Balanced binary trees in the Tamari lattice

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This work is related to [2].

Definitions and goal

Goal

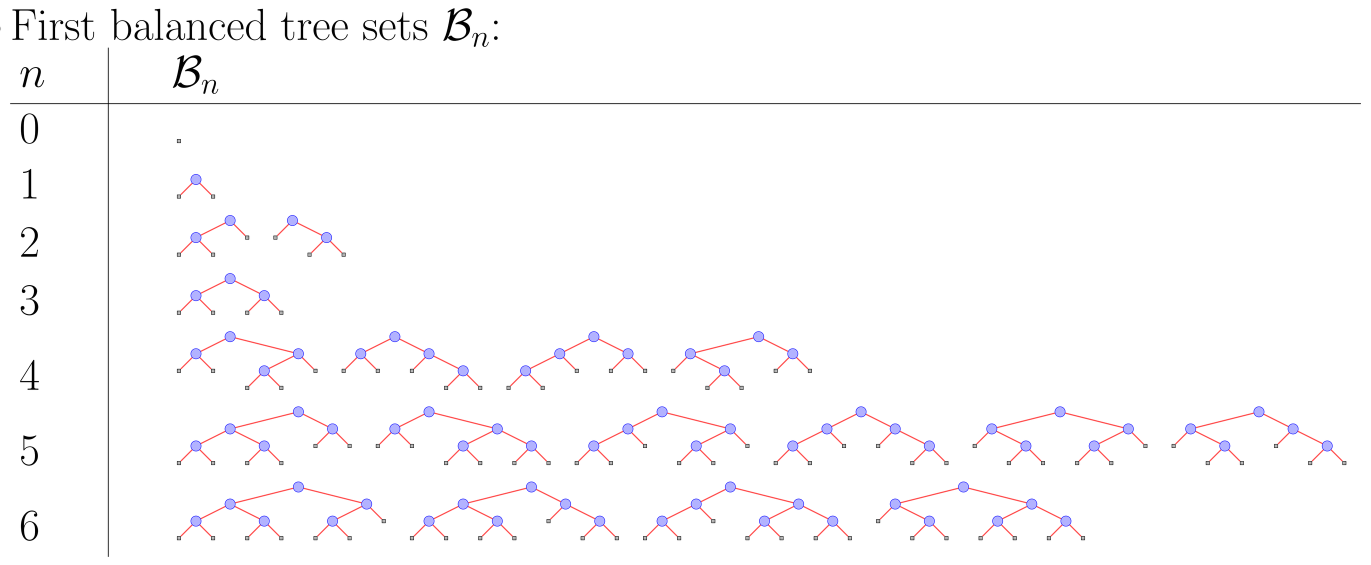
Find the particular role played by balanced binary trees in the Tamari lattice.

Balanced binary trees

- Binary search trees can represent dynamic totally ordered sets.
- Balanced binary search trees ensure the efficiency of the related algorithms.
- A tree T is balanced if for every node x of T the relation

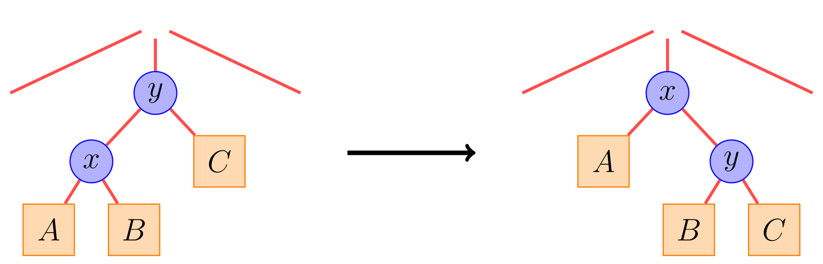
$$\mathrm{ht}(R) - \mathrm{ht}(L) \in \{-1, 0, 1\}$$

holds, where R (resp. L) is the right (resp. left) subtree of x and $\mathrm{ht}(T)$ is its height.
- The balance of the tree is kept performing rotation operations [1].
- First balanced tree sets $\mathcal{B}_n$:

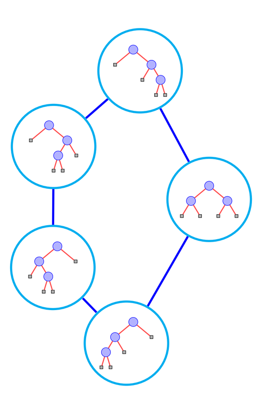


The Tamari lattice

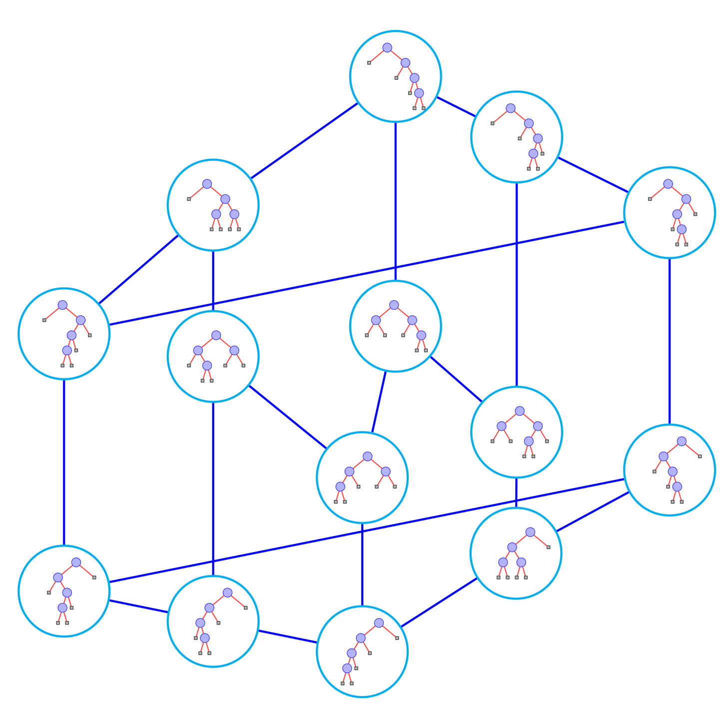
- Catalan objects of size n admit an order structure $\mathbb{T}_n$ which is rich in combinatorics: the Tamari lattice.
- On binary trees, covering relations are right rotation operations:



- The first Tamari Hasse diagrams:



$\mathbb{T}_3$



$\mathbb{T}_4$

Some new tools and concepts

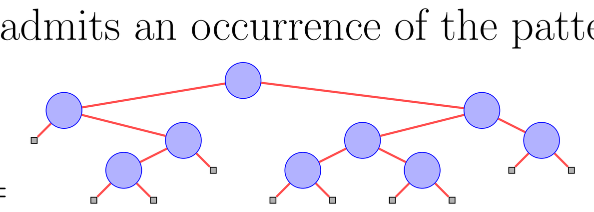
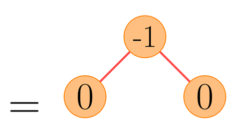
Tree patterns

- Let T be a binary tree. The tree T_γ is the $\mathbb{Z}$-labelled binary tree where the label of each node x is the value

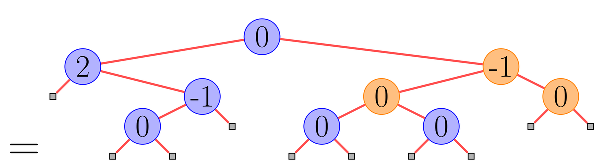
$$\mathrm{ht}(R) - \mathrm{ht}(L)$$

where R (resp. L) is the right (resp. left) subtree of x.
- A tree pattern is a nonempty non complete planar $\mathbb{Z}$-labelled binary tree.
- A tree T admits an occurrence of a tree pattern p if a connected component of T_γ has the same shape and same labels as p.

Example
The tree T admits an occurrence of the pattern p where

$T =$  and $p =$ 

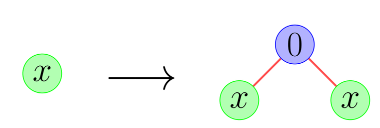
since

$T_\gamma =$  .

Synchronous grammars

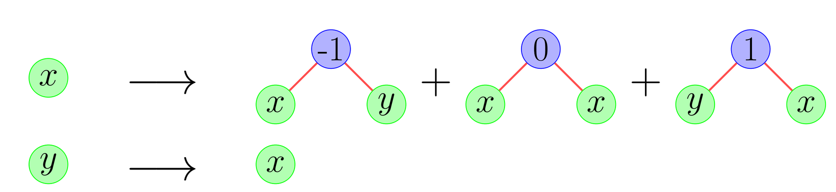
- Synchronous grammars allow to generate trees avoiding a given set of patterns.
- They are used to obtain functional equations of the generating series for various classes of binary trees.
- They span trees by simultaneously substituting all grammar axioms by new trees.

Example
The synchronous grammar



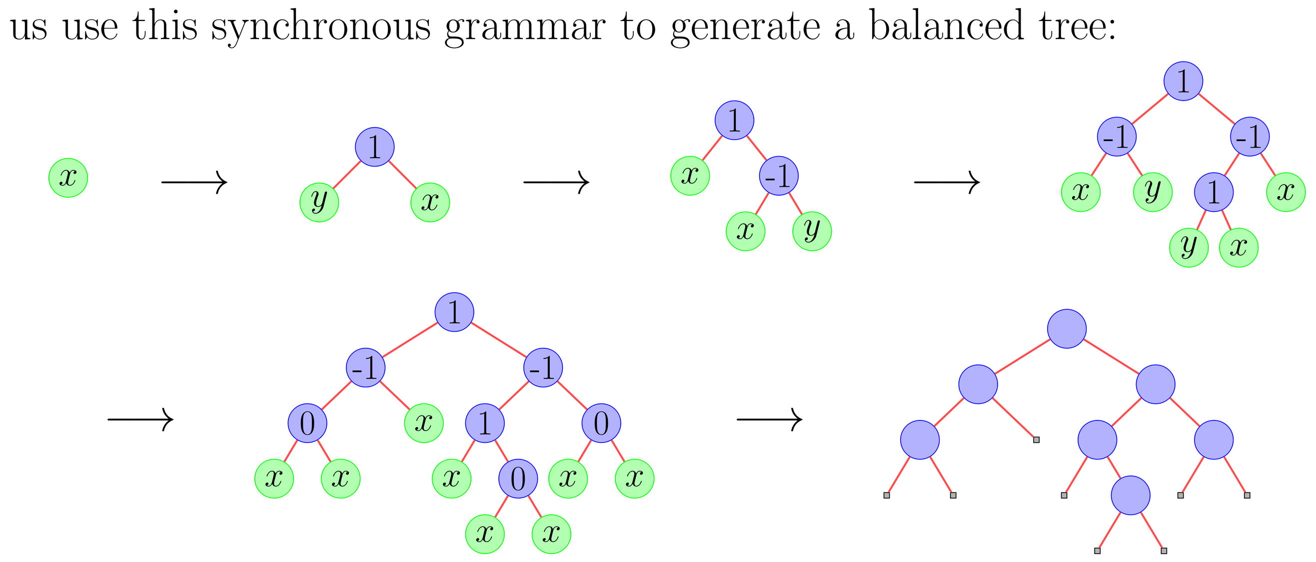
spans the perfect trees, that are the trees avoiding the set $\{ (i) \mid i \neq 0\}$.

Example
The synchronous grammar



spans the balanced trees that are the trees avoiding the set $\{ (i) \mid i \notin \{-1, 0, 1\}\}$.

Let us use this synchronous grammar to generate a balanced tree:



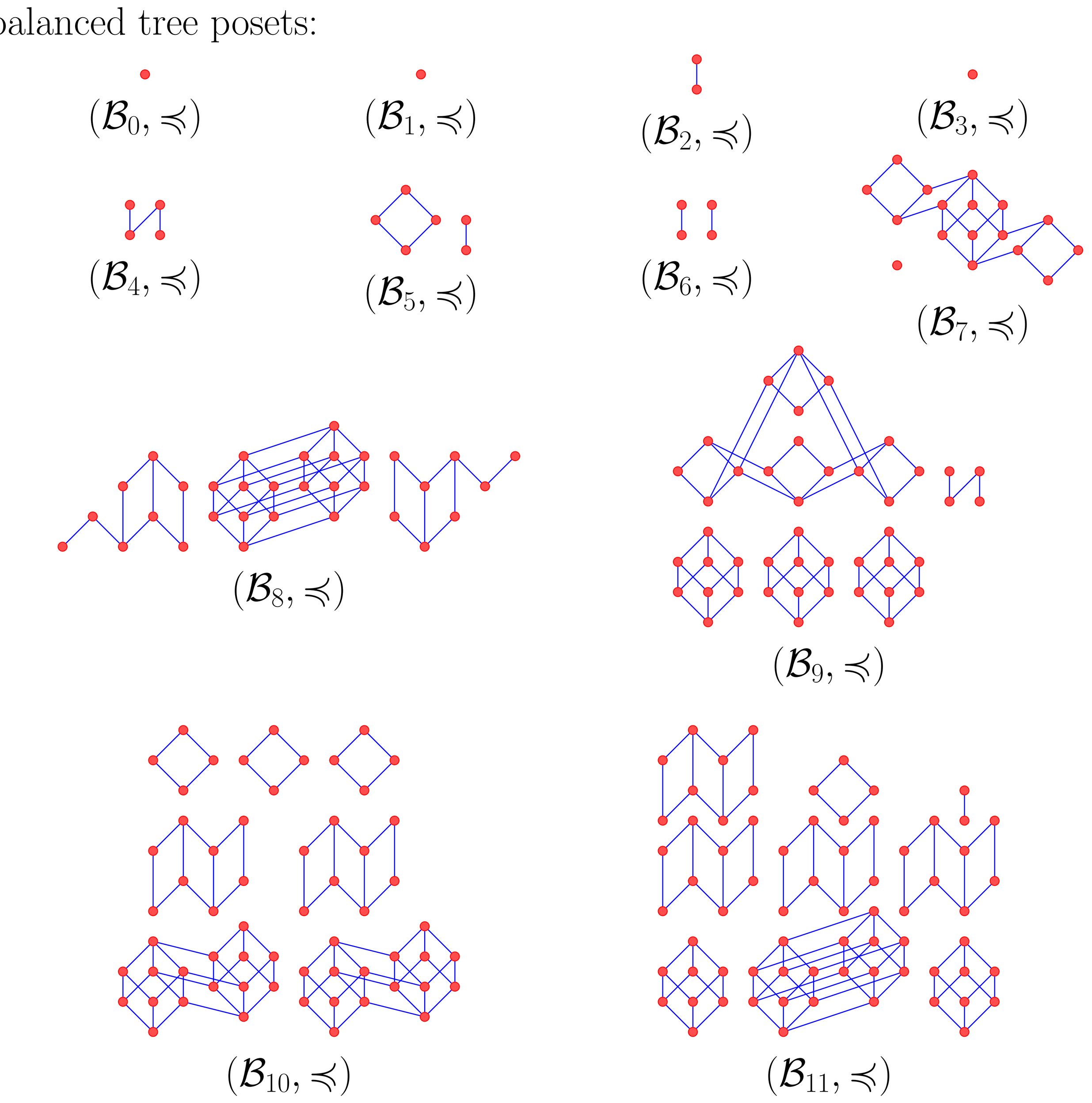
Main results and consequences

On balanced trees intervals

Theorem 1
Let T_0 and T_1 be two balanced trees such that $T_0 \preccurlyeq T_1$. Then, the Tamari interval $[T_0, T_1]$ only contains balanced trees.

Theorem 2
Let T_0 and T_1 be two balanced trees such that $T_0 \preccurlyeq T_1$. Then there exists $k \geq 0$ such that the Tamari subposet $([T_0, T_1], \preccurlyeq)$ is isomorphic to the k-dimensional hypercube.

First balanced tree posets:



Application of synchronous grammars to enumeration

From a synchronous grammar that spans objects in bijection with balanced tree intervals, we establish:

Theorem 3
The generating series enumerating balanced tree intervals in the Tamari lattice is $G_{\mathrm{inter}}(\mathbf{x}) := A(\mathbf{x}, 0, 0)$ where

$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \mathbf{x} + A(\mathbf{x}^2 + 2\mathbf{x}\mathbf{y} + \mathbf{z}, \mathbf{x}, \mathbf{x}^3 + \mathbf{x}^2\mathbf{y}).$$

First values: 1, 1, 3, 1, 7, 12, 6, 52, 119, 137, 195, 231, 1019, 3503, 6593, 12616.

Theorem 4
The generating series enumerating maximal balanced trees is $G_{\max}(x) := A(\mathbf{x}, 0, 0)$ where

$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \mathbf{x} + A(\mathbf{x}^2 + \mathbf{x}\mathbf{y} + \mathbf{y}\mathbf{z}, \mathbf{x}, \mathbf{x}\mathbf{y}).$$

First values: 1, 1, 1, 1, 2, 2, 2, 4, 6, 9, 11, 13, 22, 38, 60, 89, 128, 183, 256, 353, 512.

References

[1] G.M. Adelson-Velsky and E. M. Landis. An algorithm for the organization of information. *Soviet Mathematics Doklady, 3:1259-1263*, 1962.
[2] Samuele Giraudo. Balanced binary trees in the Tamari lattice. *Formal Power Series and Algebraic Combinatorics*, 2010.

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