Algebraic and combinatorial structures on decorated cliques

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Operads

A nonsymmetric operad (abbreviated as operad) is a graded vector space

$$\mathcal{O} := \bigoplus_{n \geqslant 1} \mathcal{O}(n)$$

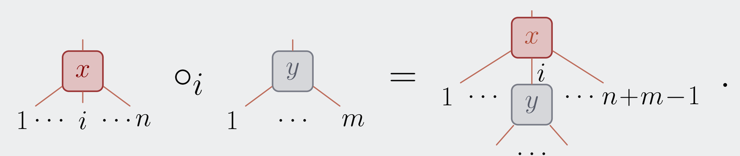
endowed with a map

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1), \qquad n, m \geqslant 1, i \in [n],$$

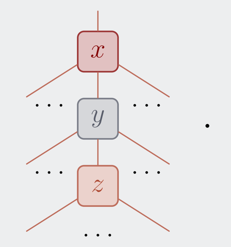
satisfying, for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $z \in \mathcal{O}$, the axioms

- for any $i \in [n]$, $j \in [m]$,
  (Asso) $\quad (x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$,
- for any $1 \leqslant i < j \leqslant n$,
  (Comm) $\quad (x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y$,
- there exists $\mathbb{1} \in \mathcal{O}(1)$ such that, for any $i \in [n]$,
  (Unit) $\quad \mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}$.

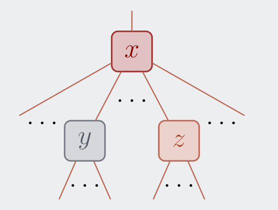
Operads provide an abstraction of the notion of operations with several inputs and their composition:



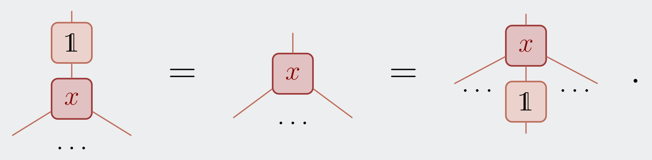
Axiom (Asso) says that there are two ways to form the operation



Axiom (Comm) says that there are two ways to form the operation



Axiom (Unit) says that the operation $\mathbb{1}$ behaves as an identity map



Some definitions:

- if $x \in \mathcal{O}(n)$, n is the arity $|x|$ of x;
- $\circ_i$ is the partial composition map of $\mathcal{O}$;
- $\mathbb{1}$ is the unit of $\mathcal{O}$;
- when all the spaces $\mathcal{O}(n)$, $n \geqslant 1$, are finite dimensional, the Hilbert series of $\mathcal{O}$ is the series
  $$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n \geqslant 1} \dim \mathcal{O}(n)\, t^n;$$
- an operad morphism is a map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ respecting the arities and commuting with the partial composition map;
- suboperads, ideals, and quotients of operads are defined in the usual algebraic way;
- a presentation of an operad $\mathcal{O}$ consists in a graded set $\mathfrak{G}_{\mathcal{O}}$ of generators and a space $\mathcal{R}_{\mathcal{O}}$ of relations between the generators.

Decorated cliques

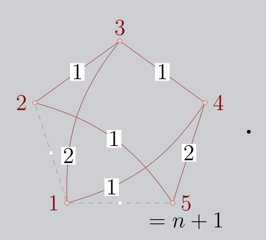
Let $(\mathcal{M}, \star, \mathbb{1}_{\mathcal{M}})$ be a unitary magma, that is a set $\mathcal{M}$ endowed with a binary operation $\star$ satisfying $a \star \mathbb{1}_{\mathcal{M}} = a = \mathbb{1}_{\mathcal{M}} \star a$.

An $\mathcal{M}$-clique $\mathfrak{p}$ of size n is a complete graph on the set $[n+1]$ of vertices such that each arc (x, y) is decorated by an element $\mathfrak{p}(x, y)$ of $\mathcal{M}$.

An arc (x, y) is solid if $\mathfrak{p}(x, y) \neq \mathbb{1}_{\mathcal{M}}$.

Example

Here is a $\mathbb{Z}/_{3\mathbb{Z}}$-clique $\mathfrak{p}$ of size 4. Only the solid arcs of $\mathfrak{p}$ are shown:

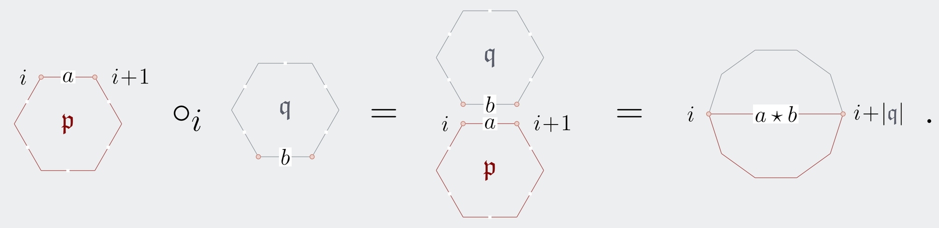


Let C be the functor from the category of unitary magmas to the category of graded vector spaces defined by

$$\mathsf{C}\mathcal{M} := \bigoplus_{n \geqslant 1} \mathsf{C}\mathcal{M}(n),$$

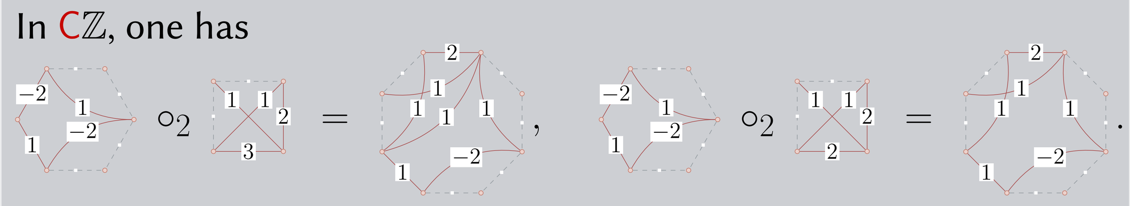
where $\mathsf{C}\mathcal{M}(1)$ is the space generated by the unique clique of size 1, and for any $n \geqslant 2$, $\mathsf{C}\mathcal{M}(n)$ is the linear span of all $\mathcal{M}$-cliques of size n.

We endow $\mathsf{C}\mathcal{M}$ with the partial composition map $\circ_i$ defined by



Example

In $\mathsf{C}\mathbb{Z}$, one has



When $\mathcal{M}$ is finite, the Hilbert series of $\mathsf{C}\mathcal{M}$ is, with $m := \#\mathcal{M}$,

$$\mathcal{H}_{\mathsf{C}\mathcal{M}}(t) = t + \sum_{n \geqslant 2} m^{\binom{n+1}{2}} t^n.$$

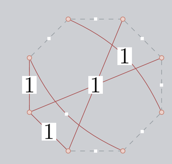
Theorem

The construction C is a functor from the category of unitary magmas to the category of nonsymmetric operads.

An $\mathcal{M}$-clique $\mathfrak{p}$ is prime if for any of its diagonal (x, y), there is a solid diagonal crossing (x, y).

Example

Here is a prime $\mathbb{Z}/_{2\mathbb{Z}}$-clique of size 7:



Proposition

The set of all prime $\mathcal{M}$-cliques is a minimal generating set of $\mathsf{C}\mathcal{M}$.

As a consequence, except when $\mathcal{M}$ is trivial, $\mathsf{C}\mathcal{M}$ is not finitely generated and has generators for any arity $n \geqslant 2$.

Substructures and quotients

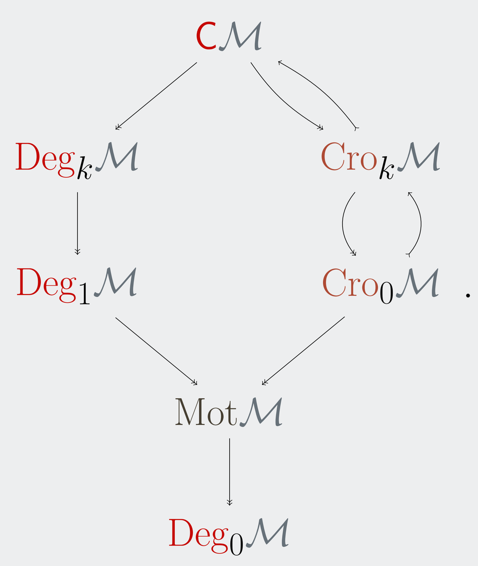
In order to construct substructures of $\mathsf{C}\mathcal{M}$ whose bases are indexed by particular $\mathcal{M}$-cliques, we use the following ideas:

Idea A. construct quotients of $\mathsf{C}\mathcal{M}$ by

1. considering a family X of $\mathcal{M}$-cliques;
2. setting $\mathcal{R}_X$ as the linear span of all the $\mathcal{M}$-cliques of $\bar{X}$;
3. when $\mathcal{R}_X$ is an operad ideal of $\mathsf{C}\mathcal{M}$, the quotient
   $$\mathsf{C}\mathcal{M}/_{\mathcal{R}_X}$$
   is a an operad on the linear span of X;

Idea B. if $\mathcal{R}_1$ and $\mathcal{R}_2$ are operad ideals of an operad $\mathcal{O}$, $\mathcal{R}_1 + \mathcal{R}_2$ is an operad ideal of $\mathcal{O}$ and $\mathcal{O}/_{\mathcal{R}_1 + \mathcal{R}_2}$ is a quotient of $\mathcal{O}$.

One obtains (among others) the operads fitting in the following diagram (arrows ↣ (resp. ↠) are injective (resp. surjective) morphisms of operads):



The degree $\deg(\mathfrak{p})$ of an $\mathcal{M}$-clique $\mathfrak{p}$ is the greatest degree of among the vertices of the graph formed by the solid arcs of $\mathfrak{p}$.

Let $\mathcal{R}_{\mathrm{Deg}_k\mathcal{M}}$ be the linear span of all $\mathcal{M}$-cliques of degrees greater than k. As a quotient space,

$$\mathrm{Deg}_k\mathcal{M} := \mathsf{C}\mathcal{M}/_{\mathrm{Deg}_k\mathcal{M}}$$

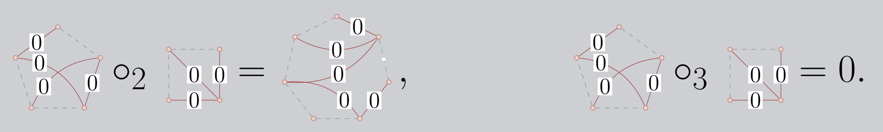
is the linear span of all $\mathcal{M}$-cliques of degrees at most k.

Proposition

When $\mathcal{M}$ has no nontrivial unit divisors, $\mathrm{Deg}_k\mathcal{M}$ is a quotient operad of $\mathsf{C}\mathcal{M}$.

Example

For $\mathbb{D} := (\{1, 0\}, \times, 1)$, one has in $\mathrm{Deg}_3\mathbb{D}$,



The crossing $\mathrm{cro}(\mathfrak{p})$ of an $\mathcal{M}$-clique $\mathfrak{p}$ is the greatest number of solid diagonals crossing (x, y) among all solid diagonals (x, y) of $\mathfrak{p}$.

Let $\mathcal{R}_{\mathrm{Cro}_k\mathcal{M}}$ be the linear span of all $\mathcal{M}$-cliques of crossing greater than k. As a quotient space,

$$\mathrm{Cro}_k\mathcal{M} := \mathsf{C}\mathcal{M}/_{\mathrm{Cro}_k\mathcal{M}}$$

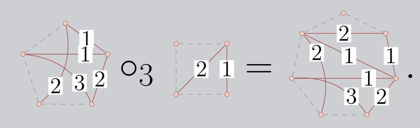
is the linear span of all $\mathcal{M}$-cliques of crossings at most k.

Proposition

The space $\mathrm{Cro}_k\mathcal{M}$ is both a suboperad and a quotient operad of $\mathsf{C}\mathcal{M}$.

Example

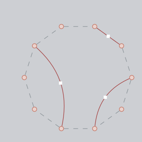
In $\mathrm{Cro}_2\mathbb{Z}/_{4\mathbb{Z}}$, one has



A Motzkin configuration is a diagram with only noncrossing and disjoint arcs. In other terms, a $\mathbb{D}$-clique is a Motzkin configuration if its degree is 0 or 1, and its crossing is 0.

Example

The $\mathbb{D}$-clique



is a Motzkin configuration of size 9.

Proposition

The space

$$\mathrm{Mot}\mathbb{D} := \mathsf{C}\mathbb{D}/_{\mathcal{R}_{\mathrm{Cro}_0\mathbb{D}} + \mathcal{R}_{\mathrm{Deg}_1\mathbb{D}}}$$

is a quotient operad of $\mathsf{C}\mathbb{D}$ whose bases are indexed by Motzkin configurations.

A bunch of other similar structures fits as substructures of $\mathsf{C}\mathcal{M}$, as *e.g.*, operads on bubbles, acyclic graphs, nesting free diagrams, forests of trees, dissections of polygons, and Lucas configurations.

Noncrossing decorated cliques

Let $\mathsf{NC}\mathcal{M} := \mathrm{Cro}_0\mathcal{M}$ be the operad of noncrossing configurations.

Proposition

The set

$$\mathfrak{G}_{\mathsf{NC}\mathcal{M}} := \left\{ \triangle_{\mathfrak{p}_0}^{\mathfrak{p}_1 \mathfrak{p}_2} : \mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{M} \right\}$$

of all $\mathcal{M}$-triangles is a minimal generating set of $\mathsf{NC}\mathcal{M}$.

Hence, unlike $\mathsf{C}\mathcal{M}$, $\mathsf{NC}\mathcal{M}$ is a binary operad. Moreover, $\mathsf{NC}\mathcal{M}$ admits two particular properties. Indeed, $\mathsf{NC}\mathcal{M}$ is

1. the smallest suboperad of $\mathsf{C}\mathcal{M}$ containing all the $\mathcal{M}$-triangles;
2. the biggest binary suboperad of $\mathsf{C}\mathcal{M}$.

Proposition

When $\mathcal{M}$ is finite, the Hilbert series of $\mathsf{NC}\mathcal{M}$ satisfies

$$t + (m^3 - 2m^2 + 2m - 1)\, t^2 + (2m^2 t - 3mt + 2t - 1)\, \mathcal{H}_{\mathsf{NC}\mathcal{M}}(t) + (m-1)\, \mathcal{H}_{\mathsf{NC}\mathcal{M}}(t)^2 = 0,$$

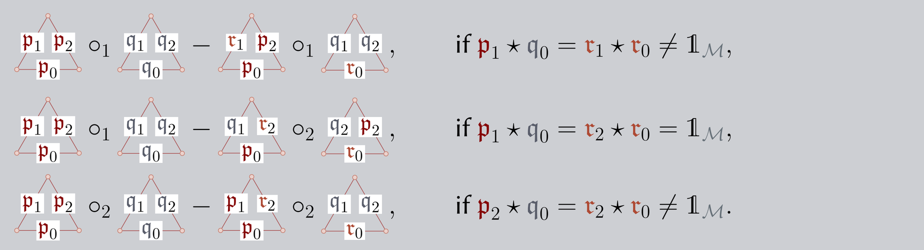
where $m := \#\mathcal{M}$.

Moreover, when $\mathcal{M}$ is finite, for all $n \geqslant 2$,

$$\dim \mathsf{NC}\mathcal{M}(n) = \sum_{0 \leqslant k \leqslant n-2} m^{n+k+1} (m-1)^{n-k-2} \frac{1}{k+1} \binom{n-2}{k} \binom{n-1}{k}.$$

Theorem

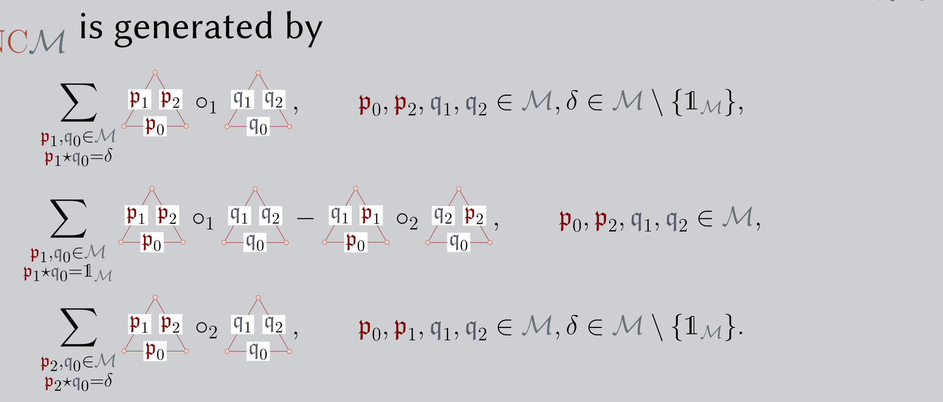
When $\mathcal{M}$ is finite, $\mathsf{NC}\mathcal{M}$ admits the presentation $(\mathfrak{G}_{\mathsf{NC}\mathcal{M}}, \mathcal{R}_{\mathsf{NC}\mathcal{M}})$ were $\mathcal{R}_{\mathsf{NC}\mathcal{M}}$ is generated by



Hence, $\mathsf{NC}\mathcal{M}$ is binary and quadratic. For this reason, $\mathsf{NC}\mathcal{M}$ admits a Koszul dual $\mathsf{NC}\mathcal{M}^!$.

Proposition

When $\mathcal{M}$ is finite, $\mathsf{NC}\mathcal{M}!$ admits the presentation $(\mathfrak{G}_{\mathsf{NC}\mathcal{M}}, \mathcal{R}^!_{\mathsf{NC}\mathcal{M}})$ where $\mathcal{R}^!_{\mathsf{NC}\mathcal{M}}$ is generated by



Proposition

When $\mathcal{M}$ is finite, the Hilbert series of $\mathsf{NC}\mathcal{M}^!$ satisfies

$$t + (m-1)t^2 + (2m^2 t - 3mt + 2t - 1)\, \mathcal{H}_{\mathsf{NC}\mathcal{M}^!}(t) + (m^3 - 2m^2 + 2m - 1)\, \mathcal{H}_{\mathsf{NC}\mathcal{M}^!}(t)^2 = 0,$$

where $m := \#\mathcal{M}$.

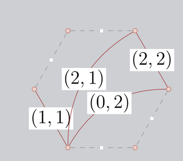
Moreover, when $\mathcal{M}$ is finite, for all $n \geqslant 2$,

$$\dim \mathsf{NC}\mathcal{M}^!(n) = \sum_{0 \leqslant k \leqslant n-2} m^{n+1} (m(m-1)+1)^k (m(m-1))^{n-k-2} \frac{1}{k+1} \binom{n-2}{k} \binom{n-1}{k}.$$

A dual $\mathcal{M}$-clique is an $\mathcal{M}^2$-clique such that all edges are decorated by pairs $(a, a) \in \mathcal{M}^2$, and all solid diagonals by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$.

Example

The $\mathbb{Z}^2$-clique



is a dual $\mathbb{Z}$-clique.

Proposition

When $\mathcal{M}$ is finite, the bases of $\mathsf{NC}\mathcal{M}^!$ are indexed by dual $\mathcal{M}$-cliques.

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