Associative algebras of cliffs

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April 24, 2020

Outline

1. Algebras and orders

2. Algebras on cliffs

3. Some open questions

1. Algebras and orders

Malvenuto-Reutenauer algebra

The Malvenuto-Reutenauer algebra [**Malvenuto, Reutenauer**, 1995] $(\mathbf{FQSym}, \cdot, 1)$ is the unital associative algebra defined as follows.

- $\mathbf{FQSym}$ is the $\mathbb{K}$-linear span $\mathbb{K}\langle\mathfrak{S}\rangle$ of all permutations.

  The set $\{\mathsf{F}_\sigma : \sigma \in \mathfrak{S}\}$ is the fundamental basis of $\mathbf{FQSym}$.

– Example –

The linear combination $\frac{1}{2}\mathsf{F}_{312} - \mathsf{F}_{43512} + 2\mathsf{F}_\epsilon - \mathsf{F}_{21}$ is an element of $\mathbf{FQSym}$.

- $\cdot$ is the shifted shuffle product, the associative product defined by

$$\mathsf{F}_\sigma \cdot \mathsf{F}_\nu := \sum_{\pi \in \sigma \overline{\sqcup\!\sqcup} \nu} \mathsf{F}_\pi .$$

- 1 is defined as F_ϵ where ϵ is the empty permutation.

– Example –

$$\mathsf{F}_{312} \cdot \mathsf{F}_{21} = \mathsf{F}_{31254} + \mathsf{F}_{31524} + \mathsf{F}_{31542} + \mathsf{F}_{35124} + \mathsf{F}_{35142} + \mathsf{F}_{35412} + \mathsf{F}_{53124} + \mathsf{F}_{53142} + \mathsf{F}_{53412} + \mathsf{F}_{54312}$$

Right weak order

The right weak order is the order relation $\preccurlyeq$ on $\mathfrak{S}(n)$ defined as the reflexive and transitive closure of the relation $\lessdot$ satisfying

$$u \, \mathsf{ab} \, v \quad \lessdot \quad u \, \mathsf{ba} \, v$$

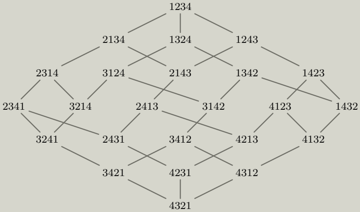
where $u, v \in \mathbb{N}^*$, and a and b are letters such that a $<$ b.

– Example –

Since $41352 \lessdot 41532 \lessdot 45132 \lessdot 45312$, one has $41352 \preccurlyeq 45312$.

– Example –

Hasse diagram of $(\mathfrak{S}(4), \preccurlyeq)$:



Product of **FQSym** and right weak order

Let $\diagup$ and $\diagdown$ be the two operations on $\mathfrak{S}$ defined by

$$\sigma \diagup \nu := \sigma \uparrow_{|\sigma|}(\nu) \qquad \text{and} \qquad \sigma \diagdown \nu := \uparrow_{|\sigma|}(\nu)\, \sigma.$$

– Example –

$312 \diagup 21 = 31254$

– Example –

$312 \diagdown 21 = 54312$

– Proposition –

For any permutations σ and ν,

$$\mathsf{F}_\sigma \cdot \mathsf{F}_\nu = \sum_{\substack{\pi \in \mathfrak{S} \\ \sigma \diagup \nu \preccurlyeq \pi \preccurlyeq \sigma \diagdown \nu}} \mathsf{F}_\pi.$$

– Example –

$\mathsf{F}_{312} \cdot \mathsf{F}_{21}$ is the formal sum of all the F_π where $\pi \in [312 \diagup 21, 312 \diagdown 21] = [31254, 54312]$.

Multiplicative bases

A basis is multiplicative if the product of two basis element is a single basis element.

The right weak order can be used to build multiplicative bases of $\mathbf{FQSym}$.

Let

$$\mathsf{E}_\sigma := \sum_{\substack{\nu \in \mathfrak{S} \\ \sigma \preccurlyeq \nu}} \mathsf{F}_\nu \qquad \text{and} \qquad \mathsf{H}_\sigma := \sum_{\substack{\nu \in \mathfrak{S} \\ \nu \preccurlyeq \sigma}} \mathsf{F}_\nu .$$

– Example –

$$\mathsf{E}_{4123} = \mathsf{F}_{4123} + \mathsf{F}_{4132} + \mathsf{F}_{4213} + \mathsf{F}_{4231} + \mathsf{F}_{4312} + \mathsf{F}_{4321}$$

– Proposition –

For any permutations σ and ν,

$$\mathsf{E}_\sigma \cdot \mathsf{E}_\nu = \mathsf{E}_{\sigma \,\diagup\, \nu} \qquad \text{and} \qquad \mathsf{H}_\sigma \cdot \mathsf{H}_\nu = \mathsf{H}_{\sigma \,\diagdown\, \nu}.$$

Generators and relations

A subset $\mathcal{G}$ of an associative algebra $\mathcal{A}$ is a minimal generating set of $\mathcal{A}$ if the smallest subalgebra of $\mathcal{A}$ containing $\mathcal{G}$ is $\mathcal{A}$ itself and $\mathcal{G}$ is minimal for set inclusion.

A permutation σ is connected if $\sigma \neq \epsilon$ and no proper prefix of σ is a permutation.

– Example –

The permutation 43257816 is connected.

– Example –

The permutation $4325176 = 43251 \,/\, 21$ is not.

– Theorem [Duchamp, Hivert, Thibon, 2002] **–**

The set $\mathcal{G}$ of all E_σ such that σ is connected is a minimal generating set of $\mathbf{FQSym}$.

Moreover, $\mathbf{FQSym}$ is free as a unital associative algebra and $\mathbf{FQSym} \simeq \mathbb{K}\langle\mathcal{G}\rangle$.

This is a consequence of the fact that any permutation σ decomposes in a unique way as $\sigma = \nu^{(1)} \,/\, \cdots \,/\, \nu^{(\ell)}$ where all $\nu^{(i)}$ are connected.

Subalgebras and subposets

FQSym admits a lot of subalgebras:

- **FSym**, the algebra of standard Young tableaux[**Poirier, Reutenauer**, 1995], [**Duchamp, Hivert, Thibon**, 2002];
- **PBT**, the algebra of binary trees [**Loday, Ronco**, 1998], [**Hivert, Novelli, Thibon**, 2005];
- **Sym**, the algebra of integer compositions [**Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon**, 1995];
- **Baxter**, the algebra of pairs of twin binary trees [**Law, Reading**, 2012], [**G.**, 2012];
- **Bell**, the algebra of set partitions [**Rey**, 2007].

Each one is constructed from a surjective map $\theta : \mathfrak{S} \to C$, where C is one of the previous sets of objects, as the subalgebra spanned by the elements

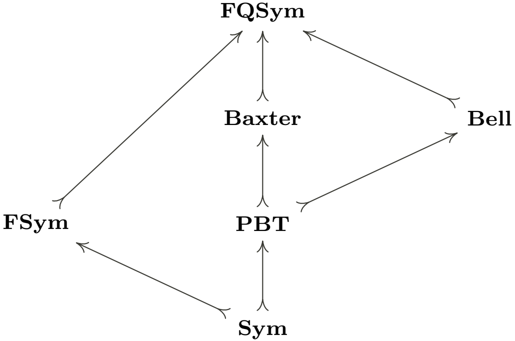
$$\mathsf{F}_x := \sum_{\substack{\sigma \in \mathfrak{S} \\ \theta(\sigma) = x}} \mathsf{F}_\sigma .$$

There are also posets $(C, \preccurlyeq)$ and operations $\diagup$ and $\diagdown$ on C such that

$$\mathsf{F}_x \cdot \mathsf{F}_y = \sum_{\substack{z \in C \\ x \diagup y \preccurlyeq z \preccurlyeq x \diagdown y}} \mathsf{F}_z .$$

Diagram of algebras

These algebras fit into the following diagram of injective algebra morphisms:



Note that these algebras are also endowed with coproducts so that they are in fact Hopf bialgebras.

Algebra of binary trees — Construction

Let $\theta : \mathfrak{S} \to \mathrm{BT}$ be the right to left insertion algorithm in binary search trees.

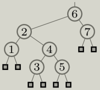
– Theorem [Hivert, Novelli, Thibon, 2005] –

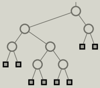
The family $\{\mathsf{F}_{\mathsf{t}} : \mathsf{t} \in \mathrm{BT}\}$ where

$$\mathsf{F}_{\mathsf{t}} := \sum_{\substack{\sigma \in \mathfrak{S} \\ \theta(\sigma) = \mathsf{t}}} \mathsf{F}_{\sigma}$$

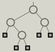
span a subalgebra of **FQSym**.

– Example –

$\theta(1354726) =$ 

$=$ 

– Example –

F  $= \mathsf{F}_{2143} + \mathsf{F}_{2413} + \mathsf{F}_{4213}$

Algebra of binary trees — Tamari order

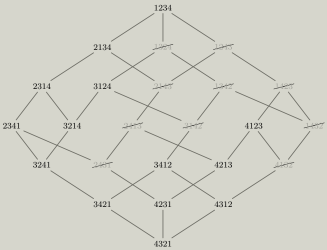
Let BT be the set of all binary trees.

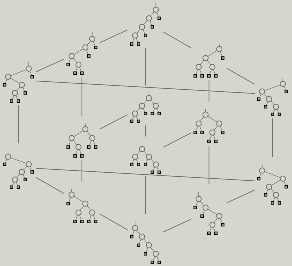
It is known that $\mathrm{BT}(n)$ is in one-to-one correspondence with the set of permutations avoiding the pattern 132.

The restriction of the right weak order on these permutations is the Tamari order [**Tamari**, 1962], [**Hivert, Novelli, Thibon**, 2005].

– Example –

Hasse diagrams of $(\mathrm{BT}(4), \preccurlyeq)$:

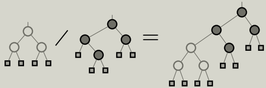




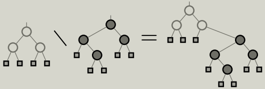
Algebra of binary trees — Product

For any $\mathsf{t}, \mathsf{s} \in \mathrm{BT}$, $\mathsf{t} \diagup \mathsf{s}$ (resp. $\mathsf{t} \diagdown \mathsf{s}$) is the binary tree obtained by grafting the root of t (resp. s) onto the first (resp. last) leaf of s (resp. t).

– Example –

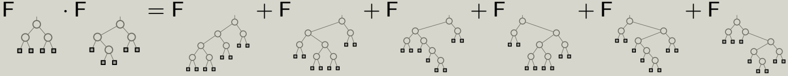


– Example –



The product in $\mathbf{PBT}$ of two basis elements F_t and F_s is the formal sum of the elements of the Tamari interval $[\mathsf{t} \diagup \mathsf{s}, \mathsf{t} \diagdown \mathsf{s}]$.

– Example –



Motivation: a new order on permutations

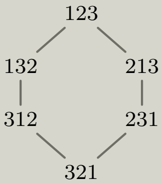
The objectives of this work are to

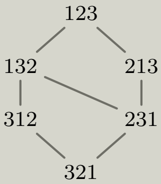
1. introduce a new order relation on permutations;
2. consider the analog of $\mathbf{FQSym}$ w.r.t. this alternative order;
3. try to construct a similar hierarchy of algebras.

For this, we consider an order extension of the right weak order and generalizations of permutations.

– Example –

Here are both the Hasse diagrams of the right weak order on permutations of size 3 and of the considered order extension:





2. Algebras on cliffs

Cliffs and posets

A range map is a map $\delta := \mathbb{N} \setminus \{0\} \to \mathbb{N}$.

A δ-cliff of size n is a word $u \in \mathbb{N}^n$ such that for all $i \in [n]$, $0 \leqslant u_i \leqslant \delta(i)$.

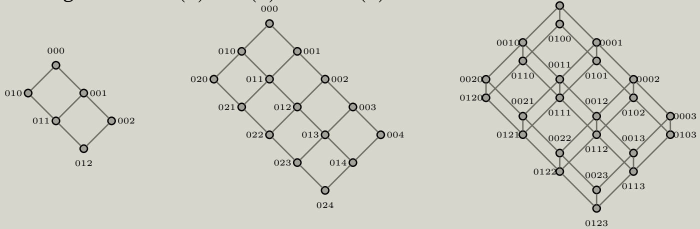
The graded collection of all δ-cliffs is denoted by Cl_δ.

Let $\preccurlyeq$ be the partial order relation on each $\mathsf{Cl}_\delta(n)$ wherein $u \preccurlyeq v$ if $u_i \leqslant v_i$ for all $i \in [n]$.

For any $m \geqslant 0$, let $\mathbf{m}$ be the map defined by $\mathbf{m}(i) := m(i-1)$.

– Example –

The Hasse diagrams of $\mathsf{Cl}_{\mathbf{1}}(3)$, $\mathsf{Cl}_{\mathbf{2}}(3)$, and $\mathsf{Cl}_{\mathbf{1}}(4)$ are



The posets $\mathsf{Cl}_{\mathbf{1}}(n)$ have been studied in [**Denoncourt**, 2013].

Lehmer codes

Let leh be the map sending any permutation σ to the **1**-cliff u wherein u_i is the number of letters a at the right of i in σ such that $i > a$. This is a variation of the Lehmer code [**Lehmer**, 1960] of a permutation.

This map $\mathrm{leh} : \mathfrak{S}(n) \to \mathsf{Cl}_{\mathbf{1}}(n)$ is a bijection.

– Example –

$$\mathrm{leh}(436512) = 002323$$

A poset $\mathcal{P}_2$ is an order extension of a poset $\mathcal{P}_1$ if there is a bijective map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ such that $x \preccurlyeq_1 y$ implies $\phi(x) \preccurlyeq_2 \phi(y)$.

– Proposition –

For any $n \geqslant 0$, the poset $(\mathsf{Cl}_{\mathbf{1}}(n), \preccurlyeq)$ is an order extension of the right weak order on $\mathfrak{S}(n)$ for the map leh.

Algebras on cliffs

Let $\mathbf{Cl}_\delta$ be the $\mathbb{K}$-linear span of all δ-cliffs and $\{\mathsf{F}_u : u \in \mathsf{Cl}_\delta\}$ a basis.

The δ-reduction of a word $w \in \mathbb{N}^n$ is the δ-cliff $\mathrm{r}_\delta(w)$ satisfying

$$(\mathrm{r}_\delta(w))_i = \min\{w_i, \delta(i)\}.$$

– Example –

$\mathrm{r}_{\mathbf{1}}(212066) = 012045$

– Example –

$\mathrm{r}_{\mathbf{2}}(212066) = 012066$

Let $\cdot$ be the product on $\mathbf{Cl}_\delta$ defined by

$$\mathsf{F}_u \cdot \mathsf{F}_v := \sum_{\substack{uv' \in \mathsf{Cl}_\delta \\ \mathrm{r}_\delta(v') = v}} \mathsf{F}_{uv'}.$$

– Example –

In $\mathbf{Cl}_{\mathbf{2}}$,

$$\mathsf{F}_{00} \cdot \mathsf{F}_{\mathbf{0}11} = \mathsf{F}_{00\mathbf{0}11} + \mathsf{F}_{00\mathbf{1}11} + \mathsf{F}_{00\mathbf{2}11} + \mathsf{F}_{00\mathbf{3}11} + \mathsf{F}_{00\mathbf{4}11}.$$

Associativity

In general, the product of $\mathbf{Cl}_\delta$ is not associative.

– Example –

For $\delta := 102^\omega$, we have

$$(\mathsf{F}_1 \cdot \mathsf{F}_0) \cdot \mathsf{F}_1 = \mathsf{F}_{10} \cdot \mathsf{F}_1 = \mathsf{F}_{101} + \mathsf{F}_{102}$$

and

$$\mathsf{F}_1 \cdot (\mathsf{F}_0 \cdot \mathsf{F}_1) = \mathsf{F}_1 \cdot 0 = 0.$$

A range map is valley-free (or unimodal) if there is no $i_1 \leqslant i_2 \leqslant i_3$ such that $\delta(i_1) > \delta(i_2) < \delta(i_3)$.

– Theorem –

The product $\cdot$ of $\mathbf{Cl}_\delta$ is associative iff δ is a valley-free range map.

Over and under operations

Let

$$\diagup,\ \diagdown : \mathsf{Cl}_\delta(n) \times \mathsf{Cl}_\delta(m) \to \mathbb{N}^{n+m}$$

be the two operations defined by $u \diagup v := uv$ and $u \diagdown v := uv'$ where v' is the word of length $|v|$ satisfying, for any $i \in [|v|]$,

$$v'_i = \begin{cases} \delta(|u| + i) & \text{if } v_i = \delta(i), \\ v_i & \text{otherwise.} \end{cases}$$

– Example –

For $\delta = 112334^\omega$, $010 \diagup 1021 = 0101021$ and $010 \diagdown \mathbf{1}0\mathbf{2}1 = 010\mathbf{3}0\mathbf{4}1$.

– Example –

For $\delta = 210^\omega$, $21 \diagdown 1\mathbf{1} = 211\mathbf{0}$. This word is not a δ-cliff.

Product and cliff posets

For $w \in \mathbb{N}^*$, let $\chi_\delta(w)$ defined as $1 \in \mathbb{K}$ if w is a δ-cliff and as $0 \in \mathbb{K}$ otherwise.

– Theorem –

For any $u, v \in \mathsf{Cl}_\delta$, we have in $\mathbf{Cl}_\delta$,

$$\mathsf{F}_u \cdot \mathsf{F}_v = \chi_\delta(u \diagup v) \sum_{\substack{w \in \mathsf{Cl}_\delta \\ u \diagup v \preccurlyeq w \preccurlyeq u \diagdown v}} \mathsf{F}_w.$$

– Example –

In $\mathbf{Cl}_{01120^\omega}$, since $01 \diagup 010 = 01010 \in \mathsf{Cl}_{01120^\omega}$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{010} = \mathsf{F}_{01010} + \mathsf{F}_{01020} + \mathsf{F}_{01110} + \mathsf{F}_{01120}.$$

– Example –

In $\mathbf{Cl}_{01120^\omega}$, since $01 \diagup 011 = 01011 \notin \mathsf{Cl}_{01120^\omega}$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{011} = 0.$$

Multiplicative bases

Let

$$\mathsf{E}_u := \sum_{\substack{v \in \mathsf{Cl}_\delta \\ u \preccurlyeq v}} \mathsf{F}_v \qquad \text{and} \qquad \mathsf{H}_u := \sum_{\substack{v \in \mathsf{Cl}_\delta \\ v \preccurlyeq u}} \mathsf{F}_v.$$

– Examples –

For $\delta := 1021^\omega$,

$$\mathsf{E}_{10010} = \mathsf{F}_{10010} + \mathsf{F}_{10011} + \mathsf{F}_{10110} + \mathsf{F}_{10111} + \mathsf{F}_{10210} + \mathsf{F}_{10211},$$

and

$$\mathsf{H}_{10010} = \mathsf{F}_{10010} + \mathsf{F}_{10000} + \mathsf{F}_{00010} + \mathsf{F}_{00000}.$$

By triangularity, $\{\mathsf{E}_u : u \in \mathsf{Cl}_\delta\}$ and $\{\mathsf{H}_u : u \in \mathsf{Cl}_\delta\}$ are bases of $\mathbf{Cl}_\delta$.

– Proposition –

For any $u, v \in \mathsf{Cl}_\delta$, we have in $\mathbf{Cl}_\delta$,

$$\mathsf{E}_u \cdot \mathsf{E}_v = \chi_\delta(u \diagup v)\mathsf{E}_{u \diagup v} \qquad \text{and} \qquad \mathsf{H}_u \cdot \mathsf{H}_v = \mathsf{H}_{\mathrm{r}_\delta(u \diagdown v)}.$$

Minimal generating set

A nonempty δ-cliff u is δ-prime if the decomposition $u = v \diagup w$ with $v, w \in \mathsf{Cl}_\delta$ implies $(v, w) \in \{(\epsilon, u), (u, \epsilon)\}$.

The set of all these elements is denoted by $\mathcal{P}_\delta$.

– Examples –

Let $\delta := 021^\omega$.

The δ-cliffs 0, 01, and 021111 are δ-prime.

The δ-cliff $0210 = 021 \diagup 0$ is not.

– Lemma –

Any nonempty δ-cliff admits exactly one suffix which is δ-prime.

– Proposition –

The set $\{\mathsf{E}_u : u \in \mathcal{P}_\delta\}$ is a minimal generating set of the magmatic algebra Cl_δ.

This is a consequence of the fact that, by the previous lemma, any δ-cliff decomposes as a **fully bracketed** expression on the described set of elements.

Nontrivial relations

– Proposition –

If δ is valley-free, any nontrivial relation of the associative algebra $\mathbf{Cl}_\delta$ express as

$$\mathsf{E}_{u^{(1)}} \cdot \ldots \cdot \mathsf{E}_{u^{(k)}} \cdot \mathsf{E}_v$$

where $k \geqslant 1$, $u^{(i)} \in \mathcal{P}_\delta$ for all $i \in [k]$, $v \in \mathcal{P}_\delta$, $u^{(1)} \mathbin{/} \ldots \mathbin{/} u^{(k)} \in \mathsf{Cl}_\delta$, and $u^{(1)} \mathbin{/} \ldots \mathbin{/} u^{(k)} \mathbin{/} v \notin \mathsf{Cl}_\delta$.

A nontrivial relation is minimal if for all $j \in [2, k]$,

$$\mathsf{E}_{u^{(j)}} \cdot \ldots \cdot \mathsf{E}_{u^{(k)}} \cdot \mathsf{E}_v$$

is not a nontrivial relation.

Let $\mathcal{R}_\delta$ be the set of all minimal nontrivial relations of $\mathbf{Cl}_\delta$.

– Example –

For $\delta = 0110^\omega$, one has $\mathcal{P}_\delta = \{0, 01, 011\}$, and $\mathcal{R}_\delta$ contains

$$\mathsf{E}_{00} \cdot \mathsf{E}_{01}, \qquad \mathsf{E}_{01} \cdot \mathsf{E}_{01}, \qquad \mathsf{E}_{011} \cdot \mathsf{E}_{0} \cdot \mathsf{E}_{01}$$

among a total of height minimal nontrivial relations.

Classification

A range map δ is 1-dominated if there is a $k \geqslant 1$ such that for all $k' \geqslant k$, $\delta(1) \geqslant \delta(k')$.

– Theorem –

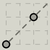
Let δ be a valley-free range map.

(A) If δ is constant, then

$\delta =$ 

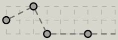
and $\mathcal{P}_\delta$ is finite and $\mathcal{R}_\delta = \emptyset$;

(B) Otherwise, if δ is weakly increasing, then

$\delta =$ 

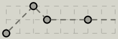
and $\mathcal{P}_\delta$ is infinite and $\mathcal{R}_\delta = \emptyset$;

(C) Otherwise, if δ is 1-dominated, then

$\delta =$ 

and $\mathcal{P}_\delta$ and $\mathcal{R}_\delta$ are both finite;

(D) Otherwise,

$\delta =$ 

and $\mathcal{P}_\delta$ and $\mathcal{R}_\delta$ are both infinite.

Examples — Types **A** and **B**

- For any $k \geqslant 0$, $\mathbf{Cl}_{k^\omega}$ is the free associative algebra over the $k+1$ generators $\mathsf{E}_0, \mathsf{E}_1, \ldots, \mathsf{E}_k$.
- $\mathbf{Cl_1}$:
  - First dimensions: $1, 1, 2, 6, 24, 120, 720, 5040$.
  - First dimensions of generators: $0, 1, 1, 3, 13, 71, 461, 3447$ (**A003319**).
  - First generators: E_0, E_{01}, $\mathsf{E}_{002}, \mathsf{E}_{011}, \mathsf{E}_{012}$, $\mathsf{E}_{0003}, \mathsf{E}_{0013}, \mathsf{E}_{0021}, \mathsf{E}_{0022}, \mathsf{E}_{0023}, \mathsf{E}_{0102}, \mathsf{E}_{0103}, \mathsf{E}_{0111}, \mathsf{E}_{0112}, \mathsf{E}_{0113}, \mathsf{E}_{0121}, \mathsf{E}_{0122}, \mathsf{E}_{0123}$.
  - Since $\mathbf{Cl_1}$ and $\mathbf{FQSym}$ are both free as associative algebras and they have the same Hilbert series, $\mathbf{Cl_1} \simeq \mathbf{FQSym}$.
- $\mathbf{Cl_2}$:
  - First dimensions: $1, 1, 3, 15, 105, 945, 10395, 135135$ (**A001147**).
  - First dimensions of generators: $0, 1, 2, 10, 74, 706, 8162, 110410$ (**A000698**).
  - First generators: E_0, $\mathsf{E}_{01}, \mathsf{E}_{02}$, $\mathsf{E}_{003}, \mathsf{E}_{004}, \mathsf{E}_{011}, \mathsf{E}_{012}, \mathsf{E}_{013}, \mathsf{E}_{014}, \mathsf{E}_{021}, \mathsf{E}_{022}, \mathsf{E}_{023}, \mathsf{E}_{024}$.

Examples — Types **C** and **D**

- $\mathbf{Cl}_{010^\omega}$:
  - Generators: $\mathsf{E}_0,\ \ \mathsf{E}_{01}.$
  - Relations: $\mathsf{E}_0 \cdot \mathsf{E}_{01},\ \ \mathsf{E}_{01} \cdot \mathsf{E}_{01}.$
- $\mathbf{Cl}_{0110^\omega}$:
  - Generators: $\mathsf{E}_0,\ \ \mathsf{E}_{01},\ \ \mathsf{E}_{011}.$
  - Relations: $\mathsf{E}_0 \cdot \mathsf{E}_0 \cdot \mathsf{E}_{01},\ \ \mathsf{E}_{01} \cdot \mathsf{E}_{01},\ \ \mathsf{E}_{01} \cdot \mathsf{E}_0 \cdot \mathsf{E}_{01},\ \ \mathsf{E}_{011} \cdot \mathsf{E}_{01},$ $\mathsf{E}_{011} \cdot \mathsf{E}_0 \cdot \mathsf{E}_{01},\ \ \mathsf{E}_0 \cdot \mathsf{E}_{011},\ \ \mathsf{E}_{01} \cdot \mathsf{E}_{011},\ \ \mathsf{E}_{011} \cdot \mathsf{E}_{011}.$
- $\mathbf{Cl}_{210^\omega}$:
  - Generators: $\mathsf{E}_0,\ \ \mathsf{E}_1,\ \ \mathsf{E}_2.$
  - Relations; $\mathsf{E}_0 \cdot \mathsf{E}_0 \cdot \mathsf{E}_1,\ \ \mathsf{E}_0 \cdot \mathsf{E}_1 \cdot \mathsf{E}_1,\ \ \mathsf{E}_1 \cdot \mathsf{E}_0 \cdot \mathsf{E}_1,\ \ \mathsf{E}_1 \cdot \mathsf{E}_1 \cdot \mathsf{E}_1,$ $\mathsf{E}_2 \cdot \mathsf{E}_0 \cdot \mathsf{E}_1,\ \ \mathsf{E}_2 \cdot \mathsf{E}_1 \cdot \mathsf{E}_1,\ \ \mathsf{E}_0 \cdot \mathsf{E}_2,\ \ \mathsf{E}_1 \cdot \mathsf{E}_2,\ \ \mathsf{E}_2 \cdot \mathsf{E}_2.$
- $\mathbf{Cl}_{021^\omega}$:
  - Generators: $\mathsf{E}_0, \mathsf{E}_{01},\ \ \mathsf{E}_{02},\ \ \mathsf{E}_{011},\ \ \mathsf{E}_{021},\ \ \mathsf{E}_{0111},\ \ \mathsf{E}_{0211},\ \ \mathsf{E}_{01111},$ $\mathsf{E}_{02111},\ \ \ldots.$
  - Relations $\mathsf{E}_{02},\ \ \mathsf{E}_{01} \cdot \mathsf{E}_{02},\ \ \mathsf{E}_{02} \cdot \mathsf{E}_{02},\ \ \mathsf{E}_{011} \cdot \mathsf{E}_{02},\ \ \mathsf{E}_{021} \cdot \mathsf{E}_{02},\ \ \mathsf{E}_0 \cdot \mathsf{E}_{021},$ $\mathsf{E}_{01} \cdot \mathsf{E}_{021},\ \ \mathsf{E}_{02} \cdot \mathsf{E}_{021},\ \ \mathsf{E}_0 \cdot \mathsf{E}_{0211},\ \ \ldots.$

Quotient algebras

Let $\mathcal{S}$ be a graded subset of Cl_δ.

Let

$$\mathbf{Cl}_\mathcal{S} := \mathbf{Cl}_\delta / \mathcal{V}_\mathcal{S}$$

be the quotient space of $\mathbf{Cl}_\delta$ where $\mathcal{V}_\mathcal{S}$ is the linear span of the set

$$\{\mathsf{F}_u : u \in \mathsf{Cl}_\delta \setminus \mathcal{S}\}.$$

By definition, the set $\{\mathsf{F}_u : u \in \mathcal{S}\}$ is a basis of $\mathbf{Cl}_\mathcal{S}$.

The set $\mathcal{S}$ is

- closed by prefix if for any $u \in \mathcal{S}$, for all prefixes u' of u, $u' \in \mathcal{S}$;
- closed by suffix reduction if for any $u \in \mathcal{S}$, for all suffixes u' of u, $\mathrm{r}_\delta(u') \in \mathcal{S}$.

– Proposition –

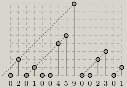
If δ is valley-free and $\mathcal{S}$ is closed by prefix and by suffix reduction, then $\mathbf{Cl}_\mathcal{S}$ is a quotient of the associative algebra $\mathbf{Cl}_\delta$.

Canyon and canyon posets

Let Ca_δ be the subset of Cl_δ containing all δ-canyons that are δ-cliffs u such that $u_{i-j} \leqslant u_i - j$, for all $i \in [|u|]$ and $j \in [u_i]$ satisfying $i - j \geqslant 1$.

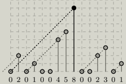
– Example –

A $\mathbf{2}$-canyon of size 15:



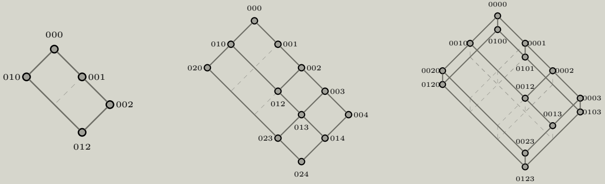
– Example –

A $\mathbf{2}$-cliff of size 15 which is not a $\mathbf{2}$-canyon:



– Example –

The Hasse diagrams of $\mathsf{Ca}_{\mathbf{1}}(3)$, $\mathsf{Ca}_{\mathbf{2}}(3)$, and $\mathsf{Ca}_{\mathbf{1}}(4)$ are



The posets $\mathsf{Ca}_{\mathbf{1}}$ are the Tamari lattices [**Pallo**, 1986].

$\mathbf{Ca}_m$ algebras

Let $\mathbf{Ca}_m$ be the quotient space $\mathbf{Cl}_{\mathsf{Ca}_\mathbf{m}}$.

Since $\mathsf{Ca}_\mathbf{m}$ is closed by prefix and by suffix reduction, $\mathbf{Ca}_m$ is an associative algebra quotient of $\mathbf{Cl}_\mathbf{m}$.

– Examples –

In $\mathbf{Ca}_1$,

$$\mathsf{F}_0 \cdot \mathsf{F}_{01} = \mathsf{F}_{001} + \mathsf{F}_{002} + \mathsf{F}_{012},$$

$$\mathsf{F}_{010} \cdot \mathsf{F}_{0020} = \mathsf{F}_{0100020} + \mathsf{F}_{0100030} + \mathsf{F}_{0101030} + \mathsf{F}_{0100050} + \mathsf{F}_{0101050} + \mathsf{F}_{0103050}$$

– Examples –

In $\mathbf{Ca}_2$,

$$\mathsf{F}_{01} \cdot \mathsf{F}_{0014} = 0,$$

$$\begin{aligned}\mathsf{F}_{020} \cdot \mathsf{F}_{02} = {} & \mathsf{F}_{02002} + \mathsf{F}_{02005} + \mathsf{F}_{02006} + \mathsf{F}_{02007} + \mathsf{F}_{02008} + \mathsf{F}_{02012} + \mathsf{F}_{02015} \\ & + \mathsf{F}_{02016} + \mathsf{F}_{02017} + \mathsf{F}_{02018} + \mathsf{F}_{02045} + \mathsf{F}_{02046} + \mathsf{F}_{02047} + \mathsf{F}_{02048} \\ & + \mathsf{F}_{02056} + \mathsf{F}_{02057} + \mathsf{F}_{02058} + \mathsf{F}_{02067} + \mathsf{F}_{02068}.\end{aligned}$$

Structure of $\mathbf{Ca}_m$ algebras

$\mathbf{Ca}_1$ is isomorphic to $\mathbf{PBT}$.

By computer exploration, minimal generating families of $\mathbf{Ca}_1$ and $\mathbf{Ca}_2$ up to respectively up to degree 5 and 4 are resp.

$$\mathsf{F}_{0}, \quad \mathsf{F}_{00}, \quad \mathsf{F}_{000}, \mathsf{F}_{001}, \quad \mathsf{F}_{0000}, \mathsf{F}_{0001}, \mathsf{F}_{0002}, \mathsf{F}_{0010}, \mathsf{F}_{0012},$$
$$\mathsf{F}_{00000}, \mathsf{F}_{00001}, \mathsf{F}_{00002}, \mathsf{F}_{00003}, \mathsf{F}_{00010}, \mathsf{F}_{00012}, \mathsf{F}_{00013}, \mathsf{F}_{00020}, \mathsf{F}_{00023}, \mathsf{F}_{00100},$$
$$\mathsf{F}_{00101}, \mathsf{F}_{00103}, \mathsf{F}_{00120}, \mathsf{F}_{00123},$$

and

$$\mathsf{F}_{0}, \quad \mathsf{F}_{00}, \mathsf{F}_{01}, \quad \mathsf{F}_{000}, \mathsf{F}_{002}, \mathsf{F}_{003}, \mathsf{F}_{010}, \mathsf{F}_{012}, \mathsf{F}_{013}, \mathsf{F}_{023},$$
$$\mathsf{F}_{0000}, \mathsf{F}_{0003}, \mathsf{F}_{0004}, \mathsf{F}_{0005}, \mathsf{F}_{0014}, \mathsf{F}_{0015}, \mathsf{F}_{0020}, \mathsf{F}_{0023}, \mathsf{F}_{0024}, \mathsf{F}_{0025}, \mathsf{F}_{0030}, \mathsf{F}_{0034},$$
$$\mathsf{F}_{0035}, \mathsf{F}_{0045}, \mathsf{F}_{0100}, \mathsf{F}_{0104}, \mathsf{F}_{0105}, \mathsf{F}_{0120}, \mathsf{F}_{0124}, \mathsf{F}_{0125}, \mathsf{F}_{0130}, \mathsf{F}_{0134}, \mathsf{F}_{0135}, \mathsf{F}_{0145},$$
$$\mathsf{F}_{0204}, \mathsf{F}_{0205}, \mathsf{F}_{0230}, \mathsf{F}_{0234}, \mathsf{F}_{0235}, \mathsf{F}_{0245}.$$

The numbers of minimal generators of $\mathbf{Ca}_2$ begin by

$$0, 1, 2, 7, 30, 149, 788, 4332.$$

$\mathbf{Ca}_0$ and $\mathbf{Ca}_1$ are free as associative algebras but $\mathbf{Ca}_m$, $m \geqslant 2$, are not.

3. Some open questions

From cliffs to increasing trees

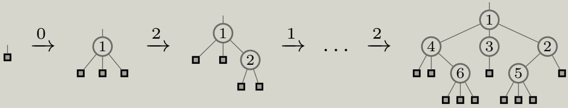
When $\delta(1) = 0$, δ is rooted.

Given $u \in \mathsf{Cl}_\delta(n)$ where δ is rooted and weakly increasing, let $\mathrm{tree}_\delta(u)$ be the δ-increasing tree defined recursively as follows:

- If $u = \epsilon$, then $\mathrm{tree}_\delta(u)$ is the leaf;
- Otherwise, $u = u'a$ and $\mathrm{tree}_\delta(u)$ is obtained by grafting on the $a{+}1$-st leaf of $\mathrm{tree}_\delta(u')$ a node labeled by n having $1 + \delta(n+1) - \delta(n)$ leaves.

– Example –

For $\delta := 0233579^\omega$ and $u := 021042$, the $\mathrm{tree}_\delta(u)$ grows as follows:



Alternative posets from increasing trees

Let δ be a rooted and weakly increasing range map.

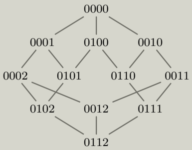
Let $\preccurlyeq'$ be the reflexive and transitive closure of the relation $\lessdot'$ on $\mathsf{Cl}_\delta(n)$ where $u \lessdot' v$ if v is obtained from u by incrementing a letter u_i when all the children of the node labeled by i in $\mathrm{tree}_\delta(u)$ are leaves excepted possibly the first one.

– Example –

$(\mathsf{Cl}_{\mathbf{1}}(n), \preccurlyeq')$ is isomorphic to the right weak order.

– Example –

The Hasse diagram of $(\mathsf{Cl}_{01122^\omega}(4), \preccurlyeq')$ is



– Conjecture –

For any rooted and weakly increasing range map δ and any $n \geqslant 0$, the poset $(\mathsf{Cl}_\delta(n), \preccurlyeq')$ is a lattice.

Dune posets

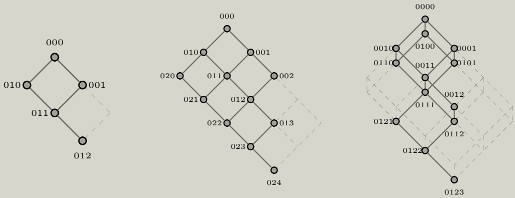
Let Du_δ be the subset of Cl_δ containing all δ-dunes that are δ-cliffs u such that for any $i \in [n-1]$, $|u_i - u_{i+1}| \leqslant |\delta(i) - \delta(i+1)|$.

Cardinalities of $\mathsf{Du}_{\mathbf{1}}(n)$: $1, 1, 2, 5, 13, 35, 96, 267, \ldots$(**A005773**, directed animals).

Cardinalities of $\mathsf{Du}_{\mathbf{2}}(n)$: $1, 1, 3, 12, 51, 226, 1025, 4724, \ldots$(**A180898**, some meanders [**Banderier et al.**, 2016]).

– Example –

The Hasse diagrams of $\mathsf{Du}_{\mathbf{1}}(3)$, $\mathsf{Du}_{\mathbf{2}}(3)$, and $\mathsf{Du}_{\mathbf{1}}(4)$ are



– Project –

Study the dune posets and their associative algebras.

Coproducts

As already mentioned, $\mathbf{FQSym}$ and its subalgebras are endowed with a coproduct.

A coproduct on a space $\mathcal{A}$ is a map

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

which, intuitively, splits any element of $\mathcal{A}$ in two smaller parts, in several different ways.

If $(\mathcal{A}, \cdot)$ is an associative algebra, a coproduct Δ is compatible with $\cdot$ if for all $f, g \in \mathcal{A}$,

$$\Delta(f \cdot g) = \Delta(f)\Delta(g).$$

– Question –

Introduce a (noncocommutative) coproduct on $\mathbf{Cl}_\delta$ compatible with its product.

Determine in what extent this coproduct is still well-defined on its quotients $\mathbf{Cl}_\mathcal{S}$.