Cliff posets and algebras

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Outline

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1. Algebras and orders

Malvenuto-Reutenauer algebra

The Malvenuto-Reutenauer algebra [**Malvenuto, Reutenauer**, 1995] $(\mathbf{FQSym}, \cdot, 1)$ is the unital associative algebra defined as follows.

- $\mathbf{FQSym}$ is the $\mathbb{K}$-linear span $\mathbb{K}\langle\mathfrak{S}\rangle$ of all permutations.

  The set $\{\mathsf{F}_\sigma : \sigma \in \mathfrak{S}\}$ is the fundamental basis of $\mathbf{FQSym}$.

- $\cdot$ is the shifted shuffle product, the associative product defined by

$$\mathsf{F}_\sigma \cdot \mathsf{F}_\nu := \sum_{\pi \in \sigma \overline{\sqcup\!\sqcup} \nu} \mathsf{F}_\pi .$$

- 1 is defined as F_ϵ where ϵ is the empty permutation.

– Example –

$$\mathsf{F}_{312} \cdot \mathsf{F}_{21} = \mathsf{F}_{31254} + \mathsf{F}_{31524} + \mathsf{F}_{31542} + \mathsf{F}_{35124} + \mathsf{F}_{35142}$$
$$+ \mathsf{F}_{35412} + \mathsf{F}_{53124} + \mathsf{F}_{53142} + \mathsf{F}_{53412} + \mathsf{F}_{54312}$$

Right weak order

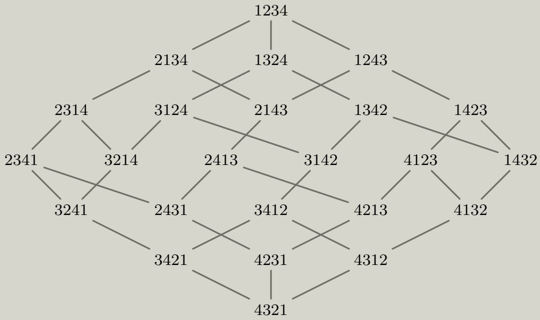
The right weak order is the order relation $\preccurlyeq$ on $\mathfrak{S}(n)$ defined as the reflexive and transitive closure of the relation $\lessdot$ satisfying

$$u\mathsf{ab}v \lessdot u\mathsf{ba}v$$

where $u, v \in \mathbb{N}^*$, and a and b are letters such that $\mathsf{a} < \mathsf{b}$.

– Example –

Hasse diagram of $(\mathfrak{S}(4), \preccurlyeq)$:



Product of $\mathbf{FQSym}$ and right weak order

Let $\diagup$ and $\diagdown$ be the two operations on $\mathfrak{S}$ defined by

$$\sigma \diagup \nu := \sigma \uparrow_{|\sigma|}(\nu) \qquad \text{and} \qquad \sigma \diagdown \nu := \uparrow_{|\sigma|}(\nu)\, \sigma.$$

– Example –

$312 \diagup 21 = 31254$

– Example –

$312 \diagdown 21 = 54312$

– Proposition –

For any permutations σ and ν,

$$\mathsf{F}_\sigma \cdot \mathsf{F}_\nu = \sum_{\substack{\pi \in \mathfrak{S} \\ \sigma \diagup \nu \preccurlyeq \pi \preccurlyeq \sigma \diagdown \nu}} \mathsf{F}_\pi .$$

– Example –

$\mathsf{F}_{312} \cdot \mathsf{F}_{21}$ is the formal sum of all the F_π where $\pi \in [312 \diagup 21, 312 \diagdown 21] = [31254, 54312]$.

Multiplicative bases

A basis is multiplicative if the product of two basis element is a single basis element.

The right weak order can be used to build multiplicative bases of $\mathbf{FQSym}$.

Let

$$\mathsf{E}_\sigma := \sum_{\substack{\nu \in \mathfrak{S} \\ \sigma \preccurlyeq \nu}} \mathsf{F}_\nu \qquad \text{and} \qquad \mathsf{H}_\sigma := \sum_{\substack{\nu \in \mathfrak{S} \\ \nu \preccurlyeq \sigma}} \mathsf{F}_\nu .$$

– Example –

$$\mathsf{E}_{4123} = \mathsf{F}_{4123} + \mathsf{F}_{4132} + \mathsf{F}_{4213} + \mathsf{F}_{4231} + \mathsf{F}_{4312} + \mathsf{F}_{4321}$$

– Proposition –

For any permutations σ and ν,

$$\mathsf{E}_\sigma \cdot \mathsf{E}_\nu = \mathsf{E}_{\sigma \,/\, \nu} \qquad \text{and} \qquad \mathsf{H}_\sigma \cdot \mathsf{H}_\nu = \mathsf{H}_{\sigma \,\backslash\, \nu}.$$

Generators and relations

A subset $\mathcal{G}$ of an associative algebra $\mathcal{A}$ is a minimal generating set of $\mathcal{A}$ if the smallest subalgebra of $\mathcal{A}$ containing $\mathcal{G}$ is $\mathcal{A}$ itself and $\mathcal{G}$ is minimal for set inclusion.

A permutation σ is connected if $\sigma \neq \epsilon$ and no proper prefix of σ is a permutation.

– Example –

The permutation 43257816 is connected.

– Example –

The permutation $4325176 = 43251 \,/\, 21$ is not.

– Theorem [Duchamp, Hivert, Thibon, 2002] **–**

The set $\mathcal{G}$ of all E_σ such that σ is connected is a minimal generating set of $\mathbf{FQSym}$.

Moreover, $\mathbf{FQSym}$ is free as a unital associative algebra and $\mathbf{FQSym} \simeq \mathbb{K}\langle\mathcal{G}\rangle$.

This is a consequence of the fact that any permutation σ decomposes in a unique way as $\sigma = \nu^{(1)} \,/\, \cdots \,/\, \nu^{(\ell)}$ where all $\nu^{(i)}$ are connected.

Subalgebras and subposets

FQSym admits a lot of subalgebras:

- **FSym**, the algebra of standard Young tableaux[**Poirier, Reutenauer**, 1995], [**Duchamp, Hivert, Thibon**, 2002];
- **PBT**, the algebra of binary trees [**Loday, Ronco**, 1998], [**Hivert, Novelli, Thibon**, 2005];
- **Sym**, the algebra of integer compositions [**Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon**, 1995];
- **Baxter**, the algebra of pairs of twin binary trees [**Law, Reading**, 2012], [**G.**, 2012];
- **Bell**, the algebra of set partitions [**Rey**, 2007].

Each one is constructed from a surjective map $\theta : \mathfrak{S} \to C$, where C is one of the previous sets of objects, as the subalgebra spanned by the elements

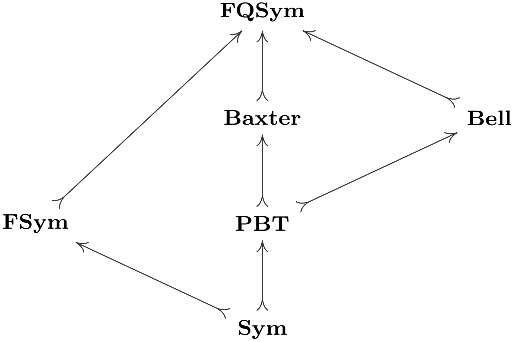
$$\mathsf{F}_x := \sum_{\substack{\sigma \in \mathfrak{S} \\ \theta(\sigma) = x}} \mathsf{F}_\sigma .$$

There are also posets $(C, \preccurlyeq)$ and operations $\diagup$ and $\diagdown$ on C such that

$$\mathsf{F}_x \cdot \mathsf{F}_y = \sum_{\substack{z \in C \\ x \diagup y \preccurlyeq z \preccurlyeq x \diagdown y}} \mathsf{F}_z .$$

Diagram of algebras

These algebras fit into the following diagram of injective algebra morphisms:



Note that these algebras are also endowed with coproducts so that they are in fact Hopf bialgebras.

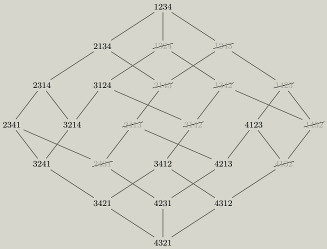
Let BT be the set of all binary trees.

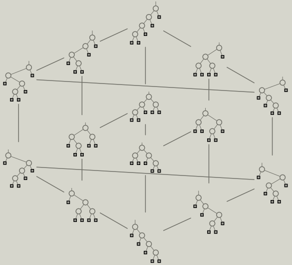
It is known that $\mathrm{BT}(n)$ is in one-to-one correspondence with the set of permutations avoiding the pattern 132.

The restriction of the right weak order on these permutations is the Tamari order [**Hivert, Novelli, Thibon**, 2005].

– Example –

Hasse diagrams of $(\mathrm{BT}(4), \preccurlyeq)$:

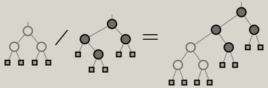




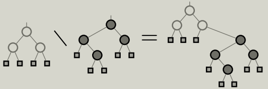
Algebra of binary trees — Product

Let $\nearrow$ and $\nwarrow$ be the two operations on BT defined as follows. For any $\mathsf{t}, \mathsf{s} \in \mathrm{BT}$, $\mathsf{t} \nearrow \mathsf{s}$ (resp. $\mathsf{t} \nwarrow \mathsf{s}$) is the binary tree obtained by grafting the root of t (resp. s) onto the first (resp. last) leaf of s (resp. t).

– Example –

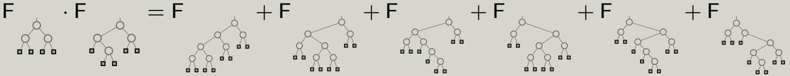


– Example –



The product in $\mathbf{PBT}$ of two basis elements F_t and F_s is the formal sum of the elements of the Tamari interval $[\mathsf{t} \nearrow \mathsf{s}, \mathsf{t} \nwarrow \mathsf{s}]$.

– Example –



Motivation: a new order on permutations

The objectives of this work are to

1. introduce a new order relation on permutations;
2. consider the analog of $\mathbf{FQSym}$ w.r.t. this alternative order;
3. try to construct a similar hierarchy of algebras.

For this, we consider an order extension of the right weak order and generalizations of permutations.

– Example –

Here are both the Hasse diagrams of the right weak order on permutations of size 3 and of the considered order extension:

2. Posets on cliffs

Cliffs and posets

A range map is a map $\delta := \mathbb{N} \setminus \{0\} \to \mathbb{N}$.

A δ-cliff of size n is a word $u \in \mathbb{N}^n$ such that for all $i \in [n]$, $0 \leqslant u_i \leqslant \delta(i)$.

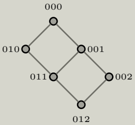
The graded collection of all δ-cliffs is denoted by Cl_δ.

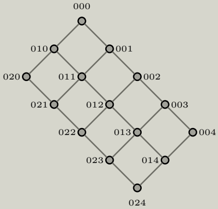
Let $\preccurlyeq$ be the partial order relation on each $\mathsf{Cl}_\delta(n)$ wherein $u \preccurlyeq v$ if $u_i \leqslant v_i$ for all $i \in [n]$.

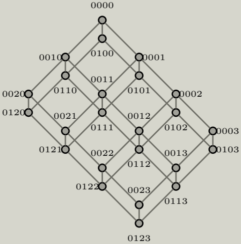
For any $m \geqslant 0$, let $\mathbf{m}$ be the map defined by $\mathbf{m}(i) := m(i-1)$.

– Example –

The Hasse diagrams of $\mathsf{Cl}_{\mathbf{1}}(3)$, $\mathsf{Cl}_{\mathbf{2}}(3)$, and $\mathsf{Cl}_{\mathbf{1}}(4)$ are







The posets $\mathsf{Cl}_{\mathbf{1}}(n)$ have been studied in [**Denoncourt**, 2013].

Lehmer codes

Let leh be the map sending any permutation σ to the $\mathbf{1}$-cliff u wherein u_i is the number of letters a at the right of i in σ such that $i > a$. This is a variation of the Lehmer code [**Lehmer**, 1960] of a permutation.

This map $\mathrm{leh} : \mathsf{Cl}_{\mathbf{1}}(n) \to \mathfrak{S}(n)$ is a bijection.

– Example –

$$\mathrm{leh}(436512) = 002323$$

A map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is a poset morphism if $x \preccurlyeq_1 y$ implies $\phi(x) \preccurlyeq_2 \phi(y)$.

A poset $\mathcal{P}_2$ is an order extension of a poset $\mathcal{P}_1$ if there is a bijective poset morphism $\phi : \mathcal{P}_1 \to \mathcal{P}_2$.

– Proposition –

For any $n \geqslant 1$, leh is a bijective poset morphism between the right weak order $(\mathfrak{S}(n), \preccurlyeq)$ and $(\mathsf{Cl}_{\mathbf{1}}(n), \preccurlyeq)$.

Subposets

The partial order $\mathsf{Cl}_\delta(n)$ has a very simple structure since

$$\mathsf{Cl}_\delta(n) \simeq [\delta(1)+1] \times \cdots \times [\delta(n)+1].$$

Its main interest lies in the fact that it contains a lot of subposets.

Let $\mathcal{S}$ be a subset of Cl, endowed with the same componentwise order relation $\preccurlyeq$.

Let us introduce the following combinatorial properties. We say that $\mathcal{S}$ is

- straight if its covering relation $\lessdot_{\mathcal{S}}$ is such that when $u \lessdot_{\mathcal{S}} v$ then u and v differ by exactly one letter;
- closed by prefix if for any $u \in \mathcal{S}$, all prefixes of u belong to $\mathcal{S}$;
- minimally (resp. maximally) extendable if for any $u \in \mathcal{S}$, $u0 \in \mathcal{S}$ (resp. $u\,\delta(|u|+1) \in \mathcal{S}$).

Geometric realizations

A geometric realization of a poset $\mathcal{P}$ refers to a way to see $\mathcal{P}$ as a geometrical object in $\mathbb{R}^k$ for a certain $k \geqslant 0$.

Let $\mathfrak{C}(\mathcal{S}(n))$ be the geometric object on the set of points

$$\{(u_1, \ldots, u_n) \in \mathbb{R}^n : u \in \mathcal{S}(n)\}$$

where there is an edge between u and v provided that $u \lessdot_{\mathcal{S}} v$.

When $\mathcal{S}$ is straight, each edge is parallel to a line passing by the origin and a point of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$. In this case, we call $\mathfrak{C}(\mathcal{S}(n))$ the cubic realization of $\mathcal{S}(n)$.

This realization raises the following questions.

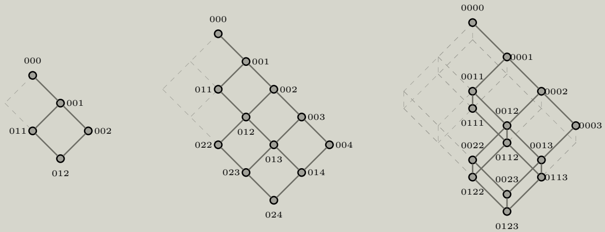
1. Describe the general shape of $\mathfrak{C}(\mathcal{S}(n))$;
2. Count the cells of $\mathfrak{C}(\mathcal{S}(n))$ of a given dimension;
3. Compute the volume $\operatorname{vol}(\mathfrak{C}(\mathcal{S}(n)))$ of $\mathfrak{C}(\mathcal{S}(n))$.

Hill posets

Let Hi_δ be the subset of Cl_δ containing all δ-hills that are weakly increasing δ-cliffs.

– Example –

The Hasse diagrams of $\text{Hi}_{\mathbf{1}}(3)$, $\text{Hi}_{\mathbf{2}}(3)$, and $\text{Hi}_{\mathbf{1}}(4)$ are



The posets $\text{Hi}_{\mathbf{1}}$ are the Stanley lattices [**Stanley**, 1975].

Properties of Hill posets

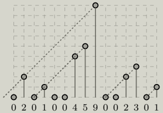
- When δ is weakly increasing, all $\mathsf{Hi}_\delta(n)$ are sublattices of $\mathsf{Cl}_\delta(n)$.
- For any $m \geqslant 0$ and $n \geqslant 0$, the cardinality of $\mathsf{Hi}_{\mathbf{m}}(n)$ is the n-th m-Fuss-Catalan number
$$\mathrm{cat}_m(n) = \frac{1}{mn+1}\binom{mn+n}{n}.$$
- For any $n \geqslant 0$, $\mathsf{Hi}_\delta(n)$ is EL-shellable.
- When δ is weakly increasing, all $\mathsf{Hi}_\delta(n)$ are constructible by interval doubling.
- For any $m \geqslant 1$ and $n \geqslant 0$, the realization $\mathfrak{C}\left(\mathsf{Hi}_{\mathbf{m}}(n)\right)$ is cubic, has dimension $n-1$, and satisfies
$$\mathrm{vol}\left(\mathfrak{C}\left(\mathsf{Hi}_{\mathbf{m}}(n)\right)\right) = \mathrm{cat}_{m-1}(n).$$

Canyon posets

Let Ca_δ be the subset of Cl_δ containing all δ-canyons that are δ-cliffs u such that $u_{i-j} \leqslant u_i - j$, for all $i \in [|u|]$ and $j \in [u_i]$ satisfying $i - j \geqslant 1$.

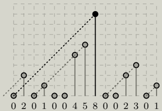
– Example –

A $\mathbf{2}$-canyon of size 15:



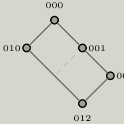
– Example –

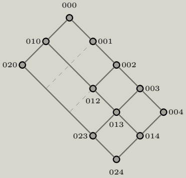
A $\mathbf{2}$-cliff of size 15 which is not a $\mathbf{2}$-canyon:

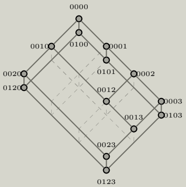


– Example –

The Hasse diagrams of $\mathsf{Ca}_{\mathbf{1}}(3)$, $\mathsf{Ca}_{\mathbf{2}}(3)$, and $\mathsf{Ca}_{\mathbf{1}}(4)$ are







Properties of Canyon posets

- The posets $\mathsf{Ca}_{\mathbf{1}}$ are the Tamari lattices [**Tamari**, 1962].
- For any $m \geqslant 0$, $\#\mathsf{Ca}_{\mathbf{m}}(n) = \mathrm{cat}_m(n)$.
- When δ is increasing, all $\mathsf{Ca}_\delta(n)$ are lattices but not sublattices of $\mathsf{Cl}_\delta(n)$.

– Example –

In $\mathsf{Ca}_{\mathbf{2}}$, there is an algorithm to compute the join:

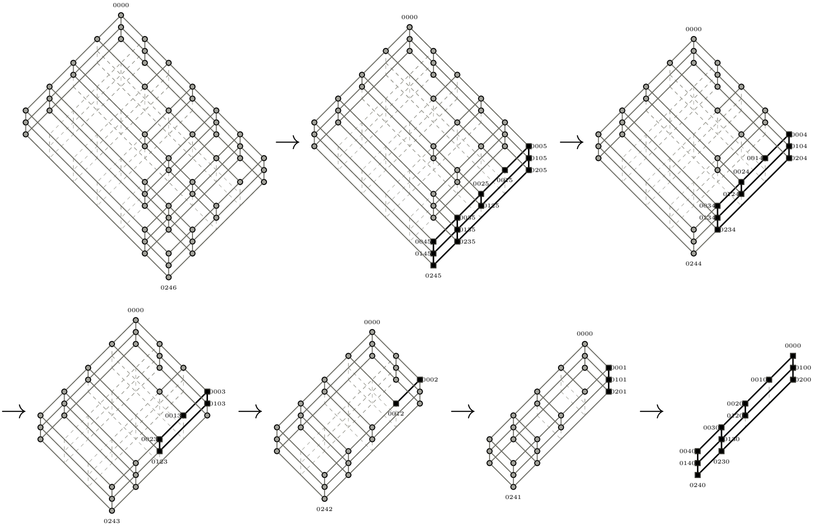
$$0124010 \vee 0205001 = (0225011)' = 0235012$$

- For any $n \geqslant 0$, $\mathsf{Ca}_\delta(n)$ is EL-shellable.
- When δ is increasing, all $\mathsf{Ca}_\delta(n)$ are constructible by interval doubling.
- For any $m \geqslant 1$ and $n \geqslant 0$, the realization $\mathfrak{C}\left(\mathsf{Ca}_{\mathbf{m}}(n)\right)$ is cubic, has dimension $n - 1$, and satisfies

$$\mathrm{vol}\left(\mathfrak{C}\left(\mathsf{Ca}_{\mathbf{m}}(n)\right)\right) = \mathrm{vol}\left(\mathfrak{C}\left(\mathsf{Cl}_{\mathbf{m}}(n)\right)\right) = m^{n-1}(n-1)!.$$

Constructibility by interval doubling

Sequence of interval contractions (reverse of interval doubling) from $\mathsf{Ca}_{\mathbf{2}}(4)$ to $\mathsf{Ca}_{\mathbf{2}}(3)$:



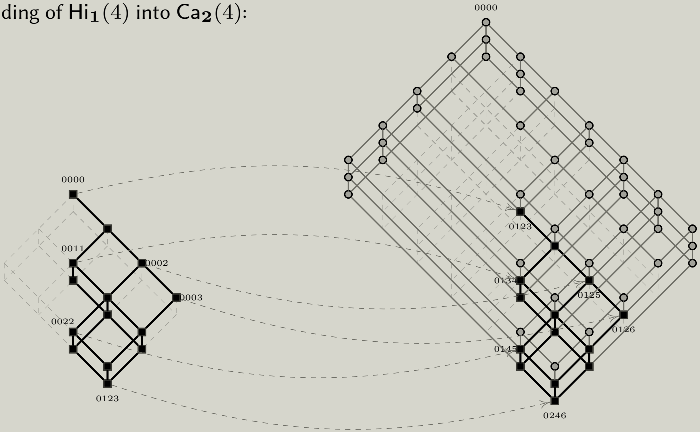
Interactions between canyon and hill posets

– Proposition –

For any $m \geqslant 1$ and $n \geqslant 0$, there is a poset embedding from $\mathsf{Hi}_{\mathbf{m}-1}(n)$ to $\mathsf{Ca}_{\mathbf{m}}(n)$.

– Example –

Embedding of $\mathsf{Hi}_{\mathbf{1}}(4)$ into $\mathsf{Ca}_{\mathbf{2}}(4)$:



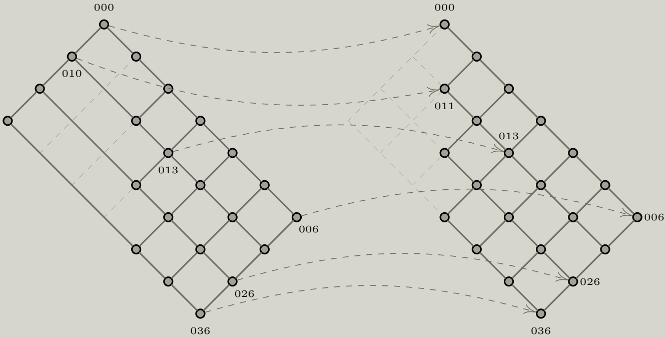
Interactions between canyon and hill posets

– Theorem –

For any $m \geqslant 1$ and $n \geqslant 0$, there is a bijective poset morphism from $\mathsf{Ca}_{\mathbf{m}}(n)$ to $\mathsf{Hi}_{\mathbf{m}}(n)$.

– Example –

Bijective morphism from $\mathsf{Ca}_{\mathbf{3}}(3)$ to $\mathsf{Hi}_{\mathbf{3}}(3)$:



3. Algebras on cliffs

Algebras on cliffs

Let $\mathbf{Cl}_\delta$ be the $\mathbb{K}$-linear span of all δ-cliffs.

The set $\{\mathsf{F}_u : u \in \mathsf{Cl}_\delta\}$ is a basis of $\mathbf{Cl}_\delta$.

Let $\mathrm{r}_\delta : \mathbb{N}^n \to \mathsf{Cl}_\delta(n)$ be the δ-reduction map defined for any $u \in \mathbb{N}^n$ and $i \in [n]$ by $(\mathrm{r}_\delta(u))_i := \min\{u_i, \delta(i)\}$.

– Example –

$\mathrm{r}_{\mathbf{1}}(212066) = 012045$

– Example –

$\mathrm{r}_{\mathbf{2}}(212066) = 012066$

Let $\cdot$ be the product on $\mathbf{Cl}_\delta$ defined by

$$\mathsf{F}_u \cdot \mathsf{F}_v := \sum_{\substack{uv' \in \mathsf{Cl}_\delta \\ \mathrm{r}_\delta(v') = v}} \mathsf{F}_{uv'}.$$

– Example –

In $\mathsf{Cl}_{\mathbf{1}}$,

$$\mathsf{F}_{00} \cdot \mathsf{F}_{011} = \mathsf{F}_{00011} + \mathsf{F}_{00021} + \mathsf{F}_{00031} + \mathsf{F}_{00111} + \mathsf{F}_{00121} + \mathsf{F}_{00131} + \mathsf{F}_{00211} + \mathsf{F}_{00221} + \mathsf{F}_{00231}.$$

Associativity

In general, the product of $\mathbf{Cl}_\delta$ is not associative.

– Example –

For $\delta := 102^\omega$, we have

$$(\mathsf{F}_1 \cdot \mathsf{F}_0) \cdot \mathsf{F}_1 = \mathsf{F}_{10} \cdot \mathsf{F}_1 = \mathsf{F}_{101} + \mathsf{F}_{102}$$

and

$$\mathsf{F}_1 \cdot (\mathsf{F}_0 \cdot \mathsf{F}_1) = \mathsf{F}_1 \cdot 0 = 0.$$

A range map is valley-free (or unimodal) if there is no $i_1 \leqslant i_2 \leqslant i_3$ such that $\delta(i_1) > \delta(i_2) < \delta(i_3)$.

– Theorem –

The product $\cdot$ of $\mathbf{Cl}_\delta$ is associative iff δ is a valley-free range map.

Over and under operations

Let

$$\diagup : \mathbb{N}^n \times \mathbb{N}^m \to \mathbb{N}^{n+m} \qquad \text{and} \qquad \diagdown : \mathbb{N}^n \times \mathbb{N}^m \to \mathbb{N}^{n+m},$$

be the two operations defined by $u \diagup v := uv$ and $u \diagdown v := uv'$ where v' is the word of length $|v|$ satisfying, for any $i \in [|v|]$,

$$v'_i = \begin{cases} \delta(|u| + i) & \text{if } v_i = \delta(i), \\ v_i & \text{otherwise.} \end{cases}$$

– Example –

For $\delta = 112334^\omega$, $010 \diagup 1021 = 0101021$ and $010 \diagdown \mathbf{1}0\mathbf{2}1 = 010\mathbf{3}0\mathbf{4}1$.

– Example –

For $\delta = 210^\omega$, $21 \diagdown 1\mathbf{1} = 211\mathbf{0}$. This word is not a δ-cliff.

Product and cliff posets

For $w \in \mathbb{N}^*$, let $\chi_\delta(w)$ defined as $1 \in \mathbb{K}$ if w is a δ-cliff and as $0 \in \mathbb{K}$ otherwise.

– Theorem –

For any $u, v \in \mathsf{Cl}_\delta$, we have in $\mathbf{Cl}_\delta$,

$$\mathsf{F}_u \cdot \mathsf{F}_v = \chi_\delta\left(u \diagup v\right) \sum_{\substack{w \in \mathsf{Cl}_\delta \\ u \diagup v \preccurlyeq w \preccurlyeq u \diagdown v}} \mathsf{F}_w.$$

– Example –

In $\mathbf{Cl}_{01120^\omega}$, since $01 \diagup 010 = 01010 \in \mathsf{Cl}_{01120^\omega}$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{010} = \mathsf{F}_{01010} + \mathsf{F}_{01020} + \mathsf{F}_{01110} + \mathsf{F}_{01120}.$$

– Example –

In $\mathbf{Cl}_{01120^\omega}$, since $01 \diagup 011 = 01011 \notin \mathsf{Cl}_{01120^\omega}$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{011} = 0.$$

Multiplicative bases

Let

$$\mathsf{E}_u := \sum_{\substack{v \in \mathsf{Cl}_\delta \\ u \preccurlyeq v}} \mathsf{F}_v \qquad \text{and} \qquad \mathsf{H}_u := \sum_{\substack{v \in \mathsf{Cl}_\delta \\ v \preccurlyeq u}} \mathsf{F}_v.$$

– Examples –

For $\delta := 1021^\omega$,

$$\mathsf{E}_{10010} = \mathsf{F}_{10010} + \mathsf{F}_{10011} + \mathsf{F}_{10110} + \mathsf{F}_{10111} + \mathsf{F}_{10210} + \mathsf{F}_{10211},$$

and

$$\mathsf{H}_{10010} = \mathsf{F}_{10010} + \mathsf{F}_{10000} + \mathsf{F}_{00010} + \mathsf{F}_{00000}.$$

By triangularity, $\{\mathsf{E}_u : u \in \mathsf{Cl}_\delta\}$ and $\{\mathsf{H}_u : u \in \mathsf{Cl}_\delta\}$ are bases of $\mathbf{Cl}_\delta$.

– Proposition –

For any $u, v \in \mathsf{Cl}_\delta$, we have in $\mathbf{Cl}_\delta$,

$$\mathsf{E}_u \cdot \mathsf{E}_v = \chi_\delta\left(u \nearrow v\right) \mathsf{E}_{u \nearrow v} \qquad \text{and} \qquad \mathsf{H}_u \cdot \mathsf{H}_v = \mathsf{H}_{\mathrm{r}_\delta\left(u \searrow v\right)}.$$

Minimal generating set

A nonempty δ-cliff u is δ-prime if the decomposition $u = v \mathbin{/} w$ with $v, w \in \mathsf{Cl}_\delta$ implies $(v, w) \in \{(\epsilon, u), (u, \epsilon)\}$.

The set of all these elements is denoted by $\mathcal{P}_\delta$.

– Examples –

Let $\delta := 021^\omega$.

The δ-cliffs 0, 01, and 021111 are δ-prime. The δ-cliff $0210 = 021 \mathbin{/} 0$ is not.

– Lemma –

Any nonempty δ-cliff admits exactly one suffix which is δ-prime.

– Proposition –

The set $\{\mathsf{E}_u : u \in \mathcal{P}_\delta\}$ is a minimal generating set of the magmatic algebra Cl_δ.

This is a consequence of the fact that, by the previous lemma, any δ-cliff decomposes as a **fully bracketed** expression on the described set of elements.

Nontrivial relations

Let the alphabet $\mathbb{A}_{\mathcal{P}_\delta} := \{a_u : u \in \mathcal{P}_\delta\}$ and $\mathbb{K}\langle\mathbb{A}_{\mathcal{P}_\delta}\rangle$ be the algebra of noncommutative polynomials on $\mathbb{A}_{\mathcal{P}_\delta}$.

Given $u \in \mathsf{Cl}_\delta$, let a^u be the monomial $a_{u^{(1)}} \dots a_{u^{(k)}}$ where $u = u^{(1)} \diagup \cdots \diagup u^{(k)}$ is the unique factorization of u on $\mathcal{P}_\delta$.

– Example –

For $\delta = 0110^\omega$, $a^{00100} = a_0 a_{01} a_0 a_0$.

– Theorem –

If δ is valley-free, then $\mathbf{Cl}_\delta$ is isomorphic to $\mathbb{K}\langle\mathbb{A}_{\mathcal{P}_\delta}\rangle /_{\mathcal{R}_\delta}$ where $\mathcal{R}_\delta$ is the associative algebra ideal of $\mathbf{Cl}_\delta$ generated by the set

$$\min_{\leqslant_\mathrm{s}} \{a^u a_v : u \in \mathsf{Cl}_\delta, v \in \mathcal{P}_\delta, \text{ and } uv \notin \mathsf{Cl}_\delta\}.$$

– Example –

For $\delta = 0110^\omega$, $\mathbb{A}_{\mathcal{P}_\delta} = \{a_0, a_{01}, a_{011}\}$ and $a^{00} a_{01}$ and $a^{01} a_{01}$ are two nontrivial relations of $\mathbf{Cl}_\delta$ (among a total of 8 nontrivial relations).

Presentation by generators and relations

A range map δ is 1-dominated if there is a $k \geqslant 1$ such that for all $k' \geqslant k$, $\delta(1) \geqslant \delta(k')$.

– Proposition –

Let δ be a valley-free range map.

(A) If δ is constant, then

$\delta =$ 

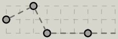
and $\mathbb{A}_{\mathcal{P}_\delta}$ is finite and $\mathcal{R}_\delta$ is the zero space;

(B) Otherwise, if δ is weakly increasing, then

$\delta =$ 

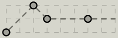
and $\mathbb{A}_{\mathcal{P}_\delta}$ is infinite and $\mathcal{R}_\delta$ is the zero space;

(C) Otherwise, if δ is 1-dominated, then

$\delta =$ 

and $\mathbb{A}_{\mathcal{P}_\delta}$ is finite and $\mathcal{R}_\delta$ is finitely generated;

(D) Otherwise,

$\delta =$ 

and $\mathbb{A}_{\mathcal{P}_\delta}$ is infinite and $\mathcal{R}_\delta$ is infinitely generated.

Examples — Types **A** and **B**

- For any $k \geqslant 0$, $\mathbf{Cl}_{k^\omega}$ is the free associative algebra over the $k+1$ generators $a_0, a_1, \ldots, a_k$.
- $\mathbf{Cl_1}$:
  - First dimensions: $1, 1, 2, 6, 24, 120, 720, 5040$.
  - First dimensions of generators: $0, 1, 1, 3, 13, 71, 461, 3447$ (**A003319**).
  - First generators: a_0, a_{01}, $a_{002}, a_{011}, a_{012}$, $a_{0003}, a_{0013}, a_{0021}, a_{0022}, a_{0023}, a_{0102}, a_{0103}, a_{0111}, a_{0112}, a_{0113}, a_{0121}, a_{0122}, a_{0123}$.
  - Since $\mathbf{Cl_1}$ and $\mathbf{FQSym}$ are both free as associative algebras and they have the same Hilbert series, $\mathbf{Cl_1} \simeq \mathbf{FQSym}$.
- $\mathbf{Cl_2}$:
  - First dimensions: $1, 1, 3, 15, 105, 945, 10395, 135135$ (**A001147**).
  - First dimensions of generators: $0, 1, 2, 10, 74, 706, 8162, 110410$ (**A000698**).
  - First generators: a_0, a_{01}, a_{02}, $a_{003}, a_{004}, a_{011}, a_{012}, a_{013}, a_{014}, a_{021}, a_{022}, a_{023}, a_{024}$.

Examples — Types **C** and **D**

- $\mathbf{Cl}_{010^\omega} \simeq \mathbb{K}\langle a_0, a_{01}\rangle /_{\mathcal{R}_{010^\omega}}$ where $\mathcal{R}_{010^\omega}$ is generated by the two monomials $a_0a_{01}, \quad a_{01}a_{01}$.

- $\mathbf{Cl}_{0110^\omega} \simeq \mathbb{K}\langle a_0, a_{01}, a_{011}\rangle /_{\mathcal{R}_{0110^\omega}}$ where $\mathcal{R}_{0110^\omega}$ is generated by the eight monomials $a_0a_0a_{01}, \quad a_{01}a_{01}, \quad a_{01}a_0a_{01}, \quad a_{011}a_{01}, \quad a_{011}a_0a_{01}, \quad a_0a_{011}, \quad a_{01}a_{011}, \quad a_{011}a_{011}$.

- $\mathbf{Cl}_{210^\omega} \simeq \mathbb{K}\langle a_0, a_1, a_2\rangle /_{\mathcal{R}_{210^\omega}}$ where $\mathcal{R}_{210^\omega}$ is generated by the seven monomials $a_0a_0a_1, \quad a_0a_1a_1, \quad a_1a_0a_1, \quad a_1a_1a_1, \quad a_2a_0a_1, \quad a_2a_1a_1, \quad a_0a_2, \quad a_1a_2, a_2a_2$.

- $\mathbf{Cl}_{021^\omega} \simeq \mathbb{K}\langle a_0, a_{01}, a_{02}, a_{011}, a_{021}, a_{0111}, a_{0211}, a_{01111}, a_{02111}, \ldots\rangle /_{\mathcal{R}_{021^\omega}}$ where $\mathcal{R}_{021^\omega}$ is generated by the infinitely many monomials $a_0a_{02}, \quad a_{01}a_{02}, \quad a_{02}a_{02}, \quad a_{011}a_{02}, \quad a_{021}a_{02}, \quad a_0a_{021}, \quad a_{01}a_{021}, \quad a_{02}a_{021}, \quad a_0a_{0211}, \ldots$

Quotient algebras

For any graded subset $\mathcal{S}$ of Cl_δ, let $\mathbf{Cl}_\mathcal{S}$ be the quotient space of $\mathbf{Cl}_\delta$ defined by

$$\mathbf{Cl}_\mathcal{S} := \mathbf{Cl}_\delta/_{\mathcal{V}_\mathcal{S}}$$

such that $\mathcal{V}_\mathcal{S}$ is the linear span of the set

$$\{\mathsf{F}_u : u \in \mathsf{Cl}_\delta \setminus \mathcal{S}\}.$$

By definition, the set $\{\mathsf{F}_u : u \in \mathcal{S}\}$ is a basis of $\mathbf{Cl}_\mathcal{S}$.

The set $\mathcal{S}$ is closed by suffix reduction if for any $u \in \mathcal{S}$, for all suffixes u' of u, $\mathrm{r}_\delta(u') \in \mathcal{S}$.

– Proposition –

If δ is valley-free and $\mathcal{S}$ is closed by prefix and by suffix reduction, then $\mathbf{Cl}_\mathcal{S}$ is a quotient of the associative algebra $\mathbf{Cl}_\delta$.

Quotient algebra products and intervals

The associative algebra $\mathbf{Cl}_{\mathcal{S}}$ has the interval condition if the support of any product $\mathsf{F}_u \cdot \mathsf{F}_v$ is empty or is an interval of a poset $\mathcal{S}(n)$, $n \geqslant 0$.

When for any $n \geqslant 0$, $\mathcal{S}(n)$ is a join semi-lattice, we denote by $\vee_{\mathcal{S}}$ its join operation.

In this case, $\mathcal{S}$ is join-stable if, for any $n \geqslant 0$ and any $u, v \in \mathcal{S}(n)$, the relation $u_i = v_i$ for an $i \in [n]$ implies that the i-th letter of $u \vee_{\mathcal{S}} v$ is equal to u_i.

– Theorem –

If δ is valley-free and $\mathcal{S}$ is closed by prefix and by suffix reduction, and at least one the following conditions is satisfied:

1. for any $n \geqslant 0$, all posets $\mathcal{S}(n)$ are sublattices of $\mathsf{Cl}_\delta(n)$;
2. for any $n \geqslant 0$, the posets $\mathcal{S}(n)$ is a meet semi-sublattice of $\mathsf{Cl}_\delta(n)$, maximally extendable, and join-stable;

then $\mathbf{Cl}_{\mathcal{S}}$ has the interval condition.

$\mathbf{Hi}_m$ algebras

let $\mathbf{Hi}_m$ be the quotient space $\mathbf{Cl}_{\mathsf{Hi}_{\mathbf{m}}}$.

Since $\mathsf{Hi}_{\mathbf{m}}$ is closed by prefix and by suffix reduction, $\mathbf{Hi}_m$ is an associative algebra quotient of $\mathbf{Cl}_{\mathbf{m}}$.

Since moreover for each $n \geqslant 0$, $\mathsf{Hi}_{\mathbf{m}}(n)$ is a sublattice of $\mathsf{Cl}_{\mathbf{m}}(n)$, $\mathbf{Hi}_m$ has the interval condition.

– Examples –

In $\mathbf{Hi}_1$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{01} = \mathsf{F}_{0111} + \mathsf{F}_{0112} + \mathsf{F}_{0113} + \mathsf{F}_{0122} + \mathsf{F}_{0123},$$

$$\mathsf{F}_{01} \cdot \mathsf{F}_{00} = 0,$$

$$\mathsf{F}_{001} \cdot \mathsf{F}_{0122} = \mathsf{F}_{0011122} + \mathsf{F}_{0011222} + \mathsf{F}_{0012222}.$$

– Examples –

In $\mathbf{Hi}_2$,

$$\mathsf{F}_{02} \cdot \mathsf{F}_{023} = \mathsf{F}_{02223} + \mathsf{F}_{02233} + \mathsf{F}_{02333},$$

$$\mathsf{F}_{011} \cdot \mathsf{F}_{01} = \mathsf{F}_{01111},$$

$$\mathsf{F}_{0015} \cdot \mathsf{F}_{014} = 0.$$

Structure of $\mathbf{Hi}_m$ algebras

By computer exploration, minimal generating families of $\mathbf{Hi}_1$ and $\mathbf{Hi}_2$ up to degree 5 and 4 are resp.

$$\mathsf{F}_{0}, \quad \mathsf{F}_{00}, \quad \mathsf{F}_{001}, \mathsf{F}_{011}, \quad \mathsf{F}_{0002}, \mathsf{F}_{0011}, \mathsf{F}_{0012}, \mathsf{F}_{0022}, \mathsf{F}_{0112}, \mathsf{F}_{0122},$$
$$\mathsf{F}_{00003}, \mathsf{F}_{00013}, \mathsf{F}_{00023}, \mathsf{F}_{00033}, \mathsf{F}_{00112}, \mathsf{F}_{00113}, \mathsf{F}_{00122}, \mathsf{F}_{00123}, \mathsf{F}_{00133}, \mathsf{F}_{00222},$$
$$\mathsf{F}_{00223}, \mathsf{F}_{00233}, \mathsf{F}_{01113}, \mathsf{F}_{01122}, \mathsf{F}_{01123}, \mathsf{F}_{01133}, \mathsf{F}_{01223}, \mathsf{F}_{01233},$$

and

$$\mathsf{F}_{0}, \quad \mathsf{F}_{00}, \mathsf{F}_{01}, \quad \mathsf{F}_{001}, \mathsf{F}_{002}, \mathsf{F}_{003}, \mathsf{F}_{012}, \mathsf{F}_{013}, \mathsf{F}_{022}, \mathsf{F}_{023},$$
$$\mathsf{F}_{0004}, \mathsf{F}_{0005}, \mathsf{F}_{0012}, \mathsf{F}_{0013}, \mathsf{F}_{0014}, \mathsf{F}_{0015}, \mathsf{F}_{0022}, \mathsf{F}_{0023}, \mathsf{F}_{0024}, \mathsf{F}_{0025}, \mathsf{F}_{0033}, \mathsf{F}_{0034},$$
$$\mathsf{F}_{0035}, \mathsf{F}_{0044}, \mathsf{F}_{0045}, \mathsf{F}_{0114}, \mathsf{F}_{0115}, \mathsf{F}_{0122}, \mathsf{F}_{0123}, \mathsf{F}_{0124}, \mathsf{F}_{0125}, \mathsf{F}_{0133}, \mathsf{F}_{0134}, \mathsf{F}_{0135},$$
$$\mathsf{F}_{0144}, \mathsf{F}_{0145}, \mathsf{F}_{0223}, \mathsf{F}_{0224}, \mathsf{F}_{0225}, \mathsf{F}_{0234}, \mathsf{F}_{0235}, \mathsf{F}_{0244}, \mathsf{F}_{0245}.$$

The number of minimal generators of $\mathbf{Hi}_1$ and $\mathbf{Hi}_2$, begin resp. by

$$0, 1, 1, 2, 6, 18, 59, 196, 669,$$

and

$$0, 1, 2, 7, 33, 168, 900, 4980.$$

For any $m \geqslant 1$, Hi_m is not free as an associative algebra.

$\mathbf{Ca}_m$ algebras

Let $\mathbf{Ca}_m$ be the quotient space $\mathbf{Cl}_{\mathsf{Ca}_\mathbf{m}}$.

Since $\mathsf{Ca}_\mathbf{m}$ is closed by prefix and by suffix reduction, $\mathbf{Ca}_m$ is an associative algebra quotient of $\mathbf{Cl}_\mathbf{m}$.

Since moreover $\mathsf{Ca}_\mathbf{m}$ is maximally extendable and join-stable, and for each $n \geqslant 0$, $\mathsf{Ca}_\mathbf{m}(n)$ is a meet semi-sublattice of $\mathsf{Cl}_\mathbf{m}(n)$, $\mathbf{Ca}_m$ has the interval condition.

– Examples –

In $\mathbf{Ca}_1$,

$$\mathsf{F}_{0} \cdot \mathsf{F}_{01} = \mathsf{F}_{001} + \mathsf{F}_{002} + \mathsf{F}_{012},$$

$$\mathsf{F}_{010} \cdot \mathsf{F}_{0020} = \mathsf{F}_{0100020} + \mathsf{F}_{0100030} + \mathsf{F}_{0101030} + \mathsf{F}_{0100050} + \mathsf{F}_{0101050} + \mathsf{F}_{0103050}$$

– Examples –

In $\mathbf{Ca}_2$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{0014} = 0,$$

$$\begin{aligned}\mathsf{F}_{020} \cdot \mathsf{F}_{02} = {} & \mathsf{F}_{02002} + \mathsf{F}_{02005} + \mathsf{F}_{02006} + \mathsf{F}_{02007} + \mathsf{F}_{02008} + \mathsf{F}_{02012} + \mathsf{F}_{02015} \\ & + \mathsf{F}_{02016} + \mathsf{F}_{02017} + \mathsf{F}_{02018} + \mathsf{F}_{02045} + \mathsf{F}_{02046} + \mathsf{F}_{02047} + \mathsf{F}_{02048} \\ & + \mathsf{F}_{02056} + \mathsf{F}_{02057} + \mathsf{F}_{02058} + \mathsf{F}_{02067} + \mathsf{F}_{02068}.\end{aligned}$$

Structure of $\mathbf{Ca}_m$ algebras

$\mathbf{Ca}_1$ is isomorphic to $\mathbf{PBT}$. The isomorphism sends F_u to F_t where t is the binary tree having u has Tamari diagram (notion introduced in [**Pallo**, 1986]).

By computer exploration, minimal generating families of $\mathbf{Ca}_1$ and $\mathbf{Ca}_2$ up to respectively up to degree 5 and 4 are resp.

$$\mathsf{F}_{0}, \quad \mathsf{F}_{00}, \quad \mathsf{F}_{000}, \mathsf{F}_{001}, \quad \mathsf{F}_{0000}, \mathsf{F}_{0001}, \mathsf{F}_{0002}, \mathsf{F}_{0010}, \mathsf{F}_{0012}, \\ \mathsf{F}_{00000}, \mathsf{F}_{00001}, \mathsf{F}_{00002}, \mathsf{F}_{00003}, \mathsf{F}_{00010}, \mathsf{F}_{00012}, \mathsf{F}_{00013}, \mathsf{F}_{00020}, \mathsf{F}_{00023}, \mathsf{F}_{00100}, \\ \mathsf{F}_{00101}, \mathsf{F}_{00103}, \mathsf{F}_{00120}, \mathsf{F}_{00123},$$

and

$$\mathsf{F}_{0}, \quad \mathsf{F}_{00}, \mathsf{F}_{01}, \quad \mathsf{F}_{000}, \mathsf{F}_{002}, \mathsf{F}_{003}, \mathsf{F}_{010}, \mathsf{F}_{012}, \mathsf{F}_{013}, \mathsf{F}_{023}, \\ \mathsf{F}_{0000}, \mathsf{F}_{0003}, \mathsf{F}_{0004}, \mathsf{F}_{0005}, \mathsf{F}_{0014}, \mathsf{F}_{0015}, \mathsf{F}_{0020}, \mathsf{F}_{0023}, \mathsf{F}_{0024}, \mathsf{F}_{0025}, \mathsf{F}_{0030}, \mathsf{F}_{0034}, \mathsf{F}_{0035}, \\ \mathsf{F}_{0045}, \mathsf{F}_{0100}, \mathsf{F}_{0104}, \mathsf{F}_{0105}, \mathsf{F}_{0120}, \mathsf{F}_{0124}, \mathsf{F}_{0125}, \mathsf{F}_{0130}, \mathsf{F}_{0134}, \mathsf{F}_{0135}, \mathsf{F}_{0145}, \mathsf{F}_{0204}, \mathsf{F}_{0205}, \\ \mathsf{F}_{0230}, \mathsf{F}_{0234}, \mathsf{F}_{0235}, \mathsf{F}_{0245}.$$

The numbers of minimal generators of $\mathbf{Ca}_2$ begins by

$$0, 1, 2, 7, 30, 149, 788, 4332.$$

$\mathbf{Ca}_0$ and $\mathbf{Ca}_1$ are free as associative algebras but $\mathbf{Ca}_m$, $m \geqslant 2$, is not.

4. Some open questions

Increasing trees

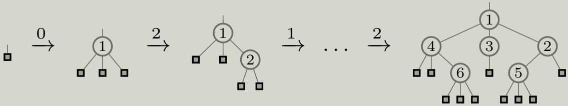
When $\delta(1) = 0$, δ is rooted.

Given $u \in \mathsf{Cl}_\delta(n)$ where δ is rooted and weakly increasing, let $\mathrm{tree}_\delta(u)$ be the δ-increasing tree defined recursively as follows:

- If $u = \epsilon$, then $\mathrm{tree}_\delta(u)$ is the leaf;
- Otherwise, $u = u'a$ with $0 \leqslant a \leqslant \delta(n)$, and $\mathrm{tree}_\delta(u)$ is obtained by grafting on the $a+1$-st leaf of $\mathrm{tree}_\delta\,(u')$ a node labeled by n having $1 + \delta(n+1) - \delta(n)$ leaves.

– Example –

For $\delta := 0233579^\omega$ and $u := 021042$, the $\mathrm{tree}_\delta(u)$ grows as follows:



Alternative posets from increasing trees

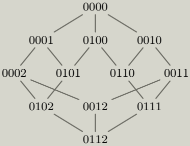
Let δ be a rooted and weakly increasing range map.

Let $\preccurlyeq'$ be the reflexive and transitive closure of the relation $\lessdot'$ on $\mathsf{Cl}_\delta(n)$ where $u \lessdot' v$ if v is obtained from u by incrementing a letter u_i when all the children of the node labeled by i in $\mathrm{tree}_\delta(u)$ are leaves excepted possibly the first one.

– Examples –

$(\mathsf{Cl}_{\mathbf{1}}(n), \preccurlyeq')$ is isomorphic to the right weak order.

The Hasse diagram of $(\mathsf{Cl}_{01122^\omega}(4), \preccurlyeq')$ is



– Conjecture –

For any rooted and weakly increasing range map δ and any $n \geqslant 0$, the poset $(\mathsf{Cl}_\delta(n), \preccurlyeq')$ is a lattice.

Dune posets

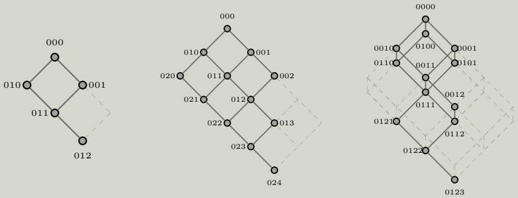
Let Du_δ be the subset of Cl_δ containing all δ-hills that are δ-cliffs u such that for any $i \in [n-1]$, $|u_i - u_{i+1}| \leqslant |\delta(i) - \delta(i+1)|$.

Cardinalities of $\mathsf{Du}_{\mathbf{1}}(n)$: $1, 1, 2, 5, 13, 35, 96, 267, \dots$ (A005773, directed animals).

Cardinalities of $\mathsf{Du}_{\mathbf{2}}(n)$: $1, 1, 3, 12, 51, 226, 1025, 4724, \dots$ (A180898, some meanders [**Banderier et al.**, 2016]).

– Example –

The Hasse diagrams of $\mathsf{Du}_{\mathbf{1}}(3)$, $\mathsf{Du}_{\mathbf{2}}(3)$, and $\mathsf{Du}_{\mathbf{1}}(4)$ are



– Project –

Study the dune posets and their associative algebras.

Coproducts

As already mentioned, $\mathbf{FQSym}$ and its subalgebras are endowed with a coproduct.

A coproduct on a space $\mathcal{A}$ is a map

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

which, intuitively, splits any element of $\mathcal{A}$ in two smaller parts, in several different ways.

If $(\mathcal{A}, \cdot)$ is an associative algebra, a coproduct Δ is compatible with $\cdot$ if for all $f, g \in \mathcal{A}$,

$$\Delta\,(f \cdot g) = \Delta(f)\Delta(g).$$

– Question –

Introduce a (noncocommutative) coproduct on $\mathbf{Cl}_\delta$ compatible with its product.

Determine in what extent this coproduct is still well-defined on its quotients $\mathbf{Cl}_\mathcal{S}$.