An introduction to operator structures: operads

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Virtual workshop on Combinatorial species, Operads, Riordan arrays and related topics

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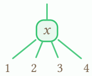
Outline

1. Introduction

Operator structures

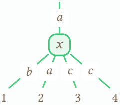
Informally, an operator structure is a set S of **operators** closed w.r.t. a set of **composition operations**.

There are a lot of kinds of operator structures, each dealing with a particular type of operators:



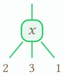
Nonsymm. operads

Planar operators



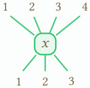
Colored operads

Typed operators



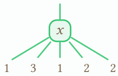
Symm. operads

Linear operators



Pros

In/out-puts



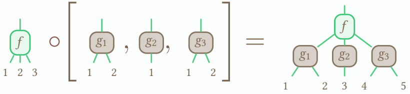
Abstract clones

General operators

Operator structures and compositions

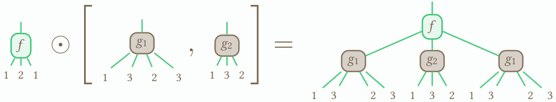
These operators can be **composed** in different ways.

- Operads:



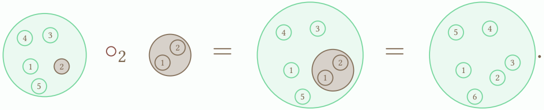
- Pros:

- Clones:



Objects as operators

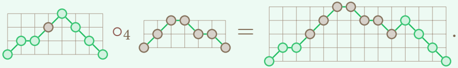
By **seeing combinatorial objects as operators**, we obtain ways to **compose** them. Schematically,



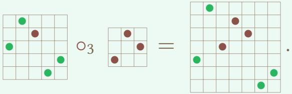
is an abstract composition of an object of size 5 with an object of size 2 at the 2-nd position.

– Examples –

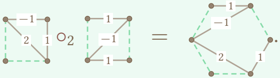
Composition of Motzkin paths:



Composition of permutations:



Composition of labeled graphs:



Main topics

In this lecture, we shall focus on **nonsymmetric operads**.

We will

- describe **free operads**;
- introduce **presentations** of operads;
- present **planar term rewrite systems** and tools to establish presentations;
- study **algebras** over operads;
- present two general **constructions** of operads;
- give some **combinatorial applications**.

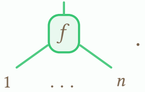
Outline

2. Elementary notions

2. Elementary notions

Planar operators and composition

A planar operator is an entity f having $n \geqslant 0$ inputs and one single output, drawn as

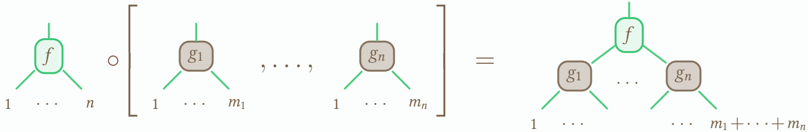


This planar operator denotes a **map** $(\mathsf{x}_1, \dots, \mathsf{x}_n) \mapsto f(\mathsf{x}_1, \dots, \mathsf{x}_n)$.

The arity $|f|$ of f is its number n of inputs, numbered from 1 to n.

Composing a planar operator f of arity n with n planar operators $g_1, \dots, g_n$ consists in **grafting** all the outputs of the g_i to the inputs i of f.

This produces the new operator $f \circ [g_1, \dots, g_n]$ of arity $m_1 + \cdots + m_n$, drawn as



and denoting the map

$$(\mathsf{x}_1, \dots, \mathsf{x}_{m_1+\cdots+m_n}) \mapsto f\left(g_1(\mathsf{x}_1, \dots, \mathsf{x}_{m_1}), \dots, g_n\left(\mathsf{x}_{m_1+\cdots+m_{n-1}+1}, \dots, \mathsf{x}_{m_1+\cdots+m_n}\right)\right).$$

Operads

Nonsymmetric operads provide a **formalization** of planar operators and their composition.

A nonsymmetric set-operad (or operad for short in this lecture) is a triple $(\mathcal{O}, \circ, \mathbf{1})$ where

- $\mathcal{O}$ is a **graded set**

$$\mathcal{O} = \bigsqcup_{n \in \mathbb{N}} \mathcal{O}(n);$$

- $\circ$ is a **map**

$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n)$$

  called full composition map;

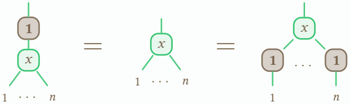
- $\mathbf{1}$ is an **element** of $\mathcal{O}(1)$ called unit.

This data has to satisfy some relations.

Operad relations

The following relations have to be satisfied:

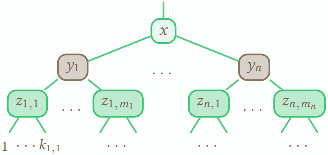
(OpU) For all $x \in \mathcal{O}$, $\mathbf{1} \circ [x] = x = x \circ [\mathbf{1}, \dots, \mathbf{1}]$.



This says that **1** is the **identity operator**.

(OpA) For all $x \in \mathcal{O}(n)$, $y_i \in \mathcal{O}(m_i)$, and $z_{i,j} \in \mathcal{O}$,

$$(x \circ [y_1, \dots, y_n]) \circ [z_{1,1}, \dots, z_{1,m_1}, \dots, z_{n,1}, \dots, z_{n,m_n}]$$
$$= x \circ [y_1 \circ [z_{1,1}, \dots, z_{1,m_1}], \dots y_n \circ [z_{n,1}, \dots, z_{n,m_n}]].$$



This says that the two ways to form an operator having **three layers** (by starting from the top or by starting from the bottom) coincide.

Partial composition maps

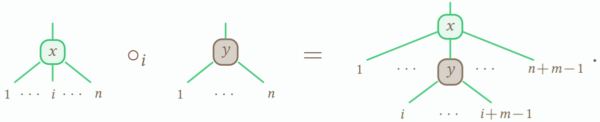
A partial composition map on $\mathcal{O}$ is any map

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1), \quad i \in [n].$$

The **partial composition map** $\circ_i$ of a full composition map $\circ$ is defined by

$$x \circ_i y := x \circ [\overbrace{\mathbf{1}, \dots, \mathbf{1}}^{i-1}, y, \overbrace{\mathbf{1}, \dots, \mathbf{1}}^{n-i}].$$

Pictorially, by using Relation (OpU),



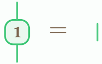
Conversely, the **full composition map** $\circ$ of a partial composition map $\circ_i$ is defined by

$$x \circ [y_1, \dots, y_n] := (\dots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \dots) \circ_1 y_1.$$

Partial composition maps and relations

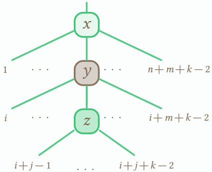
The two relations satisfied by $\circ$ lead to the following three relations for the partial composition map $\circ_i$:

(OpU') For any $x \in \mathcal{O}(n)$ and $i \in [n]$, $\mathbf{1} \circ_1 x = x = x \circ_i \mathbf{1}$.



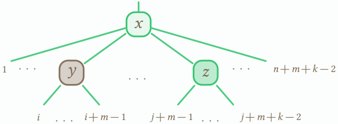
This says that **1** is the **identity operator**.

(OpAS) For any $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, $z \in \mathcal{O}(k)$, $i \in [n]$ and $j \in [m]$, $(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$.



This says that the two ways to form an operator **in series** (by starting from the top or by starting from the bottom) coincide.

(OpAP) For any $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, $z \in \mathcal{O}(k)$, $j \in [n]$ and $i \in [j-1]$, $(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y$.



This says that the two ways to form an operator **in parallel** (by starting from the left or by starting from the right) coincide.

Equivalence between full and partial composition maps

– Proposition –

Let $\mathcal{O}$ be a graded set and $\mathbf{1} \in \mathcal{O}(1)$.

1. If $\circ$ is a full composition map satisfying Relations (OpU) and (OpA), then
   (a) the partial composition map $\circ_i$ of $\circ$ satisfies Relations (OpAS), (OpAP), and (OpU');
   (b) the full composition map of $\circ_i$ is $\circ$.
2. If $\circ_i$ is a partial composition map satisfying Relations (OpAs), (OpAP), and (OpU'), then
   (a) the full composition map $\circ$ of $\circ_i$ satisfies Relations (OpU) and (OpA);
   (b) the partial composition map of $\circ$ is $\circ_i$.

– Exercise ●◐○○○ –

Show the previous proposition.

Therefore, operads can **equivalently** be defined and studied through their **full** or **partial composition maps**.

It is often important to have expressions for **both these compositions**.

Outline

2. Elementary notions

Morphisms and quotients

A map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ between two operads $(\mathcal{O}_1, \circ_i^1, \mathbf{1}_1)$ and $(\mathcal{O}_2, \circ_i^2, \mathbf{1}_2)$ is an operad morphism if

- for any $x \in \mathcal{O}_1(n)$, $\phi(x) \in \mathcal{O}_2(n)$;
- $\phi(\mathbf{1}_1) = \mathbf{1}_2$;
- for any $x, y \in \mathcal{O}_1$, $\phi(x \circ_i^1 y) = \phi(x) \circ_i^2 \phi(y)$.

An equivalence relation $\equiv$ on $(\mathcal{O}, \circ_i, \mathbf{1})$ is an operad congruence if

- by denoting by $[x]_\equiv$ the $\equiv$-equivalence class of $x \in \mathcal{O}$, for all $x' \in [x]_\equiv$, $|x'| = |x|$;
- for any $x, x', y, y' \in \mathcal{O}$ such that $x \equiv x'$ and $y \equiv y'$, $x \circ_i y \equiv x' \circ_i y'$.

Given an operad congruence $\equiv$ of $\mathcal{O}$, $(\mathcal{O}/_\equiv, \circ_i^\equiv, \mathbf{1}_\equiv)$ is the quotient operad of $\mathcal{O}$. It is defined in the following way:

- $\mathcal{O}/_\equiv(n) := \{[x]_\equiv : x \in \mathcal{O}(n)\}$;
- $[x]_\equiv \circ_i^\equiv [y]_\equiv := [x' \circ_i y']_\equiv$ where x' is any element of $[x]_\equiv$ and y' is any element of $[y]_\equiv$;
- the unit $\mathbf{1}_\equiv$ is $\equiv$-equivalence class of the unit $\mathbf{1}$ of $\mathcal{O}$.

Suboperads and generating sets

Let $(\mathcal{O}, \circ_i, \mathbf{1})$ be an operad.

When for any $n \in \mathbb{N}$, $\mathcal{O}(n)$ is finite, $\mathcal{O}$ is combinatorial. In this case, the Hilbert series of $\mathcal{O}$ is the generating series

$$\mathcal{H}_{\mathcal{O}}(z) := \sum_{n \in \mathbb{N}} \# \mathcal{O}(n) \, z^n.$$

A suboperad of $\mathcal{O}$ is any subset of $\mathcal{O}$ containing the unit $\mathbf{1}$ of $\mathcal{O}$ and closed w.r.t. $\circ_i$.

Given a subset $\mathfrak{G}$ of $\mathcal{O}$, $\mathcal{O}^{\mathfrak{G}}$ is the suboperad generated by $\mathfrak{G}$, that is the smallest suboperad of $\mathcal{O}$ containing $\mathfrak{G}$.

If $\mathfrak{G}$ is such that $\mathcal{O}^{\mathfrak{G}} = \mathcal{O}$, then $\mathfrak{G}$ is a generating set of $\mathcal{O}$. When none of the proper subsets of $\mathfrak{G}$ satisfy this property, $\mathfrak{G}$ is minimal.

– Usual questions –

Given an operad $\mathcal{O}$ and a (finite) subset $\mathfrak{G}$ of $\mathcal{O}$, **study the suboperad** $\mathcal{O}^{\mathfrak{G}}$.

Given an operad $\mathcal{O}$, **describe the minimal generating sets** of $\mathcal{O}$.

Outline

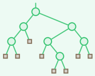
2. Elementary notions

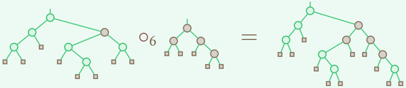
The duplicial operad

The duplicial operad **Dup** is the operad wherein

- **Dup**$(0) = \emptyset$ and for any $n \geqslant 1$, **Dup**(n) is the set of the **binary trees** with n internal nodes;
- $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by replacing the i-th internal node u of $\mathfrak{t}$ (w.r.t. the infix traversal) by a copy of $\mathfrak{s}$ and by grafting the left (resp. right) subtree of u to the first (resp. last) leaf of the copy;
- the unit **1** is the unique element of **Dup**(1).

– Examples –

 $\in$ **Dup**(7)



(first tree) $\circ_6$ (second tree) $=$ (result tree)

1 $=$ 

– Exercise ●○○○○ –

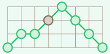
1. Show that **Dup** is an operad.
2. Describe the full composition map of **Dup**.

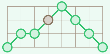
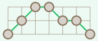
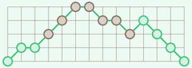
The Motz operad

The operad of Motzkin paths **Motz** is the operad wherein

- $\mathbf{Motz}(0) = \emptyset$ and for any $n \geqslant 1$, $\mathbf{Motz}(n)$ is the set of the **Motzkin paths** with n points;
- $u \circ_i v$ is obtained by replacing the i-th point of u by a copy of v;
- the unit $\mathbf{1}$ is the unique element of $\mathbf{Motz}(1)$.

– Examples –

 $\in \mathbf{Motz}(8)$

 $\circ_4$  $=$ 

$\mathbf{1} =$ 

– Exercise ●○○○○ –

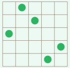
1. Show that **Motz** is an operad.
2. Describe the full composition map of **Motz**.

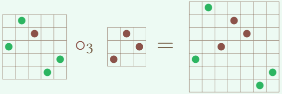
The Per operad

The operad of permutations **Per** is the operad wherein

- $\mathbf{Per}(0) = \emptyset$ and for any $n \geqslant 1$, $\mathbf{Per}(n)$ is the set of the permutations on $[n]$, seen through their **permutation matrices**;
- $\sigma \circ_i \nu$ is obtained by replacing the i-th point of σ by a copy of ν;
- the unit $\mathbf{1}$ is the unique element of $\mathbf{Per}(1)$.

– Examples –

 $\in \mathbf{Per}(5)$



$\mathbf{1} =$ 

– Exercise ●●○○○ –

1. Show that **Per** is an operad.
2. Describe the full composition map of **Per**.

Some references

Origins of operads:

- M. Gerstenhaber, **The cohomology structure of an associative ring**, 1963.
- J. P. May, **The geometry of iterated loop spaces**, 1972.
- J. M. Boardman, R. M. Vogt, **Homotopy invariant algebraic structures on topological spaces**, 1973.

General references:

- J.-L. Loday, B. Vallette, **Algebraic Operads**, 2012.
- M. A. Méndez, **Set operads in combinatorics and computer science**, 2015.
- S. Giraudo, **Nonsymmetric Operads in Combinatorics**, 2018.

About the duplicial operad:

- C. Brouder, A. Frabetti, **QED Hopf algebras on planar binary trees**, 2003.

About the operad of permutations:

- M. Aguiar, M. Livernet, **The associative operad and the weak order on the symmetric groups**, 2007.

3. Fundamental notions

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Planar terms

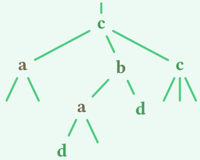
A signature is a set $\mathfrak{G} := \bigsqcup_{n \in \mathbb{N}} \mathfrak{G}(n)$ where each $\mathbf{a} \in \mathfrak{G}(n)$ is a constant of arity n.

A planar $\mathfrak{G}$-term is

- either the **leaf** $|$;
- or a **pair** $(\mathbf{a}, (\mathfrak{t}_1, \ldots, \mathfrak{t}_n))$ where $\mathbf{a} \in \mathfrak{G}(n)$ and each $\mathfrak{t}_i$ is a planar $\mathfrak{G}$-term.

The set of planar $\mathfrak{G}$-terms is $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$.

– Example –



This is the **tree representation** of the planar $\mathfrak{G}$-term

$$(\mathbf{c}, ((\mathbf{a}, (|, |)), (\mathbf{b}, ((\mathbf{a}, ((\mathbf{d}, ()), |)), (\mathbf{d}, ()))), (\mathbf{c}, (|, |, |))))$$

where $\mathfrak{G}$ is the signature such that $\mathfrak{G} = \mathfrak{G}(0) \sqcup \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ with $\mathfrak{G}(0) := \{\mathbf{d}\}$, $\mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$ and $\mathfrak{G}(3) := \{\mathbf{c}\}$.

Free operads

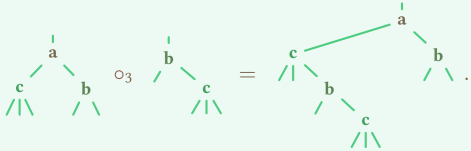
Let $\mathfrak{G}$ be a signature.

The <u>free operad</u> on $\mathfrak{G}$ is the operad $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$ where

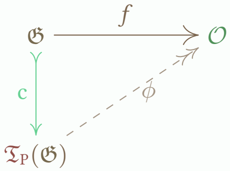
- for any $n \in \mathbb{N}$, $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})(n)$ is the set of **planar $\mathfrak{G}$-terms** with n leaves;
- the planar $\mathfrak{G}$-term $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by replacing the i-th leaf of $\mathfrak{t}$ by a copy of $\mathfrak{s}$;
- $\mathbf{1}$ is the planar $\mathfrak{G}$-term $|$.

– Example –

By setting $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ where $\mathfrak{G}(2) := \{\mathtt{a}, \mathtt{b}\}$ and $\mathfrak{G}(3) := \{\mathtt{c}\}$, we have in the free operad $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$,



For any signature $\mathfrak{G}$, any operad $\mathcal{O}$, and any map $f : \mathfrak{G} \to \mathcal{O}$ preserving the arities, there exists a unique operad morphism $\phi : \mathfrak{T}_{\mathrm{P}}(\mathfrak{G}) \to \mathcal{O}$ such that $f = \phi \circ \mathrm{c}$.



Presentation by generators and relations

Let $\mathcal{O}$ be an operad.

A presentation of $\mathcal{O}$ is a pair $(\mathfrak{G}, \mathfrak{R})$ where

- $\mathfrak{G}$ is a **signature**;
- $\mathfrak{R}$ is an **equivalence relation** on $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$;
- by denoting by $\equiv_{\mathfrak{R}}$ the smallest operad congruence of $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$ containing $\mathfrak{R}$, we have

$$\mathcal{O} \simeq \mathfrak{T}_{\mathrm{P}}(\mathfrak{G})/_{\equiv_{\mathfrak{R}}}.$$

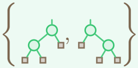
A presentation $(\mathfrak{G}, \mathfrak{R})$ is

- minimal if $\mathfrak{G}$ and $\mathfrak{R}$ are minimal w.r.t. set inclusion;
- binary if $\mathfrak{G} = \mathfrak{G}(2)$;
- quadratic if $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$ implies that both $\mathfrak{t}$ and $\mathfrak{t}'$ have exactly two internal nodes.

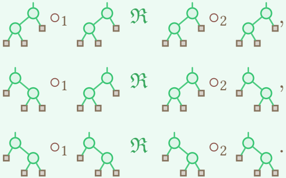
Presentation of Dup

– Proposition –

The duplicial operad **Dup** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where

$\mathfrak{G} :=$ 

and $\mathfrak{R}$ satisfies



This presentation is minimal, binary, and quadratic.

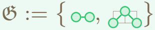
– Exercise ●●●○○ –

Show this presentation of **Dup**.

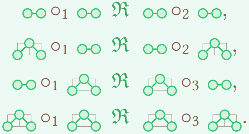
Presentation of Motz

– Proposition –

The operad **Motz** admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where



and $\mathfrak{R}$ satisfies



This presentation is minimal, not binary, and quadratic.

– Exercise ●●●○○ –

Show this presentation of **Motz**.

Realizations

On the other way round, it is possible to **define operads** through **presentations**.

In this way, a presentation specifies a quotient of a free operad.

A realization of a presentation $(\mathfrak{G}, \mathfrak{R})$ consists in

- a graded set $\mathcal{O}$;
- an element $\mathbf{1} \in \mathcal{O}(1)$;
- an explicit description of the partial compositions map $\circ_i$ on $\mathcal{O}$;

such that $(\mathcal{O}, \circ_i, \mathbf{1})$ is an operad and admits $(\mathfrak{G}, \mathfrak{R})$ as presentation.

Of course, there can be **different realizations** $\mathcal{O}$ and $\mathcal{O}'$ of $(\mathfrak{G}, \mathfrak{R})$.

In this case, $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic operads.

Realization of the diassociative operad

The diassociative operad **Dias** is the operad admitting the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G}$ is the graded set $\mathfrak{G} := \mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$ and $\mathfrak{R}$ satisfies

$$\mathbf{a} \circ_1 \mathbf{a} \;\mathfrak{R}\; \mathbf{a} \circ_2 \mathbf{a} \;\mathfrak{R}\; \mathbf{a} \circ_2 \mathbf{b},$$

$$\mathbf{a} \circ_1 \mathbf{b} \;\mathfrak{R}\; \mathbf{b} \circ_2 \mathbf{a},$$

$$\mathbf{b} \circ_1 \mathbf{a} \;\mathfrak{R}\; \mathbf{b} \circ_2 \mathbf{b} \;\mathfrak{R}\; \mathbf{b} \circ_1 \mathbf{b}.$$

This operad is realized as the graded set $\mathcal{O}$ defined as the set words on $\{0, 1\}$ having exactly one occurrence of 0, and where for any $u, v \in \mathcal{O}$, $u \circ_i v$ is obtained by replacing the i-th letter of u by v if $u_i = 0$ and by $1^{|v|}$ otherwise.

– Examples –

$$1011\mathbf{1} \circ_4 \mathbf{110} = 1011\mathbf{111}1$$

$$1\mathbf{0}111 \circ_2 \mathbf{110} = 1\mathbf{110}111$$

Outline

3. Fundamental notions

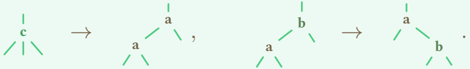
Rewrite systems on planar terms 1 / 2

A rewrite relation on $\mathfrak{T}_P(\mathfrak{G})$ is a binary relation $\to$ on $\mathfrak{T}_P(\mathfrak{G})$ such that if $\mathfrak{t} \to \mathfrak{t}'$ then $|\mathfrak{t}| = |\mathfrak{t}'|$.

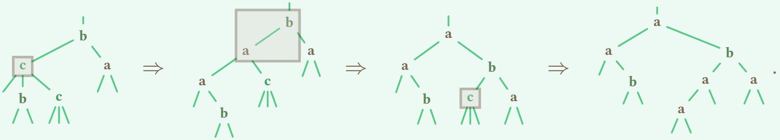
The context closure of $\to$ is the binary relation $\Rightarrow$ satisfying $\mathfrak{t} \Rightarrow \mathfrak{t}'$ if $\mathfrak{t}'$ can be obtained by **replacing** in $\mathfrak{t}$ a **connected part** (called occurrence) $\mathfrak{s}$ by $\mathfrak{s}'$ whenever $\mathfrak{s} \to \mathfrak{s}'$.

– Example –

Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ be the signature where $\mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$ and $\mathfrak{G}(3) := \{\mathbf{c}\}$. Let $\to$ be the rewrite relation defined by



We have



A planar term rewrite system (or PTRS for short) is such a pair $(\mathfrak{G}, \to)$.

Rewrite systems on planar terms 2 / 2

Let $\mathcal{S} := (\mathfrak{G}, \to)$ be a PTRS.

We define

- $\ll$ as the **reflexive and transitive closure** of $\Rightarrow$ ($\mathfrak{t} \ll \mathfrak{t}'$ iff $\mathfrak{t}'$ can be obtained from $\mathfrak{t}$ by some rewrite steps);
- $\mathrm{G}_{\mathcal{S}}(\mathfrak{t})$ as the **digraph of the binary relation** $\Rightarrow$ on $\{\mathfrak{t}' \in \mathfrak{T}_{\mathrm{P}}(\mathfrak{G}) : \mathfrak{t} \ll \mathfrak{t}'\}$, called rewrite graph of $\mathfrak{t}$;
- $\equiv$ as the **symmetric closure** of $\ll$ ($\mathfrak{t} \equiv \mathfrak{t}'$ iff $\mathfrak{t}$ and $\mathfrak{t}'$ belong to the same connected component of a rewrite graph).

A planar $\mathfrak{G}$-term $\mathfrak{t}$ is a normal form for $\mathcal{S}$ if there is no arc from $\mathfrak{t}$ in $\mathrm{G}_{\mathcal{S}}(\mathfrak{t})$.

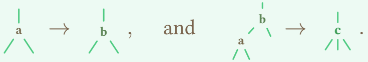
The PTRS $\mathcal{S}$ can have two important properties:

- If for any $\mathfrak{t} \in \mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$, there is **no infinite path** in $\mathrm{G}_{\mathcal{S}}(\mathfrak{t})$, then $\mathcal{S}$ is terminating;
- If for any $\mathfrak{t} \in \mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$, $\mathfrak{t} \ll \mathfrak{s}_1$ and $\mathfrak{t} \ll \mathfrak{s}_2$ implies that there exists $\mathfrak{t}'$ such that $\mathfrak{s}_1 \ll \mathfrak{t}'$ and $\mathfrak{s}_2 \ll \mathfrak{t}'$, then $\mathcal{S}$ is confluent.

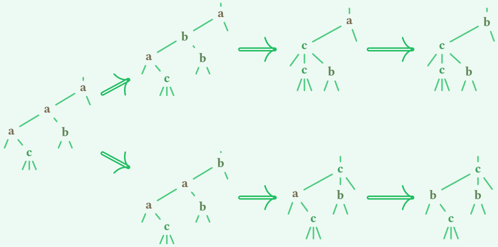
An example of a PTRS

– Example –

Let $\mathfrak{G} := \mathfrak{G}(2) \sqcup \mathfrak{G}(3)$ be the signature where $\mathfrak{G}(2) := \{\mathbf{a}, \mathbf{b}\}$ and $\mathfrak{G}(3) := \{\mathbf{c}\}$. Let $\mathcal{S} := (\mathfrak{G}, \to)$ be the PTRS where $\to$ satisfies



Here is a portion of the rewrite graph of a planar $\mathfrak{G}$-term:



This PTRS $\mathcal{S}$ is **not confluent**.

It is **terminating**. This is implied by the fact that each rewriting decreases by one the number of internal nodes labeled by **a** and there is a finite number of planar $\mathfrak{G}$-terms with a given arity.

Some properties

Let $\mathcal{S} := (\mathfrak{G}, \to)$ be a PTRS.

– Proposition (Connection with operads) –

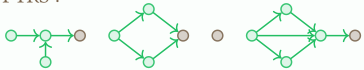
The equivalence relation $\equiv$ is an **operad congruence** of the free operad $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$.

Here are some classical properties of PTRS.

– Proposition (System of representative) –

If $\mathcal{S}$ is terminating and confluent, then the set of **normal forms** of $\mathcal{S}$ is a **system of representatives** of the quotient operad $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})/_{\equiv}$.

Typical rewrite graph of a terminating and confluent PTRS :



– Proposition (Normal forms and avoidance) –

The set of normal forms of $\mathcal{S}$ is the set of planar $\mathfrak{G}$-terms having **no occurrence** of any term appearing as **left member** of $\to$.

Evaluation map and treelike expressions

Let $\mathcal{O}$ be an operad. In particular, $\mathcal{O}$ is a signature, so that $\mathfrak{T}_{\mathrm{P}}(\mathcal{O})$ is a well-defined free operad.

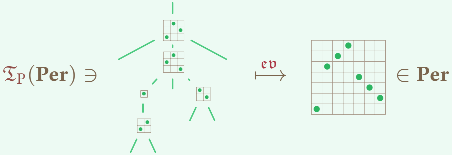
The evaluation map of $\mathcal{O}$ is the map $\mathfrak{ev} : \mathfrak{T}_{\mathrm{P}}(\mathcal{O}) \to \mathcal{O}$ defined by

$$\mathfrak{ev}(\mathfrak{t}) := \begin{cases} \mathbf{1} & \text{if } \mathfrak{t} = | \,, \\ \mathbf{a} \circ [\mathfrak{ev}(\mathfrak{t}_1), \dots, \mathfrak{ev}(\mathfrak{t}_n)] & \text{otherwise, where } \mathfrak{t} = (\mathbf{a}, (\mathfrak{t}_1, \dots, \mathfrak{t}_n)). \end{cases}$$

A treelike expression of $x \in \mathcal{O}$ is any planar $\mathcal{O}$-term $\mathfrak{t}$ such that $\mathfrak{ev}(\mathfrak{t}) = x$.

– Example –

In **Per**, we have



A relation of $\mathcal{O}$ is any pair $(\mathfrak{t}, \mathfrak{t}')$ of planar $\mathcal{O}$-terms such that $\mathfrak{ev}(\mathfrak{t}) = \mathfrak{ev}(\mathfrak{t}')$.

Links with presentations

Let $\mathcal{O}$ be an operad, $\mathfrak{G}$ be a **generating set** of $\mathcal{O}$, and $\mathfrak{R}$ be an equivalence relation on $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$ containing only **relations** of $\mathcal{O}$.

A rewrite relation $\to$ on $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$ is an orientation of $\mathfrak{R}$ if $\to$ is a **subrelation** of $\mathfrak{R}$ and for any $\mathfrak{t} \,\mathfrak{R}\, \mathfrak{t}'$, we have either $\mathfrak{t} \to \mathfrak{t}'$ or $\mathfrak{t}' \to \mathfrak{t}$.

– Theorem –

Given such $\mathcal{O}$, $\mathfrak{G}$, $\mathfrak{R}$, and $\to$, if

1. the PTRS $(\mathfrak{G}, \to)$ is terminating and confluent;
2. the set of normal forms of $(\mathfrak{G}, \to)$ of arity n are in one-to-one correspondence with $\mathcal{O}(n)$ for any $n \in \mathbb{N}$,

then $(\mathfrak{G}, \mathfrak{R})$ is a presentation of $\mathcal{O}$.

– Exercise –

Let $\mathbf{a} :=$  and $\mathbf{c} :=$  and $\mathfrak{G} := \{\mathbf{a}, \mathbf{c}\}$. Let $\to$ be the rewrite relation satisfying

and $\mathfrak{R}$ be the reflexive and symmetric closure of $\to$. Use the theorem to show that $(\mathfrak{G}, \mathfrak{R})$ is a presentation of **Motz**.

3. Fundamental notions

Algebras over an operad

An $\mathcal{O}$-algebra is a set S equipped with a map

$$\mathfrak{op} : \mathcal{O}(n) \to (S^n \to S),$$

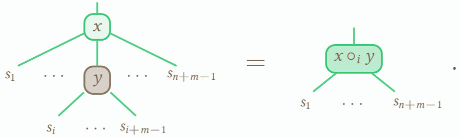
satisfying the following relations.

By writing simply $x(s_1, \dots, s_n)$ for $\mathfrak{op}(x)(s_1, \dots, s_n)$,

- for any $s \in S$, $\mathbf{1}(s) = s$,
- for any $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, $i \in [n]$, and $s_1, \dots, s_{n+m-1} \in S$,

$$(x \circ_i y)(s_1, \dots, s_{n+m-1}) = x(s_1, \dots, s_{i-1}, y(s_i, \dots, s_{i+m-1}), s_{i+m}, \dots, s_{n+m-1}).$$

On planar operators, the last relation depicts as



Algebras and presentations

If $\mathcal{O}$ admits a **presentation** $(\mathfrak{G}, \mathfrak{R})$, to specify an $\mathcal{O}$-algebra S it is enough to define $\mathfrak{op}$ on $\mathfrak{G}$ and check that for any $(\mathfrak{t}, \mathfrak{t}') \in \mathfrak{R}$,

$$\mathfrak{ev}(\mathfrak{t})(s_1, \ldots, s_n) = \mathfrak{ev}(\mathfrak{t}')(s_1, \ldots, s_n).$$

– Example –

Any **Motz**-algebra is a set S endowed with two generating operations

$$\circ\!\!-\!\!\circ : S^2 \to S \quad \text{and} \quad \pitchfork : S^3 \to S,$$

satisfying

$$\circ\!\!-\!\!\circ(\circ\!\!-\!\!\circ(s_1, s_2), s_3) = \circ\!\!-\!\!\circ(s_1, \circ\!\!-\!\!\circ(s_2, s_3)),$$
$$\pitchfork(\circ\!\!-\!\!\circ(s_1, s_2), s_3, s_4) = \circ\!\!-\!\!\circ(s_1, \pitchfork(s_2, s_3, s_4)),$$
$$\circ\!\!-\!\!\circ(\pitchfork(s_1, s_2, s_3), s_4) = \circ\!\!-\!\!\circ(s_1, \pitchfork(s_2, s_3, s_4)),$$
$$\pitchfork(\pitchfork(s_1, s_2, s_3), s_4, s_5) = \pitchfork(s_1, s_2, \pitchfork(s_3, s_4, s_5)).$$

Categories of algebras

Let $\mathcal{O}$ be an operad.

The collection of the $\mathcal{O}$-algebras forms a **category** where morphisms between two objects S and S' are maps $\phi : S \to S'$ satisfying, for any $x \in \mathcal{O}(n)$,

$$\phi(x(s_1, \ldots, s_n)) = x(\phi(s_1), \ldots, \phi(s_n)).$$

– Example –

Let **As** be the associative operad defined by $\mathbf{As}(0) := \emptyset$ and for any $n \geqslant 1$, $\mathbf{As}(n) := \{\star_n\}$, where $\star_n \circ_i \star_m := \star_{n+m-1}$.

A minimal generating set of **As** is $\{\star_2\}$.

Any **As**-algebra is a set S endowed with the generating operation $\star_2$ satisfying

$$\begin{array}{ccc} (\star_2 \circ_1 \star_2)(s_1, s_2, s_3) & = & \star_2(\star_2(s_1, s_2), s_3) \\ \| & & \| \\ (\star_2 \circ_2 \star_2)(s_1, s_2, s_3) & = & \star_2(s_1, \star_2(s_2, s_3)). \end{array}$$

Using the infix notation for the binary operation $\star_2$, we obtain the relation $(s_1 \star_2 s_2) \star_2 s_3 = s_1 \star_2 (s_2 \star_2 s_3)$, so that the category of **As**-algebras is the **category of semigroups**.

Operad morphisms and algebras

– Proposition –

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two operads and $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ be an operad morphism. If S is an $\mathcal{O}_2$-algebra, by setting for any $x \in \mathcal{O}_1(n)$ and $s_1, \dots, s_n \in S$,

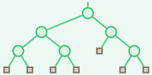
$$x(s_1, \dots, s_n) := (\phi(x))(s_1, \dots, s_n),$$

the set S becomes an $\mathcal{O}_1$-algebra.

Therefore, any operad morphism from $\mathcal{O}_1$ to $\mathcal{O}_2$ gives rise to a **functor** from the category of $\mathcal{O}_2$-algebras to the category of $\mathcal{O}_1$-algebras.

– Example –

Let ϕ : **Dup** $\to$ **Dias** be the map sending any $\mathfrak{t} \in$ **Dup**(n) to $1^{k-1}01^{n-k}$, where k is the position of the root of $\mathfrak{t}$ for the infix traversal. For instance,



$\overset{\phi}{\longmapsto} 111011.$

Since this map is an operad morphism, any **Dias**-algebra gives rise to a **Dup**-algebra.

– Exercise ●○○○○ –

Prove that ϕ is an operad morphism.

Some references

About the diassociative operad:

- J.-L. Loday, **Dialgebras**, 2001.
- F. Chapoton, **On some anticyclic operads**, 2005.

About realizations of some presentations:

- F. Chapoton, M. Livernet, **Pre-Lie algebras and the rooted trees operad**, 2001.
- S. Giraudo, **Pluriassociative algebras II: The polydendriform operad and related operads**, 2016.

About term rewrite systems:

- F. Baader, T. Nipkow, **Term rewriting and all that**, 1998.
- M. Bezem, J. W. Klop, R. de Vrijer, Terese, **Term Rewriting Systems**, 2003.

About several examples of algebras over operads:

- J.-L. Loday, **Encyclopedia of types of algebras 2010**, 2012.

4. Constructions

4. Constructions

The construction T

Let $(\mathcal{M}, \cdot, e)$ be a **monoid** and let $(\mathbf{T}\mathcal{M}, \circ_i, \mathbf{1})$ be the triple such that

- $\mathbf{T}\mathcal{M}(0) = \emptyset$ and for any $n \geqslant 1$, $\mathbf{T}\mathcal{M}(n)$ is the set $\mathcal{M}^n$;
- for any $u \in \mathbf{T}\mathcal{M}(n)$, $1 \leqslant i \leqslant n$, and $v \in \mathbf{T}\mathcal{M}$,

$$u \circ_i v := u(1, i-1) \ \ (u(i) \cdot v(1)) \dots (u(i) \cdot v(\ell(v))) \ \ u(i+1, \ell(u));$$

- $\mathbf{1}$ is the element e seen as a word of length 1.

– Examples –

Set $\mathcal{M} := (\mathbb{N}, +, 0)$. We have

$$20336 \in \mathbf{T}\mathcal{M}(5)$$

and, in $\mathbf{T}\mathcal{M}$,

$$325112 \circ_3 221 = 32\,(5+2)\,(5+2)\,(5+1)\,112 = 32776112.$$

– Theorem –

For any monoid $\mathcal{M}$, $\mathbf{T}\mathcal{M}$ is an operad.

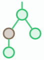
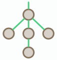
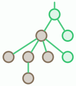
Operads from the construction T

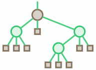
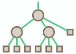
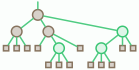
The operads $\mathsf{T}\mathcal{M}$ are large enough to contain a lot of suboperads realizable through certain combinatorial families.

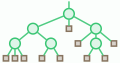
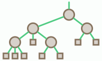
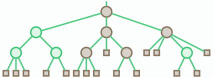
As main examples:

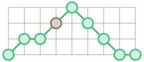
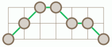
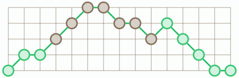
- For any $m \geqslant 0$, with $\mathcal{M} := (\mathbb{N}, +, 0)$,
    - $\mathbf{PRT}_m$, generated by $\{01, \dots, 0m\}$, on **primitive m-Dyck paths**;
    - $\mathbf{FCat}_m$, gen. by $\{00, 01, \dots, 0m\}$, on **m-trees**;
    - $\mathbf{Schr}_m$, gen. by $\{01, \dots, 0m, 00, m0, \dots, 10\}$, on some **Schröder trees**;
    - $\mathbf{Motz}_m$, gen. by $\{00, 010, \dots, 0m0\}$, on **colored Motzkin paths**.
- For any $m \geqslant 0$, with $\mathcal{M} := (\mathbb{Z}/(m+1)\mathbb{Z}, +, 0)$,
    - $\mathbf{Comp}_m$, gen. by $\{00, 01, \dots, 0m\}$, on **m-words**;
    - $\mathbf{DA}_m$, gen. by $\{00, 01, \dots, 0(m-1)\}$, on some **directed animals**.
- For any $m \geqslant 0$, $\mathcal{M} := (\mathbb{N}, \max, 0)$,
    - $\mathbf{Dias}_m$, gen. by $\{01, \dots, 0m, m0, \dots, 10\}$, is the m-pluriassociative operad;
    - $\mathbf{Trias}_m$, gen. by $\{01, \dots, 0m, 00, m0, \dots, 10\}$, is the m-pluritriassociative operad.

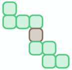
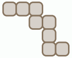
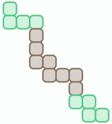
Some partial compositions on combinatorial objects

 $\circ_2$  $=$  (in $\mathbf{PRT}_1$)

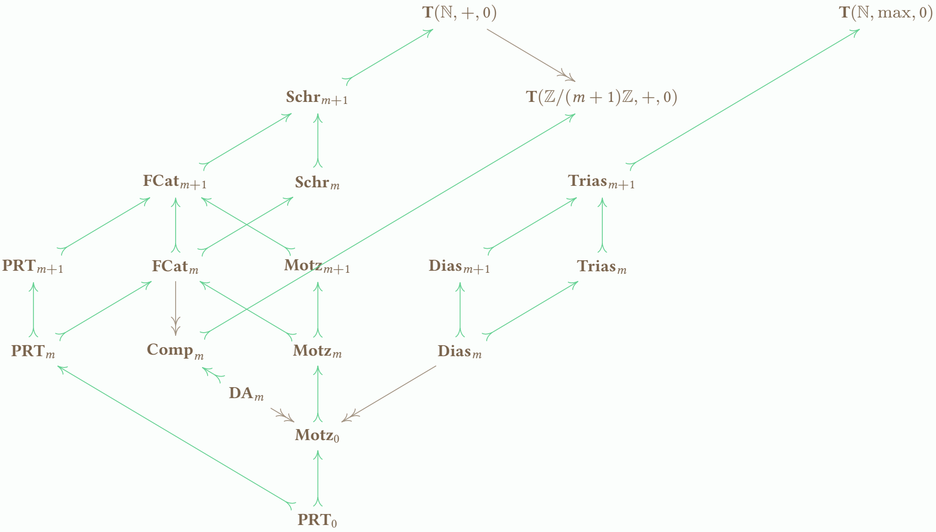
 $\circ_1$  $=$  (in $\mathbf{FCat}_2$)

 $\circ_6$  $=$  (in $\mathbf{Schr}_1$)

 $\circ_4$  $=$  (in $\mathbf{Motz}_1$)

 $\circ_5$  $=$  (in $\mathbf{Comp}_1$)

Full diagram



Translation algebras

Let $(\mathcal{M}, \cdot, e)$ be a monoid.

– Proposition –

The operad $\mathbf{T}\mathcal{M}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathcal{M} \sqcup \{ee\}$ and $\mathfrak{R}$ satisfies

$$ee \circ_1 ee \;\mathfrak{R}\; ee \circ_2 ee,$$

$$a \circ_1 b \;\mathfrak{R}\; a \cdot b, \qquad a, b \in \mathcal{M},$$

$$ee \circ [a, a] \;\mathfrak{R}\; a \circ_1 ee, \qquad a \in \mathcal{M}.$$

An $\mathcal{M}$-translation algebra is a set S endowed with a binary operation $\star : S \times S \to S$ and unary operations $\theta_a : S \to S$, $a \in \mathcal{M}$, satisfying

(TAs) $(s_1 \star s_2) \star s_3 = s_1 \star (s_2 \star s_3)$,

(TU) $\theta_e(s_1) = s_1$,

(TAc) $\theta_a(\theta_b(s_1)) = \theta_{a \cdot b}(s_1)$,

(TMo) $\theta_a(s_1 \star s_2) = \theta_a(s_1) \star \theta_a(s_2)$.

– Proposition –

Any $\mathcal{M}$-translation algebra is a $\mathbf{T}\mathcal{M}$-algebra and any $\mathbf{T}\mathcal{M}$-algebra is an $\mathcal{M}$-translation algebra.

Outline

4. Constructions

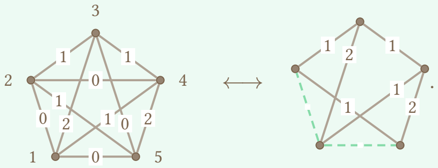
Decorated cliques

Let $(\mathcal{M}, \cdot, e)$ be a monoid.

An $\mathcal{M}$-clique $\mathfrak{p}$ is a **complete graph** on $[n+1]$ where each edge (x, y) is **decorated** by an element $\mathfrak{p}(x, y) \in \mathcal{M}$. The arity of $\mathfrak{p}$ is n.

– Example –

Set $\mathcal{M} := (\mathbb{Z}/3\mathbb{Z}, +, 0)$. Here is an $\mathcal{M}$-clique (on the right, the edges decorated by the unit e of $\mathcal{M}$ are not drawn and this convention is used in the sequel):

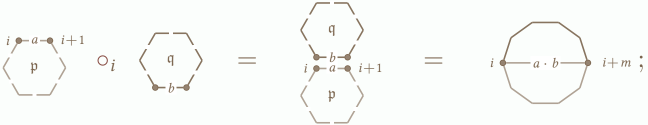


The arity of this clique is 4.

The construction C

Let $(\mathcal{M}, \cdot, e)$ be a **monoid** and let $(\mathbf{C}\mathcal{M}, \circ_i, \mathbf{1})$ be the triple such that

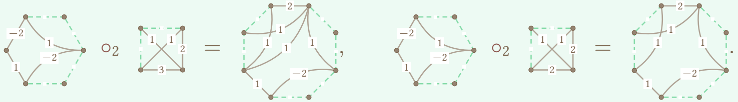
- $\mathbf{C}\mathcal{M}(0) = \emptyset$ and for any $n \geqslant 1$, $\mathbf{C}\mathcal{M}(n)$ is the set of the $\mathcal{M}$-cliques of arity n;
- For any $\mathfrak{p} \in \mathbf{C}\mathcal{M}(n)$ and $\mathfrak{q} \in \mathbf{C}\mathcal{M}(m)$, $\mathfrak{p} \circ_i \mathfrak{q}$ is defined by



- $\mathbf{1}$ is the $\mathcal{M}$-clique .

– Examples –

Set $\mathcal{M} := (\mathbb{Z}, +, 0)$. In $\mathbf{C}\mathcal{M}$, we have



– Theorem –

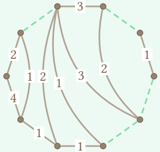
For any monoid $\mathcal{M}$, $\mathbf{C}\mathcal{M}$ is an operad.

Noncrossing cliques

An $\mathcal{M}$-clique $\mathfrak{p}$ is noncrossing if there are no edges (x, y) and (x', y') such that $\mathfrak{p}(x, y) \neq e \neq \mathfrak{p}(x, y)$ and $x < x' < y < y'$ or $x' < x < y' < y$.

– Example –

For $\mathcal{M} := (\mathbb{N}, +, 0)$, the $\mathcal{M}$-clique



is noncrossing.

– Proposition –

The set of the noncrossing $\mathcal{M}$-cliques is a suboperad of $\mathbf{C}\mathcal{M}$.

Let $\mathbf{NC}\mathcal{M}$ be this **operad of noncrossing $\mathcal{M}$-cliques**.

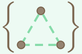
Link with translation algebras

Let $(\mathcal{M}, \cdot, e)$ be a monoid.

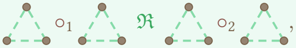
– Proposition –

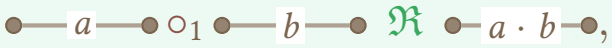
The operad $\mathbf{NC}\mathcal{M}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where

$$\mathfrak{G} := \{ \bullet\!\!-\!\!a\!\!-\!\!\bullet : a \in \mathcal{M} \} \sqcup \{ \triangle \}$$



and $\mathfrak{R}$ satisfies



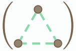


$a, b \in \mathcal{M}$.

As a consequence, any $\mathbf{NC}\mathcal{M}$-algebra is a set S endowed with a binary operation $\star : S \times S \to S$ and unary operations $\theta_a : S \to S$, $a \in \mathcal{M}$ satisfying Relations (TAs), (TAc), and (TU) of $\mathcal{M}$-**translation algebras**.

Link with the construction T

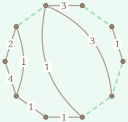
The map $\phi : \mathbf{NC}\mathcal{M} \to \mathbf{T}\mathcal{M}$ satisfying

$\phi\Big($  $\Big) = ee$ and $\phi\big($  $\big) = a$, $a \in \mathcal{M}$

extends uniquely into an operad morphism.

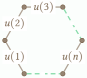
– Example –

For $\mathcal{M} := (\mathbb{N}, +, 0)$,



$\overset{\phi}{\longmapsto}$ 264185652.

Since for any $u(1) \dots u(n) \in \mathbf{T}\mathcal{M}(n)$,



$\overset{\phi}{\longmapsto} u(1)u(2)u(3)\dots u(n)$,

this morphism is surjective. Therefore, $\mathbf{T}\mathcal{M}$ is a **quotient operad** of $\mathbf{NC}\mathcal{M}$.

Some references

About the construction **T**:

- S. Giraudo, **Combinatorial operads from monoids**, 2015.

About the construction **C**:

- S. Giraudo, **Operads of decorated cliques I: Construction and quotients**, 2020.
- S. Giraudo, **Operads of decorated cliques II: Noncrossing cliques**, 2022–.

5. Combinatorics

Outline

5. Combinatorics

The operad $\mathbf{FCat}_m$

Recall that $\mathbf{FCat}_m$, $m \geqslant 0$, is the suboperad of $\mathbf{T}(\mathbb{N}, +, 0)$ generated by $\mathfrak{G}_m := \{00, 01, \ldots, 0m\}$.

– Examples –

Here are the first sets of elements of $\mathbf{FCat}_m$, for $m \in \{1, 2\}$:

$$\mathbf{FCat}_1(0) = \emptyset, \quad \mathbf{FCat}_1(1) = \{0\}, \quad \mathbf{FCat}_1(2) = \{00, 01\}, \quad \mathbf{FCat}_1(3) = \{000, 001, 010, 011, 012\},$$
$$\mathbf{FCat}_1(4) = \{0000, 0001, 0010, 0011, 0012, 0100, 0101, 0110, 0111, 0112, 0120, 0121, 0122, 0123\},$$

$$\mathbf{FCat}_2(0) = \emptyset, \quad \mathbf{FCat}_2(1) = \{0\}, \quad \mathbf{FCat}_2(2) = \{00, 01, 02\},$$
$$\mathbf{FCat}_2(3) = \{000, 001, 002, 010, 011, 012, 013, 020, 021, 022, 023, 024\},$$
$$\mathbf{FCat}_2(4) = \{0000, 0001, 0002, 0010, 0011, 0012, 0013, 0020, 0021, 0022, 0023, 0024, 0100, 0101, 0102, 0110, 0111, 0112,$$
$$0113, 0120, 0121, 0122, 0123, 0124, 0130, 0131, 0132, 0133, 0134, 0135, 0200, 0201, 0202, 0210, 0211, 0212,$$
$$0213, 0220, 0221, 0222, 0223, 0224, 0230, 0231, 0232, 0233, 0234, 0235, 0240, 0241, 0242, 0243, 0244, 0245, 0246\}.$$

– Example –

In $\mathbf{FCat}_2$, $013102 \circ_4 0232 = 013134302$.

Description of its elements

– Proposition –

The elements of $\mathbf{FCat}_m$ are exactly the words u of integers satisfying

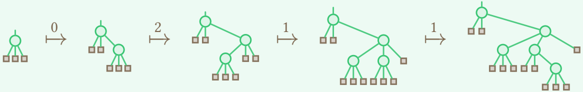
1. $u(1) = 0$;
2. $0 \leqslant u(i+1) \leqslant u(i) + m$ for all $i \in [|u| - 1]$.

Let $\mathfrak{F}_m$ be the set of the *m*-trees, that are planar rooted trees such that each internal node has exactly $m+1$ children. Let $\phi : \mathbf{FCat}_m \to \mathfrak{F}_m$ be the map recursively defined by

- $\phi(0)$ is the m-tree consisting in a single internal node;
- $\phi(ua)$ is the planar rooted tree obtained by grafting $\phi(0)$ on the a-th leaf (indexed from 0 and from the right) of $\phi(u)$.

– Example –

Computation of $\phi(00211)$ for $m := 2$:

$\phi(0) =$ 

Interpretation of $\mathbf{FCat}_m$ in terms of m-trees

– Proposition –

The map ϕ is a bijection between $\mathbf{FCat}_m$ and $\mathfrak{F}_m$.

Therefore, for any $n \geqslant 1$,

$$\#\mathbf{FCat}_m(n) = \binom{(m+1)n}{n} \frac{1}{mn+1}.$$

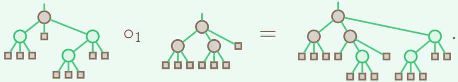
It is possible to interpret the partial composition of $\mathbf{FCat}_m$ in terms of m-trees.

– Example –

In $\mathbf{FCat}_2$, we have

$$0202 \circ_1 021 = 021202.$$

By ϕ, this translates as



Presentation

– Theorem –

The operad $\mathbf{FCat}_m$ admits the presentation $(\mathfrak{G}_m, \mathfrak{R}_m)$ where $\mathfrak{R}_m$ satisfies

$$0(a+b) \circ_1 0a \;\; \mathfrak{R}_m \;\; 0a \circ_2 0b,$$

for any $a, b \geqslant 0$ such that $a + b \leqslant m$.

– Exercise ●●○○○ –

Show this presentation by $\mathbf{FCat}_m$ by considering the orientation $\to_m$ of $\mathfrak{R}_m$ satisfying

$$0(a+b) \circ_1 0a \;\; \to_m \;\; 0a \circ_2 0b.$$

– Example –

The operad $\mathbf{FCat}_1$ admits the presentation $(\mathfrak{G}_1, \mathfrak{R}_1)$ where $\mathfrak{G}_1 := \{00, 01\}$ and $\mathfrak{R}_1$ satisfies

$$00 \circ_1 00 \;\; \mathfrak{R}_1 \;\; 00 \circ_2 00,$$

$$01 \circ_1 00 \;\; \mathfrak{R}_1 \;\; 00 \circ_2 01,$$

$$01 \circ_1 01 \;\; \mathfrak{R}_1 \;\; 01 \circ_2 00.$$

Outline

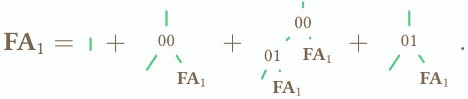
5. Combinatorics

Enumeration of the normal forms

By the previous results, there is a one-to-one correspondence between $\mathbf{FCat}_m(n)$ and the **normal forms** of arity n of the PTRS $(\mathfrak{G}_m, \rightarrow_m)$.

In the particular case where $m = 1$, these normal forms are the planar $\mathfrak{G}_1$-terms avoiding

The formal series $\mathbf{FA}_1$ of these normal form expresses as



Enumeration map

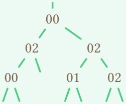
For any $m \geqslant 0$, the enumeration map is the map

$$\mathrm{en} : \mathfrak{T}_{\mathrm{P}}(\mathfrak{G}_m) \to \mathbb{Q}[[\mathrm{z}, \mathrm{q}_0, \dots, \mathrm{q}_m]]$$

such that

$$\mathrm{en}(\mathfrak{t}) := \mathrm{z}^{|\mathfrak{t}|}\ \mathrm{q}_0^{\deg_{00}(\mathfrak{t})} \dots\ \mathrm{q}_m^{\deg_{0m}(\mathfrak{t})}.$$

– Example –



$$\xmapsto{\mathrm{en}} \mathrm{z}^7 \mathrm{q}_0^2 \mathrm{q}_1 \mathrm{q}_2^3$$

This map extends by linearity on the space of formal series of planar $\mathfrak{G}$-terms.

If $\mathbf{f}$ is a formal series of $\mathfrak{G}_m$-terms, then $\mathrm{en}(\mathbf{f})$ is the **generating series** enumerating the terms appearing in $\mathbf{f}$ w.r.t. their arities (parameter z) and their numbers of internal nodes labeled by $0k$ (parameters q_k).

Generating series

Let us set $\mathrm{FA}_m := \mathrm{en}(\mathbf{FA}_m)$.

The previous expression for $\mathbf{FA}_1$ leads to the expression

$$\mathrm{FA}_1 = \mathrm{z}(\mathrm{q}_0 \mathrm{FA}_1 + 1)(\mathrm{q}_1 \mathrm{FA}_1 + 1)$$

for FA_1 so that

$$\mathrm{FA}_1 = \frac{1 - \mathrm{z}(\mathrm{q}_0 + \mathrm{q}_1) - \sqrt{1 - 2\mathrm{z}(\mathrm{q}_0 + \mathrm{q}_1) + \mathrm{z}^2(\mathrm{q}_0^2 - 2\mathrm{q}_0\mathrm{q}_1 + \mathrm{q}_1^2)}}{2\mathrm{z}\mathrm{q}_0\mathrm{q}_1}$$

and

$$\begin{aligned}\mathrm{FA}_1 = \mathrm{z} + (\mathrm{q}_0 + \mathrm{q}_1)\mathrm{z}^2 + \left(\mathrm{q}_0^2 + 3\mathrm{q}_0\mathrm{q}_1 + \mathrm{q}_1^2\right)\mathrm{z}^3 \\ + \left(\mathrm{q}_0^3 + 6\mathrm{q}_0^2\mathrm{q}_1 + 6\mathrm{q}_0\mathrm{q}_1^2 + \mathrm{q}_1^3\right)\mathrm{z}^4 \\ + \left(\mathrm{q}_0^4 + 10\mathrm{q}_0^3\mathrm{q}_1 + 20\mathrm{q}_0^2\mathrm{q}_1^2 + 10\mathrm{q}_0\mathrm{q}_1^3 + \mathrm{q}_1^4\right)\mathrm{z}^5 \\ + \left(\mathrm{q}_0^5 + 15\mathrm{q}_0^4\mathrm{q}_1 + 50\mathrm{q}_0^3\mathrm{q}_1^2 + 50\mathrm{q}_0^2\mathrm{q}_1^3 + 15\mathrm{q}_0\mathrm{q}_1^4 + \mathrm{q}_1^5\right)\mathrm{z}^6 + \cdots .\end{aligned}$$

Narayana triangle

The **specialization** $\mathrm{FA}_m(\mathrm{z}, \mathrm{q}_k)$ of all parameters of FA_m to 1 except z and the parameter q_k, $k \in [m]$, leads to a bi-indexed family of integers.

This produces **triangles of integers**, wherein the integer at position (i, j), $i \geqslant 1$, $j \geqslant 0$, is the coefficient of $\mathrm{z}^i \mathrm{q}_k^j$ in $\mathrm{FA}_m(\mathrm{z}, \mathrm{q}_k)$.

– Examples –

Coefficients of $\mathrm{FA}_1(\mathrm{z}, \mathrm{q}_0)$:

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

Coefficients of $\mathrm{FA}_1(\mathrm{z}, \mathrm{q}_1)$:

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

This is the same triangle, known as the **Triangle of Narayana numbers** (Triangle **A001263**).

It counts binary trees w.r.t. their number of internal nodes and the number of edges oriented to the left (resp. right) connecting two internal nodes.

Outline

5. Combinatorics

Alternative orientation

The presentation $(\mathfrak{G}_m, \mathfrak{R}_m)$ of $\mathbf{FCat}_m$ where $\mathfrak{R}_m$ satisfies

$$0(a+b) \circ_1 0a \;\; \mathfrak{R}_m \;\; 0a \circ_2 0b,$$

for any $a, b \geqslant 0$ such that $a + b \leqslant m$ admits the **alternative orientation**

$$\begin{aligned} 0(a+b) \circ_1 0a \;\; &\to'_m \;\; 0a \circ_2 0b, && \text{if } 0 \leqslant a \leqslant b \leqslant m \text{ and } a+b \leqslant m, \\ 0a \circ_2 0b \;\; &\to'_m \;\; 0(a+b) \circ_1 0a, && \text{if } 0 \leqslant b < a \leqslant m \text{ and } a+b \leqslant m. \end{aligned}$$

– Example –

The rewrite relation $\to'_1$ satisfies

$$\begin{aligned} 00 \circ_1 00 \;\; &\to'_1 \;\; 00 \circ_2 00, \\ 01 \circ_1 00 \;\; &\to'_1 \;\; 00 \circ_2 01, \\ 01 \circ_2 00 \;\; &\to'_1 \;\; 01 \circ_1 01. \end{aligned}$$

– Proposition –

The PTRS $(\mathfrak{G}_m, \to'_m)$ is terminating and confluent.

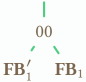
Enumeration of the normal forms

There is a one-to-one correspondence between $\mathbf{FCat}_m(n)$ and the normal forms of arity n of the PTRS $(\mathfrak{G}_m, \rightarrow'_m)$.

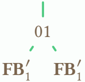
In the particular case where $m = 1$, these normal forms are the planar $\mathfrak{G}_1$-terms avoiding

The formal series $\mathbf{FB}_1$ of these normal forms expresses as

$$\mathbf{FB}_1 = | +$$



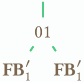
$$+$$



,

where

$$\mathbf{FB}'_1 = | +$$



.

Generating series

Let us set $\mathrm{FB}_m := \mathrm{en}(\mathbf{FB}_m)$.

The previous expression for $\mathbf{FB}_1$ leads to the generating function

$$\mathrm{FB}_1 = \frac{1 - 2zq_0 - \sqrt{1 - 4zq_1}}{2(q_1 - q_0 + zq_0^2)}$$

for FB_1 so that

$$\begin{aligned}\mathrm{FB}_1 = z + (q_0 + q_1)z^2 + \left(q_0^2 + 2q_0q_1 + 2q_1^2\right)z^3 \\ + \left(q_0^3 + 3q_0^2q_1 + 5q_0q_1^2 + 5q_1^3\right)z^4 \\ + \left(q_0^4 + 4q_0^3q_1 + 9q_0^2q_1^2 + 14q_0q_1^3 + 14q_1^4\right)z^5 \\ + \left(q_0^5 + 5q_0^4q_1 + 14q_0^3q_1^2 + 28q_0^2q_1^3 + 42q_0q_1^4 + 42q_1^5\right)z^6 + \cdots ,\end{aligned}$$

Catalan triangle

The specialization $\mathrm{FB}_m(\mathrm{z}, \mathrm{q}_k)$ of all parameters of FB_m to 1 except z and the parameter q_k, $k \in [m]$, leads to a bi-indexed family of integers.

This produces again triangles of numbers.

– Examples –

Coefficients of $\mathrm{FB}_1(\mathrm{z}, \mathrm{q}_0)$:

1					
1	1				
2	2	1			
5	5	3	1		
14	14	9	4	1	
42	42	28	14	5	1

Coefficients of $\mathrm{FB}_1(\mathrm{z}, \mathrm{q}_1)$:

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

Each triangle is the mirror of the other. They are known as the **Catalan Triangle** (Triangle **A009766**).

It counts binary trees w.r.t. their number of internal nodes and the **jump-length statistics**.

Some projects

– Research project ●●●○○ –

Extends the previous results for any $m \geqslant 2$ by providing a combinatorial interpretation of the coefficients of the obtained triangles of integers. More precisely,

1. Provide systems of equations for $\mathbf{FA}_m$ and $\mathbf{FB}_m$;
2. Provide a description of FA_m and FB_m;
3. Provide a description of the coefficients of $\mathrm{FA}_m(\mathrm{z}, \mathrm{q}_k)$ and $\mathrm{FB}_m(\mathrm{z}, \mathrm{q}_k)$ for all $k \in [m]$.

– Research project ●●●◐○ –

Develop a similar study for other operads in order to discover new triangles of integers. This includes the operad $\mathbf{Motz}_m$ of Motzkin paths, the operad $\mathbf{Schr}_m$ of Schröder trees, and the operad $\mathbf{DA}_m$ of directed animals.

Some references

About the discovery of statistics through operads:

- S. Giraudo, **Tree series and pattern avoidance in syntax trees**, 2020.

About other interactions between operads and combinatorics:

- S. Giraudo, **Colored operads, series on colored operads, and combinatorial generating systems**, 2019.
- S. Giraudo, **Generation of musical patterns through operads**, 2020.
- C. Chenavier, C. Cordero, S. Giraudo, **Quotients of the magmatic operad: lattice structures and convergent rewrite systems**, 2019.
- S. Giraudo, **Duality of graded graphs through operads**, 2021.
- C. Combe, S. Giraudo, **Cliff operads: a hierarchy of operads on words**, 2022–.
- F. Fauvet, L. Foissy, D. Manchon, **Operads of finite posets**, 2018.

6. Annex

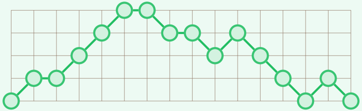
Outline

6. Annex

Enumeration of Motzkin paths

A Motzkin path is a path in $\mathbb{N}^2$ starting from $(0,0)$ and ending at $(n,0)$, made of steps $(+1,+1)$, $(+1,-1)$, and $(+1,0)$.

– Example –



With the aid of some elementary reasoning, one can prove that the generating series $\mathcal{F}(z)$ of Motzkin paths, enumerating them w.r.t. their number of points, satisfies

$$\mathcal{F}(z) = z + z\mathcal{F}(z) + z\mathcal{F}(z)^2$$

and

$$\mathcal{F}(z) = z + z^2 + 2z^3 + 4z^4 + 9z^5 + 21z^6 + 51z^7 + 127z^8 + 323z^9 + \cdots .$$

Composition of Motzkin paths and series of objects

A way to obtain the previous expression for this series consists in following both steps:

1. define a composition operation on the set of Motzkin paths;
2. express the infinite formal sum of all Motzkin paths.

If u and v are two Motzkin paths, the composition $u \circ_i v$ is obtained by replacing the i-th point of u by v.

The infinite formal sum of all Motzkin paths is

and we can prove that it satisfies the functional equation

This is a consequence of a property of the operad **Motz** of Motzkin paths (and more precisely, the fact that it is a Koszul operad).

Generating set of Per

A permutation σ is simple if σ does not admit any factor of length between 2 and $|\sigma| - 1$ which is a segment.

– Examples –

415362 is simple.

32**5786**1 is not simple.

These permutations are enumerated w.r.t. their size by Sequence **A111111**, beginning by

$$0, 2, 0, 2, 6, 46, 338, 2926, 28146, 298526.$$

– Proposition –

The set of the simple permutations is a minimal generating set of **Per**.

– Exercise ●○○○○ –

Show the previous proposition.

The operad **Per** admits neither binary nor quadratic presentation.

Factors in planar terms

Let $\mathfrak{t}, \mathfrak{s} \in \mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$.

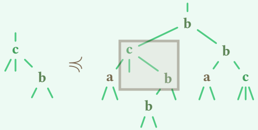
The planar term $\mathfrak{s}$ is a factor of $\mathfrak{t}$ if there exist $\mathfrak{r}, \mathfrak{r}_1, \dots, \mathfrak{r}_{|\mathfrak{s}|} \in \mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$ and $i \in [|\mathfrak{r}|]$ such that

$$\mathfrak{t} = \mathfrak{r} \circ_i \left(\mathfrak{s} \circ \left[\mathfrak{r}_1, \dots, \mathfrak{r}_{|\mathfrak{s}|}\right]\right).$$

This property is denoted by $\mathfrak{s} \preccurlyeq \mathfrak{t}$.

When $\mathfrak{s} \not\preccurlyeq \mathfrak{t}$, $\mathfrak{t}$ avoids $\mathfrak{s}$.

– Example –



– Exercise ●●○○○ –

Show that for any signature $\mathfrak{G}$, $\preccurlyeq$ is a partial order relation on $\mathfrak{T}_{\mathrm{P}}(\mathfrak{G})$.

Operad of maps and morphisms

Let S be a set. The operad of S-maps $\mathbf{Map}S$ is the operad wherein

- $\mathbf{Map}S(n)$ is the set of the maps from S^n to S;
- $f \circ_i g$ is the map satisfying

$$(f \circ_i g)(s_1, \dots, s_{n+m-1}) = f(s_1, \dots, s_{i-1}, g(s_i, \dots, s_{i+m-1}), s_{i+m}, \dots, s_{n+m-1});$$

- the unit $\mathbf{1}$ is the identity map on S.

Alternatively, any $\mathcal{O}$-algebra S can be specified by an operad morphism

$$\phi : \mathcal{O} \to \mathbf{Map}S.$$

This map ϕ is in fact the map $\mathfrak{op}$ introduced previously.

Free algebras over operads

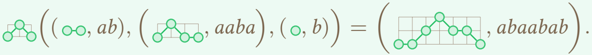
If $\mathcal{O}$ is an operad and A is a nonempty set, let $\mathcal{O}(A)$ be the set of the pairs (x, u) where $x \in \mathcal{O}(n)$ and $u \in A^n$.

Let $\mathfrak{op}$ be the map defined for any $x \in \mathcal{O}(n)$ and $(y_i, u_i) \in \mathcal{O}(A)$ by

$$x((y_1, u_1), \ldots, (y_n, u_n)) := (x \circ [y_1, \ldots, y_n], u_1 \ldots u_n).$$

– Example –

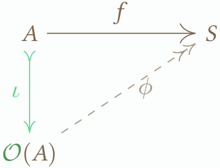
For $\mathcal{O} :=$ **Motz** and $A := \{a, b\}$,



– Proposition –

The set $\mathcal{O}(A)$ is an $\mathcal{O}$-algebra. It is moreover free as an $\mathcal{O}$-algebra.

For any set A, any $\mathcal{O}$-algebra S, and any map $f : A \to S$, there exists a unique $\mathcal{O}$-algebra morphism $\phi : \mathcal{O}(A) \to S$ such that $f = \phi \circ$ c, where $\iota : A \to \mathcal{O}(A)$ is the map $a \mapsto (\mathbf{1}, a)$.



Outline

6. Annex

The construction A

Let $(\mathcal{P}, \preccurlyeq)$ be a **poset**. Given $a, b \in \mathcal{P}$, a and b are comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$. In this case, the smallest element among a and b is denoted by $a \uparrow b$.

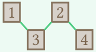
Let $\mathbf{A}\mathcal{P}$ be the operad admitting the **presentation** $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G}$ is the graded set $\mathfrak{G} := \mathfrak{G}(2) := \mathcal{P}$ and $\mathfrak{R}$ satisfies

$$a \circ_1 b \;\mathfrak{R}\; (a \uparrow b) \circ_2 (a \uparrow b), \qquad a, b \in \mathcal{P} \text{ when } a \text{ and } b \text{ are comparable,}$$

$$(a \uparrow b) \circ_1 (a \uparrow b) \;\mathfrak{R}\; a \circ_2 b, \qquad a, b \in \mathcal{P} \text{ when } a \text{ and } b \text{ are comparable.}$$

– Example –

By considering the poset $\mathcal{P}$ having the Hasse diagram on the right, the operad $\mathbf{A}\mathcal{P}$ admits the presentation $(\mathfrak{G}, \mathfrak{R})$ where $\mathfrak{G} := \mathfrak{G}(2) := \{1, 2, 3, 4\}$ and $\mathfrak{R}$ satisfies

$\mathcal{P} :=$ 

$$1 \circ_1 1 \;\mathfrak{R}\; 1 \circ_1 3 \;\mathfrak{R}\; 3 \circ_1 1 \;\mathfrak{R}\; 3 \circ_2 1 \;\mathfrak{R}\; 1 \circ_2 3 \;\mathfrak{R}\; 1 \circ_2 1,$$

$$2 \circ_1 2 \;\mathfrak{R}\; 2 \circ_1 3 \;\mathfrak{R}\; 2 \circ_1 4 \;\mathfrak{R}\; 3 \circ_1 2 \;\mathfrak{R}\; 4 \circ_1 2 \;\mathfrak{R}\; 4 \circ_2 2 \;\mathfrak{R}\; 3 \circ_2 2 \;\mathfrak{R}\; 2 \circ_2 4 \;\mathfrak{R}\; 2 \circ_2 3 \;\mathfrak{R}\; 2 \circ_2 2,$$

$$3 \circ_1 3 \;\mathfrak{R}\; 3 \circ_2 3,$$

$$4 \circ_1 4 \;\mathfrak{R}\; 4 \circ_2 4.$$

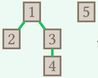
$\mathcal{P}$-alternating Schröder trees

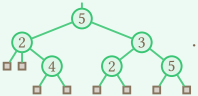
A $\mathcal{P}$-alternating Schröder tree is a planar rooted tree $\mathfrak{t}$ such that

- each internal node of $\mathfrak{t}$ has two or more children;
- each internal node of $\mathfrak{t}$ is decorated on $\mathcal{P}$;
- if u and v are two internal nodes of $\mathfrak{t}$ such that v is a child of u, then the decorations of u and of v are incomparable in $\mathcal{P}$.

– Example –

Here are a poset $\mathcal{P}$ and a $\mathcal{P}$-alternating Schröder tree $\mathfrak{t}$:

$\mathcal{P} :=$ 

, $\mathfrak{t} :=$ 

.

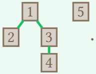
Let not denote by $\mathfrak{A}\mathcal{P}$ the graded set of the $\mathcal{P}$-alternating Schröder trees where the arity of such a tree is its number of leaves.

Forest posets and realization of $A\mathcal{P}$

A poset $\mathcal{P}$ is a forest poset if for any $a, b, c \in \mathcal{P}$, if $a \preccurlyeq c$ and $b \preccurlyeq c$, then $a = c$ or $b = c$. In other terms, the Hasse diagram of $\mathcal{P}$ is a **rooted forest**, where the roots are the minimal elements.

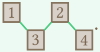
– Example –

Here is a forest poset:



– Example –

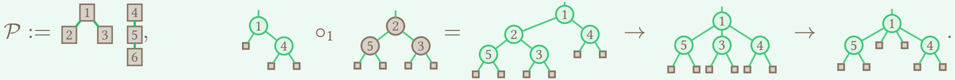
Here is a poset which is not a forest poset:



For any $\mathfrak{t}, \mathfrak{s} \in \mathfrak{A}\mathcal{P}$, $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by grafting a copy of $\mathfrak{s}$ on the i-th leaf of $\mathfrak{t}$, and by iteratively contracting each edge between two internal nodes decorated by two comparable elements a and b to form an internal node labeled by $a \uparrow b$.

– Example –

Here are a poset $\mathcal{P}$ and a partial composition in $\mathfrak{A}\mathcal{P}$:



Forest posets and realization of $\mathbf{A}\mathcal{P}$

– Theorem –

If $\mathcal{P}$ is a forest poset, then $\mathfrak{A}\mathcal{P}$ is a realization of the operad $\mathbf{A}\mathcal{P}$.

– Open question ●●●○○ –

Build a realization of $\mathbf{A}\mathcal{P}$ for any poset $\mathcal{P}$.

Reference about the construction $\mathbf{A}$:

- S. Giraudo, **Operads from posets and Koszul duality**, 2016.