LABORATOIRE D'INFORMATIQUE
GASPARD-MONGE

Sous la co-tutelle de :
CNRS
ÉCOLE DES PONTS PARISTECH
ESIEE PARIS
UPEM • UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE

Algebraic combinatorics of formal series

Samuele Giraudo

COMBI – LIGM (UMR 8049)

Combinatorial collections and enumeration

Combinatorial collections

A **combinatorial collection** is a set C wherein each element has a size and the number of elements of a given size is finite.

Primary question

Given a combinatorial collection C, **how many** objects of size n one can build?

An elementary tool

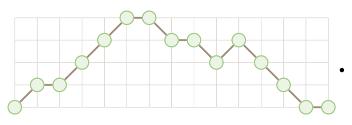
The **generating series** of a combinatorial collection C is the series

$$\mathcal{G}_C(t) = \sum_{n \geqslant 0} \# C(n)\, t^n$$

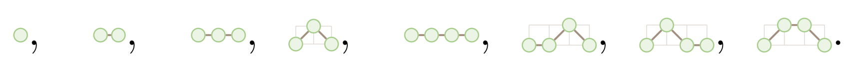
where $C(n)$ is the set of objects of size n of C.

Motzkin paths

An object of this collection M*:*



First objects:

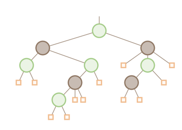


Number of objects by size (number of points):

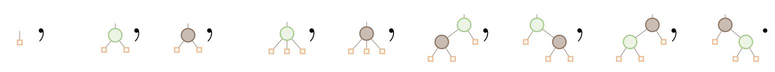
$$1, 1, 2, 4, 9, 21, 51, 127, \dots$$

Alternating Schröder trees

An object of this collection A*:*



First objects:

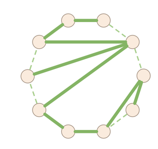


Number of objects by size (number of leaves):

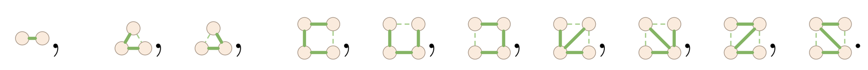
$$1, 2, 6, 22, 90, 394, 1806, 8558, \dots$$

Based noncrossing trees

An object of this collection B*:*



First objects:



Number of objects by size (number of edges):

$$1, 2, 7, 30, 143, 728, 3876, 21318, \dots$$

A toolbox from algebra

Tool 1 — Algebraic structures

Endow the linear span of C with **operations**

$$\star : \mathbb{K}\langle C\rangle^{\otimes p} \to \mathbb{K}\langle C\rangle^{\otimes q}$$

to turn C into an **algebraic structure**.

In this context, the following algebraic structures are of primary importance:

- **Hopf bialgebras** $(\mathbb{K}\langle C\rangle, \mu, \Delta)$, where μ is an associative product and Δ is a coassociative product, allowing us to iteratively assemble/disassemble the objects in a compatible way;
- **operads** $(\mathbb{K}\langle C\rangle, \circ_i)$, where $\circ_i$ is a binary composition map, allowing us to iteratively insert an object into another one.

Tool 2 — Generalized formal series

Main idea is to work in the space $\mathbb{K}\langle\langle C\rangle\rangle$ of (infinite) **formal sums** of objects of C and to describe the **characteristic series**

$$\mathbf{f}_C = \sum_{x \in C} x$$

of C, rather than its generating series. We retrieve $\mathcal{G}_C(t)$ from $\mathbf{f}_C$ as image of the linear map sending any $x \in C$ to $t^{|x|}$.

We deduce from any operation $\star$ on C having arity p and coarity q the operation

$$\bar{\star} : \mathbb{K}\langle\langle C\rangle\rangle^{\otimes p} \to \mathbb{K}\langle\langle C\rangle\rangle^{\otimes q}$$

on $\mathbb{K}\langle\langle C\rangle\rangle$ satisfying

$$\bar{\star}(\mathbf{f}_1, \dots, \mathbf{f}_p) = \sum_{x_1, \dots, x_p \in C} \prod_{i \in [p]} \langle x_i, \mathbf{f}_i\rangle \star (x_1, \dots, x_p).$$

Combinatorial consequences

Performing an algebraic study on the structure thereby deployed on C brings the following combinatorial data:

- minimal generating part
  $\rightsquigarrow$ **building blocks** forming the objects;
- relations between the generators
  $\rightsquigarrow$ **branching rules** to build objects;
- morphisms involving the structure
  $\rightsquigarrow$ transformation **algorithms** on the objects.

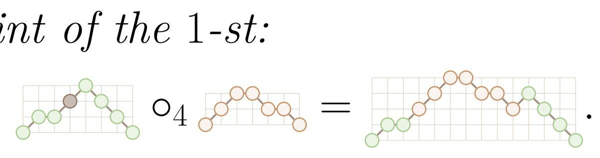
Employing the operations $\bar{\star}$ leads to

- equations describing $\mathbf{f}_C$;
- refined enumeration of C w.r.t. some statistics.

Some illustrative examples

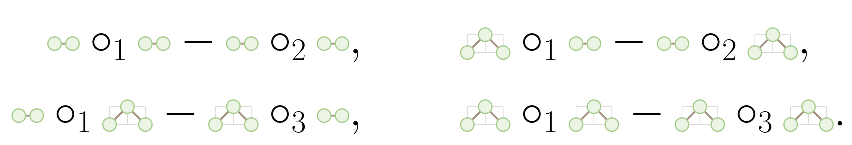
Motzkin paths

Operad structure on $\mathbb{K}\langle\mathrm{M}\rangle$*: insert the 2-nd path into a point of the 1-st:*



Minimal generating part:

Relations between the generators:



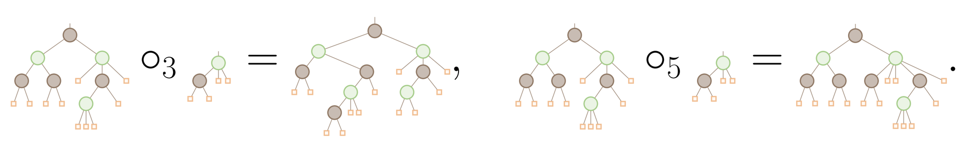
Characteristic series:

Generating series:

$$\mathcal{G}_{\mathrm{M}}(t) = t + t\mathcal{G}_{\mathrm{M}}(t) + t\mathcal{G}_{\mathrm{M}}(t)^2.$$

Alternating Schröder trees

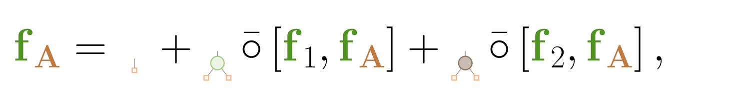
Operad structure on $\mathbb{K}\langle\mathrm{A}\rangle$*: graft the 2-nd tree on a leaf of the 1-st and contract the involved edge if the adjacents nodes have the same color:*



Minimal generating part:

Relations between the generators:

Characteristic series:



avec

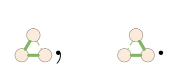
Generating series:

$$\mathcal{G}_{\mathrm{A}}(t) = \frac{t + \mathcal{G}_{\mathrm{A}}(t)^2}{1 - t}.$$

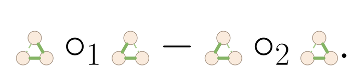
Based noncrossing trees

Operad structure on $\mathbb{K}\langle\mathrm{B}\rangle$*: glue the base of the 2-nd polygon on a side of the 1-st, where the involved diagonal of the result becomes an edge only if the considered side of the 1-st polygon in an edge:*

Minimal generating part:

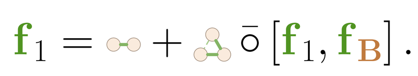


Relations between the generators:



Characteristic series:

with



Generating series:

$$\mathcal{G}_{\mathrm{B}}(t) = t + 2\mathcal{G}_{\mathrm{B}}(t)^2 - \mathcal{G}_{\mathrm{B}}(t)^3.$$

Contact

✉ samuele.giraudo@u-pem.fr