combinatorial.

Congruences on $\mathrm{W}\mathcal{M}$

Some of the previous constructions can be generalized at the level of the clone $\mathrm{W}\mathcal{M}$.

Let $\equiv_{\mathrm{st}}$, $\equiv_{\mathrm{lo}}$, $\equiv_{\mathrm{ro}}$, $\equiv_{\mathrm{ll}}$, and $\equiv_{\mathrm{rl}}$ be the equivalence relations on $\mathrm{W}\mathcal{M}$ defined in the same way as before where each color goes with its letter.

– Proposition [G., 2021–] –

For any monoid $\mathcal{M}$, the equivalence relations $\equiv_{\mathrm{st}}$, $\equiv_{\mathrm{lo}}$, $\equiv_{\mathrm{ro}}$, $\equiv_{\mathrm{ll}}$, and $\equiv_{\mathrm{rl}}$ are clone congruences of $\mathrm{W}\mathcal{M}$.

Let $\equiv_1 := \equiv_{\mathrm{st}} \circ \equiv_{\mathrm{lo}}$ and $\equiv_2 := \equiv_{\mathrm{lo}} \circ \equiv_{\mathrm{st}}$. If $\mathcal{M}$ has two different elements $\mathbf{a}$ and $\mathbf{b}$, one has

$$\begin{pmatrix} 1 & 2 & 1 \\ \mathbf{a} & \mathbf{a} & \mathbf{b} \end{pmatrix} \not\equiv_1 \begin{pmatrix} 1 & 2 \\ \mathbf{b} & \mathbf{a} \end{pmatrix}$$

but

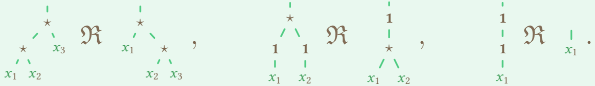
$$\begin{pmatrix} 1 & 2 & 1 \\ \mathbf{a} & \mathbf{a} & \mathbf{b} \end{pmatrix} \equiv_2 \begin{pmatrix} 1 & 2 \\ \mathbf{b} & \mathbf{a} \end{pmatrix}.$$

For this reason, in this general case, $\equiv_1$ and $\equiv_2$ are not clone congruences of $\mathrm{W}\mathcal{M}$.

Monoids on two elements

The two monoids on two elements are $\mathcal{M}_1 := (\mathbb{Z}/2\mathbb{Z}, +, 0)$ and $\mathcal{M}_2 := (\{0, 1\}, \max, 0)$.

The clone $\mathrm{W}\mathcal{M}_1$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(1) \sqcup \mathfrak{G}(2)$, $\mathfrak{G}(1) := \{\mathbf{1}\}$, $\mathfrak{G}(2) := \{\star\}$, and $\mathfrak{R}$ satisfies



Any algebra on this clone is a semigroup endowed with an **involutive semigroup endomorphism**.

The clone $\mathrm{W}\mathcal{M}_2$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(1) \sqcup \mathfrak{G}(2)$, $\mathfrak{G}(1) := \{\mathbf{1}\}$, $\mathfrak{G}(2) := \{\star\}$, and $\mathfrak{R}$ satisfies

Any algebra on this clone is a semigroup endowed with an **idempotent semigroup endomorphism**.

Conclusion and future work

In this work,

- we use **clones** as a framework to study **varieties of algebras**;
- we use **rewrite systems on terms** to build term realizations of varieties;
- we introduce a new functorial **construction** W from monoids to clones;
- we build **quotients** of the clone of monochrome words providing **clone realizations** of special classes of semigroups.

Future work include

- the discovery of **other congruences** of $\mathsf{W}\mathcal{M}$;
- the exploration of the previous constructions for **colored words**;
- the study of **subclones** of $\mathsf{W}\mathcal{M}$ generated by finite sets of colored words.

4. Appendix

Bi-rooted multisets

Let $\equiv := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{ll}} \cap \equiv_{\mathrm{rl}}$ and **BRMSet** $:=$ **Word**$/_{\equiv}$.

For any $n \geqslant 0$, the elements of $\mathbf{BRMSet}(n)$ can be seen as triples (M, i, i') where M is a nonempty multiset on $[n]$ and $i,' i \in M$.

– Proposition [G., 2021–] –

The clone $\mathbf{RMSet}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies

Therefore, **BRMSet** is a clone realization of the variety of **medial semigroups**, that are semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_3 \star x_1 = x_1 \star x_3 \star x_2 \star x_1$.

Integers

Let $\mathbf{Int}_{\mathrm{l}} := \mathbf{Word}/_{\equiv_{\mathrm{ll}}}$.

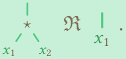
For any $n \geqslant 0$, the set $\mathbf{Int}_{\mathrm{l}}(n)$ can be identified with $[n]$. The superposition of $\mathbf{Int}_{\mathrm{l}}$ expresses as

$$i \circledcirc [j_1, \ldots, j_n] = j_i.$$

Moreover, $\#\mathbf{Int}_{\mathrm{l}}(n) = n$.

– Proposition [G., 2021–] –

The clone $\mathbf{Int}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



This is the trivial clone.

Therefore, $\mathbf{Int}_{\mathrm{l}}$ is a clone realization of the variety of **left-zero bands**, that are semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 = x_1$.

Analog properties hold for the quotient $\mathbf{Int}_{\mathrm{r}} := \mathbf{Word}/_{\equiv_{\mathrm{rl}}}$, leading to **right-zero bands**.

Rooted arrangements

Let $\equiv := \equiv_{\mathrm{lo}} \cap \equiv_{\mathrm{rl}}$ and $\mathbf{RArr}_{\mathrm{l}} := \mathbf{Word}/_{\equiv}$.

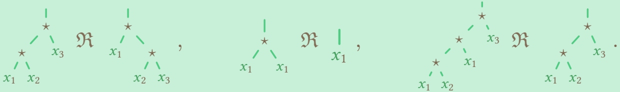
For any $n \geqslant 0$, the elements of $\mathbf{RArr}_{\mathrm{l}}(n)$ can be seen as pairs $(\boldsymbol{a}, i)$ where $\boldsymbol{a}$ is a nonempty arrangement on $[n]$ and i occurs in $\boldsymbol{a}$. Moreover,

$$\#\mathbf{RArr}_{\mathrm{l}}(n) = \sum_{0 \leqslant k \leqslant n-1} \frac{n!(n-k)}{k!}$$

and this sequence starts by $0, 1, 6, 33, 196, 1305, 9786, 82201, 767208$ (Sequence **A093964**).

– Proposition [G., 2021–] –

The clone $\mathbf{RArr}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, $\mathbf{RArr}_{\mathrm{l}}$ is a clone realization of the variety of idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_1 \star x_3 = x_1 \star x_2 \star x_3$.

Analog properties hold for the quotient $\mathbf{RArr}_{\mathrm{r}} := \mathbf{Word}/_{\equiv'}$ where $\equiv' := \equiv_{\mathrm{ro}} \cap \equiv_{\mathrm{ll}}$

Rooted sets

Let $\equiv := \equiv_{\mathrm{in}} \cap \equiv_{\mathrm{ll}}$ and $\mathbf{RSet}_{\mathrm{l}} := \mathbf{Word}/_{\equiv}$.

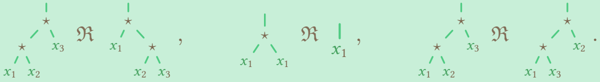
For any $n \geqslant 0$, the elements of $\mathbf{RSet}_{\mathrm{l}}(n)$ can be seen as pairs (S, i) where S is a nonempty subset of $[n]$ and $i \in S$. Moreover,

$$\#\mathbf{RSet}_{\mathrm{l}}(n) = n\, 2^{n-1}$$

and this sequence starts by 0, 1, 4, 12, 32, 80, 192, 448, 1024 (Sequence **A001787**).

– Proposition [G., 2021–] –

The clone $\mathbf{RSet}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, $\mathbf{RSet}_{\mathrm{l}}$ is a clone realization of the variety of idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_3 = x_1 \star x_3 \star x_2$.

Analog properties hold for the quotient $\mathbf{RSet}_{\mathrm{r}} := \mathbf{Word}/_{\equiv'}$, where $\equiv' := \equiv_{\mathrm{in}} \cap \equiv_{\mathrm{rl}}$.

Bi-rooted sets

Let $\equiv := \equiv_{\text{in}} \cap \equiv_{\text{ll}} \cap \equiv_{\text{rl}}$ and **BRSet** $:=$ **Word**$/_{\equiv}$.

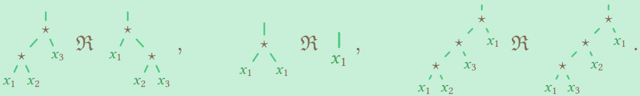
For any $n \geqslant 0$, the elements of $\mathbf{BRSet}(n)$ can be seen as triples (S, i, i') where S is a nonempty subset of $[n]$ and $i, i' \in S$. Moreover,

$$\#\mathbf{BRSet}(n) = n(n+1)\, 2^{n-2}$$

and this sequence starts by 0, 1, 6, 24, 80, 240, 672, 1792, 4608 (Sequence **A001788**).

– Proposition [G., 2021–] –

The clone **BRSet** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, **BRSet** is a clone realization of the variety of **normal bands**, that are idempotent semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_3 \star x_1 = x_1 \star x_3 \star x_2 \star x_1$.

Arrangements of runs

Let $\equiv := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{lo}}$ (**stalactite congruence** [Hivert, Novelli, Thibon, 2007]) and $\mathbf{ArrR}_{\mathrm{l}} := \mathbf{Word}/_{\equiv}$.

For any $n \geqslant 0$, the elements of $\mathbf{ArrR}_{\mathrm{l}}(n)$ can be seen as nonempty arrangements of runs on $[n]$.

– Examples –

The word 33 1111 55 2 6 is an element of $\mathbf{ArrR}_{\mathrm{l}}(9)$.

The word **22222** 33311 **2** is not an element of $\mathbf{ArrR}_{\mathrm{l}}$.

– Proposition [G., 2020–] –

The clone $\mathbf{ArrR}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies

$$(x_1 \star x_2) \star x_3 \;\mathfrak{R}\; x_1 \star (x_2 \star x_3), \qquad (x_1 \star x_2) \star x_1 \;\mathfrak{R}\; (x_1 \star x_1) \star x_2.$$

Therefore $\mathbf{ArrR}_{\mathrm{l}}$ is a clone realization of semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_2 \star x_1 = x_1 \star x_1 \star x_2$.

Analog properties hold for the quotient $\mathbf{ArrR}_{\mathrm{r}} := \mathbf{Word}/_{\equiv'}$, where $\equiv' := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{ro}}$.

Pairs of compatible arrangements

Let $\equiv := \equiv_{\mathrm{lo}} \cap \equiv_{\mathrm{ro}}$ and $\mathbf{PArr} := \mathbf{Word}/_{\equiv}$.

For $n \geqslant 0$, the elements of $\mathbf{PArr}(n)$ can be seen as pairs (u, v) such that u and v are nonempty arrangements on $[n]$ with j appears in u iff j appears in v.

– Example –

$(3261, 1263)$ is an element of $\mathbf{PArr}(6)$.

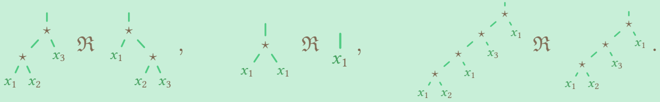
Moreover,

$$\#\mathbf{PArr}(n) = \sum_{k \in [n]} \frac{n!k!}{(n-k)!}$$

and this sequence starts by 0, 1, 6, 51, 748, 17685, 614226, 29354311, 1844279256 (linked with Sequence **A046662**).

– Proposition [G., 2020–] –

The clone **PArr** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies



Therefore, **PArr** is a clone realization of the variety of **regular bands**.

Pairs of compatible arrangements of runs

Let $\equiv := \equiv_{\text{st}} \cap \equiv_{\text{lo}} \cap \equiv_{\text{ro}}$ and **PArrR** $:=$ **Word**$/_{\equiv}$.

For any $n \geqslant 0$, the elements of $\mathbf{PArrR}(n)$ can be seen as pairs (u, v) such that u and v are nonempty arrangements of runs of repeated letters on $[n]$, with u and v having the same number of occurrences of any letter.

– Example –

$(3222611, 22211263)$ is an element of $\mathbf{PArrR}(6)$.

$(221, 12)$ is not.

– Proposition [G., 2020–] –

The clone **PArrR** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies

Therefore, **PArrR** is a clone realization of semigroups wherein the operation $\star$ satisfies the relation $x_1 \star x_1 \star x_2 \star x_3 \star x_1 = x_1 \star x_2 \star x_1 \star x_3 \star x_1 = x_1 \star x_2 \star x_3 \star x_1 \star x_1$.

Rooted arrangements of runs

Let $\equiv := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{lo}} \cap \equiv_{\mathrm{rl}}$ and $\mathbf{RArrR}_{\mathrm{l}} := \mathbf{Word}/_{\equiv}$.

For any $n \geqslant 0$, the elements of $\mathbf{RArrR}_{\mathrm{l}}(n)$ can be seen as nonempty arrangements of runs on $[n]$ wherein the rightmost run is marked or a run of length two or more is marked.

– Proposition [G., 2021–] –

The clone $\mathbf{RArrR}_{\mathrm{l}}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{\star\}$ and $\mathfrak{R}$ satisfies

Therefore, $\mathbf{RArrR}_{\mathrm{l}}$ is a clone realization of semigroups wherein the operation $\star$ satisfies the relation $x_1 x_2 x_1 x_3 = x_1 x_1 x_2 x_3$.

Analog properties hold for the quotient $\mathbf{RArrR}_{\mathrm{r}} := \mathbf{Word}/_{\equiv'}$, where $\equiv := \equiv_{\mathrm{st}} \cap \equiv_{\mathrm{ro}} \cap \equiv_{\mathrm{ll}}$.